Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **51**, 1–10; doi: 10.14232/ejqtde.2016.1.51 http://www.math.u-szeged.hu/ejqtde/

Oscillation criteria for neutral half-linear differential equations without commutativity in deviating arguments

Simona Fišnarová ✉

Department of Mathematics, Mendel University in Brno,
Zemědělská 1, Brno, CZ–613 00, Czech Republic

Received 16 March 2016, appeared 15 July 2016
Communicated by Michal Fečkan

Abstract. We study the half-linear neutral differential equation

$$\Big[r(t)\Phi(z'(t))\Big]' + c(t)\Phi(x(\sigma(t))) = 0, \qquad z(t) = x(t) + b(t)x(\tau(t)),$$

where $\Phi(t) = |t|^{p-2}t$. We present new oscillation criteria for this equation in case when $\sigma(\tau(t)) \neq \tau(\sigma(t))$ and $\int^\infty r^{1-q}(t)\mathrm{d}t < \infty$, $q = p/(p-1)$, $p \geq 2$ is a real number. The results of this paper complement our previous results in case when the above integral is divergent and/or the deviations τ, σ commute with respect to their composition.

Keywords: half-linear differential equation, delay equation, oscillation criteria, neutral equation.

2010 Mathematics Subject Classification: 34K11, 34K40, 34C10.

1 Introduction

In this paper we study the second order half-linear neutral differential equation

$$\Big[r(t)\Phi(z'(t))\Big]' + c(t)\Phi(x(\sigma(t))) = 0, \qquad z(t) = x(t) + b(t)x(\tau(t)), \tag{1.1}$$

where $\Phi(t) = |t|^{p-2}t$, $p \geq 2$.

We suppose that the coefficients r, c and b satisfy the conditions $r \in C([t_0,\infty),\mathbb{R}^+)$, $c \in C([t_0,\infty),\mathbb{R}^+)$, and $b \in C^1([t_0,\infty),\mathbb{R}_0^+)$, $b(t) \leq b_0$ for some $b_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. Further we suppose that the deviating arguments are increasing, unbounded and sufficiently smooth, i.e., $\tau \in C^2([t_0,\infty),\mathbb{R})$, $\tau'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \infty$, $\sigma \in C^1([t_0,\infty),\mathbb{R})$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\sigma(t) = \infty$. Finally, by q we mean the conjugate number to p, $q = \frac{p}{p-1}$.

By the solution of (1.1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi(z'(t))$ is differentiable and (1.1) holds for

✉ Email: fisnarov@mendelu.cz

large t. Equation (1.1) is said to be *oscillatory* if it does not have a solution which is eventually positive or negative (i.e., it does not have a zero for large t).

The criteria presented in this paper are derived using the so called comparison method which is based on comparison of the studied neutral second order equation with a certain linear first order delay or advanced differential equation or inequality. The method has been frequently used in oscillation theory of the second order neutral equations, see e.g. [2–6, 8–11] and the references therein. In most of the papers equation (1.1) has been studied under the condition

$$\int^{\infty} r^{1-q}(t)\mathrm{d}t = \infty. \tag{1.2}$$

The reason is that in this case the eventually positive solutions of (1.1) behave such that the corresponding function z is increasing (more precisely, all eventually positive solutions satisfy condition (2.1)) in contrast to the case when the above integral is convergent and the function z associated to an eventually positive solution can be either increasing or decreasing, see Lemma 2.1 below. Note that in the commutative case

$$\sigma(\tau(t)) = \tau(\sigma(t)) \tag{1.3}$$

some oscillation criteria for (1.1) have been obtained using the comparison method under the condition

$$\int^{\infty} r^{1-q}(t)\mathrm{d}t < \infty, \tag{1.4}$$

see [6, 10]. Comparing results of those papers, in [6] we have used a refined version of the comparison method, which enabled us to obtain better oscillation criteria then those in [10]. This improved method has been then adjusted for the non-commutative case in [8], where we studied (1.1) under the condition (1.2). Note also that this kind of improvement has been used for the first time in our paper [7], where equation (1.1) has been studied using the Riccati method.

In this paper we study the complementary case – we study equation (1.1) under condition (1.4) and we suppose that the condition on commutativity (1.3) is broken. This means that we extend the present results in two directions – we extend results from [6] to non-commutative case and, at the same time, we extend results from [8] to the case when (1.4) holds. We use the above mentioned refinement of the comparison method, which is based on introducing new parameters in estimates and inequalities which are then used in the proofs of the oscillation criteria, see ε in Lemma 2.2 and Lemma 2.3 and also φ in (1.5) below and compare with the method used e.g. in [5, 10], where $\varepsilon = \frac{1}{2}$ and $\varphi = 1$.

As a main result of this paper we prove a version of the following statement from [8], where we replace condition (1.2) by condition (1.4).

Define

$$Q(t,\varphi) := \min\{c(\sigma^{-1}(t)), \varphi c(\sigma^{-1}(\tau(t)))\}. \tag{1.5}$$

Theorem A. *Suppose that* $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$ *and condition* (1.2) *holds. Let* φ *be an arbitrary positive real number and* $\eta(t) \leq t$ *be a smooth increasing function which satisfies* $\lim_{t\to\infty} \eta(t) = \infty$ *and one of the following conditions be satisfied:*

(i) $\sigma(\eta(t)) < \tau(t) \leq t$ *and*

$$\liminf_{T\to\infty} \int_{\tau^{-1}(\sigma(\eta(T)))}^{T} Q(t,\varphi) \left(\int_{t_1}^{\eta(t)} r^{1-q}(s)\mathrm{d}s\right)^{p-1} \mathrm{d}t > \frac{1}{e\sigma_0}\left(1 + b_0\left(\frac{\varphi}{\tau_0}\right)^{q-1}\right)^{p-1}, \tag{1.6}$$

(ii) $\sigma(\eta(t)) < t \le \tau(t)$ *and*

$$\liminf_{T\to\infty} \int_{\sigma(\eta(T))}^{T} Q(t,\varphi) \left(\int_{t_1}^{\eta(t)} r^{1-q}(s)\mathrm{d}s \right)^{p-1} \mathrm{d}t > \frac{1}{e\sigma_0} \left(1 + b_0 \left(\frac{\varphi}{\tau_0} \right)^{q-1} \right)^{p-1}. \tag{1.7}$$

Then equation (1.1) *does not have an eventually positive solution, i.e., is oscillatory.*

The paper is organized as follows. In the next section we present the preliminary results, Section 3 contains the main results, i.e., oscillation criteria for (1.1) and in the last section we show how the obtained results can be applied to the Euler-type equation.

2 Preliminary statements

In this section we present some preliminary results which are used in the proofs of the main results. Note that every inequality is assumed to be valid eventually, if not stated explicitly otherwise.

The following lemma can be found e.g. in [6].

Lemma 2.1. *Let* (1.4) *hold. If* $x(t)$ *is an eventually positive solution of* (1.1)*, then the corresponding function* $z(t) = x(t) + b(t)x(\tau(t))$ *satisfies either*

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\Phi(z'(t)) \right)' < 0 \tag{2.1}$$

or

$$z(t) > 0, \quad z'(t) < 0, \quad \left(r(t)\Phi(z'(t)) \right)' < 0 \tag{2.2}$$

eventually.

The next two lemmas can be found in [8].

Lemma 2.2. *Let* $\varepsilon \in (0,1)$*. Then*

$$\varepsilon^{2-p}\Phi(x) + (1-\varepsilon)^{2-p}\Phi(y) \ge \Phi(x+y).$$

Lemma 2.3. *For* $\alpha > 0$ *we have*

$$\min_{\varepsilon\in(0,1)} \left\{ \varepsilon^{2-p} + \alpha(1-\varepsilon)^{2-p} \right\} = \left(1 + \alpha^{q-1} \right)^{p-1}.$$

The last statement of this section is a criterion for the first order advanced inequality which appears in the proofs of our main results and is compared with (1.1). The proof can be found in [1, Lemma 2.2.10].

Lemma 2.4. *Let* $q(t) \ge 0$, $\sigma(t) > t$ *and*

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)\mathrm{d}s > \frac{1}{e}.$$

Then the inequality

$$y'(t) - q(t)y(\sigma(t)) \ge 0$$

has no eventually positive solution.

3 Oscillation criteria

In the following statement we give sufficient conditions for nonexistence of eventually positive solutions satisfying (2.2).

Theorem 3.1. *Suppose that $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$ and condition (1.4) holds. Let φ be an arbitrary positive real number and $\zeta(t) \geq t$ be a smooth increasing function satisfying $\lim_{t\to\infty} \zeta(t) = \infty$ and one of the following conditions be satisfied:*

(i) $\tau(t) \leq t < \sigma(\zeta(t))$ *and*

$$\liminf_{T\to\infty} \int_T^{\sigma(\zeta(T))} Q(t,\varphi) \left(\int_{\zeta(t)}^{\infty} r^{1-q}(s) \mathrm{d}s \right)^{p-1} \mathrm{d}t > \frac{1}{e\sigma_0} \left(1 + b_0 \left(\frac{\varphi}{\tau_0} \right)^{q-1} \right)^{p-1}, \tag{3.1}$$

(ii) $t \leq \tau(t) < \sigma(\zeta(t))$ *and*

$$\liminf_{T\to\infty} \int_T^{\tau^{-1}(\sigma(\zeta(T)))} Q(t,\varphi) \left(\int_{\zeta(t)}^{\infty} r^{1-q}(s) \mathrm{d}s \right)^{p-1} \mathrm{d}t > \frac{1}{e\sigma_0} \left(1 + b_0 \left(\frac{\varphi}{\tau_0} \right)^{q-1} \right)^{p-1}. \tag{3.2}$$

Then equation (1.1) does not have an eventually positive solution such that $z'(t) < 0$.

Proof. By contradiction, suppose that x is an eventually positive solution of (1.1) satisfying condition (2.2). Shifting equation (1.1) from t to $\sigma^{-1}(t)$ and $\sigma^{-1}(\tau(t))$, respectively, and using conditions $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$, we obtain the following inequalities:

$$\frac{1}{\sigma_0} \left[r(\sigma^{-1}(t)) \Phi(z'(\sigma^{-1}(t))) \right]' + c(\sigma^{-1}(t)) \Phi(x(t)) \leq 0, \tag{3.3}$$

$$\frac{1}{\sigma_0 \tau_0} \left[r(\sigma^{-1}(\tau(t))) \Phi(z'(\sigma^{-1}(\tau(t)))) \right]' + c(\sigma^{-1}(\tau(t))) \Phi(x(\tau(t))) \leq 0. \tag{3.4}$$

Denote $w(t) = r(t)\Phi(z'(t))$ and take the linear combination of inequalities (3.3) and (3.4) with the coefficients ε^{2-p} and $b_0^{p-1}\varphi(1-\varepsilon)^{2-p}$, where $\varepsilon \in (0,1)$. Then

$$\begin{aligned} &\left[\frac{\varepsilon^{2-p}}{\sigma_0} w(\sigma^{-1}(t)) + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0} w(\sigma^{-1}(\tau(t))) \right]' \\ &\qquad + \varepsilon^{2-p} c(\sigma^{-1}(t))\Phi(x(t)) + b_0^{p-1}\varphi(1-\varepsilon)^{2-p} c(\sigma^{-1}(\tau(t)))\Phi(x(\tau(t))) \leq 0 \end{aligned}$$

and consequently, using the definition of $Q(t,\varphi)$ and Lemma 2.2 we have

$$\left[\frac{\varepsilon^{2-p}}{\sigma_0} w(\sigma^{-1}(t)) + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0} w(\sigma^{-1}(\tau(t))) \right]' + Q(t,\varphi)\Phi(z(t)) \leq 0.$$

Next, since $\zeta(t) \geq t$ and since z is decreasing, we obtain

$$\left[\frac{\varepsilon^{2-p}}{\sigma_0} w(\sigma^{-1}(t)) + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0} w(\sigma^{-1}(\tau(t))) \right]' + Q(t,\varphi)\Phi(z(\zeta(t))) \leq 0. \tag{3.5}$$

Since w is decreasing, we have from definition of w:

$$z'(s) \leq \Phi^{-1}(r(t)) z'(t) r^{1-q}(s) \quad \text{for } s \geq t.$$

Integrating this inequality from t to T, letting $T \to \infty$ and since $z(T) > 0$ we obtain

$$-z(t) \le \Phi^{-1}(r(t))z'(t) \int_t^\infty r^{1-q}(s)\mathrm{d}s.$$

Shifting t to $\zeta(t)$, we have

$$-z(\zeta(t)) \le \Phi^{-1}(r(\zeta(t)))z'(\zeta(t)) \int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s. \tag{3.6}$$

Combining inequalities (3.5) and (3.6) and using the notation $u(t) = -w(t)$, we obtain

$$\left[\frac{\varepsilon^{2-p}}{\sigma_0}u(\sigma^{-1}(t)) + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}u(\sigma^{-1}(\tau(t)))\right]' - Q(t,\varphi)\left(\int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s\right)^{p-1} u(\zeta(t)) \ge 0. \tag{3.7}$$

Denote

$$y(t) = \frac{\varepsilon^{2-p}}{\sigma_0}u(\sigma^{-1}(t)) + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}u(\sigma^{-1}(\tau(t))). \tag{3.8}$$

Now we distinguish cases (i) and (ii) of the theorem.

Suppose that (i) holds. Since $\tau(t) \le t$ and σ and u are increasing, we have $u(\sigma^{-1}(\tau(t))) \le u(\sigma^{-1}(t))$. Hence, from (3.8)

$$y(t) \le \left(\frac{\varepsilon^{2-p}}{\sigma_0} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}\right) u(\sigma^{-1}(t)).$$

Replacing t with $\sigma(\zeta(t))$ in the last inequality we obtain

$$y(\sigma(\zeta(t))) \le \left(\frac{\varepsilon^{2-p}}{\sigma_0} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}\right) u(\zeta(t)). \tag{3.9}$$

Substituting $u(\zeta(t))$ from (3.9) to (3.7) we find that u is a positive solution of the inequality

$$y'(t) - Q(t,\varphi)\left(\int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s\right)^{p-1} \sigma_0 \left(\varepsilon^{2-p} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\tau_0}\right)^{-1} y(\sigma(\zeta(t))) \ge 0. \tag{3.10}$$

On the other hand, since $\sigma(\zeta(t)) > t$, condition (3.1), Lemma 2.3 and Lemma 2.4 imply that (3.10) has no positive solution. We have a contradiction. Statement (i) is proved.

Suppose that (ii) holds. Since $\tau(t) \ge t$, we have $u(\sigma^{-1}(\tau(t))) \ge u(\sigma^{-1}(t))$. Hence

$$y(t) \le \left(\frac{\varepsilon^{2-p}}{\sigma_0} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}\right) u(\sigma^{-1}(\tau(t))).$$

Replacing t with $\tau^{-1}(\sigma(\zeta(t)))$ in the last inequality we obtain

$$y(\tau^{-1}(\sigma(\zeta(t)))) \le \left(\frac{\varepsilon^{2-p}}{\sigma_0} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\sigma_0\tau_0}\right) u(\zeta(t)) \tag{3.11}$$

and substituting $u(\zeta(t))$ from (3.11) to (3.7) we find that u is a positive solution of the inequality

$$y'(t) - Q(t,\varphi)\left(\int_{\zeta(t)}^{\infty} r^{1-q}(s)\mathrm{d}s\right)^{p-1}\sigma_0\left(\varepsilon^{2-p} + \frac{b_0^{p-1}\varphi(1-\varepsilon)^{2-p}}{\tau_0}\right)^{-1} y(\tau^{-1}(\sigma(\zeta(t)))) \geq 0. \tag{3.12}$$

Since $\tau^{-1}(\sigma(\zeta(t))) > t$, by Lemma 2.3, Lemma 2.4 and condition (3.2) we have contradiction with the existence a positive solution of (3.12). Statement (ii) is proved. □

Remark 3.2. The proof of Theorem 3.1 is based on comparing equation (1.1) with first order inequalities (3.10) and (3.12) and then the particular criterion from Lemma 2.4 is applied to this first order inequalities to obtain conditions (3.1) and (3.2). The statement can be formulated as a more general comparison result as follows.

(i) If $\tau(t) \leq t$ and (3.10) does not have an eventually positive solution, then (1.1) does not have an eventually positive solution such that $z'(t) < 0$.

(ii) If $\tau(t) \geq t$ and (3.12) does not have an eventually positive solution, then (1.1) does not have an eventually positive solution such that $z'(t) < 0$.

Note that if (1.2) holds, then all eventually positive solutions of (1.1) satisfy condition (2.1) from Lemma 2.1, see, e.g. [8, Lemma 3]. This is used in the proof of Theorem A and it is the only reason, why condition (1.2) is used in Theorem A. This means that under the conditions of Theorem A, where we replace condition (1.2) by condition (1.4), equation (1.1) does not have an eventually positive solution such that $z'(t) > 0$. Hence, combining Lemma 2.1, Theorem A and Theorem 3.1 and the fact that equation (1.1) is homogeneous (from which it follows that if it does not have an eventually positive solution, it also does not have an eventually negative solution), we can formulate the following oscillation criterion.

Theorem 3.3. *Suppose that $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$ and condition (1.4) holds. Let φ be an arbitrary positive real number, $\eta(t) \leq t$ and $\zeta(t) \geq t$ be smooth increasing functions which satisfy $\lim_{t\to\infty}\eta(t) = \infty$, $\lim_{t\to\infty}\zeta(t) = \infty$ and one of the following conditions be satisfied:*

(i) $\sigma(\eta(t)) < \tau(t) \leq t < \sigma(\zeta(t))$ and both conditions (1.6), (3.1) hold.

(ii) $\sigma(\eta(t)) < t \leq \tau(t) < \sigma(\zeta(t))$ and both conditions (1.7), (3.2) hold.

Then equation (1.1) is oscillatory.

4 Euler-type equation

In the following we apply the results to the Euler-type equation of the form

$$\left[t^{\alpha}\Phi(z'(t))\right]' + \frac{\gamma}{t^{p-\alpha}}\Phi(x(\sigma(t))) = 0, \qquad z(t) = x(t) + b(t)x(\tau(t)), \tag{4.1}$$

where $\sigma(t) = \lambda_1 t + \lambda_2$, $\tau(t) = \lambda_3 t + \lambda_4$, the coefficients λ_1, λ_2, λ_3, λ_4 are real numbers such that $\lambda_1 > 0$, $\lambda_3 > 0$ and $b(t) \leq b_0$.

If $\alpha < p-1$, then condition (1.2) holds for this equation and, as a consequence of Theorem A, we have proved in [8] that (4.1) oscillates if

$$\gamma > \frac{\left(1+b_0\lambda_3^{(p-\alpha-1)(q-1)}\right)^{p-1}(\alpha(1-q)-1)^{p-1}}{e\max\limits_{\lambda\in J}(\lambda_1\lambda)^{p-\alpha-1}\ln\frac{\min\{1,\lambda_3\}}{\lambda_1\lambda}},$$

where $J=\{\lambda\in(0,1]:\lambda_1\lambda<\min\{1,\lambda_3\}\}$ and either

$$\lambda_1\lambda<\lambda_3\le 1 \quad \text{with } \lambda_4\le 0 \text{ if } \lambda_3=1$$

or

$$\lambda_1\lambda<1\le\lambda_3 \quad \text{with } \lambda_4\ge 0 \text{ if } \lambda_3=1.$$

If $\alpha > p-1$, then condition (1.4) holds and we obtain the following result.

Corollary 4.1. *Let $b(t)\le b_0$, $\alpha>p-1$ and put $\sigma(t)=\lambda_1 t+\lambda_2$, $\tau(t)=\lambda_3 t+\lambda_4$, where λ_1, λ_2, λ_3, λ_4 are real numbers such that $\lambda_1>0$, $\lambda_3>0$. Put $\bar J=\{\bar\lambda\ge 1:\lambda_1\bar\lambda>\max\{1,\lambda_3\}\}$ and suppose that either*

$$\lambda_3\le 1<\lambda_1\bar\lambda \quad \textit{with } \lambda_4\le 0 \textit{ if } \lambda_3=1 \tag{4.2}$$

or

$$1\le\lambda_3<\lambda_1\bar\lambda \quad \textit{with } \lambda_4\ge 0 \textit{ if } \lambda_3=1. \tag{4.3}$$

If

$$\gamma > \frac{\left(1+b_0\lambda_3^{(p-\alpha-1)(q-1)}\right)^{p-1}(\alpha(q-1)-1)^{p-1}}{e\max\limits_{\bar\lambda\in\bar J}(\lambda_1\bar\lambda)^{p-\alpha-1}\ln\frac{\lambda_1\bar\lambda}{\max\{1,\lambda_3\}}}, \tag{4.4}$$

then equation (4.1) is oscillatory.

Proof. We apply Theorem 3.3 with $\eta(t)=\lambda t$, $\lambda\in(0,1]$ and $\zeta(t)=\bar\lambda t$, $\bar\lambda\ge 1$. First we deal with the case $z'(t)<0$, i.e., we apply Theorem 3.1. Since $\sigma(\zeta(t))=\lambda_1\bar\lambda t+\lambda_2$, we have the following conditions on τ and σ:

$$\lambda_3 t+\lambda_4\le t<\lambda_1\bar\lambda t+\lambda_2 \quad \text{in case (i)}$$

or

$$t\le\lambda_3 t+\lambda_4<\lambda_1\bar\lambda t+\lambda_2 \quad \text{in case (ii),}$$

i.e., conditions (4.2), (4.3). Both this conditions give $\lambda_1\bar\lambda>\max\{1,\lambda_3\}$ and note that the case $\lambda_1\bar\lambda=\max\{1,\lambda_3\}$ is excluded because of the logarithmic term in (4.4). Condition (4.4) implies that there exist $\bar\lambda\in\bar J$ and $\varepsilon>0$ such that $\varepsilon<\lambda_3$ and

$$\gamma > \frac{\left(1+b_0\left(\frac{(\lambda_3+\delta)^{p-\alpha}}{\lambda_3}\right)^{q-1}\right)^{p-1}(\alpha(q-1)-1)^{p-1}}{e(\lambda_1\bar\lambda)^{p-\alpha-1}\ln\frac{\lambda_1\bar\lambda}{\max\{1,\lambda_3\}}}(1+\varepsilon), \tag{4.5}$$

where

$$\delta=\begin{cases}\varepsilon & \text{if } p-\alpha\ge 0\\ -\varepsilon & \text{if } p-\alpha<0.\end{cases}$$

We have $r(t) = t^\alpha$, $c(t) = \frac{\gamma}{t^{p-\alpha}}$, $\tau^{-1}(t) = \frac{t-\lambda_4}{\lambda_3}$, $\sigma^{-1}(t) = \frac{t-\lambda_2}{\lambda_1}$. Hence we take $\tau_0 = \lambda_3$, $\sigma_0 = \frac{1}{\lambda_1}$. Consequently, $c(\sigma^{-1}(t)) = \gamma\big(\frac{\lambda_1}{t-\lambda_2}\big)^{p-\alpha}$, $c(\sigma^{-1}\tau((t))) = \gamma\big(\frac{\lambda_1}{\lambda_3 t+\lambda_4-\lambda_2}\big)^{p-\alpha}$, hence

$$\frac{c(\sigma^{-1}(t))}{c(\sigma^{-1}\tau((t)))} = \left(\lambda_3 + \frac{\lambda_4 + \lambda_2(\lambda_3-1)}{t-\lambda_2}\right)^{p-\alpha} \le (\lambda_3+\delta)^{p-\alpha}$$

for sufficiently large t. We take $\varphi = (\lambda_3+\delta)^{p-\alpha}$ and from (1.5) we have $Q(t,\varphi) = \gamma\big(\frac{\lambda_1}{t-\lambda_2}\big)^{p-\alpha}$. Next,

$$\begin{aligned}
Q(t,\varphi)\left(\int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s\right)^{p-1} &= \gamma\left(\frac{\lambda_1}{t-\lambda_2}\right)^{p-\alpha}\left(\int_{\bar\lambda t}^\infty s^{\alpha(1-q)}\mathrm{d}s\right)^{p-1}\\
&= \gamma\left(\frac{\lambda_1}{t-\lambda_2}\right)^{p-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}(\bar\lambda t)^{[\alpha(1-q)+1](p-1)}\\
&= \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{t}\left(\frac{t}{t-\lambda_2}\right)^{p-\alpha}\\
&> \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{t}\frac{1}{1+\varepsilon}
\end{aligned}$$

for sufficiently large t. The left-hand side of (3.1) satisfies then

$$\begin{aligned}
&\liminf_{T\to\infty}\int_T^{\sigma(\zeta(T))} Q(t,\varphi)\left(\int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s\right)^{p-1}\mathrm{d}t\\
&\quad> \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\liminf_{T\to\infty}\left[\ln(\lambda_1\bar\lambda T+\lambda_2)-\ln T\right]\\
&\quad= \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\ln(\lambda_1\bar\lambda).
\end{aligned}$$

Similarly, since $\tau^{-1}(\sigma(\zeta(T))) = \frac{\lambda_1\bar\lambda t+\lambda_2-\lambda_4}{\lambda_3}$, the left-hand side of (3.2) satisfies

$$\begin{aligned}
&\liminf_{T\to\infty}\int_T^{\tau^{-1}(\sigma(\zeta(T)))} Q(t,\varphi)\left(\int_{\zeta(t)}^\infty r^{1-q}(s)\mathrm{d}s\right)^{p-1}\mathrm{d}t\\
&\quad> \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\liminf_{T\to\infty}\left[\ln\left(\frac{\lambda_1\bar\lambda T+\lambda_2-\lambda_4}{\lambda_3}\right)-\ln T\right]\\
&\quad= \gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\ln\left(\frac{\lambda_1\bar\lambda}{\lambda_3}\right).
\end{aligned}$$

Conditions (3.1) and (3.2) together with (4.2) and (4.3) give

$$\begin{aligned}
&\gamma\lambda_1^{p-\alpha}\bar\lambda^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\ln\left(\frac{\lambda_1\bar\lambda}{\max\{1,\lambda_3\}}\right)\\
&\quad> \frac{\lambda_1}{e}\left(1+b_0\left(\frac{(\lambda_3+\delta)^{p-\alpha}}{\lambda_3}\right)^{q-1}\right)^{p-1},
\end{aligned}$$

which is equivalent with (4.5). Since (4.5) is guaranteed by (4.4), we have proved that under the conditions of the corollary, equation (4.1) does not have an eventually positive solution such that $z'(t) < 0$. Finally, we show that the conditions of the corollary are sufficient also for oscillation, since conditions (1.6) and (1.7) hold for any function $\eta(t) = \lambda t$, $\lambda \in (0,1]$ such that $\lambda_1\lambda < \min\{1, \lambda_3\}$. Indeed, we have

$$\begin{aligned}
&Q(t,\varphi)\left(\int_{t_1}^{\eta(t)} r^{1-q}(s)\mathrm{d}s\right)^{p-1} \\
&= \gamma\left(\frac{\lambda_1}{t-\lambda_2}\right)^{p-\alpha}\left(\int_{t_1}^{\lambda t} s^{\alpha(1-q)}\mathrm{d}s\right)^{p-1} \\
&= \gamma\left(\frac{\lambda_1}{t-\lambda_2}\right)^{p-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\left(t_1^{\alpha(1-q)+1} - (\lambda t)^{\alpha(1-q)+1}\right)^{p-1} \\
&= \gamma\lambda_1^{p-\alpha}t_1^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}t^{\alpha-p}\left(\frac{t}{t-\lambda_2}\right)^{p-\alpha}\left(1-\left(\frac{t_1}{\lambda t}\right)^{\alpha(q-1)-1}\right)^{p-1} \\
&> \gamma\lambda_1^{p-\alpha}t_1^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}t^{\alpha-p}\frac{1}{1+\varepsilon}
\end{aligned}$$

for sufficiently large t. Hence, since $\lambda_1\lambda < \lambda_3$,

$$\begin{aligned}
&\liminf_{T\to\infty}\int_{\tau^{-1}(\sigma(\eta(T)))}^{T} Q(t,\varphi)\left(\int_{t_1}^{\eta(t)} r^{1-q}(s)\mathrm{d}s\right)^{p-1}\mathrm{d}t \\
&> \gamma\lambda_1^{p-\alpha}t_1^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\frac{1}{\alpha-p+1} \\
&\quad\times\liminf_{T\to\infty}\left(T^{\alpha-p+1} - \left(\frac{\lambda_1\lambda T+\lambda_2-\lambda_4}{\lambda_3}\right)^{\alpha-p+1}\right) = \infty,
\end{aligned}$$

i.e., condition (1.6) holds. Similarly, since $\lambda_1\lambda < 1$,

$$\begin{aligned}
&\liminf_{T\to\infty}\int_{\sigma(\eta(T))}^{T} Q(t,\varphi)\left(\int_{t_1}^{\eta(t)} r^{1-q}(s)\mathrm{d}s\right)^{p-1}\mathrm{d}t \\
&> \gamma\lambda_1^{p-\alpha}t_1^{p-1-\alpha}\left(\frac{1}{\alpha(q-1)-1}\right)^{p-1}\frac{1}{1+\varepsilon}\frac{1}{\alpha-p+1} \\
&\quad\times\liminf_{T\to\infty}\left(T^{\alpha-p+1} - (\lambda_1\lambda T+\lambda_2)^{\alpha-p+1}\right) = \infty,
\end{aligned}$$

i.e., condition (1.7) holds.

□

Remark 4.2. Denote

$$F(\bar\lambda) := (\lambda_1\bar\lambda)^{p-\alpha-1}\ln\frac{\lambda_1\bar\lambda}{\max\{1,\lambda_3\}}.$$

By a direct computation we find that this function has a local maximum at

$$\bar\lambda_{\max} = \frac{\max\{1,\lambda_3\}}{\lambda_1}e^{\frac{1}{\alpha-p+1}}$$

and the value of F at this local maximum is

$$F(\bar\lambda_{\max}) = \frac{(\max\{1,\lambda_3\})^{p-\alpha-1}}{e(\alpha-p+1)}.$$

Since $\bar{J} = \{\bar{\lambda} \geq 1 : \lambda_1 \bar{\lambda} > \max\{1, \lambda_3\}\}$, we have

$$\max_{\bar{\lambda} \in \bar{J}} F(\bar{\lambda}) = \begin{cases} F(\bar{\lambda}_{\max}) & \text{if } \bar{\lambda}_{\max} \geq 1 \\ F(1) & \text{if } \bar{\lambda}_{\max} \leq 1. \end{cases}$$

Consequently, condition (4.4) can be written in the form

$$\gamma > \bar{\gamma} := \begin{cases} \dfrac{\left(1+b_0 \lambda_3^{(p-\alpha-1)(q-1)}\right)^{p-1} (\alpha(q-1)-1)^{p-1} (\alpha-p+1)}{(\max\{1,\lambda_3\})^{p-\alpha-1}} & \text{if} \quad \frac{\max\{1,\lambda_3\}}{\lambda_1} e^{\frac{1}{\alpha-p+1}} \geq 1 \\[2ex] \dfrac{\left(1+b_0 \lambda_3^{(p-\alpha-1)(q-1)}\right)^{p-1} (\alpha(q-1)-1)^{p-1}}{e \lambda_1^{p-\alpha-1} \ln \frac{\lambda_1}{\max\{1,\lambda_3\}}} & \text{if} \quad \frac{\max\{1,\lambda_3\}}{\lambda_1} e^{\frac{1}{\alpha-p+1}} \leq 1. \end{cases}$$

From this condition we can see, that e.g. [6, Example 19] can be obtained as a special case of Corollary 4.1.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for difference and functional differential equations*, Kluwer Academic Publishers, Dordrecht, 2000. MR1774732; url

[2] B. Baculíková, J. Džurina, Oscillation theorems for second-order neutral differential equations, *Comput. Math. Appl.* **61**(2011), 94–99. MR2739438; url

[3] B. Baculíková, J. Džurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.* **62**(2011), 4472–4478. MR2855589; url

[4] B. Baculíková, T. Li, J. Džurina, Oscillation theorems for second order neutral differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 74, 1–13. MR2838502

[5] B. Baculíková, T. Li, J. Džurina, Oscillation theorems for second-order superlinear neutral differential equations, *Math. Slovaca* **63**(2013), No. 1, 123–134. MR3015412; url

[6] S. Fišnarová, R. Mařík, On eventually positive solutions of quasilinear second order neutral differential equations, *Abstr. Appl. Anal.* **2014**, Art. ID 818732, 1–11. MR3212450; url

[7] S. Fišnarová, R. Mařík, Oscillation criteria for neutral second-order half-linear differential equations with applications to Euler type equations, *Bound. Value Probl.* **2014**, 2014:83, 1–14. MR3347718; url

[8] S. Fišnarová, R. Mařík, Oscillation of neutral second order half-linear differential equations without commutativity in deviating arguments, *Math. Slovaca.*, accepted.

[9] Z. Han, T. Li, S. Sun, W. Chen, Oscillation criteria for second-order nonlinear neutral delay differential equations, *Adv. Difference Equ.* **2010**, Art. ID 763278, 1–23. MR2683910

[10] T. Li, Y. V. Rogovchenko, Ch. Zhang, Oscillation of second order neutral differential equations, *Funkcial. Ekvac.* **56**(2013), 111–120. MR3099036; url

[11] T. Li, Comparison theorems for second-order neutral differential equations of mixed type, *Electron. J. Differential Equations* **2010**, No. 167, 1–7. MR2740608