Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **54**, 1–16; doi: 10.14232/ejqtde.2015.1.54 http://www.math.u-szeged.hu/ejqtde/

Asymptotic properties of solutions of Riccati matrix equations and inequalities for discrete symplectic systems

Roman Šimon Hilscher[✉]

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic

Received 29 May 2015, appeared 24 August 2015
Communicated by Josef Diblík

Abstract. In this paper we study the asymptotic properties of the distinguished solutions of Riccati matrix equations and inequalities for discrete symplectic systems. In particular, we generalize the inequalities known for symmetric solutions of Riccati matrix equations to Riccati matrix inequalities. We also justify the definition and properties of the distinguished solution and the recessive solution at minus infinity by relating them to their counterparts at plus infinity.

Keywords: symplectic system, Riccati matrix equation, Riccati matrix inequality, recessive solution, distinguished solution, nonoscillation.

2010 Mathematics Subject Classification: 39A21, 39A22.

1 Introduction

In this paper we consider a discrete symplectic system

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \qquad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k \tag{S}$$

for $k \in [0,\infty)_{\mathbb{Z}} := [0,\infty) \cap \mathbb{Z}$, where X_k, U_k and $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are real $n \times n$ matrices such that the $2n \times 2n$ coefficient matrix in (S) is symplectic. This means that

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \qquad \mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \qquad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.1}$$

With the notation $Z = (X^T,\, U^T)^T$ we can then write system (S) as the first order linear system $Z_{k+1} = \mathcal{S}_k Z_k$. The main goal of this paper is to study the asymptotic properties at $+\infty$ and $-\infty$ of the symmetric solutions of the associated discrete Riccati matrix equation

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0 \tag{RE}$$

✉ Email: hilscher@math.muni.cz

and more generally of the discrete Riccati matrix inequalities

$$R[Q]_k\,(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\le 0, \tag{RI$_\le$}$$

$$R[Q]_k\,(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\ge 0. \tag{RI$_\ge$}$$

Solutions $\{(X_k,U_k)\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ will be abbreviated by (X,U) and Q, respectively. Symplectic systems (symplectic integrators) arise in the numerical analysis of Hamiltonian differential equations [20, 21]. Equation (RE) or its equivalent forms have many applications e.g. in the discrete control theory, Kalman filtering, and other discrete optimization problems, see [3, 4, 25] and in particular [5, Section 3.17].

In this paper we provide an overview of some classical and recent results about inequalities for symmetric solutions of the Riccati equation (RE) and the Riccati inequalities (RI$_\le$) and (RI$_\ge$). In [12, Theorem 3.2] it is stated that for a system (S), which is eventually controllable and nonoscillatory at $+\infty$, there exists a unique symmetric solution Q^∞ of the Riccati equation (RE) which is eventually minimal at $+\infty$, i.e., every symmetric solution Q of (RE) eventually satisfies $Q_k\ge Q_k^\infty$ for all k near $+\infty$ (see Proposition 3.1). The solution Q^∞ then corresponds to the recessive solution of (S) at $+\infty$. The minimal property of Q^∞ was recently generalized in [14, Lemma 4.4] to the Riccati equations (RE) corresponding to two symplectic systems (S) satisfying a Sturmian majorant condition (see Theorem 3.4). Following our previous work in [23] on discrete Riccati inequalities for system (S), we derive in this paper inequalities for symmetric solutions of the Riccati matrix inequalities (RI$_\le$) and (RI$_\ge$). Our further results include the corresponding study of distinguished solutions at $-\infty$. For this case we present a transformation to an equivalent problem at $+\infty$, to which the previously developed theory can be applied. These results then clarify and justify the definition of a recessive solution of (S) at $-\infty$, which was recently used in [13, 14, 17].

We note that the results of this paper apply also to special discrete symplectic systems, such as to the second order matrix Sturm–Liouville difference equations

$$\Delta(R_k\Delta X_k)+P_kX_{k+1}=0 \tag{1.2}$$

with symmetric R_k and P_k and invertible R_k, to higher order Sturm–Liouville difference equations, or to the linear Hamiltonian difference systems

$$\Delta X_k=A_kX_{k+1}+B_kU_k,\qquad \Delta U_k=C_kX_{k+1}-A_k^TU_k \tag{1.3}$$

with $n\times n$ matrices A_k, B_k, C_k such that B_k and C_k are symmetric and $I-A_k$ is invertible, see e.g. [6, 9, 5].

The paper is organized as follows. In Section 2 we recall basic properties of discrete symplectic systems and review some inequalities for symmetric solutions of (RE), (RI$_\le$), and (RI$_\ge$) on a finite interval. In Sections 3 and 4 we study the distinguished solution of the Riccati equation (RE) at $+\infty$ and $-\infty$, respectively.

2 Riccati matrix equations and inequalities

Symplectic matrices (of a given dimension $2n$) form a group with respect to the matrix multiplication. From (1.1) it follows that $\mathcal{S}_k^T$ and $\mathcal{S}_k^{-1}$ are also symplectic and $\mathcal{S}_k^{-1}=-\mathcal{J}\mathcal{S}_k^T\mathcal{J}$. This then yields the following properties of the coefficients of system (S):

$$\mathcal{A}_k^T\mathcal{C}_k,\ \mathcal{B}_k^T\mathcal{D}_k,\ \mathcal{A}_k\mathcal{B}_k^T,\ \mathcal{D}_k\mathcal{C}_k^T\ \text{ are symmetric},\quad \mathcal{A}_k^T\mathcal{D}_k-\mathcal{C}_k^T\mathcal{B}_k=I=\mathcal{D}_k\mathcal{A}_k^T-\mathcal{C}_k\mathcal{B}_k^T. \tag{2.1}$$

These properties immediately imply the following surprising result about the solutions of the Riccati equation (RE), see [25, Lemma 3.2].

Lemma 2.1. *For any matrices Q_k and Q_{k+1} we have the identity*

$$(\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\,(\mathcal{A}_k + \mathcal{B}_k Q_k) = I - \mathcal{B}_k^T R[Q]_k. \tag{2.2}$$

Consequently, $\mathcal{B}_k^T R[Q]_k = 0$ if and only if $\mathcal{A}_k + \mathcal{B}_k Q_k$ and $\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1}$ are invertible with

$$(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1} = \mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1}. \tag{2.3}$$

Identity (2.2) yields that solutions Q of the Riccati equation (RE) automatically satisfy the invertibility condition (2.3). Moreover, for symmetric solutions Q of (RE) the matrix $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k$ is symmetric, too. The following result about the solvability of (RE) is classical and can be found e.g. in [5, Theorem 3.57] or [16, Theorem 3].

Proposition 2.2. *Riccati equation (RE), $k \in [0,N]_{\mathbb{Z}}$, has a symmetric solution Q on $[0,N+1]_{\mathbb{Z}}$ if and only if system (S), $k \in [0,N]_{\mathbb{Z}}$, has a matrix solution (X,U) on $[0,N+1]_{\mathbb{Z}}$ such that X_k is invertible and $X_k^T U_k$ is symmetric for all $k \in [0,N+1]_{\mathbb{Z}}$. In this case $Q_k = U_k X_k^{-1}$ on $[0,N+1]_{\mathbb{Z}}$ and $\mathcal{A}_k + \mathcal{B}_k Q_k = X_{k+1}X_k^{-1}$ is invertible on $[0,N]_{\mathbb{Z}}$.*

The above result shows an intimate connection between equation (RE) and system (S). The solution (X,U) in Proposition 2.2 is an example of a conjoined basis of (S). More generally, a solution (X,U) of (S) is called a conjoined basis if $X_k^T U_k$ is symmetric and $\operatorname{rank}(X_k^T,\ U_k^T)^T = n$ for some (and hence for any) index $k \in [0,N+1]_{\mathbb{Z}}$. Following [7, pg. 715 and 719] we say that a conjoined basis (X,U) has no forward focal points in the interval $(k,k+1]$ if

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0, \tag{2.4}$$

and (X,U) has no backward focal points in the interval $[k,k+1)$ if

$$\operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1} \quad \text{and} \quad X_{k+1} X_k^{\dagger} \mathcal{B}_k^T \geq 0. \tag{2.5}$$

Here the dagger means the Moore–Penrose pseudoinverse. We refer to Section 4 for an explanation of the relationship between (2.4) and (2.5). We note that in [26] and [11, 18] the multiplicities of the forward and backward focal points of (X,U) are defined. However, these more advanced concepts will not be needed in this paper.

Remark 2.3. When (X,U) is a conjoined basis of (S) such that X_k and X_{k+1} are invertible, then (2.4) and (2.5) read as

$$(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0, \qquad (\mathcal{A}_k + \mathcal{B}_k Q_k)\,\mathcal{B}_k^T \geq 0, \tag{2.6}$$

respectively, where $Q_k = U_k X_k^{-1}$ is the associated symmetric solution of (RE). In this case we can easily see that the two conditions in (2.6) are equivalent.

For any solutions (X,U) and $(\hat{X},\hat{U})$ of (S) we define their Wronskian by $W_k := X_k^T \hat{U}_k - U_k^T \hat{X}_k$. Then from (2.1) it follows that $W_k \equiv W$ is constant on any interval where (X,U) and $(\hat{X},\hat{U})$ are defined.

In the following result we provide a comparison of two symmetric solutions of the Riccati equation (RE). Although it can be obtained from a more general statement (see Proposition 2.6), we present its proof for completeness and future reference, see also the proof of [12, Theorem 3.2].

Proposition 2.4. *Assume that Q and $\hat{Q}$ be symmetric solutions of the Riccati equation (RE) on $[0,N]_{\mathbb{Z}}$ such that $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k \geq 0$ on $[0,N]_{\mathbb{Z}}$. If $Q_0 \geq \hat{Q}_0$ ($Q_0 > \hat{Q}_0$), then $Q_k \geq \hat{Q}_k$ ($Q_k > \hat{Q}_k$) on $[0,N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\mathcal{B}_k \geq 0$ on $[0,N]_{\mathbb{Z}}$ as well.*

Proof. Let (X,U) and $(\hat{X},\hat{U})$ be the conjoined bases of (S), which are associated with Q and $\hat{Q}$ through Proposition 2.2. Then $Q_k = U_kX_k^{-1}$ and $\hat{Q}_k = \hat{U}_k\hat{X}_k^{-1}$ on $[0,N+1]_{\mathbb{Z}}$. Let W be the (constant) Wronskian of (X,U) and $(\hat{X},\hat{U})$. Then by direct calculations we get the identities $Q_k-\hat{Q}_k = -X_k^{T-1}(W\hat{X}_k^{-1}X_k)\,X_k^{-1}$ and $\Delta(\hat{X}_k^{-1}X_k) = -\hat{X}_{k+1}^{-1}\mathcal{B}_k\hat{X}_k^{T-1}W^T$. This implies that

$$Q_k-\hat{Q}_k = X_k^{T-1}\big[X_0^T(Q_0-\hat{Q}_0)\,X_0+W\hat{H}_kW^T\big]X_k^{-1}, \tag{2.7}$$

where the symmetric matrix $\hat{H}_k$ is defined by

$$\hat{H}_k := \sum_{j=0}^{k-1}\hat{X}_{j+1}^{-1}\mathcal{B}_j\hat{X}_j^{T-1} = \sum_{j=0}^{k-1}\hat{X}_j^{-1}(\mathcal{A}_j+\mathcal{B}_j\hat{Q}_j)^{-1}\mathcal{B}_j\,\hat{X}_j^{T-1}.$$

The assumptions on $\hat{Q}$ imply that $\hat{H}_k \geq 0$ on $[0,N]_{\mathbb{Z}}$. Therefore, from $Q_0 \geq \hat{Q}_0$ ($Q_0 > \hat{Q}_0$) and (2.7) we obtain $Q_k \geq \hat{Q}_k$ ($Q_k > \hat{Q}_k$) on $[0,N+1]_{\mathbb{Z}}$. Moreover, from Remark 2.3 (applied to $\hat{Q}$) and from the estimate $\mathcal{B}_k(\mathcal{A}_k+\mathcal{B}_kQ_k)^T \geq \mathcal{B}_k(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^T \geq 0$ it now follows that $(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\mathcal{B}_k \geq 0$ on $[0,N]_{\mathbb{Z}}$ as well. $\square$

Remark 2.5. We note that without the assumption $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k \geq 0$ on $[0,N]_{\mathbb{Z}}$ the conclusion of Proposition 2.4 does not hold in general. For example, the Riccati equation (RE) for system (S) with $\mathcal{S}_k := \mathcal{J}$, i.e., for $\mathcal{A}_k = 0 = \mathcal{D}_k$ and $\mathcal{B}_k = I = -\mathcal{C}_k$, has the form $Q_{k+1}Q_k+I=0$. This implies that $Q_{k+1} = -Q_k^{-1}$, which for the initial conditions $Q_0 = I$ and $\hat{Q}_0 = -I$ yields the solutions $Q_k = (-1)^kI$ and $\hat{Q}_k = (-1)^{k+1}I$. Then $Q_0 > \hat{Q}_0$, but $Q_k < \hat{Q}_k$ for each odd index k. In this case we have $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k = \hat{Q}_k^{-1} = (-1)^{k+1}I \ngeq 0$ on $[0,N]_{\mathbb{Z}}$ when $N \geq 1$.

In the next statement we present an extension of Proposition 2.4 to two systems. Thus, consider with (S) another symplectic system

$$x_{k+1} = \underline{\mathcal{A}}_kx_k+\underline{\mathcal{B}}_ku_k, \qquad u_{k+1} = \underline{\mathcal{C}}_kx_k+\underline{\mathcal{D}}_ku_k \tag{$\underline{\mathrm{S}}$}$$

where $\underline{\mathcal{A}}_k$, $\underline{\mathcal{B}}_k$, $\underline{\mathcal{C}}_k$, $\underline{\mathcal{D}}_k$ are real $n\times n$ matrices such that the $2n\times 2n$ coefficient matrix $\underline{\mathcal{S}}_k$ in system ($\underline{\mathrm{S}}$) is symplectic. Define the symmetric $n\times n$ matrix $\mathcal{E}_k$ and $2n\times 2n$ matrix $\mathcal{G}_k$ by

$$\mathcal{B}_k^T\mathcal{E}_k\mathcal{B}_k = \mathcal{B}_k^T\mathcal{D}_k, \qquad \mathcal{E}_k^T = \mathcal{E}_k, \qquad \mathcal{G}_k := \begin{pmatrix} \mathcal{A}_k^T\mathcal{E}_k\mathcal{A}_k-\mathcal{C}_k^T\mathcal{A}_k & \mathcal{C}_k^T-\mathcal{A}_k^T\mathcal{E}_k \\ \mathcal{C}_k-\mathcal{E}_k\mathcal{A}_k & \mathcal{E}_k \end{pmatrix}. \tag{2.8}$$

For example, we may choose $\mathcal{E}_k := \mathcal{B}_k\mathcal{B}_k^\dagger\mathcal{D}_k\mathcal{B}_k^\dagger$. Note that $\mathcal{G}_k$ is a symmetric solution of

$$\begin{pmatrix} I & 0 \\ \mathcal{A}_k & \mathcal{B}_k \end{pmatrix}^T\mathcal{G}_k\begin{pmatrix} I & 0 \\ \mathcal{A}_k & \mathcal{B}_k \end{pmatrix} = \begin{pmatrix} I & 0 \\ \mathcal{A}_k & \mathcal{B}_k \end{pmatrix}^T\begin{pmatrix} 0 & -I \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} = \begin{pmatrix} \mathcal{A}_k^T\mathcal{C}_k & \mathcal{C}_k^T\mathcal{B}_k \\ \mathcal{B}_k^T\mathcal{C}_k & \mathcal{B}_k^T\mathcal{D}_k \end{pmatrix}.$$

The matrices $\underline{\mathcal{E}}_k$ and $\underline{\mathcal{G}}_k$ are defined analogously to (2.8) via the coefficients of system ($\underline{\mathrm{S}}$).

The following majorant conditions

$$\operatorname{Im}(\mathcal{A}_k-\underline{\mathcal{A}}_k,\ \mathcal{B}_k) \subseteq \operatorname{Im}\underline{\mathcal{B}}_k, \qquad \begin{pmatrix} I & 0 \\ \mathcal{A}_k & \mathcal{B}_k \end{pmatrix}^T(\mathcal{G}_k-\underline{\mathcal{G}}_k)\begin{pmatrix} I & 0 \\ \mathcal{A}_k & \mathcal{B}_k \end{pmatrix} \geq 0 \tag{2.9}$$

were introduced in [14, Formula (2.12)], or in a slightly stronger form in [15, Theorem 10.38] or [22, Section 3]. In this context we may say that system ($\underline{\mathrm{S}}$) is a Sturm majorant for (S) on an interval J, or that system (S) is a Sturm minorant for ($\underline{\mathrm{S}}$) on J, when (2.9) holds for all $k \in J$. This terminology is justified by the fact that under (2.9) the oscillation of system (S), measured by the existence of forward focal points, implies the oscillation of the majorant system ($\underline{\mathrm{S}}$), see Proposition 3.3. Also, for the Sturm–Liouville difference equations (1.2) the conditions in (2.9) reduce to the well known majorant relations $R_k \geq \underline{R}_k$ and $P_k \leq \underline{P}_k$ on J, see also Remark 2.15.

With system ($\underline{\mathrm{S}}$) we consider the corresponding discrete Riccati matrix equation

$$\underline{R}[Q]_k := Q_{k+1}(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k Q_k) - (\underline{\mathcal{C}}_k + \underline{\mathcal{D}}_k Q_k) = 0 \tag{$\underline{\mathrm{RE}}$}$$

and the discrete Riccati matrix inequalities

$$\underline{R}[Q]_k\,(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k Q_k)^{-1} \leq 0, \tag{$\underline{\mathrm{RI}}_{\leq}$}$$

$$\underline{R}[Q]_k\,(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k Q_k)^{-1} \geq 0. \tag{$\underline{\mathrm{RI}}_{\geq}$}$$

The following result is a consequence of [24, Theorem 7.1] or [14, Lemma 3.7].

Proposition 2.6. *Let* (2.9) *be satisfied for all $k \in [0, N]_{\mathbb{Z}}$. Assume that a symmetric Q solves* (RE) *on $[0, N]_{\mathbb{Z}}$ and that a symmetric $\underline{Q}$ solves* ($\underline{\mathrm{RE}}$) *on $[0, N]_{\mathbb{Z}}$ with $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{Q}_k)^{-1}\underline{\mathcal{B}}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$. If $Q_0 \geq \underline{Q}_0$, then $Q_k \geq \underline{Q}_k$ on $[0, N+1]_{\mathbb{Z}}$.*

Based on Proposition 2.6, we will now derive further comparison results for solutions of the Riccati equations and inequalities, which extend Proposition 2.4 as well as Proposition 2.6 itself. For this we utilize the following general statement about a Sturm majorant/minorant system for (S). We note that the matrix $R[Q]_k(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}$ is indeed symmetric on $[0, N]_{\mathbb{Z}}$ when Q_k is symmetric on $[0, N+1]_{\mathbb{Z}}$, see [23].

Lemma 2.7. *Let $\mathcal{E}_k$ and F_k be symmetric matrices, $\mathcal{B}_k^T \mathcal{E}_k \mathcal{B}_k = \mathcal{B}_k^T \mathcal{D}_k$, and define for $k \in [0, N]_{\mathbb{Z}}$ the coefficients*

$$\underline{\mathcal{A}}_k := \mathcal{A}_k, \quad \underline{\mathcal{B}}_k := \mathcal{B}_k, \quad \underline{\mathcal{C}}_k := \mathcal{C}_k + F_k\,\mathcal{A}_k, \quad \underline{\mathcal{D}}_k := \mathcal{D}_k + F_k\,\mathcal{B}_k, \quad \underline{\mathcal{E}}_k := \mathcal{E}_k + F_k. \tag{2.10}$$

If $F_k \leq 0$ on $[0, N]_{\mathbb{Z}}$, then (2.9) *holds for all $k \in [0, N]_{\mathbb{Z}}$. Moreover, in this case for any symmetric matrices Q_k on $[0, N+1]_{\mathbb{Z}}$ we have the equality*

$$\underline{R}[Q]_k(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k Q_k)^{-1} = R[Q]_k(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1} - F_k. \tag{2.11}$$

Proof. The result follows by verifying the majorant conditions in (2.9). While the first condition in (2.9) is under (2.10) satisfied trivially, the second condition in (2.9) follows from the calculation $\mathcal{G}_k - \underline{\mathcal{G}}_k = \operatorname{diag}\{0, -F_k\}$. Therefore, if $F_k \leq 0$ on $[0, N]_{\mathbb{Z}}$, then ($\underline{\mathrm{S}}$) is a Sturm majorant for (S). Formula (2.11) now follows by a direct calculation. $\square$

Lemma 2.8. *Let Q be a symmetric function defined on $[0, N+1]_{\mathbb{Z}}$. The Q solves the Riccati inequality* ($\mathrm{RI}_{\leq}$)*, resp.* ($\mathrm{RI}_{\geq}$)*, if and only if there exist symmetric functions $F_k \leq 0$, resp. $F_k \geq 0$, on $[0, N]_{\mathbb{Z}}$ such that Q solves the majorant, resp. minorant, Riccati equation $\underline{R}[Q]_k = 0$ on $[0, N]_{\mathbb{Z}}$, whose coefficients are given in* (2.10)*.*

Proof. Assume that a symmetric Q solves ($\mathrm{RI}_{\leq}$) on $[0, N]_{\mathbb{Z}}$, the proof for ($\mathrm{RI}_{\geq}$) is exactly the same. Define $F_k := R[Q]_k(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}$ on $[0, N]_{\mathbb{Z}}$. Then F_k is symmetric and $F_k \leq 0$ on $[0, N]_{\mathbb{Z}}$. With the coefficients in (2.10) we then have $\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k Q_k = \mathcal{A}_k + \mathcal{B}_k Q_k$ invertible, and by (2.11) we obtain $\underline{R}[Q]_k(\underline{\mathcal{A}}_k+\underline{\mathcal{B}}_k Q_k)^{-1}=0$. This means that Q solves $\underline{R}[Q]_k=0$ on $[0,N]_{\mathbb{Z}}$. Conversely, assume that for some symmetric matrices $F_k\leq 0$ the symmetric function Q solves $\underline{R}[Q]_k=0$ on $[0,N]_{\mathbb{Z}}$. Then by Lemma 2.1 the matrix $\underline{\mathcal{A}}_k+\underline{\mathcal{B}}_k Q_k$ is invertible on $[0,N]_{\mathbb{Z}}$ and from (2.11) we obtain $R[Q]_k(\mathcal{A}_k+\mathcal{B}_k Q_k)^{-1}=F_k\leq 0$ on $[0,N]_{\mathbb{Z}}$. The proof is complete. □

From Lemma 2.8 we know that solutions Q of the Riccati inequalities (RI$_\leq$) and (RI$_\geq$) correspond to solutions of certain Riccati equations ($\underline{\text{RE}}$). This means, in view of Lemma 2.1, that the matrix $\mathcal{A}_k+\mathcal{B}_k Q_k=\underline{\mathcal{A}}_k+\underline{\mathcal{B}}_k Q_k$ is automatically invertible, which justifies the presence of its inverse in the inequalities (RI$_\leq$) and (RI$_\geq$).

In the next result we present an extension of Proposition 2.4 to the solutions of the Riccati inequalities (RI$_\leq$) and (RI$_\geq$) for one system (S).

Theorem 2.9. *Assume that a symmetric $\hat{Q}$ solves (RI$_\leq$) on $[0,N]_{\mathbb{Z}}$ with $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ and that a symmetric $\tilde{Q}$ solves (RI$_\geq$) on $[0,N]_{\mathbb{Z}}$. If $\tilde{Q}_0\geq\hat{Q}_0$, then $\tilde{Q}_k\geq\hat{Q}_k$ on $[0,N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ as well.*

Proof. Let $\hat{Q}$ and $\tilde{Q}$ be as in the theorem and define on $[0,N]_{\mathbb{Z}}$ the symmetric matrices

$$\hat{F}_k := R[\hat{Q}]_k(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\leq 0,\qquad \tilde{F}_k := R[\tilde{Q}]_k(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\geq 0.$$

By Lemma 2.8, the function $\hat{Q}$ solves the Riccati equation $\underline{\hat{R}}[Q]_k=0$ on $[0,N]_{\mathbb{Z}}$, whose coefficients $\hat{\mathcal{A}}_k$, $\hat{\mathcal{B}}_k$, $\hat{\mathcal{C}}_k$, $\hat{\mathcal{D}}_k$ are given by (2.10) with $F_k:=\hat{F}_k$. Moreover, $(\underline{\hat{\mathcal{A}}}_k+\underline{\hat{\mathcal{B}}}_k\hat{Q}_k)^{-1}\underline{\hat{\mathcal{B}}}_k=(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$. Similarly, the function $\tilde{Q}$ solves the Riccati equation $\underline{\tilde{R}}[Q]_k=0$ on $[0,N]_{\mathbb{Z}}$, whose coefficients $\tilde{\mathcal{A}}_k$, $\tilde{\mathcal{B}}_k$, $\tilde{\mathcal{C}}_k$, $\tilde{\mathcal{D}}_k$ are given by (2.10) with $F_k:=\tilde{F}_k$. It follows that the associated symplectic systems – denoted by ($\underline{\hat{\text{S}}}$) and ($\underline{\tilde{\text{S}}}$) – have the property that ($\underline{\hat{\text{S}}}$) is a Sturm majorant for ($\underline{\tilde{\text{S}}}$) on $[0,N]_{\mathbb{Z}}$. Therefore, the inequality $\tilde{Q}_k\geq\hat{Q}_k$ on $[0,N+1]_{\mathbb{Z}}$ follows from Proposition 2.6 applied to the two Riccati equations $\underline{\hat{R}}[Q]_k=0$ and $\underline{\tilde{R}}[Q]_k=0$ on $[0,N]_{\mathbb{Z}}$. Finally, since $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ is equivalent with $\mathcal{B}_k(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^T\geq 0$ and $\tilde{Q}_k\geq\hat{Q}_k$ is already proven, we obtain $\mathcal{B}_k(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^T\geq\mathcal{B}_k(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^T\geq 0$. In turn, this means that $(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$, which completes the proof. □

The result in Theorem 2.9 allows to compare solutions of the Riccati equation (RE) with the solutions of the Riccati inequality (RI$_\leq$) or (RI$_\geq$).

Corollary 2.10. *Assume that a symmetric $\hat{Q}$ solves (RI$_\leq$) on $[0,N]_{\mathbb{Z}}$ with $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ and that a symmetric Q solves (RE) on $[0,N]_{\mathbb{Z}}$. If $Q_0\geq\hat{Q}_0$, then $Q_k\geq\hat{Q}_k$ on $[0,N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_k Q_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ as well.*

Proof. We apply Theorem 2.9 with $\tilde{Q}_k:=Q_k$. □

Corollary 2.11. *Assume that a symmetric Q solves (RE) on $[0,N]_{\mathbb{Z}}$ with $(\mathcal{A}_k+\mathcal{B}_k Q_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ and that a symmetric $\tilde{Q}$ solves (RI$_\geq$) on $[0,N]_{\mathbb{Z}}$. If $\tilde{Q}_0\geq Q_0$, then $\tilde{Q}_k\geq Q_k$ on $[0,N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ as well.*

Proof. We apply Theorem 2.9 with $\hat{Q}_k:=Q_k$. □

Combining the statements in Corollaries 2.10 and 2.11 yields the following.

Corollary 2.12. *Assume that a symmetric $\hat{Q}$ solves (RI$_\leq$) on $[0,N]_{\mathbb{Z}}$ with $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ and that a symmetric $\tilde{Q}$ solves (RI$_\geq$) on $[0,N]_{\mathbb{Z}}$. Then for any symmetric solution Q of (RE) such that $\tilde{Q}_0\geq Q_0\geq\hat{Q}_0$ the inequalities $\tilde{Q}_k\geq Q_k\geq\hat{Q}_k$ hold on $[0,N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_k Q_k)^{-1}\mathcal{B}_k\geq 0$ and $(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[0,N]_{\mathbb{Z}}$ as well.*

Proof. First we apply Corollary 2.10 to obtain $Q_k \geq \hat{Q}_k$ on $[0, N+1]_{\mathbb{Z}}$ and $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$. This then allows to apply Corollary 2.11 to get $\tilde{Q}_k \geq Q_k$ on $[0, N+1]_{\mathbb{Z}}$ and $(\mathcal{A}_k + \mathcal{B}_k \tilde{Q}_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$. □

In the last part of this section we generalize Proposition 2.6 and Theorem 2.9 to Riccati inequalities for two symplectic systems (S) and ($\underline{\text{S}}$) satisfying the majorant condition in (2.9).

Theorem 2.13. *Let (2.9) be satisfied for all $k \in [0, N]_{\mathbb{Z}}$. Assume that a symmetric $\underline{\hat{Q}}$ solves the Riccati inequality ($\underline{\text{RI}}_{\leq}$) on $[0, N]_{\mathbb{Z}}$ with $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{\hat{Q}}_k)^{-1}\underline{\mathcal{B}}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$, and that a symmetric $\tilde{Q}$ solves the Riccati inequality ($\text{RI}_{\geq}$) on $[0, N]_{\mathbb{Z}}$. If $\tilde{Q}_0 \geq \underline{\hat{Q}}_0$, then $\tilde{Q}_k \geq \underline{\hat{Q}}_k$ on $[0, N+1]_{\mathbb{Z}}$.*

Proof. By Lemma 2.8 applied to the Riccati inequality ($\underline{\text{RI}}_{\leq}$), the function $\underline{\hat{Q}}$ is a symmetric solution of a Riccati equation, which is majorant to ($\underline{\text{RE}}$). Similarly, the function $\tilde{Q}$ is a symmetric solution of a Riccati equation, which is minorant to ($\text{RI}_{\geq}$). Therefore, the statement follows from Proposition 2.6 applied to these two Riccati equations. □

Corollary 2.14. *Let (2.9) be satisfied for all $k \in [0, N]_{\mathbb{Z}}$. Assume that a symmetric $\tilde{Q}$, Q, $\underline{Q}$, $\underline{\hat{Q}}$ solve respectively the Riccati equations and inequalities ($\text{RI}_{\geq}$), (RE), ($\underline{\text{RE}}$), ($\underline{\text{RI}}_{\leq}$) on $[0, N]_{\mathbb{Z}}$ and that $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{\hat{Q}}_k)^{-1}\underline{\mathcal{B}}_k \geq 0$ and $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$. If $\tilde{Q}_0 \geq Q_0 \geq \underline{Q}_0 \geq \underline{\hat{Q}}_0$, then $\tilde{Q}_k \geq Q_k \geq \underline{Q}_k \geq \underline{\hat{Q}}_k$ on $[0, N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k + \mathcal{B}_k \tilde{Q}_k)^{-1}\mathcal{B}_k \geq 0$ and $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{Q}_k)^{-1}\underline{\mathcal{B}}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$ as well.*

Proof. The statement follows from Corollary 2.10 applied to the solutions $\underline{Q}$ and $\underline{\hat{Q}}$ of ($\underline{\text{RE}}$) and ($\underline{\text{RI}}_{\leq}$), from Proposition 2.6 applied to the solutions Q and $\underline{Q}$ of (RE) and ($\underline{\text{RE}}$), and from Corollary 2.11 applied to the solutions $\tilde{Q}$ and Q of ($\text{RI}_{\geq}$) and (RE). □

We conclude this section with a comment on the scalar Riccati equation and Riccati inequalities for Sturm–Liouville difference equations (1.2).

Remark 2.15. Consider the scalar (i.e., $n = 1$) second order Sturm–Liouville difference equation (1.2) with $R_k \neq 0$. It is known that by setting $U_k := R_k \Delta X_k$ we can write (1.2) as the symplectic system (S) with $\mathcal{A}_k := 1$, $\mathcal{B}_k := 1/R_k$, $\mathcal{C}_k := -P_k$, $\mathcal{D}_k := 1 - P_k/R_k$. Moreover, the substitution $Q_k := U_k/X_k = (R_k \Delta X_k)/X_k$ leads to the associated Riccati equation

$$R_{\mathrm{SL}}[Q]_k := \Delta Q_k + P_k + \frac{Q_k^2}{R_k + Q_k} = 0. \tag{2.12}$$

Note that with the above coefficients $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ we have $R_{\mathrm{SL}}[Q]_k = R[Q]_k/(\mathcal{A}_k + \mathcal{B}_k Q_k)$. Therefore, all the results in this section apply to the solutions of the Riccati equation (2.12) and the solutions of the Riccati inequalities $R_{\mathrm{SL}}[Q]_k \leq 0$ and $R_{\mathrm{SL}}[Q]_k \geq 0$. We remark that the equation

$$\Delta(\underline{R}_k \Delta X_k) + \underline{P}_k X_{k+1} = 0 \tag{2.13}$$

with nonzero $\underline{R}_k$ is a Sturm majorant for equation (1.2) on the interval J if

$$R_k \geq \underline{R}_k, \qquad P_k \leq \underline{P}_k$$

holds for all $k \in J$. In this case we have $\mathcal{E}_k = \mathcal{D}_k/\mathcal{B}_k = R_k - P_k$ and $\underline{\mathcal{E}}_k = \underline{\mathcal{D}}_k/\underline{\mathcal{B}}_k = \underline{R}_k - \underline{P}_k$.

3 Distinguished solution at $+\infty$

In this section we study the minimal properties of certain symmetric solutions of the Riccati equation (RE) and the Riccati inequalities ($\mathrm{RI}_{\leq}$) and ($\mathrm{RI}_{\geq}$) at $+\infty$. First we recall the following terminology, see e.g. [13, 17]. For an index $M \in [0,\infty)_{\mathbb{Z}}$ the solution $(\hat{X}^{[M]}, \hat{U}^{[M]})$ of (S) satisfying the initial conditions $\hat{X}^{[M]}_M = 0$ and $\hat{U}^{[M]}_M = I$ is called the *principal solution at* M. System (S) is *nonoscillatory at* $+\infty$ if there exists $M \in [0,\infty)_{\mathbb{Z}}$ such that the principal solution at M has no forward focal points in the interval $(0,\infty)$, i.e., condition (2.4) holds for all $k \in [M,\infty)_{\mathbb{Z}}$. This means by the Sturmian comparison theorem in [10, Theorem 1.3] or [27, Proposition 3.2] that every conjoined basis of (S) has eventually no forward focal points near $+\infty$.

System (S) is said to be *controllable near* $+\infty$ if for every $M \in [0,\infty)_{\mathbb{Z}}$ there exists $N \in [M,\infty)_{\mathbb{Z}}$ such that the principal solution $(\hat{X}^{[M]}, \hat{U}^{[M]})$ of (S) at $k = M$ has $\hat{X}^{[M]}_N$ invertible. The nonoscillation and controllability of system (S) at $+\infty$ then imply that for any conjoined basis (X,U) of (S) the matrix X_k is invertible for all k near $+\infty$.

When system (S) is nonoscillatory at $+\infty$ and controllable near $+\infty$, then there exists a special conjoined basis (X^∞, U^∞) with the property that X^∞_k is invertible and $X^\infty_k (X^\infty_{k+1})^{-1} \mathcal{B}_k \geq 0$ for all $k \in [N,\infty)_{\mathbb{Z}}$ for some $N \in [0,\infty)_{\mathbb{Z}}$ and

$$\lim_{k\to\infty} \Big(\sum_{i=N}^{k-1} (X^\infty_{i+1})^{-1} \, \mathcal{B}_i \, (X^\infty_i)^{T-1} \Big)^{-1} = 0. \tag{3.1}$$

The conjoined basis (X^∞, U^∞) is called the *recessive solution at* $+\infty$ (or the principal solution at $+\infty$). The latter terminology is justified by the fact the recessive solution is the smallest conjoined basis of (S) at $+\infty$ when it is compared with any other linearly independent conjoined basis (X,U) of (S), i.e., we have the limit property

$$\lim_{k\to\infty} X_k^{-1} X^\infty_k = 0 \tag{3.2}$$

for every conjoined basis (X,U) of (S) such that the (constant) Wronskian W of (X,U) and (X^∞, U^∞) is invertible, see [1, 2, 5, 8, 12]. Moreover, the recessive solution (X^∞, U^∞) at $+\infty$ is unique up to a constant invertible multiple.

Since the recessive solution of (S) at $+\infty$ has X^∞_k invertible for large k, we may associate with (X^∞, U^∞) the corresponding solution Q^∞ of the Riccati matrix equation (RE), where $Q^\infty_k = U^\infty_k (X^\infty_k)^{-1}$, see Proposition 2.2. The function Q^∞ is called the *distinguished solution at* $+\infty$ of (RE). From the properties of (X^∞, U^∞) on $[N,\infty)_{\mathbb{Z}}$ it then follows that Q^∞_k is symmetric and $(\mathcal{A}_k + \mathcal{B}_k Q^\infty_k)^{-1} \mathcal{B}_k \geq 0$ for all $k \in [N,\infty)_{\mathbb{Z}}$. The following minimal property of the distinguished solution Q^∞ of (RE) at $+\infty$ is stated in [12, Theorem 3.2].

Proposition 3.1. *Let Q^∞ be the distinguished solution at $+\infty$ of (RE) on $[N,\infty)_{\mathbb{Z}}$. Then for any symmetric solution Q of (RE) defined on $[N,\infty)_{\mathbb{Z}}$ there exists $M \in [N,\infty)_{\mathbb{Z}}$ such that $Q_k \geq Q^\infty_k$ on $[M,\infty)_{\mathbb{Z}}$.*

Proof. As in the proof of Proposition 2.4 we have the inequality

$$\begin{aligned} Q_k - Q^\infty_k &= X_k^{T-1} \big[X_N^T (Q_N - Q^\infty_N)\, X_N + W H^\infty_k \, W^T \big] X_k^{-1} \\ &= X_k^{T-1} \big[-X_N^T (X^\infty_N)^{T-1} W^T + W H^\infty_k \, W^T \big] X_k^{-1} \end{aligned} \tag{3.3}$$

where (X,U) is the conjoined basis of (S) from Proposition 2.2 such that $Q_k = U_k X_k^{-1}$ on $[N,\infty)_{\mathbb{Z}}$, W is the Wronskian of (X,U) and (X^∞, U^∞), and where the positive semidefinite matrix H_k^∞ is defined by

$$H_k^\infty := \sum_{j=N}^{k-1} (X_{j+1}^\infty)^{-1}\, \mathcal{B}_j\, (X_j^\infty)^{T-1} = \sum_{j=N}^{k-1} (X_j^\infty)^{-1} (\mathcal{A}_j + \mathcal{B}_j Q_j^\infty)^{-1} \mathcal{B}_j\, (X_j^\infty)^{T-1}.$$

Since (X^∞, U^∞) is the recessive solution at $+\infty$, it follows that the eigenvalues of H_k^∞ tend monotonically to $+\infty$ as $k \to \infty$. Therefore, for every $d \in \mathbb{R}^n$, $d \neq 0$, we have $d^T H_k^\infty d \to \infty$ as $k \to \infty$. Define the symmetric matrix $T_k := -X_N^T (X_N^\infty)^{T-1} W^T + W H_k^\infty W^T$ and let $c \in \mathbb{R}^n$ be arbitrary. If $W^T c = 0$, then $c^T T_k\, c = 0$ for all $k \in [N, \infty)_{\mathbb{Z}}$. On the other hand, if $d := W^T c \neq 0$, then $c^T T_k\, c = -c^T X_N^T (X_N^\infty)^{T-1} d + d^T H_k^\infty d \to \infty$ as $k \to \infty$. This implies that there exists $M \in [N, \infty)_{\mathbb{Z}}$ such that $T_k \geq 0$ for all $k \in [M, \infty)_{\mathbb{Z}}$. Inequality (3.3) now yields $Q_k - Q_k^\infty \geq 0$ for all $k \in [M, \infty)_{\mathbb{Z}}$, which completes the proof. □

Remark 3.2. Observe that the proof of [12, Theorem 3.2] utilizes the limit property in (3.2), hence it necessarily requires the invertibility of $Q_k - Q_k^\infty$. This assumption is however not stated in [12, Theorem 3.2]. The above proof of Proposition 3.1 does not require this extra condition.

Now we will consider the distinguished solutions at $+\infty$ of two Riccati equations (RE) and ($\underline{\text{RE}}$) satisfying the majorant conditions in (2.9). First we recall a statement which justifies the terminology of being a Sturm majorant system.

Proposition 3.3. *Let (2.9) be satisfied for all $k \in [N, \infty)_{\mathbb{Z}}$ for some $N \in [0, \infty)_{\mathbb{Z}}$. If ($\underline{\text{S}}$) is nonoscillatory at $+\infty$, then (S) is nonoscillatory at $+\infty$ as well.*

Proof. The statement follows from [10, Theorem 1.3]. □

Based on Proposition 3.3, the nonoscillation of system ($\underline{\text{S}}$) at $+\infty$ and the controllability of (S) and ($\underline{\text{S}}$) near $+\infty$ imply the existence of the distinguished solutions of the Riccati equations (RE) and ($\underline{\text{RE}}$). The following result extends [9, Theorem 2] from linear Hamiltonian difference systems (1.3) to symplectic systems (S). Also, the same statement was recently obtained in [14, Lemma 4.4] via the comparative index theory.

Theorem 3.4. *Let (2.9) be satisfied for all $k \in [N, \infty)_{\mathbb{Z}}$ for some $N \in [0, \infty)_{\mathbb{Z}}$ and let Q^∞ and $\underline{Q}^\infty$ be the distinguished solutions at $+\infty$ of the Riccati equations (RE) and ($\underline{\text{RE}}$) on $[N, \infty)_{\mathbb{Z}}$. Then there exists $M \in [N, \infty)_{\mathbb{Z}}$ such that $Q_k^\infty \geq \underline{Q}_k^\infty$ on $[M, \infty)_{\mathbb{Z}}$.*

Proof. By the definition of the distinguished solution, the matrices Q_k^∞ and $\underline{Q}_k^\infty$ are symmetric with $(\mathcal{A}_k + \mathcal{B}_k Q_k^\infty)^{-1} \mathcal{B}_k \geq 0$ and $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{Q}_k^\infty)^{-1} \underline{\mathcal{B}}_k \geq 0$ on $[N, \infty)_{\mathbb{Z}}$. Then we can represent $\underline{Q}_k^\infty = \underline{U}_k^\infty (\underline{X}_k^\infty)^{-1}$ on $[N, \infty)_{\mathbb{Z}}$, where $(\underline{X}^\infty, \underline{U}^\infty)$ is the recessive solution of ($\underline{\text{S}}$) on $[N, \infty)_{\mathbb{Z}}$. Consider the conjoined basis $(\underline{X}, \underline{U})$ of ($\underline{\text{S}}$) with the initial conditions $\underline{X}_N = I$ and $\underline{U}_N = Q_N^\infty$. By [19, Corollary 2.1] or [14, Corollary 3.4], the conjoined basis $(\underline{X}, \underline{U})$ has no focal points in the interval (N, ∞). Hence, $\underline{X}_k$ is invertible on $[N, \infty)_{\mathbb{Z}}$ and the symmetric matrix $\underline{Q}_k := \underline{U}_k \underline{X}_k^{-1}$ solves on $[N, \infty)_{\mathbb{Z}}$ the Riccati equation ($\underline{\text{RE}}$) with $(\underline{\mathcal{A}}_k + \underline{\mathcal{B}}_k \underline{Q}_k)^{-1} \underline{\mathcal{B}}_k \geq 0$ on $[N, \infty)_{\mathbb{Z}}$. From Proposition 2.4 we obtain that $Q_k^\infty \geq \underline{Q}_k$ on $[N, \infty)_{\mathbb{Z}}$. On the other hand, since $\underline{Q}$ solves the same Riccati equation ($\underline{\text{RE}}$) as $\underline{Q}^\infty$, it follows from Proposition 3.1 that there exists $M \in [N, \infty)_{\mathbb{Z}}$ such that $\underline{Q}_k \geq \underline{Q}_k^\infty$ on $[M, \infty)_{\mathbb{Z}}$. Therefore, we have $Q_k^\infty \geq \underline{Q}_k \geq \underline{Q}_k^\infty$ on $[M, \infty)_{\mathbb{Z}}$, which completes the proof. □

4 Distinguished solution at $-\infty$

In this section we present the proper concept of a distinguished solution of the Riccati equation (RE) at $-\infty$. The situation is not completely symmetric to the concept of a distinguished solution of (RE) at $+\infty$. The reasons are analogous to those why the definitions of the forward focal points in $(k, k+1]$ in (2.4) and the backward focal points in $[k, k+1)$ in (2.5) are not completely symmetric. In this section we provide an explanation of these differences and a connection between the definitions of the distinguished solutions of (RE) at $+\infty$ and $-\infty$, which were recently used in [13, 14, 17].

Consider a discrete symplectic system (S) on the interval $(-\infty, 0]_{\mathbb{Z}}$. Suppose we have no prior knowledge about the concepts of backward focal points for conjoined bases of (S), about the recessive solution of (S) at $-\infty$, about the associated Riccati equation and inequalities on $(-\infty, 0]_{\mathbb{Z}}$, or about the distinguished solution at $-\infty$. We wish to define these concepts in a correct way which would be in agreement with their corresponding notions at $+\infty$. One possible way is to consider the following transformation of system (S) on $(-\infty, 0]_{\mathbb{Z}}$ into a symplectic system on $[0, \infty)_{\mathbb{Z}}$. We define the $2n \times 2n$ matrices

$$\mathcal{K} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \mathcal{L} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{4.1}$$

and for $j \in [0, \infty)_{\mathbb{Z}}$ the transformed quantities $k := -j$ and

$$\left.\begin{aligned} \tilde{\mathcal{S}}_j &= \begin{pmatrix} \tilde{\mathcal{A}}_j & \tilde{\mathcal{B}}_j \\ \tilde{\mathcal{C}}_j & \tilde{\mathcal{D}}_j \end{pmatrix} := \mathcal{K}\, \mathcal{S}_{-j}^T\, \mathcal{K} = \mathcal{K}\, \mathcal{S}_k^T\, \mathcal{K} = \begin{pmatrix} \mathcal{D}_k^T & \mathcal{B}_k^T \\ \mathcal{C}_k^T & \mathcal{A}_k^T \end{pmatrix}, \\ \tilde{Z}_j &= \begin{pmatrix} \tilde{X}_j \\ \tilde{U}_j \end{pmatrix} := \mathcal{L}\, Z_{1-j} = \mathcal{L}\, Z_{k+1} = \begin{pmatrix} X_{k+1} \\ -U_{k+1} \end{pmatrix}. \end{aligned}\right\} \tag{4.2}$$

This way we obtain the transformed system

$$\tilde{Z}_{j+1} = \tilde{\mathcal{S}}_j\, \tilde{Z}_j \tag{S̃}$$

on the interval $[0, \infty)_{\mathbb{Z}}$. The exact relationship between the systems (S) and (S̃) is described in the following result.

Lemma 4.1. *The transformation in* (4.2) *transforms a symplectic system* (S) *on* $(-\infty, 0]_{\mathbb{Z}}$ *into a symplectic system* (S̃) *on* $[0, \infty)_{\mathbb{Z}}$.

Proof. From the identities $\mathcal{K}\mathcal{L} = -\mathcal{J}$, $\mathcal{L}\mathcal{K} = \mathcal{J}$, $\mathcal{K}\mathcal{J} = -\mathcal{L}$, $\mathcal{J}\mathcal{K} = \mathcal{L}$, $\mathcal{K}\mathcal{J}\mathcal{K} = -\mathcal{J}$, and $\mathcal{S}_k \mathcal{J} \mathcal{S}_k^T = \mathcal{J}$ we obtain that $\tilde{\mathcal{S}}_j^T \mathcal{J} \tilde{\mathcal{S}}_j = \mathcal{K}\mathcal{S}_k\mathcal{K}\mathcal{J}\mathcal{K}\mathcal{S}_k^T\mathcal{K} = -\mathcal{K}\mathcal{S}_k\mathcal{J}\mathcal{S}_k^T\mathcal{K} = -\mathcal{K}\mathcal{J}\mathcal{K} = \mathcal{J}$, i.e., the matrix $\tilde{\mathcal{S}}_j$ is symplectic for all $j \in [0, \infty)_{\mathbb{Z}}$. Moreover, if z solves system (S) on $(-\infty, 0]_{\mathbb{Z}}$, then for $j \in [0, \infty)_{\mathbb{Z}}$ we have

$$\tilde{\mathcal{S}}_j \tilde{Z}_j = \mathcal{K}\mathcal{S}_k^T\mathcal{K}\mathcal{L}Z_{k+1} = \mathcal{K}\mathcal{J}\mathcal{J}\mathcal{S}_k^T\mathcal{J}Z_{k+1} = \mathcal{L}\mathcal{S}_k^{-1}Z_{k+1} = \mathcal{L}Z_k = \mathcal{L}Z_{1-(1-k)} = \tilde{Z}_{j+1}.$$

Therefore, (S̃) is a discrete symplectic system on $[0, \infty)_{\mathbb{Z}}$. □

In view of Lemma 4.1 and the formula $\tilde{X}_j = X_{k+1}$ (with $j = -k$), condition (2.4) on no forward focal points of the conjoined basis $(\tilde{X}, \tilde{U})$ of (S̃) in $(j, j+1]$ transforms exactly to condition (2.5) on no backward focal points of the conjoined basis (X, U) of (S) in $[k, k+1)$. Therefore, the nonoscillation of (S) at $-\infty$ must be considered in terms of the nonexistence of the *backward* focal points. More precisely, let $(\tilde{X}^{[M]}, \tilde{U}^{[M]})$ be the conjoined basis of (S) satisfying the initial conditions $\tilde{X}^{[M]}_M = 0$ and $\tilde{U}^{[M]}_M = -I$ (sometimes it is called the *associated solution* at $k = M$). System (S) is *nonoscillatory at* $-\infty$ if there exists $M \in (-\infty, 0]_{\mathbb{Z}}$ such that the associated solution at $k = M$ has no backward focal points in the interval $(-\infty, M)$, i.e., condition (2.5) holds for all $k \in (-\infty, M-1]_{\mathbb{Z}}$. Then every conjoined basis of (S) has eventually no backward focal points near $-\infty$.

System (S) is *controllable near* $-\infty$ if for every $M \in (-\infty, 0]_{\mathbb{Z}}$ there exists $N \in (-\infty, M]_{\mathbb{Z}}$ such that the associated solution $(\tilde{X}^{[M]}, \tilde{U}^{[M]})$ at $k = M$ has $\tilde{X}^{[M]}_N$ invertible. The nonoscillation and controllability of system (S) at $-\infty$ then imply that for any conjoined basis (X, U) of (S) the matrix X_k is invertible for all k near $-\infty$.

Definition 4.2 (Recessive solution at $-\infty$)**.** A conjoined basis (X^{∞}, U^{∞}) of (S) is called the *recessive solution at* $-\infty$ (or the principal solution at $-\infty$) if for some $N \in (-\infty, 0]_{\mathbb{Z}}$ the matrix $X^{-\infty}_k$ is invertible for all $k \in (-\infty, N+1]_{\mathbb{Z}}$, $X^{-\infty}_{k+1}(X^{-\infty}_k)^{-1}\mathcal{B}^T_k \geq 0$ for all $k \in (-\infty, N]_{\mathbb{Z}}$, and

$$\lim_{k\to-\infty} \left(-\sum_{i=k+1}^{N} (X^{-\infty}_i)^{-1}\, \mathcal{B}^T_i\, (X^{-\infty}_{i+1})^{T-1} \right)^{-1} = 0. \tag{4.3}$$

The definition of the recessive solution at $-\infty$ of (S) is made in such a way that it corresponds exactly to the recessive solution at $+\infty$ of the transformed system ($\tilde{\text{S}}$). Indeed, changing the summation index in (4.3) from i to $-i$ and then using $j = -k$ with (4.2), i.e., $X^{-\infty}_{-i} = \tilde{X}^{\infty}_{\ell+1}$, $X^{-\infty}_{-i+1} = \tilde{X}^{\infty}_{\ell}$, $\mathcal{B}^T_{-i} = \tilde{\mathcal{B}}_{\ell}$, yields

$$\lim_{k\to-\infty} \left(-\sum_{i=-k-1}^{-N} (X^{-\infty}_{-i})^{-1}\, \mathcal{B}^T_{-i}\, (X^{-\infty}_{-i+1})^{T-1} \right)^{-1} = \lim_{j\to\infty} \left(-\sum_{\ell=j-1}^{-N} (\tilde{X}^{\infty}_{\ell+1})^{-1}\, \tilde{\mathcal{B}}_{\ell}\, (\tilde{X}^{\infty}_{\ell})^{T-1} \right)^{-1}.$$

Upon interchanging the summation limits in the last expression we then obtain the formula in (3.1), i.e., $(\tilde{X}^{\infty}, \tilde{U}^{\infty})$ is the recessive solution of the transformed system ($\tilde{\text{S}}$) at $+\infty$.

It then follows that the nonoscillation and controllability of (S) near $-\infty$ implies the existence of the (unique up to a constant nonsingular multiple) recessive solution $(X^{-\infty}, U^{-\infty})$ at $-\infty$. Moreover, the recessive solution $(X^{-\infty}, U^{-\infty})$ at $-\infty$ has the limit property

$$\lim_{k\to-\infty} X^{-1}_k X^{-\infty}_k = 0$$

for every conjoined basis (X, U) of (S) such that the (constant) Wronskian W of (X, U) and $(X^{-\infty}, U^{-\infty})$ is invertible.

Now we analyze the associated Riccati equation and inequality. From (4.2) we have the relations $\tilde{X}_j := X_{k+1}$ and $\tilde{U}_j := -U_{k+1}$, so that $\tilde{Q}_j := \tilde{U}_j\tilde{X}^{-1}_j = -U_{k+1}X^{-1}_{k+1} = -Q_{k+1}$ and $\tilde{Q}_{j+1} = -Q_k$. Hence, using the definition of the Riccati operator for system ($\tilde{\text{S}}$) we get

$$\begin{aligned} \tilde{R}[\tilde{Q}]_j &= \tilde{Q}_{j+1}(\tilde{\mathcal{A}}_j + \tilde{\mathcal{B}}_j\tilde{Q}_j) - (\tilde{\mathcal{C}}_j + \tilde{\mathcal{D}}_j\tilde{Q}_j) \\ &= -Q_k(\mathcal{D}^T_k - \mathcal{B}^T_k Q_{k+1}) - (\mathcal{C}^T_k - \mathcal{A}^T_k Q_{k+1}) = (R[Q]_k)^T. \end{aligned}$$

Moreover, from $\tilde{\mathcal{A}}_j + \tilde{\mathcal{B}}_j\tilde{Q}_j = \mathcal{D}^T_k - \mathcal{B}^T_k Q_{k+1}$ and $\tilde{\mathcal{D}}^T_j - \tilde{\mathcal{B}}^T_j\tilde{Q}_{j+1} = \mathcal{A}_k + \mathcal{B}_k Q_k$ and (2.3) we have

$$\tilde{R}[\tilde{Q}]_j(\tilde{\mathcal{A}}_j + \tilde{\mathcal{B}}_j\tilde{Q}_j)^{-1} = (\mathcal{A}_k + \mathcal{B}_k Q_k)^T \{R[Q]_k(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\}^T (\mathcal{A}_k + \mathcal{B}_k Q_k)$$

These calculations show that both the Riccati equation (RE) and the Riccati inequalities ($\mathrm{RI}_{\leq}$) and ($\mathrm{RI}_{\geq}$) are preserved for the system (S) on $(-\infty,0]_{\mathbb{Z}}$. In other words (with $j=-k$),

$$\begin{aligned} \tilde{R}[\tilde{Q}]_j=0 \quad &\Leftrightarrow \quad R[Q]_k=0,\\ \tilde{R}[\tilde{Q}]_j(\tilde{\mathcal{A}}_j+\tilde{\mathcal{B}}_j\tilde{Q}_j)^{-1}\leq 0\ \ (\geq 0) \quad &\Leftrightarrow \quad R[Q]_k(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\leq 0\ \ (\geq 0),\\ (\tilde{\mathcal{A}}_j+\tilde{\mathcal{B}}_j\tilde{Q}_j)^{-1}\tilde{\mathcal{B}}_j\geq 0 \quad &\Leftrightarrow \quad (\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\mathcal{B}_k\geq 0. \end{aligned}$$

Finally, the majorant conditions in (2.9) translate for system ($\tilde{\text{S}}$) as

$$\operatorname{Im}(\mathcal{D}_k^T-\underline{\mathcal{D}}_k^T,\ \mathcal{B}_k^T)\subseteq \operatorname{Im}\underline{\mathcal{B}}_k^T,\qquad \begin{pmatrix} I & 0\\ \mathcal{D}_k^T & \mathcal{B}_k^T\end{pmatrix}^T(\mathcal{G}_k^- - \underline{\mathcal{G}}_k^-)\begin{pmatrix} I & 0\\ \mathcal{D}_k^T & \mathcal{B}_k^T\end{pmatrix}\geq 0, \tag{4.4}$$

where

$$\mathcal{B}_k\mathcal{E}_k^-\mathcal{B}_k^T=\mathcal{B}_k\mathcal{A}_k^T,\qquad (\mathcal{E}_k^-)^T=\mathcal{E}_k^-,\qquad \mathcal{G}_k^- := \begin{pmatrix} \mathcal{D}_k\mathcal{E}_k^-\mathcal{D}_k^T-\mathcal{D}_k\mathcal{C}_k^T & \mathcal{C}_k-\mathcal{D}_k\mathcal{E}_k^-\\ \mathcal{C}_k^T-\mathcal{E}_k^-\mathcal{D}_k^T & \mathcal{E}_k^-\end{pmatrix}. \tag{4.5}$$

For example, we may take $\mathcal{E}_k^- := \mathcal{B}_k^\dagger\mathcal{A}_k\mathcal{B}_k^\dagger\mathcal{B}_k$. Note that $\mathcal{G}_k^-$ is a symmetric solution of

$$\begin{pmatrix} I & 0\\ \mathcal{D}_k^T & \mathcal{B}_k^T\end{pmatrix}^T\mathcal{G}_k^-\begin{pmatrix} I & 0\\ \mathcal{D}_k^T & \mathcal{B}_k^T\end{pmatrix}=\begin{pmatrix} I & 0\\ \mathcal{D}_k^T & \mathcal{B}_k^T\end{pmatrix}^T\begin{pmatrix} 0 & -I\\ \mathcal{C}_k^T & \mathcal{A}_k^T\end{pmatrix}=\begin{pmatrix} \mathcal{D}_k\mathcal{C}_k^T & \mathcal{C}_k\mathcal{B}_k^T\\ \mathcal{B}_k\mathcal{C}_k^T & \mathcal{B}_k\mathcal{A}_k^T\end{pmatrix}.$$

The matrices $\underline{\mathcal{E}}_k^-$ and $\underline{\mathcal{G}}_k^-$ associated with the system ($\underline{\text{S}}$) are defined analogously to (4.5) via the coefficients of ($\underline{\text{S}}$). Indeed, set $\tilde{\mathcal{E}}_j := \mathcal{E}_k^-$ and then according to (2.8) and (4.2) we have

$$\tilde{\mathcal{B}}_j^T\tilde{\mathcal{E}}_j\tilde{\mathcal{B}}_j=\tilde{\mathcal{B}}_j^T\tilde{\mathcal{D}}_j,\qquad \tilde{\mathcal{E}}_j^T=\tilde{\mathcal{E}}_j,\qquad \tilde{\mathcal{G}}_k := \begin{pmatrix} \tilde{\mathcal{A}}_k^T\tilde{\mathcal{E}}_k\tilde{\mathcal{A}}_k-\tilde{\mathcal{C}}_k^T\tilde{\mathcal{A}}_k & \tilde{\mathcal{C}}_k^T-\tilde{\mathcal{A}}_k^T\tilde{\mathcal{E}}_k\\ \tilde{\mathcal{C}}_k-\tilde{\mathcal{E}}_k\tilde{\mathcal{A}}_k & \tilde{\mathcal{E}}_k\end{pmatrix}=\mathcal{G}_k^-. \tag{4.6}$$

Similar calculations hold for $\underline{\tilde{\mathcal{E}}}_j=\underline{\mathcal{E}}_k^-$ and $\underline{\tilde{\mathcal{G}}}_j=\underline{\mathcal{G}}_k^-$.

Therefore, applying the results in Sections 2 and 3 to the transformed systems ($\tilde{\text{S}}$) and ($\underline{\tilde{\text{S}}}$) and using the sign changing relation

$$\tilde{Q}_{j+1}=-Q_k \tag{4.7}$$

yields the inequalities for solutions of (RE), ($\mathrm{RI}_{\leq}$), ($\mathrm{RI}_{\geq}$) on $[N,0]_{\mathbb{Z}}$ or $(-\infty,N]_{\mathbb{Z}}$, which are opposite to those on $[0,-N]_{\mathbb{Z}}$ or $[-N,\infty)_{\mathbb{Z}}$.

Theorem 4.3. *Let (4.4) be satisfied for all $k\in[N,0]_{\mathbb{Z}}$. Assume that a symmetric Q solves (RE) on $[N,0]_{\mathbb{Z}}$ and that a symmetric $\underline{Q}$ solves ($\underline{\text{RE}}$) on $[N,0]_{\mathbb{Z}}$ with $(\underline{\mathcal{A}}_k+\underline{\mathcal{B}}_k\underline{Q}_k)^{-1}\underline{\mathcal{B}}_k\geq 0$ on $[N,0]_{\mathbb{Z}}$. If $Q_0\leq \underline{Q}_0$, then $Q_k\leq \underline{Q}_k$ on $[N-1,0]_{\mathbb{Z}}$.*

Proof. We apply Proposition 2.6 to the Riccati equations $\tilde{R}[\tilde{Q}]_j=0$ and $\underline{\tilde{R}}[\tilde{Q}]_j=0$ on $[0,-N]_{\mathbb{Z}}$, which correspond to the transformed systems ($\tilde{\text{S}}$) and ($\underline{\tilde{\text{S}}}$). □

Theorem 4.4. *Assume that a symmetric $\hat{Q}$ solves ($\mathrm{RI}_{\leq}$) on $[N,0]_{\mathbb{Z}}$ with $(\mathcal{A}_k+\mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[N,0]_{\mathbb{Z}}$ and that a symmetric $\tilde{Q}$ solves ($\mathrm{RI}_{\geq}$) on $[N,0]_{\mathbb{Z}}$. Then for any symmetric solution Q of (RE) such that $\tilde{Q}_0\leq Q_0\leq \hat{Q}_0$ the inequalities $\tilde{Q}_k\leq Q_k\leq \hat{Q}_k$ hold on $[N-1,0]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\mathcal{B}_k\geq 0$ and $(\mathcal{A}_k+\mathcal{B}_k\tilde{Q}_k)^{-1}\mathcal{B}_k\geq 0$ on $[N,0]_{\mathbb{Z}}$ as well.*

Proof. We apply Corollary 2.12 to the transformed Riccati inequalities $\tilde{R}[\tilde{Q}]_j(\tilde{\mathcal{A}}_j+\tilde{\mathcal{B}}_j\tilde{Q}_j)^{-1}\leq 0$ and $\tilde{R}[\tilde{Q}]_j(\tilde{\mathcal{A}}_j+\tilde{\mathcal{B}}_j\tilde{Q}_j)^{-1}\geq 0$ and to the transformed Riccati equation $\tilde{R}[\tilde{Q}]_j=0$ on $[0,-N]_{\mathbb{Z}}$. □

Denote by $Q^{-\infty}$ the distinguished solution at $-\infty$ of the Riccati equation (RE), that is, $Q_k^{-\infty} = U_k^{-\infty}(X_k^{-\infty})^{-1}$ on $(-\infty, N]_{\mathbb{Z}}$, where $(X^{-\infty}, U^{-\infty})$ is the recessive solution of (S) at $-\infty$ according to Definition 4.2. Then $Q^{-\infty}$ is the maximal symmetric solution of (RE).

Theorem 4.5. *Let $Q^{-\infty}$ be the distinguished solution at $-\infty$ of* (RE) *on $(-\infty, N]_{\mathbb{Z}}$. Then for any symmetric solution Q of* (RE) *defined on $(-\infty, N]_{\mathbb{Z}}$ there exists $M \in (-\infty, N]_{\mathbb{Z}}$ such that $Q_k \le Q_k^{-\infty}$ on $(-\infty, M]_{\mathbb{Z}}$.*

Proof. The result follows from Proposition 3.1, Lemma 4.1, and (4.7). □

Recall now that the nonoscillation of (S) at $-\infty$ is defined via the nonexistence of backward focal points in the interval $(-\infty, N)$.

Proposition 4.6. *Let* (4.4) *be satisfied for all $k \in (-\infty, N]_{\mathbb{Z}}$ for some $N \in (-\infty, 0]_{\mathbb{Z}}$. If* ($\underline{\text{S}}$) *is nonoscillatory at $-\infty$, then* (S) *is nonoscillatory at $-\infty$ as well.*

Proof. The statement follows from Proposition 3.3 and Lemma 4.1. □

Theorem 4.7. *Let* (4.4) *be satisfied for all $k \in (-\infty, N]_{\mathbb{Z}}$ for some $N \in (-\infty, 0]_{\mathbb{Z}}$ and let $Q^{-\infty}$ and $\underline{Q}^{-\infty}$ be the distinguished solutions at $-\infty$ of the Riccati equations* (RE) *and* ($\underline{\text{RE}}$) *on $(-\infty, N]_{\mathbb{Z}}$. Then there exists $M \in (-\infty, N]_{\mathbb{Z}}$ such that $Q_k^{-\infty} \le \underline{Q}_k^{-\infty}$ on $(-\infty, M]_{\mathbb{Z}}$.*

Proof. The result follows from Theorem 3.4 and Lemma 4.1. □

Remark 4.8. The above comparison of the distinguished solutions at $-\infty$ of the Riccati equation (RE) and its majorant Riccati equation ($\underline{\text{RE}}$) is known in [14, Lemma 4.4]. However, that result uses the majorant conditions from (2.9) instead of the conditions in (4.4). Below we show that the conditions in (2.9) and (4.4) are equivalent.

For simplicity we first consider the situation when the matrices $\mathcal{B}_k$ and $\underline{\mathcal{B}}_k$ are invertible – this is the case of the Sturm–Liouville difference equations (1.2) and (2.13) or the symmetric three-term recurrence equations [28]. In this case, from (2.8) and (4.5) we have

$$\mathcal{E}_k = \mathcal{D}_k \mathcal{B}_k^{-1}, \qquad \mathcal{G}_k = \begin{pmatrix} \mathcal{B}_k^{-1}\mathcal{A}_k & -\mathcal{B}_k^{-1} \\ -\mathcal{B}_k^{T-1} & \mathcal{D}_k \mathcal{B}_k^{-1} \end{pmatrix}, \qquad \mathcal{E}_k^{-} = \mathcal{B}_k^{-1}\mathcal{A}_k, \qquad \mathcal{G}_k^{-} = \begin{pmatrix} \mathcal{D}_k \mathcal{B}_k^{-1} & -\mathcal{B}_k^{T-1} \\ -\mathcal{B}_k^{-1} & \mathcal{B}_k^{-1}\mathcal{A}_k \end{pmatrix},$$

and similarly for $\underline{\mathcal{E}}_k$, $\underline{\mathcal{G}}_k$, $\underline{\mathcal{E}}_k^{-}$, $\underline{\mathcal{G}}_k^{-}$. We can see that in this case $\mathcal{G}_k^{-} = \mathcal{K}\,\mathcal{G}_k\,\mathcal{K}$ and $\underline{\mathcal{G}}_k^{-} = \mathcal{K}\,\underline{\mathcal{G}}_k\,\mathcal{K}$ with the matrix $\mathcal{K}$ given in (4.1). Hence, $\mathcal{G}_k \ge \underline{\mathcal{G}}_k$ if and only if $\mathcal{G}_k^{-} \ge \underline{\mathcal{G}}_k^{-}$.

In the general situation for noninvertible $\mathcal{B}_k$ or $\underline{\mathcal{B}}_k$ one can use the properties of the comparative index for two matrices developed in [18, 19] in order to prove that (2.9) and (4.4) are really equivalent. For the definition of the *comparative index* $\mu(\underline{Y}, Y) := \mu_1(\underline{Y}, Y) + \mu_2(\underline{Y}, Y)$ of two matrices $\underline{Y}$ and Y we refer to [18, pg. 448] or [14, pg. 1271]. The *dual comparative index* $\mu^*(\underline{Y}, Y) := \mu_1^*(\underline{Y}, Y) + \mu_2^*(\underline{Y}, Y)$ of $\underline{Y}$ and Y is defined by $\mu_1^*(\underline{Y}, Y) := \mu_1(\underline{Y}, Y)$ and $\mu_2^*(\underline{Y}, Y) := \mu_2(\mathcal{L}\,\underline{Y}, \mathcal{L}\,Y)$ with $\mathcal{L}$ given in (4.1). Now by [19, Remark 2.1], condition (2.9) is equivalent with

$$\mu(\langle \underline{\mathcal{S}}_k \rangle, \langle \mathcal{S}_k \rangle) = 0, \qquad \langle \mathcal{S}_k \rangle := \begin{pmatrix} I & \mathcal{A}_k^T & 0 & \mathcal{C}_k^T \\ 0 & \mathcal{B}_k^T & -I & \mathcal{D}_k^T \end{pmatrix}^T, \tag{4.8}$$

while [19, Lemma 2.3 (iii)] yields that (4.8) is equivalent with

$$\mu^*(\langle \underline{\mathcal{S}}_k^{-1} \rangle, \langle \mathcal{S}_k^{-1} \rangle) = 0, \qquad \langle \mathcal{S}_k^{-1} \rangle := \begin{pmatrix} I & \mathcal{D}_k & 0 & -\mathcal{C}_k \\ 0 & -\mathcal{B}_k & -I & \mathcal{A}_k \end{pmatrix}^T, \tag{4.9}$$

where the matrices $\langle \underline{\mathcal{S}}_k \rangle$ and $\langle \underline{\mathcal{S}}_k^{-1} \rangle$ are defined as above through $\underline{\mathcal{A}}_k$, $\underline{\mathcal{B}}_k$, $\underline{\mathcal{C}}_k$, $\underline{\mathcal{D}}_k$. From the definition of $\mu^*(\langle \underline{\mathcal{S}}_k^{-1} \rangle, \langle \mathcal{S}_k^{-1} \rangle) = 0$ we then get that (4.9) is equivalent with $\mu_1^*(\langle \underline{\mathcal{S}}_k^{-1} \rangle, \langle \mathcal{S}_k^{-1} \rangle) = 0$ together with $\mu_2^*(\langle \underline{\mathcal{S}}_k^{-1} \rangle, \langle \mathcal{S}_k^{-1} \rangle) = 0$. By [19, Remark 2.1], these are equivalent with

$$\operatorname{Im}(\mathcal{D}_k^T - \underline{\mathcal{D}}_k^T,\ -\mathcal{B}_k^T) \subseteq \operatorname{Im}(-\underline{\mathcal{B}}_k^T), \qquad \begin{pmatrix} I & 0 \\ \mathcal{D}_k^T & -\mathcal{B}_k^T \end{pmatrix}^T (\underline{\mathcal{H}}_k - \mathcal{H}_k) \begin{pmatrix} I & 0 \\ \mathcal{D}_k^T & -\mathcal{B}_k^T \end{pmatrix} \geq 0, \tag{4.10}$$

where $\mathcal{H}_k$ is a symmetric solution of the equation

$$\begin{pmatrix} I & 0 \\ \mathcal{D}_k^T & -\mathcal{B}_k^T \end{pmatrix}^T \mathcal{H}_k \begin{pmatrix} I & 0 \\ \mathcal{D}_k^T & -\mathcal{B}_k^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ \mathcal{D}_k^T & -\mathcal{B}_k^T \end{pmatrix}^T \begin{pmatrix} 0 & -I \\ -\mathcal{C}_k^T & \mathcal{A}_k^T \end{pmatrix} = \begin{pmatrix} -\mathcal{D}_k\mathcal{C}_k^T & \mathcal{C}_k\mathcal{B}_k^T \\ \mathcal{B}_k\mathcal{C}_k^T & -\mathcal{B}_k\mathcal{A}_k^T \end{pmatrix} \tag{4.11}$$

and similarly for $\underline{\mathcal{H}}_k$. It is now easy to see that the first condition in (4.10) is equivalent with the first condition in (4.4). Multiplying (4.11) and the second equality in (4.10) from both left and right sides by the (symmetric and invertible) matrix $\mathcal{L}$, we obtain that with $\mathcal{H}_k := -\mathcal{G}_k^-$ and $\underline{\mathcal{H}}_k := -\underline{\mathcal{G}}_k^-$ the second condition in (4.10) is equivalent with the second condition in (4.4). This completes the proof of the equivalence of (2.9) and (4.4).

Remark 4.9. In this paper we considered the distinguished solutions at $+\infty$ and $-\infty$ of the Riccati equations (RE) and ($\underline{\text{RE}}$) under the controllability assumption near $+\infty$ and $-\infty$. Recently in [27], we succeeded to construct the recessive solution at $+\infty$ (and $-\infty$) of system (S) without the controllability assumption. In this case the matrix X_k^∞ is not necessarily invertible near $+\infty$ and the limit in (3.1) involves the Moore–Penrose pseudoinverses. We expect, that some of the present results can be obtained also in this more general situation. This research is the subject of our present work.

Acknowledgements

The author thanks the anonymous referee for pointing out the equivalence of (2.9) and (4.4) for noninvertible $\mathcal{B}_k$ and $\underline{\mathcal{B}}_k$ discussed in the last part of Remark 4.8.

References

[1] C. D. Ahlbrandt, Continued fraction representations of maximal and minimal solutions of a discrete matrix Riccati equation, *SIAM J. Math. Anal.* **24**(1993), No. 6, 1597–1621. MR1241160

[2] C. D. Ahlbrandt, Dominant and recessive solutions of symmetric three term recurrences, *J. Differential Equations* **107**(1994), 238–258. MR1264521

[3] C. D. Ahlbrandt, M. Heifetz, Discrete Riccati equations of filtering and control, in: *Proceedings of the First International Conference on Difference Equations (San Antonio, TX, 1994)*, S. Elaydi, J. Graef, G. Ladas, and A. Peterson (editors), pp. 1–16, Gordon and Breach, Newark, NJ, 1996. MR1678625

[4] C. D. Ahlbrandt, M. Heifetz, J. W. Hooker, W. T. Patula, Asymptotics of discrete time Riccati equations, robust control, and discrete linear Hamiltonian systems, *Panamer. Math. J.* **5**(1996), No. 2, 1–39. MR1336792

[5] C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian systems. Difference equations, continued fractions, and Riccati equations*, Kluwer Texts in the Mathematical Sciences, Vol. 16, Kluwer Academic Publishers Group, Dordrecht, 1996. MR1423802

[6] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, *J. Math. Anal. Appl.* **199**(1996), No. 3, 804–826. MR1386607

[7] M. Bohner, O. Došlý, Disconjugacy and transformations for symplectic systems, *Rocky Mountain J. Math.* **27**(1997), No. 3, 707–743. MR1490271

[8] M. Bohner, O. Došlý, R. Hilscher, Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution, *Nonlinear Anal.* **47**(2001), 849–860. MR1970703

[9] M. Bohner, O. Došlý, W. Kratz, Inequalities and asymptotics for Riccati matrix difference operators, *J. Math. Anal. Appl.* **221**(1998), No. 1, 262–286. MR1619144

[10] M. Bohner, O. Došlý, W. Kratz, Sturmian and spectral theory for discrete symplectic systems, *Trans. Amer. Math. Soc.* **361**(2009), No. 6, 3109–3123. MR2485420

[11] O. Došlý, Oscillation theory of symplectic difference systems, in: *Advances in Discrete Dynamical Systems, Proceedings of the Eleventh International Conference on Difference Equations and Applications (Kyoto, 2006)*, S. Elaydi, K. Nishimura, M. Shishikura, and N. Tose (editors), Adv. Stud. Pure Math., Vol. 53, pp. 41–50, Mathematical Society of Japan, Tokyo, 2009. MR2582403

[12] O. Došlý, Principal and nonprincipal solutions of symplectic dynamic systems on time scales, in: *Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations, (Szeged, Hungary, 1999)*, No. 5, 1–14. (electronic), *Electron. J. Qual. Theory Differ. Equ.*, Szeged, 2000. MR1798655

[13] O. Došlý, Focal points and recessive solutions of discrete symplectic systems, *Appl. Math. Comput.* **243**(2014), 963–968. MR3244543

[14] O. Došlý, J. Elyseeva, Singular comparison theorems for discrete symplectic systems, *J. Difference Equ. Appl.* **20**(2014), No. 8, 1268–1288. MR3216899

[15] O. Došlý, S. Hilger, R. Hilscher, Symplectic dynamic systems, in: *Advances in Dynamic Equations on Time Scales*, M. Bohner and A. Peterson (editors), pp. 293–334, Birkhäuser, Boston, 2003. MR1962552

[16] O. Došlý, R. Hilscher, Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales, *J. Differ. Equations Appl.* **7**(2001), 265–295. MR1923624

[17] O. Došlý, W. Kratz, A remark on focal points of recessive solutions of discrete symplectic systems, *J. Math. Anal. Appl.* **363**(2010), No. 1, 209–213. MR2559054

[18] Yu. V. Eliseeva, Comparative index for solutions of symplectic difference systems, *Differential Equations* **45**(2009), No. 3, 445–459; translated from *Differencial'nyje Uravnenija* **45**(2009), No. 3, 431–444. MR2596770

[19] Yu. V. Eliseeva, Comparison theorems for symplectic systems of difference equations, *Differential Equations* **46**(2010), No. 9, 1339–1352; translated from *Differencial'nyje Uravnenija* **46**(2010), No. 9, 1329–1342. MR2798697

[20] K. Feng, M. Qin, *Symplectic geometric algorithms for Hamiltonian systems*, Translated and revised from the Chinese original. With a foreword by Feng Duan. Zhejiang Science and Technology Publishing House, Hangzhou. Springer, Heidelberg, 2010. MR2839393

[21] E. Hairer, C. Lubich, G. Wanner, *Geometric numerical integration. Structure-preserving algorithms for ordinary differential equations*, Reprint of the second (2006) edition. Springer Series in Computational Mathematics, Vol. 31, Springer, Heidelberg, 2010. MR2221614

[22] R. Hilscher, Disconjugacy of symplectic systems and positive definiteness of block tridiagonal matrices, *Rocky Mountain J. Math.* **29**(1999), No. 4, 1301–1319. MR1743373

[23] R. Hilscher, V. Růžičková, Riccati inequality and other results for discrete symplectic systems, *J. Math. Anal. Appl.* **322**(2006), No. 2, 1083–1098. MR2250637

[24] R. Hilscher, V. Růžičková, Implicit Riccati equations and quadratic functionals for discrete symplectic systems, *Int. J. Difference Equ.* **1**(2006), No. 1, 135–154. MR2296502

[25] R. Hilscher, V. Zeidan, Extension of discrete LQR-problem to symplectic systems, *Int. J. Difference Equ.* **2**(2007), No. 2, 197–208. MR2493598

[26] W. Kratz, Discrete oscillation, *J. Difference Equ. Appl.* **9**(2003), No. 1, 135–147. MR1958308

[27] P. Šepitka, R. Šimon Hilscher, Recessive solutions for nonoscillatory discrete symplectic systems, *Linear Algebra Appl.* **469**(2015), 243–275. MR3299064

[28] R. Šimon Hilscher, V. Zeidan, Symmetric three-term recurrence equations and their symplectic structure, *Adv. Difference Equ.* **2010**, Art. ID 626942, 1–17. MR2669704