Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **15**, 1–18; doi: 10.14232/ejqtde.2016.1.15 http://www.math.u-szeged.hu/ejqtde/

Asymptotic behaviour of a system of micropolar equations

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Received 10 September 2015, appeared 22 March 2016
Communicated by Gabriele Bonanno

Abstract. This work is concerned with three-dimensional micropolar fluids flows in a bounded domain with boundary of class C^∞. Based on the theory of dissipative systems, we prove the existence of restricted global attractors for local semiflows on suitable fractional phase spaces $\mathbf{Z}_p^\alpha$, namely for $p \in (3, +\infty)$ and $\alpha \in [1/2, 1)$. Moreover, we prove that all these attractors are in fact the same set. Previously, it is shown that the Lamé operator is a sectorial operator in each $L_p(\Omega)$ with $1 < p < +\infty$, $p \neq 3/2$ and therefore, it generates an analytic semigroup in these spaces.

Keywords: micropolar fluids, local semiflows and restricted global attractors.

2010 Mathematics Subject Classification: 35Q35, 76D03.

1 Introduction and notation

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with smooth boundary $\partial\Omega$, namely of class C^∞; we consider the system of equations for the motion of micropolar fluid

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\nu+\chi)\Delta\mathbf{u} + \nabla p = \chi \operatorname{rot}\mathbf{w} + \mathbf{f}, \quad \text{in } \Omega\times(0,T), \\ \operatorname{div}\mathbf{u} = 0, \quad \text{in } \Omega\times(0,T), \\ \dfrac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} - \mu\Delta\mathbf{w} - (\mu+\sigma)\nabla\operatorname{div}\mathbf{w} + 2\chi\mathbf{w} = \chi\operatorname{rot}\mathbf{u} + \mathbf{g}, \quad \text{in } \Omega\times(0,T), \end{cases} \tag{1.1}$$

together with the following boundary and initial conditions

$$\begin{cases} \mathbf{u} = 0 & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{in } \Omega, \\ \mathbf{w} = 0 & \text{on } \partial\Omega\times(0,T), \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

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where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, and $\mathbf{w} = (w_1, w_2, w_3)$ is the micro-rotational interpreted as the angular velocity field of rotational of particles. The fields $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ are external forces and moments respectively. The positive constants ν, χ, μ, σ represent viscosity coefficients, ν is the usual Newtonian viscosity and χ is called the micro-rotational viscosity. We will assume that these constants satisfy $\mu > 0$ and $3\sigma + 2\mu > 0$. In the last ten years, much effort has been devoted to study the long time behaviour for the micropolar equations (see for instance [15,16,20]). Most of these papers deal with problems assuming that Ω is a bounded domain in $\mathbb{R}^2$ and they conclude the existence of global attractor in an $L_2(\Omega)$-framework by standard techniques; whereas the case of Ω a subset in $\mathbb{R}^3$ and $L_p(\Omega)$-theory with $1 < p < +\infty$ has not received so much attention. In the 3D problem, two issues appear to be possible obstructions to build a global attractor. For general data $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{f}, \mathbf{g})$, there always exists a weak solution $\mathbf{U}(t) = (\mathbf{u}, \mathbf{w})$, that is defined for all time $t \geq 0$. However, it is not known whether this solution is uniquely determined by the data. As a result, one cannot conclude that the mapping $S(t) : (\mathbf{u}_0, \mathbf{w}_0) \to S(t)(\mathbf{u}_0, \mathbf{w}_0) = \mathbf{U}(t)$ satisfies the semigroup property required for a semiflow. On the other hand, for "good" data $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{f}, \mathbf{g})$, the initial value problem does have a unique strong solution on some interval $[0, T)$. However, it is not known whether this strong solution continues to exist for all time.

Nevertheless, with an alternative point of view, Carvalho, Cholewa, and Dlokto in a series of works (see [5–7]), propose bypasses to these issues for the Navier–Stokes equations and other initial boundary value problems for semilinear parabolic equations. In fact, the authors look at the problems as a sectorial equation in relevant Banach spaces ($L_p(\Omega)$, $1 < p < +\infty$), and then discuss to generate a local semiflow $S(t)$ on a fractional phase space $\mathbf{Z}^\alpha_p$, afterwards applying adequate estimates, they choose a suitable metric space $V \subset \mathbf{Z}^\alpha_p$ on which $S(t)$ becomes a dissipative compact semigroup of global solutions. As a consequence, the existence of a global attractor $\mathcal{A}$ for $S(t)$ restricted to V will follow from a suitable estimate of the solutions in a Sobolev space.

The goal of this paper is to prove, following the ideas contained in the above references, that the system (1.1) has a restricted global attractor for a local semiflow on $\mathbf{Z}^\alpha_p$ (defined below) for $\alpha \in [1/2, 1)$ and $p \in (3, +\infty)$.

The structure of the paper is as follows. After some notations introduced in this section, we recall some preliminary notions on the abstract formulation of the problem (Section 2), on conditions ensuring the existence of (local) solutions, discussions on how to turn them global, and on the concept and existence of (restricted) global attractor for a suitable local semiflow. The subsequent sections are devoted to follow this scheme. Namely, in Section 3 it is shown that the Lamé operator is a sectorial operator in $L_p(\Omega)$ with $1 < p < +\infty$ and therefore, it generates analytic semigroups in these spaces. Then, in Section 4 the study of local and global solutions is carried out. Finally, our main result on existence of attractors in different spaces and the relation among them is stated in Section 5.

In this paper we use the following notations, for $1 \leq p \leq +\infty$, $L_p(\Omega)$ denotes the usual Lebesgue space over Ω, $W^{m,p}(\Omega)$ the usual L_p-Sobolev space of order m, and $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions in Ω with compact support in Ω. For function spaces of vector fields, we use the following symbols

$$\mathbf{L}_p \equiv \mathbf{L}_p(\Omega) = [L_p(\Omega)]^3,$$
$$\mathbf{W}^{m,p} \equiv \mathbf{W}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^3.$$

We define

$$\mathbf{C}^\infty_{0,\sigma}(\Omega) := \{\mathbf{v} \in \mathbf{C}^\infty_0(\Omega) : \operatorname{div} \mathbf{v} = 0 \ \text{ in } \ \Omega\},$$

and denote by $\mathbf{L}_p^\sigma$ the closure of $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ in $\mathbf{L}_p(\Omega)$.

For notational simplicity, we denote the norms $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. For the differentiation of the vector field $\mathbf{u} = (u_1, u_2, u_3)$ and the scalar field p we use the following symbols: $\partial_j p = \frac{\partial p}{\partial x_j}, \partial_t p = \frac{\partial p}{\partial t}, \nabla p = (\partial_1 p, \partial_2 p, \partial_3 p)$, and

$$\begin{aligned} \operatorname{div} \mathbf{u} &= \sum_{j=1}^{3} \partial_j u_j, \\ \operatorname{rot} \mathbf{u} &= (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1). \end{aligned}$$

The identity operator will be denoted by I.

In order to give an operator interpretation of the problem (1.1)–(1.2), we shall introduce the well known Helmholtz and Weyl decomposition. Let $1 < p < +\infty$. Then, the Banach space $\mathbf{L}_p(\Omega)$ admits the Helmholtz and Weyl decomposition (cf. [9])

$$\mathbf{L}_p = \mathbf{L}_p^\sigma \oplus G^p(\Omega),$$

where $\oplus$ denotes direct sum and

$$G^p(\Omega) = \{\nabla\psi \,:\, \psi \in W^{1,p}(\Omega)\}.$$

Let $P = P_p$ be a continuous projection from $\mathbf{L}_p(\Omega)$ into $\mathbf{L}_p^\sigma$ along with $G^p(\Omega)$. Then the projection P has the L_p-boundedness property

$$\|P\mathbf{u}\|_p \leq C_p \|\mathbf{u}\|_p.$$

Denote $\mathbf{U} = (\mathbf{u}, \mathbf{w})^\top$ and let us define the linear operator

$$\mathrm{A}_p = \begin{pmatrix} -(\nu+\chi)P\Delta & 0 \\ 0 & -\mu\Delta - (\mu+\sigma)\nabla \operatorname{div} \end{pmatrix} \tag{1.3}$$

with domain

$$D(\mathrm{A}_p) = \left\{ U = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} \;:\; \begin{array}{l} \mathbf{u} \in \mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p} \cap \mathbf{L}_p^\sigma, \\ \mathbf{w} \in \mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega) \end{array} \right\}.$$

Here we have used the fact that $P \operatorname{rot} \mathbf{w} = \operatorname{rot} \mathbf{w}$, since $\operatorname{div} \operatorname{rot} \mathbf{w} = 0$ in Ω.

These facts imply that $\operatorname{rot} \mathbf{w} \in \mathbf{L}_p^\sigma$, because the space $\mathbf{L}_p^\sigma$ is characterized (cf. [9]) as

$$\mathbf{L}_p^\sigma = \{\mathbf{u} \in \mathbf{L}_p(\Omega) : \operatorname{div}\, \mathbf{u} = 0 \;\text{ in } \Omega, \;\; \mathbf{n}\cdot\mathbf{u} = 0 \;\text{ on } \Omega\}.$$

We set $\mathbf{Z}_p \equiv \mathbf{L}_p^\sigma \times \mathbf{L}_p$, using the notation above, and state the following Cauchy problem in the Banach space $\mathbf{Z}_p$,

$$\begin{cases} \mathbf{U}_t + A_p\mathbf{U} = \mathbf{N}\mathbf{U} + \mathbf{F} =: G(\mathbf{U}), \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{1.4}$$

where $\mathbf{U}_0 = (\mathbf{u}_0, \mathbf{w}_0)^\top$, $\mathbf{F} = (P\mathbf{f}, \mathbf{g})^\top$, and the nonlinear term is given by

$$\mathbf{N}\mathbf{U} = \begin{pmatrix} -P(\mathbf{u}\cdot\nabla)\mathbf{u} + \chi \operatorname{rot} \mathbf{w} \\ -(\mathbf{u}\cdot\nabla\mathbf{w}) + \chi \operatorname{rot} \mathbf{u} - 2\chi\mathbf{w} \end{pmatrix}.$$

Finally, suppose that A is a sectorial operator and $\operatorname{Re}\sigma(A) > 0$ in a Banach space $\mathbf{Z}$, and define for each $\alpha \geq 0$, $\mathbf{Z}^\alpha = D(A^\alpha)$ with the graph norm $\|z\|_\alpha = \|A^\alpha z\|$, $z \in \mathbf{Z}^\alpha$. In the next sections, the role of A operator will be played by A_p.

2 Preliminaries and background

With the above notation, consider the Cauchy problem (1.4), and concerning this, let us enumerate some assumptions.

(i) $\mathrm{A}_p : D(\mathrm{A}_p) \to \mathbf{Z}_p$ is a sectorial and positive operator ($\mathrm{Re}\,\sigma(\mathrm{A}_p) > a > 0$) in $\mathbf{Z}_p$.

(ii) For certain $\alpha \in [0,1)$, $G : \mathbf{Z}_p^\alpha \to \mathbf{Z}_p$ is Lipschitz continuous on bounded subsets of $\mathbf{Z}_p^\alpha = D(\mathrm{A}_p^\alpha)$.

(iii) The resolvent of A_p is compact.

It is known from the results of Henry [13] and Hale [12] that, under the assumptions (i) and (ii), to each $\mathbf{U}_0 \in \mathbf{Z}_p^\alpha$, a unique local $\mathbf{Z}_p^\alpha$-solution $\mathbf{U} = \mathbf{U}(t, \mathbf{U}_0)$ corresponds to (1.4) defined on a maximal interval of existence $[0, \tau_{max}(\mathbf{U}_0))$. More precisely, $\mathbf{U}$ belongs to $C([0, \tau_{max}(\mathbf{U}_0)), \mathbf{Z}_p^\alpha) \cap C^1((0, \tau_{max}(\mathbf{U}_0)), \mathbf{Z}_p)$, $\mathbf{U}(0) = \mathbf{U}_0$, $\mathbf{U}(t)$ belongs to $D(\mathbf{A}_p)$ for each $t \in (0, \tau_{max}(\mathbf{U}_0))$, and the first equation in (1.4) holds in $\mathbf{Z}_p$ for all $t \in (0, \tau_{max}(\mathbf{U}_0))$.

The assumptions (i), (ii) and (iii) are satisfied by the Stokes operator $-(\nu + \chi)P\Delta$ (cf. Giga–Miyakawa [11]). Therefore, in order to show that A_p satisfies these assumptions, one only needs to prove that Lamé operator $\mathcal{L}_p\mathbf{w} = -\mu\Delta\mathbf{w} - (\mu + \sigma)\nabla\,\mathrm{div}(\mathbf{w})$ is a sectorial, positive operator with compact resolvent. We will prove these facts in Section 3.

After local existence has been proved (in Section 4), one must discuss the global existence of solutions. For instance, a subset $V_p^\alpha \subset \mathbf{Z}_p^\alpha$, for $p \in (3, +\infty)$ and $\alpha \in [1/2, 1)$, will be distinguished such that fractional solutions $S(t)\mathbf{U}_0 = \mathbf{U}(t, \mathbf{U}_0)$ of (1.4) with $\mathbf{U}_0 \in V_p^\alpha$ are defined globally in the time. In addition, the existence of a restricted global attractor for the semigroup $\{S(t)\}$ restricted to V_p^α (see the definition just below) will be established.

Definition 2.1. Let $p \in (3, +\infty)$, $\alpha \in [1/2, 1)$, and $\{S(t)\}$ be a local semiflow defined on $\mathbf{Z}_p^\alpha$. We say that $\mathcal{A} \subset \mathbf{Z}_p^\alpha$ is a restricted global attractor for $\{S(t)\}$ in $\mathbf{Z}_p^\alpha$ if for some closed, nonempty subset V of $\mathbf{Z}_p^\alpha$, $S(t) : V \to V$, $(t \geq 0)$ is a global semiflow on V such that $\mathcal{A}$ is a global attractor for $\{S(t)\}$ restricted to V as stated in [12], that is, (i) $S(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$, (ii) $\mathcal{A}$ is compact, (iii) $\mathcal{A}$ attracts all trajectories starting at bounded subsets of V.

The following result, given in [6,7] provides a useful criterion for the existence of restricted global attractor.

Lemma 2.2. *Let $p \in (3, +\infty)$, $\alpha \in [1/2, 1)$, $\{S(t)\}$ be a local semiflow on $\mathbf{Z}_p^\alpha$, and the resolvent of A_p be compact. Then, in order to prove the existence of a restricted global attractor for $\{S(t)\}$ in $\mathbf{Z}_p^\alpha$, it suffices to show that there exists a Banach space $Y \supset D(A_p)$ and a nondecreasing function $g : [0, +\infty) \to [0, +\infty)$ for which the conjunction of conditions* (a) *and* (b) *stated below holds with some closed and positively invariant nonempty subset V of $\mathbf{Z}_p^\alpha$, where*

(a) $\exists\, C > 0, \forall\, \mathbf{U}_0 \in V, \forall\, t \in (0, \tau_{max}(\mathbf{U}_0))$,

$$\|S(t)\mathbf{U}_0\|_Y \leq C.$$

(b) $\exists\, \theta \in [0, 1), \forall\, \mathbf{U}_0 \in V, \forall\, t \in (0, \tau_{max}(\mathbf{U}_0))$,

$$\|G(S(t)\mathbf{U}_0)\|_p \leq g(\|S(t)\mathbf{U}_0\|_Y)(1 + \|S(t)\mathbf{U}_0\|_{\mathbf{Z}_p^\alpha}^\theta).$$

Now, let us recall some definitions (e.g. cf. [12, Chapter 3]) concerning the asymptotic behaviour of dynamical systems. Let $\mathbf{V}$ be a complete metric space, and $S(t) : \mathbf{V} \to \mathbf{V}$ be a C^0-semigroup on $\mathbf{V}$. Denoting by $[B]_V$ the closure of a set B in the space $\mathbf{V}$, for any set $B \subset \mathbf{V}$ the two sets $\gamma^+(B)$ and $\omega(B)$ defined by

$$\begin{aligned} \gamma^+(B) &= \cup_{t \geq 0} S(t)B, \\ \omega(B) &= \cap_{s \geq 0} [\cup_{t \geq s} S(t)B]_{\mathbf{V}}, \end{aligned}$$

are called, respectively, the positive orbit and the ω-limit set of B. Thus, an ω-limit set consists of all points $v \in \mathbf{V}$ for which there exist positive numbers $t_n \nearrow +\infty$ and points $v_n \in B$ with $S(t_n)v_n \to v$ as $n \to +\infty$.

Remark 2.3. The above requirements (a) and (b) ensure that local solutions corresponding to $\mathbf{U}_0 \in V$ exist for all $t \geq 0$. If, in addition, $\mathbf{U}_0 \in V$ implies that $\mathbf{U}(t, \mathbf{U}_0) \in V$ as long as it exists, then the relation $S(t)\mathbf{U}_0 = \mathbf{U}(t, \mathbf{U}_0)$, defines on V a C^0-semigroup of global $\mathbf{Z}_p^\alpha$ solutions. By (a), $\{S(t)\}$ is point dissipative (that is, there exists a nonempty, bounded set $B \subset V$ which attracts every point of V); and since the resolvent of A_p is compact, $S(t) : V \to V$ is a compact map for each $t > 0$, whence the existence of the attractor follows (cf. [12]).

3 On the Lamé operator

The main goal of this section is to show that the Lamé operator $\mathcal{L}_p\mathbf{v} = -\mu\Delta\mathbf{v} - (\mu+\sigma)\nabla\mathrm{div}(\mathbf{v})$, defined on domain $D(\mathcal{L}_p) = \mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega)$ $(1 < p < +\infty)$, is a sectorial, positive operator with compact resolvent in $\mathbf{L}_p$.

Namely, we consider the following problem:

$$\begin{aligned} -\mu\Delta\mathbf{v} - (\mu+\sigma)\nabla\,\mathrm{div}\,\mathbf{v} + \lambda\mathbf{v} &= \mathbf{h}, \\ \mathbf{v}|_{\partial\Omega} &= 0. \end{aligned} \tag{3.1}$$

For this purpose, let us denote by $\Delta_\varphi(\eta)$, where $\eta \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$, the sector of the complex plane given by

$$\Delta_\varphi(\eta) = \left\{ \lambda \in \mathbb{C} \,:\, |\arg(\lambda - \eta)| < \frac{\pi}{2} + \varphi,\ \lambda \neq \eta \right\}.$$

Before establishing our main result of this section, we state a useful lemma with a priori estimates of solutions.

Lemma 3.1. *Let be given $p > 1$ with $p \neq 3/2$, $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, and $\mathbf{h} \in \mathbf{L}_p$. Then, there exists a constant $0 < C = C(\mu, \sigma)$ such that for any solution $\mathbf{v} \in D(\mathcal{L}_p)$ of (3.1), the following inequalities hold:*

$$(|\lambda|^p - C)\|\mathbf{v}\|_p^p + \frac{1}{C+1}\|\mathbf{v}\|_{2,p}^p \leq C\|\mathbf{h}\|_p^p, \quad \text{for } p > 1,\ p \neq 3/2 \tag{3.2}$$

and

$$\|\mathbf{v}\|_{2,p} \leq C(|\lambda|\|\mathbf{h}\|_2 + \|\mathbf{h}\|_p), \qquad p \geq 2. \tag{3.3}$$

Proof. Define $\mathbf{w}(x,t) = e^{\lambda t}\mathbf{v}(x)$ and $\mathbf{q}(x,t) = e^{\lambda t}\mathbf{h}(x)$. Multiplying the equation (3.1) by $e^{\lambda t}$ we have that $\mathbf{w}$ satisfies

$$\begin{aligned} \partial_t\mathbf{w} - \mu\Delta\mathbf{w} - (\mu+\sigma)\nabla\,\mathrm{div}\,\mathbf{w} &= \mathbf{q}(x,t), \\ \mathbf{w}|_{\partial\Omega} &= 0, \\ \mathbf{w}(x,0) &= \mathbf{v}(x), \quad \text{in } \Omega. \end{aligned} \tag{3.4}$$

Let us denote by $W^{2-2/p,p} = W^{2-2/p,p}(\Omega)$ the Slobodetskii–Besov space. As proved in [19], we have

$$\|\mathbf{w}\|^{(2,1)}_{L_p(Q_T)} + \sup_{\tau \le T} \|\mathbf{w}(x,\tau)\|_{W^{2-2/p,p}} \le C(\|\mathbf{q}\|_{L_p(Q_T)^3} + \|\mathbf{v}\|_{W^{2-2/p,p}}), \tag{3.5}$$

where $Q_T = \Omega \times (0,T)$, and

$$\|\mathbf{w}\|^{(2,1)}_{L_p(Q_T)} = \|\partial_t \mathbf{w}\|_{L_p(Q_T)^3} + \|\nabla(\nabla \mathbf{w})\|_{L_p(Q_T)^{27}} + \|\nabla \mathbf{w}\|_{L_p(Q_T)^9} + \|\mathbf{w}\|_{L_p(Q_T)^3}.$$

Let us recall that $W^{(2,1)}_p(Q_T)$ is the space of distributions $\mathbf{w} \in L_p(0,T;\mathbf{W}^{2,p}(\Omega))$ such that $\partial_t \mathbf{w} \in \mathbf{L}_p(Q_T)$. This space, endowed with the norm $\|\mathbf{w}\|^{(2,1)}_{L_p(Q_T)}$, is a Banach space.

The constant $C = C(\mu,\sigma,T)$ in (3.5) still depends on T and it has the property that $C(\mu,\sigma,T_1) \le C(\mu,\sigma,T_2)$ if $T_1 \le T_2$. By standard arguments, we can replace this constant by a quantity $C(\mu,\sigma)$, which does not depend on T, and we arrive at

$$\begin{aligned}\int_0^T (\|\partial_t \mathbf{w}\|_p^p + \|\mathbf{w}\|_{2,p}^p)ds + \sup_{\tau \le T} \|\mathbf{w}(x,\tau)\|^p_{W^{2-2/p,p}} \\ \le C(\mu,\sigma)\left(\int_0^T (\|\mathbf{q}\|_p^p + \|\mathbf{w}\|_p^p)ds + \|\mathbf{v}\|^p_{W^{2-2/p,p}}\right), \qquad \forall T > 0.\end{aligned} \tag{3.6}$$

Taking into account the change of variables done at the beginning, we deduce that $\mathbf{v}$ and $\mathbf{h}$ appearing in (3.1) satisfy

$$\begin{aligned}|\lambda|^p \int_0^T e^{p\,\mathrm{Re}(\lambda)t}\|\mathbf{v}\|_p^p dt + \int_0^T e^{p\,\mathrm{Re}(\lambda)t}\|\mathbf{v}\|_{2,p}^p dt \\ \le C\left(\|\mathbf{v}\|^p_{W^{2,p}} + \int_0^T e^{p\,\mathrm{Re}(\lambda)t}\|\mathbf{h}\|_p^p dt + \int_0^T e^{p\,\mathrm{Re}(\lambda)t}\|\mathbf{v}\|_p^p dt\right).\end{aligned}$$

Then, we obtain

$$\frac{e^{p\,\mathrm{Re}(\lambda)T} - 1}{p\,\mathrm{Re}(\lambda)}\left(|\lambda|^p\|\mathbf{v}\|_p^p + \|\mathbf{v}\|_{2,p}^p\right) \le C\left[\|\mathbf{v}\|_{2,p}^p + \frac{e^{p\,\mathrm{Re}(\lambda)T} - 1}{p\,\mathrm{Re}(\lambda)}\left(\|\mathbf{h}\|_p^p + \|\mathbf{v}\|_p^p\right)\right].$$

Taking T such that $C + 1 \le \frac{e^{p\,\mathrm{Re}(\lambda)T}-1}{p\,\mathrm{Re}(\lambda)}$, we deduce (3.2).

In the case $p = 3/2$, to the norm $\|\mathbf{v}\|_{W^{2-2/p,p}}$ that appears on the right side of (3.6) we must add the term

$$\left(\int_\Omega \frac{|\mathbf{v}|^p}{(\delta(x))^{2p-2}}dx\right)^{1/p}$$

($\delta(x)$ is the distance from x to $\partial\Omega$). Therefore it is not possible to obtain the estimate (3.2) (see [14] for details).

Next, for $p \ge 2$, we multiply the equation in (3.4) by $\mathbf{w}$ and integrate over Q_t $(0 \le t \le T)$ to obtain

$$\int_0^t ((\mathbf{w}_t, \mathbf{w}) + \mu\|\nabla \mathbf{w}\|_2)ds + (\mu+\sigma)\int_0^t \|\operatorname{div}(\mathbf{w})\|_2^2 ds = \int_0^t (\mathbf{q}, \mathbf{w})ds,$$

hence

$$\mathbf{w}_t \in L^2(0,T;\mathbf{L}_2(\Omega)) \subset L^2(0,T;\mathbf{H}^{-1,2}(\Omega)), \qquad \mathbf{w} \in L^2(0,T;\mathbf{W}^{1,2}_0(\Omega)).$$

From this we obtain, with $\delta > 0$,

$$\|\mathbf{w}(t)\|_2^2 + (2\mu - \delta)\int_0^t \|\nabla \mathbf{w}\|_2 ds \le \|\mathbf{v}\|_2^2 + C(\delta)\int_0^t \|\mathbf{q}\|_2^2 ds. \tag{3.7}$$

On the other hand, from Gagliardo–Nirenberg inequality we have

$$\|\mathbf{w}(t)\|_p \le \delta \|\mathbf{w}(t)\|_{2,p} + C(\delta)\|\mathbf{w}(t)\|_1. \tag{3.8}$$

Returning to (3.6), using (3.8) and inserting (3.7) we have

$$\int_0^T (\|\partial_t \mathbf{w}\|_p^p + \|\mathbf{w}\|_{2,p}^p) ds + \sup_{\tau \le T} \|\mathbf{w}(x,\tau)\|^p_{W^{2-2/p,p}}$$
$$\le C(\mu,\sigma)\left(\int_0^t \|\mathbf{q}\|_p^p ds + \left(\int_0^t \|\mathbf{q}\|_2 ds\right)^{p/2} + \|\mathbf{v}\|^p_{W^{2-2/p,p}}\right).$$

From this estimate and using the same arguments above, we conclude (3.3). □

Lemma 3.2. *The problem* (3.1) *has a unique weak solution* $\mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega)$.

Proof. We observe that the problem (3.1) is equivalent to the variational problem

$$\begin{cases} \text{To find } \mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega) \quad \text{such that} \\ \mu(\mathbf{v},\mathbf{z}) + (\mu+\sigma)(\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{z}) + \lambda(\mathbf{v},\mathbf{z}) = (\mathbf{h},\mathbf{z}), \quad \forall\, \mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega). \end{cases} \tag{3.9}$$

To prove that this problem has a solution, since $\mathbf{W}_0^{1,2}(\Omega)$ is a separable Hilbert space, we can consider a sequence of elements $\{z_m\}_{m\ge1}$ of $\mathbf{W}_0^{1,2}(\Omega)$ which is free and total in $\mathbf{W}_0^{1,2}(\Omega)$. For each fixed integer $m \ge 1$, we would like to define an approximation solution $\mathbf{v}_m$ of (3.9) by

$$\begin{cases} \mathbf{v}_m = \sum_{i=1}^m \xi_{i,m}\mathbf{z}_i, \quad \xi_{i,m} \in \mathbb{R}, \\ \mu(\nabla \mathbf{v}_m, \nabla \mathbf{z}_k) + (\mu+\sigma)(\operatorname{div} \mathbf{v}_m, \operatorname{div} \mathbf{z}_k) + \lambda(\mathbf{v}_m,\mathbf{z}_k) = (\mathbf{h},\mathbf{z}_k), \quad \forall\, k = 1,\dots,m. \end{cases} \tag{3.10}$$

The equations (3.10) are the system of linear equations for $\xi_{1,m},\dots,\xi_{m,m}$, and the existence of a solution follows easily. The passage to the limit is a consequence of the following argument.

We multiply (3.10) by $\xi_{k,m}$ and sum from $k = 1,\dots,m$; this gives

$$\mu\|\nabla \mathbf{v}_m\|^2 + (\mu+\sigma)\|\operatorname{div} \mathbf{v}_m\|^2 + \lambda\|\mathbf{v}_m\|^2 = (\mathbf{h},\mathbf{v}_m),$$

or

$$\mu\|\nabla \mathbf{v}_m\|^2 + (\mu+\sigma)\|\operatorname{div} \mathbf{v}_m\|^2 + \lambda\|\mathbf{v}_m\|^2 \le \frac{1}{2\mu}\|\mathbf{h}\|^2_{H^{-1}} + \frac{\mu}{2}\|\nabla \mathbf{v}_m\|^2.$$

Thus, we obtain the a priori estimate

$$\|\nabla \mathbf{v}_m\|^2 \le C\|\mathbf{h}\|^2_{H^{-1}}. \tag{3.11}$$

Since the sequence $\mathbf{v}_m$ remains bounded in $H_0^1(\Omega)$, there exist some $\mathbf{v} \in H_0^1(\Omega)$ and a subsequence $m' \to \infty$ such that

$$\mathbf{v}_{m'} \to \mathbf{v} \quad \text{in the weak topology of } H_0^1(\Omega). \tag{3.12}$$

The injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, so we have also

$$\mathbf{v}_{m'} \to \mathbf{v} \quad \text{in the norm of } L^2(\Omega). \tag{3.13}$$

With the convergences (3.12)–(3.13) it is easy to pass to the limit in (3.10) and thus to obtain the existence of a weak solution of (3.9). The uniqueness is proved in the standard way. □

Theorem 3.3. *Consider $p > 1$, $p \neq 3/2$. There exists a value $\eta > 0$ such that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\eta$, and each $\mathbf{h} \in \mathbf{L}_p$, the problem (3.1) has a unique solution $\mathbf{v} \in D(\mathcal{L}_p)$.*

Moreover, there also exist constants $\varphi_\epsilon \in (0, \pi/2)$ and $M > 0$ such that the resolvent set $\rho(-\mathcal{L}_p)$ contains the sector $\Delta_{\varphi_\epsilon}(-\eta)$, and the following estimate holds

$$\|(\lambda I + \mathcal{L}_p)^{-1}\|_p \leq \frac{M}{|\lambda| + 1}, \qquad \lambda \in \Delta_{\varphi_\epsilon}(-\eta).$$

In the terminology of Henry [13], $\mathcal{L}_p$ is a sectorial operator.

Proof. Before we proceed with the proof of all statements, let us observe some good properties of the Lamé operator when $p = 2$. From Agranovich et al. [3], it is well known that $\mathcal{L}_2$ is a closed densely defined self-adjoint operator with compact resolvent. Using the Gårding inequality, we can see that $\mathcal{L}_2$ is a positive operator since

$$\delta \|\mathbf{v}\|^2_{H^1(\Omega)} \leq (\mathcal{L}_2 \mathbf{v}, \mathbf{v}), \qquad \forall \mathbf{v} \in D(\mathcal{L}_2),$$

with $\delta > 0$. From this, we deduce that the $\mathcal{L}_2$ is a sectorial operator. The spectrum of $\mathcal{L}_2$ consists of isolated positive eigenvalues of finite multiplicity. Numbering them in nondecreasing order taking into account their multiplicities, we obtain a sequence $\{\lambda_j\}_{j=1}^{+\infty}$ with the asymptotic behaviour $\lambda_j \sim \Lambda_0 j^{2/3}$ (Λ_0 a positive constant).

Next, for a small enough constant $\epsilon > 0$ and suitable values η and φ_ϵ (to be specified later) we show that the subset $\Delta_{\varphi_\epsilon}(-\eta)$ is contained in the resolvent set $\rho(-\mathcal{L}_p)$ in several steps. For this, we define the following sectors in the complex plane (where the constant C is given in Lemma 3.1):

$$\Delta' = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0, |\lambda| \geq \sqrt[p]{C+1}\},$$
$$\Delta'' = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0, |\lambda| \leq \sqrt[p]{C+1}\},$$
$$\Delta_\epsilon = \left\{\lambda \in \mathbb{C} : \frac{|\mathrm{Re}\,\lambda|}{|\mathrm{Im}\,\lambda|} < \frac{1}{(1+\epsilon)M}\right\}.$$

Step I: We prove that $\Delta' \subset \rho(-\mathcal{L}_p)$.

First, assume that $p \geq 2$. To solve $(\lambda I + \mathcal{L}_p)\mathbf{v} = \mathbf{h}$ for any $\mathbf{h} \in \mathbf{L}_p(\Omega)$, we use the continuous injection of $\mathbf{L}_p$ into $\mathbf{L}_2$. By Lemma 3.2, we can see that (3.1) has a unique weak solution in $\mathbf{W}^{1,2}(\Omega)$. In fact $\mathbf{v} \in \mathbf{W}^{1,2}_0(\Omega)$, and it satisfies

$$\lambda \int_\Omega \mathbf{v} \cdot \mathbf{u} + \int_\Omega \Xi(\mathbf{v}) : \varepsilon(\mathbf{u}) = \int_\Omega \mathbf{h} \cdot \mathbf{u}, \qquad \forall \mathbf{u} \in \mathbf{W}^{1,2}_0(\Omega),$$

where

$$\Xi(\mathbf{v}) = \sigma tr(\varepsilon(\mathbf{v}))I + 2\mu\varepsilon(\mathbf{v})$$

and

$$\varepsilon(\mathbf{v}) = 1/2\left(\partial_j \mathbf{v}_i + \partial_i \mathbf{v}_j\right).$$

As long as Ω has smooth boundary, then $\mathbf{v} \in \mathbf{W}^{2,2}(\Omega)$. Thus, $\mathbf{v}$ is a strong solution for the system. Using the Sobolev Lemma and interpolation, we have that $\mathbf{v} \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{L}_p(\Omega)$, $\|\mathbf{v}\|_{C(\overline{\Omega})} \leq C\|\mathcal{L}_2 \mathbf{v}\|$, and therefore $\mathcal{L}_2 \mathbf{v} = -\lambda \mathbf{v} + \mathbf{h} \in \mathbf{L}_p(\Omega)$. From this, and using [18, Theorem 5.1, p. 301], we conclude that $\mathbf{v} \in D(\mathcal{L}_p)$.

Next we consider the case $1 < p < 2$ with $p \neq 3/2$ and $\mathbf{h} \in \mathbf{L}_p(\Omega)$. We approximate $\mathbf{h}$ by a sequence $(\mathbf{h}_n) \in \mathbf{C}_0^\infty(\Omega)$. Each problem $(\lambda I + \mathcal{L}_p)\mathbf{v}_n = \mathbf{h}_n$ has a solution, applying the inequality (3.2) to $(\lambda I + \mathcal{L}_p)(\mathbf{v}_n - \mathbf{v_m}) = \mathbf{h}_n - \mathbf{h}_m$, we have

$$(|\lambda|^p - C)\|\mathbf{v}_n - \mathbf{v}_m\|_p + \frac{1}{C+1}\|\mathbf{v}_n - \mathbf{v}_m\|_{2,p} \leq C\|\mathbf{h}_n - \mathbf{h}_m\|_p.$$

Thus, $\mathbf{v}_n \to \mathbf{v} \in \mathbf{W}^{2,p}(\Omega)$ as $n \nearrow +\infty$. Therefore, from the inequality (3.2) we deduce

$$\|(\lambda I + \mathcal{L}_p)\mathbf{v}\|_p \geq \sqrt[p]{\frac{|\lambda|^p - C}{C}}\|\mathbf{v}\|_p,$$

from which, the resolvent $(\lambda I + \mathcal{L}_p)^{-1}$ exists for all $p > 1$ with $p \neq 3/2$. Using the resolvent series we obtain that Δ' is contained in the resolvent set of $-\mathcal{L}_p$ and

$$\|(\lambda I + \mathcal{L}_p)^{-1}\|_p \leq \frac{2^{(p-1)/p}\widetilde{c}}{|\lambda| - \widetilde{c}}, \qquad \forall \lambda \in \Delta',$$

where $C = \widetilde{c}^p$ ($\widetilde{c}$ is a positive constant).
Step II: Now we prove that $\Delta'' \subset \rho(-\mathcal{L}_p)$.

For $p \geq 2$, we solve as before $(\lambda I + \mathcal{L}_p)\mathbf{v} = \mathbf{h}$ and from (3.3), the following estimate holds

$$\|(\lambda I + \mathcal{L}_p)^{-1}\|_p \leq C, \qquad \operatorname{Re}\lambda > 0.$$

When $1 < p < 2$, $p \neq 3/2$, let us suppose that $\mathbf{v}_1, \mathbf{v}_2 \in D(\mathcal{L}_p)$ are two solutions for $(\lambda I + \mathcal{L}_p)\mathbf{v} = \mathbf{h}$. Using the Sobolev embedding we obtain that $\mathbf{v}_2 - \mathbf{v}_1 \in \mathbf{W}^{2,1}(\Omega) \subset \mathbf{L}_3(\Omega)$, $(\lambda I + \mathcal{L}_p)(\mathbf{v}_2 - \mathbf{v}_1) = 0$. From this we have

$$\mathcal{L}_p(\mathbf{v}_2 - \mathbf{v}_1) = -\lambda(\mathbf{v}_2 - \mathbf{v}_1) \quad \text{in } \mathbf{L}_2(\Omega).$$

By a regularity result in [18], we conclude that $\mathbf{v}_2 - \mathbf{v}_1 \in D(\mathcal{L}_2)$, where uniqueness holds, which implies that $\mathbf{v}_2 = \mathbf{v}_1$.

Now, we claim that there exists a constant $\gamma = \gamma(\lambda) > 0$ such that

$$\|(\lambda I + \mathcal{L}_p)\mathbf{v}\|_p \geq \gamma(\lambda)\|\mathbf{v}\|_p, \qquad \forall \mathbf{v} \in D(\mathcal{L}_p). \tag{3.14}$$

Suppose this is false. Then there exists $\mathbf{v}_n \in D(\mathcal{L}_p)$, $\mathbf{h}_n = (\lambda I + \mathcal{L}_p)\mathbf{v}_n \in \mathbf{L}_p(\Omega)$ such that, for all n, $\|\mathbf{v}_n\|_p = 1$, $\|\mathbf{h}_n\|_p \leq 1/n$.

Since $\mathcal{L}_p\mathbf{v}_n = -\lambda\mathbf{v}_n + \mathbf{h}_n \in \mathbf{L}_p$, using [18, Lemma 4.4, p. 301], we have that (for certain C' and C'')

$$\|\mathbf{v}_n\|_{2,p} \leq C'(|\lambda|\,\|\mathbf{v}_n\|_p + |\lambda|\,\|\mathbf{h}_n\|_p + \|\mathbf{v}_n\|_p) \leq C''.$$

Thus, $\|\mathbf{v}_n\|_{W^{2,p}}$ is bounded. Taking a subsequence $\{\mathbf{v}_{n'}\}$ converging weakly to some $\mathbf{v} \in D(\mathcal{L}_p)$, we obtain that $(\lambda I + \mathcal{L}_p)\mathbf{v} = 0$. By the uniqueness of solution to any problem of type (3.1), it must be $\mathbf{v} = \mathbf{0}$, which contradicts $\|\mathbf{v}\|_p = 1$. So, (3.14) holds.

Now, approximate again $\mathbf{h} \in \mathbf{L}_p$ by $\mathbf{h}_n \in C_0^\infty(\Omega)$, and denote by $\mathbf{v}_n$ the unique solution of $(\lambda I + \mathcal{L}_2)\mathbf{v} = \mathbf{h}_n$ and write

$$\mathcal{L}_2(\mathbf{v}_m - \mathbf{v}_n) = (\mathbf{h}_m - \mathbf{h}_n) - \lambda(\mathbf{v}_m - \mathbf{v}_n).$$

Using again [18, Lemma 4.4, p. 301], we have that

$$\|\mathbf{v}_m - \mathbf{v}_n\|_{2,p} \leq C'(\|\mathbf{h}_m - \mathbf{h}_n\|_p + |\lambda|\,\|\mathbf{v}_m - \mathbf{v}_n\|_p). \tag{3.15}$$

Since by (3.14), it holds that

$$\gamma(\lambda)\|\mathbf{v}_m-\mathbf{v}_n\|_p \le \|(\lambda I+\mathcal{L}_p)(\mathbf{v}_m-\mathbf{v}_n)\|_p,$$

we conclude that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}_p(\Omega)$. From this, together with the inequality (3.15), the sequence $\mathbf{v}_n$ converges in $\mathbf{W}^{2,p}(\Omega)$ to the unique solution $\mathbf{v} \in D(\mathcal{L}_p)$ of $(\lambda I+\mathcal{L}_p)\mathbf{v}=\mathbf{h}$, which concludes the proof of existence of solution to (3.1).

Finally, using the estimate [18, Lemma 4.4, p. 301] together with (3.14), we have

$$\|\mathbf{v}\|_{2,p} \le C\|\mathbf{h}\|_p + \frac{\sqrt[p]{C+1}}{\delta}\|\mathbf{h}\|_p$$

Thus $(\lambda I+\mathcal{L}_p)^{-1}$ exists in Δ'' and using the compactness of this set, we conclude

$$\|(\lambda I+\mathcal{L}_p)^{-1}\|_p \le C.$$

Step III: We prove now that $\{\lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda)=0\} \subset \rho(-\mathcal{L}_p)$.

For one such λ we proceed as follows. Indeed, we only need to take care to apply Lemma 3.1, where the condition $\mathrm{Re}(\lambda)>0$ appears. Consider a sequence of positive real numbers $\{\varepsilon_n\}_n$ with $\lim_{n\to+\infty}\varepsilon_n \downarrow 0$. Consider the problem $((\lambda+\varepsilon_n)I+\mathcal{L}_p)\mathbf{v}_n=\mathbf{h}$. By the above steps I and II, we have that $\{\mathbf{v}_n\}$ is bounded in $W^{2,p}$, whence a subsequence with a weak limit $\mathbf{v}$ exists, being $\mathbf{v}$ solution of $(\lambda I+\mathcal{L}_p)\mathbf{v}=\mathbf{h}$.

Finally, collecting all estimates in these steps we arrive at

$$\|(\lambda I+\mathcal{L}_p)^{-1}\|_p \le \frac{M}{|\lambda|+1}, \qquad \mathrm{Re}\,\lambda \ge 0.$$

Step IV: Now we prove that $\Delta_\epsilon \subset \rho(-\mathcal{L}_p)$. The following argument shows that $\rho(-\mathcal{L}_p) \supset \Delta_\epsilon$ for some $\varphi_\epsilon \in (0,\frac{\pi}{2})$. We consider the resolvent series and choose μ such that $\mathrm{Im}\,\mu=\mathrm{Im}\,\lambda$, $|\mathrm{Re}\,\mu| \le |\mathrm{Im}\,\lambda|\frac{1}{(1+\epsilon)M}$ where λ is given in $\rho(-\mathcal{L}_p)$ with $\mathrm{Re}\,\lambda=0$, $\epsilon>0$. Moreover,

$$\begin{aligned}
\|(\mu I+\mathcal{L}_p)^{-1}\|_p &\le \sum_{n=0}^{+\infty}|\mu-\lambda|^n\|(\lambda I+\mathcal{L}_p)^{-1}\|_p^{n+1}\\
&= \sum_{l=1}^{+\infty}|\mu-\lambda|^{l-1}\|(\lambda I+\mathcal{L}_p)^{-1}\|_p^{l}\\
&\le \sum_{l=1}^{+\infty}\|(\lambda I+\mathcal{L}_p)^{-1}\|_p(|\mu-\lambda|\|(\lambda I+\mathcal{L}_p)^{-1}\|_p)^{l-1}\\
&\le \|(\lambda I+\mathcal{L}_p)^{-1}\|_p\frac{1}{1-|\mu-\lambda|\|(\lambda I+\mathcal{L}_p)^{-1}\|_p}\\
&\le \frac{M}{|\lambda|+1}\cdot\frac{1}{1-|\mu-\lambda|\frac{M}{|\lambda|+1}}.
\end{aligned}$$

Observing that $|\lambda-\mu|=|\mathrm{Re}\,\mu|$, we have

$$\frac{M}{|\lambda|+1}|\lambda-\mu| \le \frac{M|\mathrm{Re}\,\mu|}{|\mathrm{Im}\,\lambda|+1} \le \frac{1}{1+\epsilon},$$

so

$$\frac{1}{1-\frac{|\lambda-\mu|M}{|\lambda|+1}} \le \frac{1+\epsilon}{\epsilon}.$$

Using this estimate, we deduce that

$$\|(\mu I+\mathcal{L}_p)^{-1}\|_p \leq \frac{M(1+\epsilon)}{(|\lambda|+1)\epsilon},$$

but

$$|\mu| \leq |\lambda|\sqrt{1+\left(\frac{1}{(1+\epsilon)M}\right)^2}.$$

From this, we conclude that

$$\|(\mu I+\mathcal{L}_p)^{-1}\|_p \leq \frac{\sqrt{1+(\frac{1}{(1+\epsilon)M})^2}\frac{M(1+\epsilon)}{\epsilon}}{|\mu|+\sqrt{1+(\frac{1}{(1+\epsilon)M})^2}}.$$

Thus, the resolvent set $\rho(-\mathcal{L}_p)$ contains the sector Δ_ϵ for some $\varphi_\epsilon \in (0,\frac{\pi}{2})$, i.e.,

$$\sin\varphi_\epsilon \leq \frac{\frac{1}{(1+\epsilon)M}}{\sqrt{1+(\frac{1}{(1+\epsilon)M})}}.$$

So far, we have shown that the resolvent $\rho(-\mathcal{L}_p)$ contains the sector $\Delta_{\varphi_\epsilon}(0)=\{\lambda\in\mathbb{C} : |\arg(\lambda)|\leq\frac{\pi}{2}+\varphi_\epsilon\}$.

Since the imaginary axis lies in the resolvent set of $-\mathcal{L}_p$ (Step III), we claim that for some $\eta>0$, the strip $|\operatorname{Re}(\xi)|<\eta$, $\xi\in\mathbb{C}$, also lies the resolvent set of $-\mathcal{L}_p$.

Indeed, if there would be no such $\eta>0$, then there would exist a sequence ξ_k in $\sigma(-\mathcal{L}_p)$ (spectrum of $-\mathcal{L}_p$) such that $\operatorname{Re}\xi_k\to 0$ and since

$$|\operatorname{Im}\xi_k| \leq |\operatorname{Re}\xi_k|\tan(\varphi_\epsilon),$$

a subsequence of $\{\xi_k\}$ should converge to ξ with $\operatorname{Re}\xi=0$. This is not possible, since $\sigma(-\mathcal{L}_p)$ is closed. Thus, there exists a value $\eta>0$ such that $\Delta_{\varphi_\epsilon}(-\eta)\subset\rho(-\mathcal{L}_p)$. By [17, Chapter 3, p. 78], the following equivalence holds

$$\Delta_{\varphi_\epsilon}(-\eta)\subset\rho(-\mathcal{L}_p) \Longleftrightarrow \Sigma_\gamma(\eta)\subset\rho(\mathcal{L}_p),$$

where

$$\Sigma_\gamma(\eta)=\{\lambda\in\mathbb{C} : |\arg(\lambda-\eta)|>\gamma,\, \lambda\neq\eta\}, \qquad \gamma=\frac{\pi}{2}-\varphi_\epsilon.$$

From [17, Lemma 31.6] one finds

$$\|(\lambda I-\mathcal{L}_p)^{-1}\|_p \leq \frac{M}{|\lambda-\eta|}, \qquad \lambda\in\Sigma_\gamma(\eta).$$

Hence, $\mathcal{L}_p$ is sectorial on $\mathbf{L}_p$. $\square$

Remark 3.4. Theorem 3.3 shows that $-\mathcal{L}_p$ generates an analytic semigroup $e^{-t\mathcal{L}_p}$ in $\mathbf{L}_p(\Omega)$ where $1<p<\infty$, $p\neq 3/2$. Taking into account that $\operatorname{Re}\sigma(\mathcal{L}_p)>\delta>0$ for some δ, from Henry [13, Theorem 1.3.4], we also have the following estimates

$$\|e^{-t\mathcal{L}_p}\|_p \leq Ce^{-\delta t}, \qquad t\geq 0,$$
$$\|\mathcal{L}_p e^{-t\mathcal{L}_p}\|_p \leq C_1 te^{-\delta}, \qquad t>0,$$

where C and C_1 are positive constants.

Remark 3.5.

(i) In the proof of the above theorem, we suppose that $p \neq 3/2$ since we do not know if the estimate (3.2) remains valid for $p = 3/2$.

(ii) Let us observe that as a consequence of the facts in the Step I, it follows that if $\lambda \in \rho(-\mathcal{L}_2)$, then $\lambda \in \rho(-\mathcal{L}_p)$ for each $p > 2$. Moreover, from [4, Corollary IX.14] we deduce that the eigenfunctions of $-\mathcal{L}_2$ belong to $\mathbf{L}_p(\Omega)$. Thus, the spectrum $\sigma(-\mathcal{L}_2)$ is contained in $\sigma(-\mathcal{L}_p)$ and therefore $\sigma(-\mathcal{L}_2) = \sigma(-\mathcal{L}_p)$. Following the argument in Agmon [2, pp. 131–132], we conclude that it is true for any $p > 1$.

Lemma 3.6. *For $0 < \alpha < 1$, $1 < p < \infty$, $p \neq 3/2$, we have*

$$\mathbf{Y}^\alpha = D(\mathcal{L}_p^\alpha) \subset [\mathbf{L}_p(\Omega), D(\mathcal{L}_p)]_\alpha \subset \mathbf{W}^{2\alpha,p}(\Omega),$$

where $[\cdot,\cdot]_\alpha$ denotes a complex interpolation space.

Proof. Since $\partial\Omega$ is C^∞, by [8] we conclude that $\mathcal{L}_p^{it}$, $-\infty < t < \infty$, are bounded operators in $\mathbf{L}_p(\Omega)$. It is also known (see [21, Section 4.3.1]) that, for bounded set Ω satisfying the cone condition (in particular for the case under study here)

$$[\mathbf{L}_p(\Omega), \mathbf{W}^{2,p}(\Omega)]_\alpha = \mathbf{W}^{2\alpha,p}(\Omega), \qquad \alpha \in (0,1).$$

Using the definition of the complex interpolation space, we have the embedding

$$[\mathbf{L}_p(\Omega), D(\mathcal{L}_p)]_\alpha \subset [\mathbf{L}_p(\Omega), \mathbf{W}^{2,p}(\Omega)]_\alpha, \qquad \alpha \in (0,1),$$

concluding the proof. □

Remark 3.7. Compactness of the resolvent $(\lambda I + \mathcal{L}_p)^{-1} : \mathbf{L}_p(\Omega) \to \mathbf{L}_p(\Omega)$ follows from the estimate in [18, Theorem 5.1] and the compactness of the embedding $W^{2,p}(\Omega) \hookrightarrow L_p(\Omega)$.

4 Existence of solutions and additional estimates

In this section we prove a series of lemmas, which will be required below to ensure the existence of local solutions and moreover, later for the existence of the attractors.

First at all, observe that from the results of the last section, we have that the operator A_p defined in (1.3) is a sectorial operator on $\mathbf{Z}_p = \mathbf{L}_p^\sigma \times \mathbf{L}_p$. For convenience, we use the notation $\mathcal{S}_p$ for the Stokes operators in $\mathbf{L}_p(\Omega)$. From [7], we have that $\mathrm{Re}(\sigma(\mathcal{S}_2)) \geq (\nu + \chi)\lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $\mathbf{L}_2(\Omega)$ under homogeneous Dirichlet boundary conditions. Similarly, from [3] we also have that $\mathrm{Re}(\sigma(\mathcal{L}_2)) \geq \mu\lambda_1$. Using the elliptic regularity theory we conclude that $(\sigma(\mathcal{S}_p)) \geq (\nu + \chi)\lambda_1$ and $\mathrm{Re}(\sigma(\mathcal{L}_p)) \geq \mu\lambda_1$ for $p > 1$. Thus, we can define

$$A_p^\alpha \mathbf{U} = [\mathcal{S}_p^\alpha \mathbf{u}, \mathcal{L}_p^\alpha \mathbf{w}],$$

the fractional powers of A_p with $\alpha \in [0,1]$, on the domains $\mathbf{Z}_p^\alpha = \mathbf{X}_p^\alpha \times \mathbf{Y}_p^\alpha$, where $\mathbf{X}_p^\alpha = D(\mathcal{S}_p^\alpha)$ and $\mathbf{Y}_p^\alpha = D(\mathcal{L}_p^\alpha)$, and for each $p \in (1, +\infty)$, $\alpha \in (0,1]$ we have

$$\|A_p^\alpha e^{-tA_p}\|_{\mathcal{L}(\mathbf{Z}_p^\alpha, \mathbf{Z_p})} \leq C_{\alpha,p} t^{-\alpha} e^{-\lambda_1 \delta t}, \tag{4.1}$$

where $\delta = \min\{(\nu + \chi), \mu\}$.

Since $\mathcal{S}_p$ and $\mathcal{L}_p$ have compact resolvents, then the embeddings $\mathbf{X}_p^\beta \subset \mathbf{X}_p^\alpha$, $\mathbf{Y}_p^\beta \subset \mathbf{Y}_p^\alpha$ $(0 < \alpha < \beta < 1,\ 1 < p < +\infty)$ are compact.

Lemma 4.1. *Consider the problem* (1.4) *and assume that* $p \in (3, +\infty)$, $\alpha \in [1/2, 1)$, $\mathbf{F} \in \mathbf{L}_p$. *Then, the nonlinear term* $\mathbf{N} : \mathbf{Z}_p^\alpha \to \mathbf{Z}_p$ *is Lipschitz continuous on bounded sets. In particular, to each* $\mathbf{U}_0 \in \mathbf{Z}_p^\alpha$ *corresponds a unique local solution* $\mathbf{U} = \mathbf{U}(t, \mathbf{U}_0)$ *to* (1.4) *on a maximal interval of existence* $[0, \tau_{max}(\mathbf{U}_0))$.

Proof. Firstly, we prove that the nonlinear term $\mathbf{N} : \mathbf{Z}_p^\alpha \to \mathbf{Z}_p$ is Lipschitz continuous on bounded sets. For this, we follow the argument in Cholewa et al. [5–7]. Let be $\mathbf{U} = (\mathbf{u}, \mathbf{w}), \mathbf{V} = (\mathbf{v}, \overline{\mathbf{w}}) \in \mathcal{O}$, where $\mathcal{O}$ is a bounded set in $\mathbf{Z}_p^\alpha$, then

$$\begin{aligned}\|\mathbf{N}(\mathbf{U}) - \mathbf{N}(\mathbf{V})\|_p \leq \; & \|P[(\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{u}]\|_{L_p} + \chi\|\operatorname{rot}(\overline{\mathbf{w}} - \mathbf{w})\|_{L_p} \\ & + \|(\mathbf{v}\cdot\nabla)\overline{\mathbf{w}} - (\mathbf{u}\cdot\nabla)\mathbf{w}\|_{L_p} + \chi\|\operatorname{rot}(\mathbf{v} - \mathbf{u})\|_{L_p} \\ & + 2\chi\|\mathbf{w} - \overline{\mathbf{w}}\|_{L_p}.\end{aligned}$$

Using the Sobolev embedding (e.g. cf. [1]), if $mp > N = 3$, then $W^{m,p}(\Omega) \hookrightarrow L_q(\Omega)$, for $p \leq q \leq +\infty$; choosing $m = 1$ we have

$$\|P(\mathbf{u}\cdot\nabla)\mathbf{v}\|_p \leq C\|\mathbf{u}\|_\infty\|\nabla\mathbf{v}\|_p \leq c\|\mathbf{u}\|_{1,p}\|\mathbf{v}\|_{1,p},$$

and consequently

$$\|P[(\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{u}]\|_{L_p} \leq C\|\mathbf{v}\|_{1,p}\|\mathbf{v} - \mathbf{u}\|_{1,p} + C\|\mathbf{v} - \mathbf{u}\|_{1,p}\|\mathbf{u}\|_{1,p}. \tag{4.2}$$

By a similar argument, we obtain

$$\|(\mathbf{v}\cdot\nabla\overline{\mathbf{w}} - (\mathbf{u}\cdot\nabla)\mathbf{w}\|_p \leq C\|\overline{\mathbf{w}}\|_{1,p}\|\mathbf{v} - \mathbf{u}\|_{1,p} + C\|\overline{\mathbf{w}} - \mathbf{w}\|_{1,p}\|\mathbf{u}\|_{1,p}. \tag{4.3}$$

From (4.2) and (4.3) we deduce

$$\|\mathbf{N}(\mathbf{U}) - \mathbf{N}(\mathbf{V})\|_p \leq C(\|\mathbf{u}\|_{1,p}, \|\mathbf{v}\|_{1,p}, \|\overline{\mathbf{w}}\|_{1,p}, \chi)\left(\|\mathbf{v} - \mathbf{u}\|_{1,p} + \|\mathbf{w} - \overline{\mathbf{w}}\|_{1,p}\right).$$

Now, we estimate the terms $\|\mathbf{v} - \mathbf{u}\|_{1,p}$ and $\|\mathbf{w} - \overline{\mathbf{w}}\|_{1,p}$. It is known from general results for the Stokes operator (e.g. see [10,11]) that $D(\mathcal{S}_p^\alpha) \hookrightarrow D(\mathcal{S}_p^{1/2})$ with $\alpha \in [1/2, 1)$. Since $D(\mathcal{S}_p^{1/2})$ is continuously injected in $\mathbf{L}_p^\sigma \cap \mathbf{W}^{1,p}(\Omega)$ (cf. Giga–Miyakawa [11, Proposition 1.4]), we conclude that $\|\mathbf{v} - \mathbf{u}\|_{1,p} \leq C\|\mathbf{v} - \mathbf{u}\|_{D(\mathcal{S}_p^\alpha)}$, $\forall\alpha \in [1/2, 1)$. Using Lemma 3.6 with $\alpha = 1/2$, we have $\|\mathbf{w} - \overline{\mathbf{w}}\|_{1,p} \leq C\|\mathbf{w} - \overline{\mathbf{w}}\|_{D(\mathcal{L}_p^{1/2})}$ and therefore we conclude that $\mathbf{N}$ is Lipschitz continuous on bounded sets of $\mathbf{Z}_p^\alpha$.

Now, the existence of local solutions follows from the general results in [13, Chapter 3], mentioned in Section 2. So, we have a local semiflow $\{S(t)\}$ (where $S(t)\mathbf{U}_0 = \mathbf{U}(t, \mathbf{U}_0)$) for $t \in [0, \tau_{max}(\mathbf{U}_0))$ of maximal fractional solutions of (1.4) defined on $\mathbf{Z}_p^\alpha$. □

Lemma 4.2. *Under the above assumptions and notation, the fractional local solution satisfies the estimate* (b) *in Lemma* 2.2.

Proof. Observe that

$$\begin{aligned}\|\mathbf{N}(\mathbf{U}) + \mathbf{F}\|_p &\leq \|\mathbf{u}\|_{1,p}\|\mathbf{U}\|_{1,p} + 2\chi\|\mathbf{U}\|_{1,p} + C\|\mathbf{F}\|_p \\ &\leq \|\mathbf{U}\|_{1,p}^2 + 2\chi\|\mathbf{U}\|_{1,p} + C\|\mathbf{F}\|_p.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\mathbf{U}\|_{1,p} &= \|(\mathbf{u}, \mathbf{w})\|_{1,p}^{1/4}\|(\mathbf{u}, \mathbf{w})\|_{1,p}^{3/4} \\ &\leq C_p\|\mathbf{U}\|_{\mathbf{Z}_p^\alpha}^{1/4}\|\mathbf{U}\|_{\mathbf{Z}_p^{1/2}}^{3/4}.\end{aligned}$$

From this, we deduce that

$$\|\mathbf{N}(\mathbf{U})+\mathbf{F}\|_p \leq C_p\|\mathbf{U}\|^{1/2}_{\mathbf{Z}^\alpha_p}\|\mathbf{U}\|^{3/2}_{\mathbf{Z}^{1/2}_p} + 2C_p\chi\|\mathbf{U}\|^{1/4}_{\mathbf{Z}^\alpha_p}\|\mathbf{U}\|^{3/4}_{\mathbf{Z}^{1/2}_p} + C\|\mathbf{F}\|_p,$$

and applying the Young inequality, we deduce

$$\|\mathbf{N}(\mathbf{U})+\mathbf{F}\|_p \leq (\widetilde{C}_p\|\mathbf{U}\|^{3/2}_{\mathbf{Z}^{1/2}_p} + C\|\mathbf{F}\|_p)(1+\|\mathbf{U}\|^{1/2}_{\mathbf{Z}^\alpha_p}), \tag{4.4}$$

which concludes the proof. □

To complete the assumptions in Lemma 2.2, we need an additional estimate of the term $\|\mathbf{U}\|_{Z^\alpha_p}$. This will lead to the existence of a restricted global attractor for the semiflow on $\mathbf{Z}^\alpha_p$. In next lemma we are going to obtain the required estimation of type (a) in Lemma 2.2.

Lemma 4.3 (Estimate of the $\mathbf{Z}^{1/2}_p$-norm of fractional solutions)**.** *Let $p \in (3,+\infty)$ and $\{S(t)\}$ be a local semiflow on $\mathbf{Z}^{1/2}_p$ defined by Lemma 4.1. If the norm $\|\mathbf{F}\|_p$ fulfils the smallness restriction* (4.11) *(below) and the viscosities satisfy the restriction* (4.9) *(below), then there exist $R>0$ and $\kappa>0$ such that*

$$\|S(t)\mathbf{U}_0\|_{\mathbf{Z}^{1/2}_p} \leq R \quad \textit{for each} \quad \mathbf{U}_0 \in B_{\mathbf{Z}^{1/2}_p}(0,\kappa), \tag{4.5}$$

where $B_{\mathbf{Z}^{1/2}_p}(0,\kappa)$ denotes the open ball in $\mathbf{Z}^{1/2}_p$ centered at 0 with radius κ.

Proof. Since (1.4) is equivalent to the integral equations

$$\begin{cases} \mathbf{u}(t) = e^{-t\mathcal{S}_p}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathcal{S}_p}P(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \chi\,\mathrm{rot}\,\mathbf{w} + \mathbf{f})ds, \\ \mathbf{w}(t) = e^{-t\mathcal{L}_p}\mathbf{w}_0 + \int_0^t e^{-(t-s)\mathcal{L}_p}(-(\mathbf{u}\cdot\nabla)\mathbf{w} + \chi\,\mathrm{rot}\,\mathbf{u} - 2\chi\mathbf{w} + \mathbf{g})ds, \end{cases} \tag{4.6}$$

taking the $X^{1/2}_p$-norm and $Y^{1/2}_p$-norm respectively in both sides in (4.6) and using the estimate (4.1), we obtain

$$\|\mathbf{u}(t)\|_{X^{1/2}_p} \leq e^{-\lambda_1\delta t}\|\mathbf{u}_0\|_{X^{1/2}_p} + C_{1/2}\int_0^t (t-s)^{-1/2}e^{-\lambda_1\delta(t-s)}(\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_p + \chi\|\nabla\mathbf{w}\|_p + \|\mathbf{f}\|_p)ds,$$

$$\|\mathbf{w}(t)\|_{Y^{1/2}_p} \leq e^{-\lambda_1\delta t}\|\mathbf{w}_0\|_{Y^{1/2}_p} + C_{1/2}\int_0^t (t-s)^{-1/2}e^{-\lambda_1\delta(t-s)}(\|(\mathbf{u}\cdot\nabla)\mathbf{w}\|_p + \chi\|\nabla\mathbf{u}\|_p + 2\chi\|\mathbf{w}\|_p + \|\mathbf{g}\|_p)ds.$$

By summing these inequalities, we have

$$\begin{aligned}\|\mathbf{U}\|_{\mathbf{Z}^{1/2}_p} \leq \|\mathbf{U}_0\|_{\mathbf{Z}^{1/2}_p} &+ C_{1/2}\int_0^t (t-s)^{-1/2}e^{-\lambda_1\delta(t-s)} \\ &\times \{\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_p + \|(\mathbf{u}\cdot\nabla)\mathbf{w}\|_p + \chi\|\nabla\mathbf{u}\|_p + \chi\|\nabla\mathbf{w}\|_p + 2\chi\|\mathbf{w}\|_p + \|\mathbf{F}\|_p\}ds,\end{aligned} \tag{4.7}$$

where $\|\mathbf{F}\|_p = \|\mathbf{f}\|_p + \|\mathbf{g}\|_p$. Since $\|\mathbf{P}(u\cdot\nabla)v\|_p \leq c\|u\|_{1,p}\|v\|_{1,p}$, and $D(\mathbf{S}^{1/2}_p)$ is continuously embedding in $\mathbf{L}^\sigma_p(\Omega)\cap\mathbf{W}^{1,p}(\Omega)$, one has

$$\|\mathbf{P}(\mathbf{u}\cdot\nabla)\mathbf{u}\|_p \leq \frac{C}{\nu+\chi}\|\mathbf{U}\|^2_{\mathbf{Z}^{1/2}_p}.$$

By the similar argument, we have that

$$\|(\mathbf{u}\cdot\nabla)\mathbf{w}\|_p \leq \frac{C}{\nu+\chi}\|\mathbf{U}\|^2_{\mathbf{Z}^{1/2}_p}.$$

Using these estimates, we arrive at

$$\begin{aligned}&\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_p+\|(\mathbf{u}\cdot\nabla)\mathbf{w}\|_p+\chi\|\nabla\mathbf{u}\|_p+\chi\|\nabla\mathbf{w}\|_p+2\chi\|\mathbf{w}\|_p+C_p\|\mathbf{F}\|_p\\&\quad\le\frac{2C}{\chi+\nu}\|\mathbf{U}\|^2_{\mathbf{Z}_p^{1/2}}+\frac{3\chi C}{\chi+\nu}\|\mathbf{U}\|_{\mathbf{Z}_p^{1/2}}+C_p\|\mathbf{F}\|_p.\end{aligned}$$

From this and the estimate (4.7), we have the condition

$$\|\mathbf{U}\|_{\mathbf{Z}_p^{1/2}}\le\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}+\left(\frac{2C}{\chi+\nu}\sup_{s\in[0,t]}\|\mathbf{U}\|^2_{\mathbf{Z}_p^{1/2}}+\frac{3\chi C}{\chi+\nu}\sup_{s\in[0,t]}\|\mathbf{U}\|_{\mathbf{Z}_p^{1/2}}+C_p\|\mathbf{F}\|_p\right)\frac{\Gamma(1/2)C_{1/2}}{(\lambda_1\delta)^{1/2}},\tag{4.8}$$

where

$$\int_0^{+\infty}y^{-1/2}e^{-\lambda_1\delta y}dy=\frac{\Gamma(1/2)}{(\lambda_1\delta)^{1/2}}.$$

Choosing the viscosities such that

$$\frac{3\chi C\Gamma(1/2)C_{1/2}}{(\chi+\nu)(\lambda_1\delta)^{1/2}}\le 1/2\tag{4.9}$$

and setting

$$A=\frac{2C\Gamma(1/2)C_{1/2}}{(\chi+\nu)(\lambda_1\delta)^{1/2}},\qquad D=\frac{C_p\Gamma(1/2)C_{1/2}}{(\lambda_1\delta)^{1/2}},$$

$$B=D\|\mathbf{F}\|_p,\qquad r(t)=\sup_{s\in[0,t]}\|\mathbf{U}\|_{\mathbf{Z}_p^{1/2}},$$

from (4.8) we conclude that

$$r(t)\le 2Ar^2(t)+2\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}+2B.\tag{4.10}$$

If further $\|\mathbf{F}\|_p$-norm is required to satisfy

$$\|\mathbf{F}\|_p\le\frac{(\chi+\nu)(\lambda_1\delta)^{1/2}}{32CC_pC_{1/2}\Gamma(1/2)^2C_{1/2}^2},\tag{4.11}$$

the determinant $1-16A(\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}+B)$ of the quadratic inequality (4.10) will be positive, provided that

$$\begin{aligned}\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}&<\frac{1}{16A}-B\\&=\frac{(\chi+\nu)(\lambda_1\delta)^{1/2}}{32C\Gamma(1/2)C_{1/2}}-\frac{C_p\Gamma(1/2)C_{1/2}}{(\lambda_1\delta)^{1/2}}\|\mathbf{F}\|_p=:\kappa.\end{aligned}$$

Thus, we observed that if $\mathbf{U}_0$ belongs to $B_{\mathbf{Z}_p^{1/2}}(0,\kappa)$, then according to the inequality (4.10) and to the continuity of $S(t)\mathbf{U}_0$ in $\mathbf{Z}_p^{1/2}$, the norm of $\|S(t)\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}$ is never going to exceed the value of the smallest root of the equation $2Az^2-z+2B+2\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}=0$. Thus, for $\mathbf{U}_0\in B_{\mathbf{Z}_p^{1/2}}(0,\kappa)$, $S(t)\mathbf{U}_0$ is a global solution and satisfies for all $t\ge 0$ the estimation

$$\begin{aligned}\|S(t)\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}&\le\frac{1-\sqrt{1-16A(\|\mathbf{U}_0\|_{\mathbf{Z}_p^{1/2}}+B)}}{4A}\\&<\frac{1}{4A}=\frac{(\chi+\nu)(\lambda_1\delta)^{1/2}}{8C\Gamma(1/2)C_{1/2}}=:R.\end{aligned}$$

$\square$

5 Restricted global attractors in Z_p^α

The purpose of this section is to show the existence of a restricted global attractor for $\{S(t)\}$. The construction of this attractor is based on the lemmas from the above section.

Theorem 5.1. *Let $p \in (3, +\infty)$, $\alpha \in [1/2, 1)$ and $\mathbf{F} \in \mathbf{L}_p$ be sufficiently small, according to assumptions in Lemma 4.3. Denote by $\{S(t)\}$ the local semiflow (where $S(t)\mathbf{U}_0 = \mathbf{U}(t, \mathbf{U}_0)$ for $t \in [0, \tau_{max}(\mathbf{U}_0)))$ of maximal fractional solutions of (1.4) defined on $\mathbf{Z}_p^\alpha$, and consider the constant κ given in Lemma 4.3. Then the following statements hold.*

(i) For

$$\mathbf{V}_p^\alpha = [\gamma^+(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)) \cap \mathbf{Z}_p^\alpha]_{\mathbf{Z}_p^\alpha},$$

all fractional solutions $S(t)\mathbf{U}_0$, with $\mathbf{U}_0 \in \mathbf{V}_p^\alpha$, are globally defined, $S(t)\mathbf{V}_p^\alpha \subset \mathbf{V}_p^\alpha$ $(t \geq 0)$, and the semigroup $\{S(t)\}$ restricted to $\mathbf{V}_p^\alpha$ has a global attractor $\mathcal{A}_{\alpha,p}$.

(ii) All attractors $\mathcal{A}_{\alpha,p}$ with $p \in (3, +\infty)$, $\alpha \in [1/2, 1)$ coincide, i.e.,

$$\mathcal{A}_{\alpha,p} = \omega_{\mathbf{Z}_p^{1/2}}(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)).$$

Proof. Due to the continuity of the semigroup, the estimate (4.5) remains valid for all $\mathbf{U}_0$ in $[\gamma^+(B_{\mathbf{Z}_p^{1/2}}(0,\kappa))]_{\mathbf{Z}_p^{1/2}}$. Using Lemma 2.2 with $\mathbf{Y} = \mathbf{Z}_p^{1/2}$, we can choose

$$\mathbf{V}_p^\alpha = [\gamma^+(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)) \cap \mathbf{Z}_p^\alpha]_{\mathbf{Z}_p^\alpha},$$

and as (4.4) and (4.5) are the estimations required by Lemma 2.2, that is, conditions (b) and (a) respectively, we obtain the existence of the global restricted attractor $\mathcal{A}_{\alpha,p}$ and therefore (i) is proved.

Observe that $\mathbf{V}_p^\alpha$ for $\alpha \in (1/2, 1)$ and $p \in (3, +\infty)$ is an unbounded, complete metric subspace of $\mathbf{Z}_p^\alpha$. When $\alpha = 1/2$ and $p \in (3, +\infty)$, then $\mathbf{V}_p^{1/2} = [\gamma^+(B_{\mathbf{Z}_p^{1/2}}(0,\kappa))]_{\mathbf{Z}_p^{1/2}}$ is a bounded, closed subset of $\mathbf{Z}_p^{1/2}$, and since $S(t)$ is compact for $t \geq 0$ then, $\omega_{\mathbf{Z}_p^{1/2}}(\mathbf{V}_p^{1/2})$ is nonempty, compact, invariant and attracts $\mathbf{V}_p^{1/2}$ (cf. [12, Theorem 4.2.2, Lemma 3.2.1]). Thus, the restricted attractor $\mathcal{A}_{1/2,p}$ claimed in part (i) coincides with $\omega_{\mathbf{Z}_p^{1/2}}(\mathbf{V}_p^{1/2})$.

In order to prove (ii), observe that

$$\begin{aligned}
\omega_{\mathbf{Z}_p^{1/2}}(\mathbf{V}_p^{1/2}) &= \cap_{s\geq 0} \left[\cup_{t\geq s} S(t)\gamma^+(B_{\mathbf{Z}_p^{1/2}}(0,\kappa))\right]_{\mathbf{Z}_p^{1/2}} \\
&= \cap_{s\geq 0} \left[\cup_{t\geq s} \cup_{r\geq t} S(t)B_{\mathbf{Z}_p^{1/2}}(0,\kappa)\right]_{\mathbf{Z}_p^{1/2}} \\
&= \cap_{s\geq 0} \left[\cup_{r\geq s} S(r)B_{\mathbf{Z}_p^{1/2}}(0,\kappa)\right]_{\mathbf{Z}_p^{1/2}} \\
&= \omega_{\mathbf{Z}_p^{1/2}}(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)).
\end{aligned}$$

For $p \in (3, +\infty)$, $\omega_{\mathbf{Z}_p^{1/2}}(B_{\mathbf{Z}_p^{1/2}}(0,\kappa))$ is a bounded set in $\mathbf{V}_p^{1/2}$. Using [13, Theorem 3.3.6] we have that $S(t)(\omega_{\mathbf{Z}_p^{1/2}}(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)))$ is a bounded set in $\mathbf{V}_p^\alpha$ with $\alpha \in [1/2, 1)$, and by invariance $\omega_{\mathbf{Z}_p^{1/2}}(B_{\mathbf{Z}_p^{1/2}}(0,\kappa)) \subset \mathcal{A}_{\alpha,p}$ for $1/2 \leq \alpha < 1$. To obtain the converse inclusion, it suffices to use the compactness of the embedding $\mathbf{Z}_p^\alpha \subset \mathbf{Z}_p^{1/2}$ for $\alpha \in (1/2, 1)$. □

Acknowledgements

P. Marín-Rubio was partially supported by MCI (Spain) MTM2015-63723-P and Junta de Andalucía Grant P12-FQM-1492. M. Poblete-Cantellano was partially supported by Universidad de Atacama, project 22256 DIUDA. M. Rojas-Medar was partially supported by Project MTM2012-32325, Spain and Fondecyt-Chile, Grant 1120260. F. Torres-Cerda was partially supported by Universidad de Atacama, project 22294 DIUDA.

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