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SOME REMARKS ON THREE-POINT AND FOUR-POINT BVP'S FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We are interested in the existence of a positive solution to the four-point boundary value problem

$$\begin{cases} y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1, \\ y(0) = \alpha y(\xi), \quad y(1) = \beta y(\eta), \end{cases} \tag{$*$}$$

where $0 < \xi \le \eta < 1$, $0 < \alpha < 1/(1-\xi)$, $0 < \beta < 1/\eta$ and $\alpha\beta(1-\beta) + (1-\alpha)(1-\beta\eta) > 0$. A result of B. Liu [22] is improved with an alternative, simplified proof. The same method is used to obtain extensions of earlier results by Ma [26], [27], Liu [20] [21], Liu and Yu [24], and others, on three-point boundary value problems, i.e, with $\alpha = 0$ in (*),

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1 Introduction

We are interested in the existence of positive solutions to the second-order boundary value problems (BVP) on the interval $[0, 1]$ with multi-point boundary conditions, i.e.

boundary conditions at one or both endpoints are specified in relation to some interior points in $[0,1]$.

In [22], Liu proved the existence of a positive solution of the following second-order nonlinear differential equation:

$$y''(t)+a(t)f(y(t))=0, \quad 0<t<1, \tag{1.1}$$

subject to the four-point boundary conditions

$$y(0)=\alpha y(\xi), \quad y(1)=\beta y(\eta), \tag{1.2}$$

where $0<\xi\le\eta<1$. Here $f(y)$ is a continuous nonnegative function on $[0,\infty)$, i.e. $f\in C([0,\infty),[0,\infty))$, and $a(t)$ is a continuous nonnegative function on $(0,1)$. It is also assumed that the following limits exist as extended real numbers:

$$f_0=\lim_{u\to 0}\frac{f(u)}{u}, \quad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}. \tag{1.3}$$

The main result of [22] is

Theorem A. (Liu [22]) *Suppose that*

(H_1) $0<\alpha<\dfrac{1}{1-\xi}, \quad 0<\beta<\dfrac{1}{\eta},$

(H_2) $\Lambda=\alpha\xi(1-\beta)+(1-\alpha)(1-\beta\eta)>0,$

(H_3) *There exists* $t_0\in[1/4,3/4]$ *such that* $a(t_0)>0$.

Then the BVP (1.1), (1.2) has at least one positive solution if either

$$f_0=0, \quad f_\infty=\infty, \tag{1.4}$$

or

$$f_0=\infty, \quad f_\infty=0. \tag{1.5}$$

The two inequalities in (H_1) were shown by Liu to be necessary conditions for the existence of a positive solution to the BVP in question. They are simple consequences of the concavity of the function $y(t)$. Condition (H_2), on the other hand, has always been presumed in all subsequent known criteria for the existence of positive solutions of four-point problems. It is needed to make the proof work, in particular, in defining the integral operator K discussed below and in showing that it maps positive functions to positive functions. In Section 3 we will prove that, in fact, the condition is also a necessary condition for the existence of positive solution.

The main purpose of this note is to give an alternative proof which simplifies the laborious calculations given in [22] that totaled 7 pages plus another 4 more pages on preliminary lemmas. Furthermore, our proof will show that the somewhat awkward assumption (H_3) can be replaced by the more natural condition that $a(t) \not\equiv 0$.

Theorem A above was further improved in [22] as follows:

Theorem B. (Liu [22]) *Assume that (H_1), (H_2), and (H_3) hold. Then the BVP (1.1), (1.2) has at least one positive solution if either*

$$f_0 < \Lambda_1, \quad f_\infty > \Lambda_2, \tag{1.6}$$

or

$$f_0 > \Lambda_2, \quad f_\infty < \Lambda_1, \tag{1.7}$$

where

$$\Lambda_1 = \Lambda \left((|1-\alpha| + \alpha\xi) \int_0^1 (1-s)a(s)\,ds \right)^{-1}, \tag{1.8}$$

$$\Lambda_2 = 16\Lambda \left(\min[1-\xi-\beta(\eta-\xi), 4\xi(1-\eta), 4\xi(1-\beta\eta), \xi]a_0\right)^{-1}, \tag{1.9}$$

and

$$a_0 = \int_{1/4}^{3/4} a(s)\,ds. \tag{1.10}$$

Our main result is

Theorem 1 *Suppose that (H_1), (H_2) hold and $a(t) \not\equiv 0$. Then the BVP (1.1), (1.2) has at least one positive solution if either*

$$f_0 < \overline{\Lambda}_1, \quad f_\infty > \overline{\Lambda}_2, \tag{1.11}$$

or

$$f_0 > \overline{\Lambda}_2, \quad f_\infty < \overline{\Lambda}_1, \tag{1.12}$$

where

$$\overline{\Lambda}_1 = \Lambda \left((1-\alpha+\alpha\xi) \int_0^1 (1-s)a(s)\,ds \right)^{-1}, \tag{1.13}$$

$$\overline{\Lambda}_2 = \Lambda \left((1-\alpha+\alpha\xi)\gamma\eta \int_\eta^1 (1-s)a(s)\,ds \right)^{-1}, \tag{1.14}$$

and

$$\gamma = \min\left(\eta, \beta\eta, \frac{\beta(1-\eta)}{1-\beta\eta}\right) < 1. \tag{1.15}$$

A corollary of Theorem 1, obtained by choosing $\alpha = 0$, $\xi = \eta$, is the following result of Liu [20] for the three-point BVP:

Theorem C. (Liu [20]) *Let $\Delta = 1 - \beta\eta > 0$ and suppose that either*

$$f_0 < \Delta_1, \quad f_\infty > \Delta_2, \tag{1.16}$$

or

$$f_0 > \Delta_1, \quad f_\infty < \Delta_2, \tag{1.17}$$

where

$$\Delta_1 = \Delta \left(\int_0^1 (1-s)a(s)\,ds \right)^{-1}, \tag{1.18}$$

and

$$\Delta_2 = \Delta \left(\gamma\eta \int_\eta^1 (1-s)a(s)\,ds \right)^{-1}, \tag{1.19}$$

Then the three-point BVP (1.1) with boundary condition

$$y(0) = 0, \quad y(1) = \beta y(\eta) \tag{1.20}$$

has a positive solution.

The three-point BVP (1.1) (1.20) is the most studied topic in this subject area due to its simplicity and its origin from the study of semi-linear elliptic equations. See, for example, Hai [12], Gupta [8], [9], [10], Gupta and Trofimchuk [11], Ma [26], [27], [28], Marano [31], and Ren and Ge [32].

We next use the method of proof of Theorem 1 to give an extension of Theorem C to nonhomogeneous boundary conditions of the form

$$y(0) = 0, \quad y(1) = \beta y(\eta) + b,\ b \geq 0. \tag{1.21}$$

Theorem 2 *Suppose that $\Delta = 1 - \beta\eta > 0$ and either*

$$f_0 < \frac{1}{2}\Delta_1, \quad f_\infty > \Delta_2, \tag{1.22}$$

or

$$f_0 > \Delta_2, \quad f_\infty < \frac{1}{2}\Delta_1, \tag{1.23}$$

where Δ_1 and Δ_2 are given by (1.18) and (1.19). Then the three-point BVP (1.1), (1.21) has a positive solution if $b \geq 0$ is sufficiently small.

Using the same method of proof of Theorem 2, we can also prove a similar result when the Dirichlet boundary condition at the left endpoint is replaced by the Neumann boundary condition

$$y'(0) = 0, \quad y(1) = \beta y(\eta) + b, \ b \geq 0. \tag{1.24}$$

Theorem 3 *Suppose that $0 < \beta < 1$ and either*

$$f_0 < \frac{1}{2}\Delta_3, \quad f_\infty > \Delta_4, \tag{1.25}$$

or

$$f_0 > \Delta_4, \quad f_\infty < \frac{1}{2}\Delta_3, \tag{1.26}$$

where Δ_3 and Δ_4 are given by

$$\Delta_3 = (1-\beta)\left(\int_0^T 1(1-s)a(s)\,ds\right)^{-1}, \tag{1.27}$$

$$\Delta_4 = \frac{1-\beta}{\mu}\left(\int_\eta^1 (1-s)a(s)\,ds + (1-\eta)\int_0^\eta a(s)\,ds\right)^{-1}, \tag{1.28}$$

and

$$\mu = \frac{\beta(1-\eta)}{1-\beta\eta}. \tag{1.29}$$

Then the three-point BVP (1.1), (1.24) has a positive solution if $b \geq 0$ is sufficiently small.

When $b = 0$, Theorem 3 reduces to another result of Liu [21], the proof of which is 9 pages of laborious computations.

2 Proofs of Theorems

The proof of the theorems in this paper is based upon the well-known fixed point theorem on cones due to Krasnoselskii and Guo [5] which is cited below.

Theorem D. (Krasnoselskii and Guo [5]) *Let E be a Banach space and $P \subseteq E$ be a positive cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\Omega_1 \subseteq \Omega_2$ and $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that*

(i) $||Au|| \leq ||u||$, $u \in P \cap \partial\Omega_1$ *and*
$||Au|| \geq ||u||$, $u \in P \cap \partial\Omega_2$; *or*

(ii) $||Au|| \geq ||u||$, $u \in P \cap \partial\Omega_1$ *and*
$||Au|| \leq ||u||$, $u \in P \cap \partial\Omega_2$.

Here $\partial\Omega_1$ and $\partial\Omega_2$ denote the boundaries of Ω_1 and Ω_2, respectively. Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Krasnoselskii's fixed point theorem is based on appropriate applications of topological degree theory, a favorite subject in nonlinear functional analysis. We refer the interested reads to Krasnoselskii [16], Guo [5], Guo and Lakshmikantham [6], Kwong [17], Webb [34], and Ge [3].

In addition to the Guo-Krasnoselskii theorem above, we need a technical lemma concerning lower bounds of positive solutions of the BVP (1.1), (1.2). The proof of this lemma is given in Ma [26, Lemma 4]. We give here an alternative and more intuitive geometric proof for the sake of completeness.

Lemma 1 *Let $y(t) \in C[0,1]$, $y(t) \geq 0$ and $y''(t) \leq 0$ satisfying $y(1) = \beta y(\eta)$. Then*

$$\min\{y(t) : \eta \leq t \leq 1\} \geq \gamma ||y||, \tag{2.1}$$

where $||\cdot||$ denotes the sup-norm of $C[0,1]$ and γ is given by (1.15).

Proof. Let O be the origin $(0,0)$, and A, B denote the points $(\eta, y(\eta))$ and $(1, y(1))$ that lie on the solution curve, respectively; see Figure 1 below. Let the straight line BA, when extended, cut the x-axis at the point C, and the straight line OA cuts the line $x = 1$ at the point D. Due to concavity, the solution curve lies below the the line segments CA and AD.

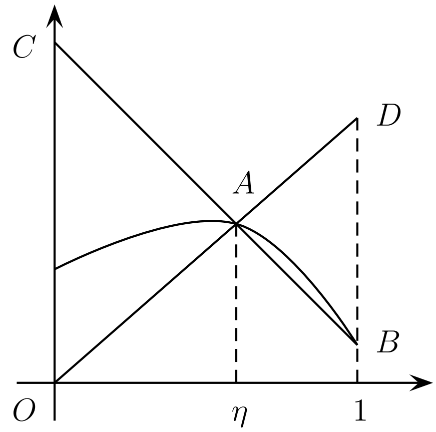

Figure 1.

It is easy to compute the coordinates of the points $C = (0, (1-\beta\eta)y(\eta)/(1-\eta))$ and $D = (1, y(\eta)/\eta)$. Again by concavity, the minimum of $y(t)$ in $[\eta, 1]$ is attained either at $x = \eta$ or at $x = 1$. We consider two cases.

Case 1: $\beta \geq 1$. It is easy to see that the third number on the righthand side of (1.15) is greater than 1 and so it is greater than η. The second number $\beta\eta$ is also greater than η. Hence, $g = \eta$. The point B is now higher than A which is in turn higher than C. Thus, $\min\{y(t) : \eta \leq t \leq 1\} = y(\eta)$ and $||y||$ is less than the height of D, which is $y(\eta)/\eta$, from which (2.1) follows.

Case 2: $\beta < 1$. Then $\min\{y(t) : \eta \leq t \leq 1\} = y(1) = \beta y(\eta)$ and $||y||$ is less than the larger of the heights of C and D:

$$||y|| \leq \max\left\{\frac{1}{\eta}, \frac{1-\beta\eta}{1-\eta}\right\} y(\eta). \tag{2.2}$$

In the definition of γ, since $\eta > \beta\eta$, we can throw away the first number on the righthand side of (1.15). Thus,

$$\gamma = \min\left\{\eta, \frac{(1-\eta)}{1-\beta\eta}\right\}\beta = \frac{\beta}{\max\left\{\frac{1}{\eta}, \frac{1-\beta\eta}{1-\eta}\right\}} \leq \frac{\beta y(\eta)}{||y||}, \tag{2.3}$$

from which (2.1) follows. ∎

The proof originated in Ma's paper [26] is based upon an integral operator $I[y]$ defined for any $y \in C[0,1]$ by

$$I(t) = I[y](t) = \int_0^t (t-s)a(s)f(y(s))\,ds. \tag{2.4}$$

Clearly $I(0) = I'(0) = 0$, and it satisfies the differential equation $I''(t) = a(t)f(y(t))$. So a solution of the operator equation $y + I[y] = 0$ is the solution of the initial value problem of equation (1.1) satisfying zero initial conditions $y(0) = y'(0) = 0$.

Following the approach in Gupta [5] for the three-point BVP's, Liu [21] defined the nonlinear operator A in terms of I as

$$Ay(t) = -I(t) - \frac{\alpha}{\Lambda}l_1(t)I(\xi) - \frac{\beta}{\Lambda}l_2(t)I(\eta) + \frac{1}{\Lambda}l_2(t)I(1), \tag{2.5}$$

where

$$l_1(t) = (\beta - 1)t + (1 - \beta\eta) \tag{2.6}$$

and

$$l_2(t) = (1-\alpha)t + \alpha\xi. \tag{2.7}$$

Using $-I(t)$ instead of $I(t)$ means that now we are dealing with the operator equation $y = A[y]$, thus enabling us to adopt the fixed point methodology. Linear functions l_1 and l_2 are used to modify $I(t)$ so that $A(t)$ now satisfies the four-point boundary condition (1.2). These functions serve the purpose because they are the fundamental solutions of the homogeneous differential equation $y'' = 0$. Adding an arbitrary linear function to $-I(t)$ amounts to finding the general solution. The specific linear functions used in (2.5) are determined by choosing the appropriate arbitrary constants so that the resulting solution satisfies the desired boundary conditions.

We thus conclude that a fixed point of the operator A furnishes a solution (not necessarily positive) to the four-point BVP (1.1), (1.2). The fixed point does produce a positive solution if it lies within the positive cone

$$P = \{y(t) : y \in C[0,1], y(t) \geq 0\} \tag{2.8}$$

and $y(t) \not\equiv 0$. In particular, a fixed point of A resulted from an application of the Guo-Krasnoselskii theorem cited above will yield a positive solution to the BVP.

We now wish to introduce an alternative formulation of the operator A. For a given $y \in C[0,1]$, define the operator $K : C[0,1] \to C[0,1]$ by

$$K(t) = K[y](t) = \int_0^1 k(t,s)a(s)f(y(s))\,ds, \tag{2.9}$$

where

$$k(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.10}$$

is the Green's function of the Dirichlet BVP, namely, (1.1) with the boundary conditions $y(0) = y(1) = 0$. Note that

$$K''(t) = -I''(t), \quad K(0) = K(1) = 0, \quad K'(0) = I(1). \tag{2.11}$$

It thus follows from (2.9) that

$$-I(t) = K(t) - K'(0)t = K(t) - I(1)t. \tag{2.12}$$

Using (2.12) in (2.5), we find

$$Ay(t) = K(t) + \frac{\alpha}{\Lambda} l_1(t)K(\xi) + \frac{\beta}{\Lambda} l_2(t)K(\eta). \tag{2.13}$$

Observe that both $I(t)$ and K(t) are positive operators in the sense that for any $y \in C[0,1]$, $y(t) \geq 0$, $t \in [0,1]$, we have $I(t) \geq 0$ and $K(t) \geq 0$. Hence, the same is true for the operator A if we can show that $l_1(t)$ and $l_2(t)$ are nonnegative over $[0,1]$. This is a well-known fact which is stated in the following lemma for easy reference.

Lemma 2 *For $t \in [0,1]$, $l_1(t) \geq 0$ and $l_2(t) \geq 0$. As a corollary, the operator A maps nonnegative functions to nonnegative functions. Furthermore, the operator is completely continuous.*

Proof. Let us first prove that $l_1(t) \geq 0$ for $t \in [0,1]$. We consider two cases. First, when $\beta \leq 1$. Then the first term on the righthand side of (2.6) is negative. So $l_1(t)$, over $[0,1]$, attains its minimum at $t = 1$, and we have

$$l_1(t) \geq (\beta - 1) + (1 - \beta\eta) = \beta(1 - \eta) \geq 0. \tag{2.14}$$

Case 2 is when $\beta > 1$, then $l_1(t)$, over $[0,1]$ attains its minimum at $t = 0$, and we have

$$l_1(t) \geq (1 - \beta\eta) \geq 0 \tag{2.15}$$

by the second assumption in (H_1). The proof for $l_2(t) \geq 0$ is similar.

The complete continuity of A is a well-known fact. ∎

In other words, the operator A maps the cone P of positive functions into itself. We remark that both definitions of the operator A as defined by (2.5) and (2.13) give rise to a solution of the BVP. However, the formulation of (2.13) using the operator $K(t)$ has the advantage of showing directly that A is a positive operator. This simplifies the subsequent proofs considerably.

Proof of Theorem 1. Let

$$\Omega_1 = \{y \in C[0,1] : ||y|| < \rho_1\} \tag{2.16}$$

and

$$\Omega_2 = \{y \in C[0,1] : ||y|| < \rho_2\} \tag{2.17}$$

with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. When $y \in \partial\Omega_i$, $||y|| = \rho_i$, $i = 1, 2$.

Let $y \in P \cap \partial\Omega_1 = \{y \in P : ||y|| = \rho_1\}$. From the definition of A as given by (2.5) and the fact that $I(t) \geq 0$ in $[0,1]$, we have

$$\begin{aligned} Ay(t) &\leq \frac{1}{\Lambda}(1-\alpha+\alpha\xi)I(1) \\ &= \frac{1}{\Lambda}(1-\alpha+\alpha\xi)\int_0^1 (1-s)a(s)f(y(s))\,ds. \end{aligned} \tag{2.18}$$

Assumption (1.11) of Theorem 1 now ensures that there exists $\rho_1 > 0$ sufficiently small so that $f(y) \leq f_0 + \varepsilon \leq \overline{\Lambda}_1$ for all y, $0 \leq y \leq \rho_1$ where $\varepsilon = \overline{\Lambda}_1 - f_0 > 0$. Using this in (2.18) and the definition of $\overline{\Lambda}_1$ by (1.13), we obtain for all $t \in [0,1]$ that

$$\begin{aligned} Ay(t) &\leq \frac{1}{\Lambda}(1-\alpha+\alpha\xi)\overline{\Lambda}_1\rho_1\int_0^1 (1-s)a(s)\,ds \\ &\leq ||y||. \end{aligned} \tag{2.19}$$

Hence, $||Ay|| \leq ||y||$ for all $y \in P \cap \partial\Omega_1$.

Now let $y \in P\cap\partial\Omega_2 = \{y \in P : ||y|| = \rho_2\}$. Again by the assumption that $f_\infty > \overline{\Lambda}_2$, there exists $\rho_2 > \rho_1 > 0$ such that $f(y) \geq (f_\infty - \varepsilon)y \geq \overline{\Lambda}_2 y$ for all $y \in P \cap \partial\Omega_2$ where $\delta = f_\infty - \overline{\Lambda}_2 > 0$. Now use (2.13) for $Ay(t)$ (instead of (2.5)) given in terms of the operator K and set $t = \eta$, we find, by (2.1), that

$$\begin{aligned} Ay(\eta) &= \frac{1-\alpha+\alpha\xi}{\Lambda}K(\eta) + \frac{\alpha(1-\eta)}{\Lambda}K(\xi) \\ &\geq \frac{1-\alpha+\alpha\xi}{\Lambda}\eta\int_\eta^1 (1-s)a(s)f(y(s))\,ds \\ &\geq \frac{1-\alpha+\alpha\xi}{\Lambda}\eta\gamma\overline{\Lambda}_2||y||\int_\eta^1 (1-s)a(s)\,ds \\ &= ||y||. \end{aligned} \tag{2.20}$$

So $||Ay|| \geq ||y||$ for all $y \in P \cap \partial\Omega_2$. Hence A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, establishing the existence of a positive solution of the four-point BVP (1.1), (1.2).

The proof of the second part of Theorem 1 is similar and uses the second part of the Guo-Krasnoselskii theorem. ∎

Proof of Theorem 2. In case of the three-point BVP (1.1), (1.21), the operator A defined by (2.5) with $\alpha = 0$ becomes

$$Ay(t) = -I(t) + \frac{t}{1-\beta\eta}\left(I(1) - \beta I(\eta) + b\right). \tag{2.21}$$

Since $I(t) \geq 0$ for all $t \in [0,1]$, we obtain from (2.21)

$$Ay(t) \leq \frac{1}{\Delta}\left(I(1) + b\right). \tag{2.22}$$

Assumption (1.23) implies that there exists $\rho_3 > 0$ such that $f(y) \leq f_0 + \varepsilon \leq \frac{1}{2}\Delta_1$ for all $y \in \Omega_3 = \{y \in P : ||y|| = \rho_3\}$ where $\varepsilon = \frac{1}{2}\Delta_1 - f_0$. From (2.22), we note that

$$\begin{aligned} Ay(t) &\leq \frac{1}{\Delta}\left(\frac{1}{2}\Delta_1\rho_3 \int_0^1 (1-s)a(s)\, ds + b\right) \\ &= \frac{1}{2}\rho_3 + \frac{b}{\Delta} \\ &\leq \rho_3, \end{aligned} \tag{2.23}$$

if b is small, i.e. $0 \leq b \leq \Delta\rho_3/2$. Since (2.23) holds for all $t \in [0,1]$, we obtain $||Ay|| \leq ||y||$.

When $f_\infty > \Delta_2$, there exists $\rho_4 > \rho_3 > 0$ such that $f(y) \geq (f_\infty - \varepsilon)y = \Delta_2 y$, where $\varepsilon = f_\infty - \Delta_2$ for all $y \in \Omega_4 = \{y \in P : ||y|| = \rho_4\}$. We rewrite the operator A given by (2.21) in terms of the operator $K(t)$ as

$$Ay(t) = K(t) + \frac{\beta t}{1-\beta\eta}\left(K(\eta) + \frac{b}{\beta}\right). \tag{2.24}$$

Setting $t = \eta$ in (2.24) and noting that $b \geq 0$, we obtain, from (2.1) and Lemma 1,

$$\begin{aligned} Ay(\eta) &\geq \frac{1}{\Delta}K(\eta) \\ &\geq \frac{1}{\Delta}\left(\eta \int_\eta^1 (1-s)a(s)f(y(s))\, ds\right) \\ &\geq \frac{1}{\Delta}\left(\Delta_2\eta\gamma \int_\eta^1 (1-s)a(s)\, ds\right)||y||. \end{aligned} \tag{2.25}$$

By the definition of Δ_2, we conclude from the above that $||Ay|| \geq ||y||$ for all $y \in P \cap \partial\Omega_4 = \{y \in P : ||y|| = \rho_4\}$. Now apply the Guo-Krasnoselskii theorem, we conclude that the operator A has a fixed point in $P \cap (\overline{\Omega}_4 \setminus \Omega_3)$ which is the desired positive solution of BVP (1.1), (1.21).

The proof of the second part of Theorem 2 is similar in this so-called sublinear case. We apply the compression part of the Guo-Krasnoselskii theorem as in the case of the second part of Theorem 1. ∎

We now turn to the BVP (1.1), (1.24), a result proved by Ma [27] using different methods.

Proof of Theorem 3. We introduce the operator B defined by

$$By(t) = -I(t) + \frac{1}{1-\beta}\left(I(1) - I(\eta)\beta + b\right). \tag{2.26}$$

Also like the operator A given in terms of $I(t)$ or $K(t)$, the operator B is completely continuous. It is easy to verify that a fixed point of B is a solution of (1.1), (1.24).

Note that $(By(t))' = -I'(t) \leq 0$. Since $By(t)$ is monotone decreasing in $[0, 1]$, we have $By(0) \geq B(y(t) \geq By(1)$. Setting $t = 1$ in (2.26), we have

$$By(t) \geq B(1) = \frac{\beta}{1-\beta}\left(I(1) - I(\eta) + \frac{b}{\beta}\right), \tag{2.27}$$

because $I(t)$ is nondecreasing and $b \geq 0$. From (2.27), we conclude that $B : P \to P$.

Let $y(t)$ be a solution of $By = y$ in P. Since $y''(t) = -I(t) \leq 0$, $y'(t)$ is monotone decreasing. So $y'(t) \leq y'(0) = 0$ and $y(t)$ is also monotone decreasing. If $y(1) < 0$, then $0 < \beta < 1$, implying $y(1) = \beta y(\eta) > y(\eta)$ which contradicts the fact that $y(t)$ is decreasing in $[0, 1]$. Thus,

$$y(0) = ||y|| = \max\{y(t) : 0 \leq t \leq 1\} > \min\{y(t) : 0 \leq t \leq 1\} = y(1) \geq 0. \tag{2.28}$$

We now show that by the concavity of $y(t)$, in fact, $y(1) > 0$. For $0 < \eta < 1$, we have

$$\frac{y(1)}{\beta} = y(\eta) \geq (1-\eta)y(0) + \eta y(1), \tag{2.29}$$

which reduces to

$$y(1)(1 - \beta\eta) \geq \beta(1-\eta)||y|| \tag{2.30}$$

or

$$\min\{y(t) : 0 \leq t \leq 1\} \geq \mu||y||, \tag{2.31}$$

where $\mu = \beta(1-\eta)/(1-\beta\eta)$ as given in (1.29).

Suppose that $f_0 > \Delta_3$. Then there exists $\rho_5 > 0$ such that $f(y) \geq (f_0 - \varepsilon)y = \Delta_3 y$ where $\varepsilon = \Delta_3 - f_0 > 0$ for all $y \in \Omega_5 = \{y \in P : ||y|| \leq \rho_5\}$. Evaluate $B(\eta)$ by (2.26) and apply (2.31), and we find that

$$\begin{aligned} B(\eta) &\geq \frac{1}{1-\beta}(I(1) - I(\eta) + b) \\ &\geq \frac{1}{1-\beta}(I(1) - I(\eta)) \\ &\geq \frac{1}{1-\beta}\Delta_3\mu||y||\left\{\int_\eta^1 (1-s)a(s)\,ds + (1-\eta)\int_0^\eta a(s)\,ds\right\}. \end{aligned} \tag{2.32}$$

Using the definition of Δ_3 in (1.27) and by (2.32), we conclude that $||By|| \geq ||y||$ for all $y \in P \cap \partial\Omega_5$.

Next we consider $f_\infty < \Delta_4/2$. There exists ρ_6, $0 < \rho_5 < \rho_6$ such that $f(y) \leq (f_\infty + \varepsilon)y = \Delta_4 y/2$ where $\varepsilon = \Delta_4/2 - f_\infty$ for all $y \in \Omega_6 = \{y \in P : ||y|| \geq \rho_6\}$. We now estimate $By(t)$ using (2.26) for $y \in P \cap \partial\Omega_6$.

$$\begin{aligned} B(y) &\leq \frac{1}{1-\beta}I(1) + \frac{b}{1-\beta} \\ &\leq \frac{1}{1-\beta}\left(\frac{1}{2}\Delta_4||y||\int_0^1 (1-s)a(s)\,ds + b\right). \end{aligned} \tag{2.33}$$

By the definition of Δ_4, we have from (2.33)

$$By(t) \leq \frac{\rho_6}{2} + \frac{b}{1-\beta} \leq \rho_6 \tag{2.34}$$

if $0 \leq 2b \leq (1-\beta)\rho_6$. Hence, for $y \in P \cap \partial\Omega_6$, (2.34) shows that $||By|| \leq ||y||$.

An application of the Guo-Krasnoselskii theorem to the operator B now gives a fixed point $y \in P \cap \left(\overline{\Omega}_6 \setminus \Omega_5\right)$. Since $y \notin \Omega_6$, by (2.31), we obtain a positive solution to the BVP (1.1) (1.24). ∎

3 The Necessity of Condition (H_2)

In this section, we show that condition (H_2) holds for any concave nonlinear function. We assume that ξ and η are any two numbers in the open interval $(0,1)$, without requiring that $\xi < \eta$.

Lemma 3 *Let $y(t)$ be any concave nonlinear function defined on $[0,1]$ such that the four-point (1.2) is satisfied, with $\xi, \eta \in (0,1)$. Then (H_2) holds.*

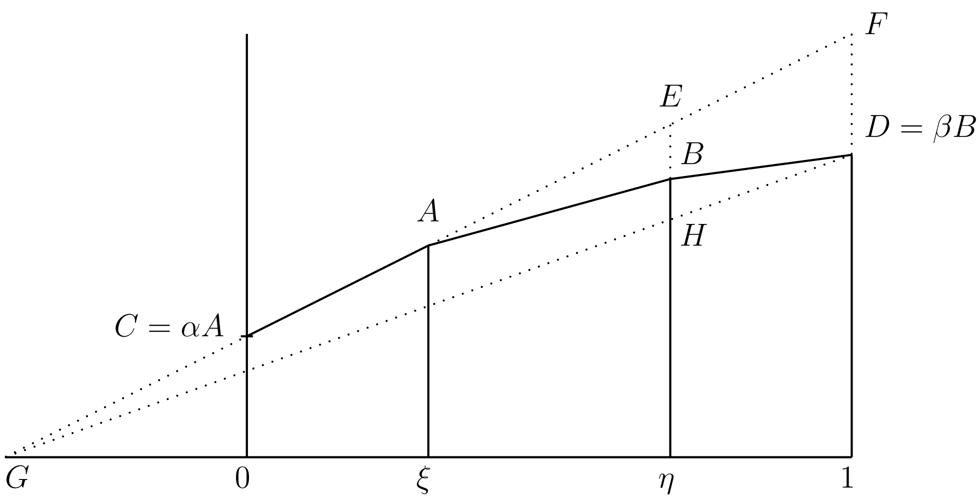

Figure 2.

Proof. It may not seem obvious, but it is not difficult to show that condition (H_2) is invariant under a reflection of $[0,1]$, in the sense that if (H_2) holds for $y(t)$, then it holds for $\bar{y}(t) = y(1-t)$, with the new $\bar{\xi}$ and $\bar{\eta}$ defined by $1-\xi$ and $1-\eta$, respectively, and the new $\bar{\alpha} = \beta$, $\bar{\beta} = \alpha$.

Concavity implies that if one of α and β is larger than or equal to 1, the other must be strictly less than 1. Using a reflection if necessary, we may assume without loss of generality that $\alpha < 1$. In Figure 2, this means that the line joining the points A and C will cut the x-axis to the left of the origin O.

Let us first consider the case $\xi < \eta$ as in Figure 2. Extend the line joining $C = \alpha A$ and A, to cut the x-axis at G and the vertical lines $x = \xi$ and $x = 1$ at E and F, respectively. Using the two known points $(0, \alpha A)$ and (ξ, A) on the line, we can find the equation of the straight line as

$$y = \frac{(\alpha\xi + (1-\alpha)x)A}{\xi}. \tag{3.1}$$

Putting $x = \eta$ and $x = 1$ into this equation, respectively, we see that

$$E = \frac{(\alpha\xi + (1-\alpha)\eta)A}{\xi}, \tag{3.2}$$

and

$$F = \frac{(\alpha\xi + (1-\alpha))A}{\xi}. \tag{3.3}$$

Draw the line GD, cutting the line ηB at H. Due to concavity of the piecewise linear curve $CABD$, we see that B lies above H. Hence,

$$\frac{1D}{\eta B} \leq \frac{1D}{\eta H} = \frac{1G}{\eta G} = \frac{1F}{\eta E}, \tag{3.4}$$

where $1D$ represents the length of the line $1 \to D$, etc. The last two equalities are due to the similarity of the two pairs of triangles $G1D$, $G\eta H$, and $G1F$, $G\eta E$.

Hence,

$$\beta \leq \frac{F}{E} = \frac{\alpha\xi + 1 - \alpha}{\alpha\xi + (1-\alpha)\eta}. \tag{3.5}$$

After simplification, this is equivalent to $\Lambda \geq 0$.

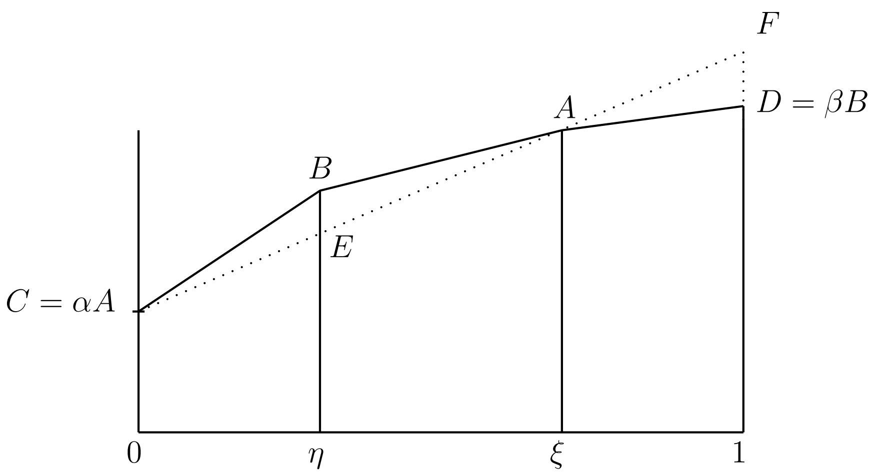

Figure 3.

The case $\xi > \eta$ (shown in Figure 3) is actually even simpler. Equations (3.2) and (3.3) still hold. It is obvious that inequality $1D/\eta B \leq 1F/\eta E$ (compare (3.4)) holds, by simply noticing $\eta E \leq \eta B$ and $1F \geq 1D$. Hence, (3.5) holds and so does the inequality $\Lambda \geq 0$. ∎

4 Examples and Remarks

In this last section, we discuss two examples studied by Liu [21], [22] and make a few remarks relating our results to those in existing literature.

Example 1. (Liu [21]) Consider the four-point boundary value problem:

$$y'' + (1-t)^2 aye^{2y}(b + e^y + e^{2y}) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$y(0) = y\left(\frac{1}{4}\right), \quad y(1) = \frac{1}{2}y\left(\frac{1}{2}\right), \tag{4.2}$$

where a and b are positive numbers. Here $a(t) = (1-t)^2$, $\alpha = 1$, $\xi = 1/4$, $\beta = \eta = 1/2$ and $f(u) = aue^{2u}(b + e^u + e^{2u})^{-1}$ We find $\Lambda = 1/8$, $\Lambda_1 = \overline{\Lambda}_1 = 2$ and

$$\overline{\Lambda}_2 = 48 < \Lambda_2 = 192. \tag{4.3}$$

(Note that in [21], a different value is given for Λ_2) Also, we have $f_0 = a(2+b)^{-1,}$ $f_\infty = a$. So for $a = 49$, $b > 23$, we have $f_0 < \overline{\Lambda}_1 = \Lambda_1 = 2$ and $f_\infty > \overline{\Lambda}_2$. By Theorem 1, we conclude that the BVP (4.1) (4.2) has a positive solution whereas for this same example, the results in [21] require $a > 192$ and $b > 190$. (The values as stated in [21], namely, $a = 64$ and $b > 30$, have been corrected.) ∎

Example 2. Consider the three-point problem

$$y'' + 3ty\left(\frac{1}{2} + \frac{c}{1+y}\right) = 0, \quad 0 < t < 1, \tag{4.4}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{4}y\left(\frac{1}{3}\right), \tag{4.5}$$

where c is a positive constant. A similar example was given by Liu [22]. Here $a(t) = 3t$, $\beta = 1/4$, $\eta = 1/3$ and $\mu = 2/11$. Using (1.27) and (1.28), we find $\Delta_3 = 3/2$ and $\Delta_4 = 297/104$. Here $f_0 = 1 + c$, $f_\infty = 1/2$. So $f_\infty < \Delta_3/2$ and $f_0 > \Delta_4$ if $c > 2$. By Theorem 3, we conclude that the above boundary value problem has a positive solution. ∎

We now give a few remarks relating our results to the existing literature.

1. As we have mentioned earlier, Liu [22] showed that assumption (H_1) is necessary for the existence of positive solution to the BVP (1.1) (1.2). In this paper, we have also shown that (H_2) is necessary. The requirement that $\Lambda \neq 0$ corresponds to the non-resonance of BVP (1.1) (1.2) and is equivalent to the solvability of the linear functions $l_1(t)$ and $l_2(t)$ as given by (2.6) and (2.7). When $y(t) = Ay(t)$ is positive as given in (2.13), it is necessary that $\Lambda^{-1}l_1(t)$ and $\Lambda^{-1}l_2(t)$ are positive. Since assumption (H_1) implies that $l_1(t)$, $l_2(t)$ are positive in $[0, 1]$, so $\Lambda > 0$.

2. Theorems 2 and 3 deal with three-point BVP with Dirichlet and Neumann boundary conditions at the left endpoint. They extend results of Ma [26] and Liu [21] when $b = 0$. In the proof of Theorems 1 and 2, we have used the lower estimate given by (1.15) which is valid only if $y \in C[0, 1]$, $y''(t) \le 0$ and $y(1) = \beta y(\eta)$ so that Lemma 1 is applicable. In fact the positivity of operators A and B given by (2.5) and (2.26) implies that $A, B : P \to P$. Hence, strictly speaking, the application of the Guo-Krasnoselskii fixed point theorem is not to P but to the subcone P_1 defined by
$$P_1 = \{y \in P : y''(t) \le 0, y(1) = \beta y(\eta)\}. \tag{4.6}$$
This fact was also tacitly assumed in the proofs by Liu [20], [21], [22], whereas in [26], Ma applied the fixed point theorem to the cone $Q = \{y \in P : \min_{0\le t\le 1} y(t) \ge \gamma ||y||\}$.

3. Theorem 1 can also be formulated to allow nonhomogeneous boundary conditions at the right endpoint $t = 1$ for the four-point BVP's similar to Theorems 2 and 3 in the three-point case. We leave the details to the readers.

4. We call the readers' attention to that the study of three-point BVP's has been extended to that of multi-point BVP's with Dirichlet and Neumann boundary conditions at the left endpoint, i.e. $y(0) = 0$ and $y'(0) = 0$, respectively. See, for example, Ma [25], [30], Graef and Kong [4], and Ge [3].

5. In our earlier paper [19] (see also [18]), we have used the "shooting method" to treat multi-point BVP's and obtain sharp conditions in the three-point case with $\Delta_1 = \Delta_2 = \lambda$, where λ is the smallest eigenvalue of the linear problem, i.e. when $f(y) = y$. The simple classical shooting method is not adequate to yield a similar result for the four-point case and we hope to return to the question at a later date.

6. Our method can also be adopted to simplify the proofs of known results on existence of multiple solutions, such as those in the papers of He and Ge [13], Henderson [14], Henderson and Thompson [15], and Raffoul [33].

7. We have not discussed the case when the nonlinear term in equation (1.1) contains the first order derivative, i.e. considering $f(y(t), y'(t))$ instead of $f(y(t))$. When

the dependence on $y'(t)$ is bounded above and below, similar conclusions can be formulated (see our earlier work [19]). If the dependence is of the Nagumo type, other methods are required (see e.g. Guo and Ge [7] and Ge [3]).

8. Our results and others related to ours are concerned with the so-called non-resonance cases, namely, when the boundary conditions (1.2) and (1.20) satisfy $\Lambda \neq 0$ and $\Delta \neq 0$, respectively. For existence results on the resonance case, i.e. $\Lambda = \Delta = 0$, see, for example, Feng and Webb [2], Ma [29], Liu and Yu [24], [23] and Bai, Li and Ge [1].

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