PRINCIPAL AND NONPRINCIPAL SOLUTIONS OF SYMPLECTIC DYNAMIC SYSTEMS ON TIME SCALES

Ondřej Došlý

Abstract. We establish the concept of the principal and nonprincipal solution for the so-called symplectic dynamic systems on time scales. We also present a brief survey of the history of these concept for differential and difference equations.

1. Introduction.

The aim of this paper is to establish the concepts of the principal and nonprincipal solutions of the so-called *symplectic dynamic systems* on time scales.

The concept of the *principal solution* appeared for the first time in the paper [28] and it concerned the Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \quad r(t) > 0, \tag{1.1}$$

and was used when investigating singular quadratic functionals associated with (1.1), see also [24,25]. In the fifties of this century Hartman (see [18, Chap. XI] and the references given therein) investigated properties of this solution, introduced the concept of the *nonprincipal solution* and offered several equivalent characterizations of principal and nonprincipal solutions. Later, principal and nonprincipal solutions were extended to more general equations and systems and finally Reid [31,32] unified these definitions in the scope of the qualitative theory of linear Hamiltonian systems

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{1.2}$$

see also [19].

Concerning the Sturm-Liouville difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad r_k \neq 0, \tag{1.3}$$

the concept of the principal solution, named in the difference equations theory *recessive solution*, and of the nonprincipal solution (= *dominant solution*), appeared e.g. in [17,29]. These concepts were extended to the three term symmetric matrix recurrence relation

$$R_{k+1}x_{k+2} + P_k x_{k+1} + R_k x_k = 0$$

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in [4] and to more general difference systems – the so-called *symplectic difference systems* – in [5].

The similarity between qualitative theories of differential and difference equations and systems suggests to look for a unifying theory. The first attempt to establish such a theory was made in [33] (see also [27]), where both (1.1) and (1.3) are written as an integral equation with Riemann-Stieltjes integrals. However, this approach requires the sequence r_k in (1.3) to be positive and this assumption is by no means necessary as it is shown e.g. in [5,7]. Another approach which we also follow in this paper was used in [16] and is based on the concept of *time scale* (an alternative terminology is *measure chain*). Our investigation leans on results of the recent paper [15], where we established basic properties of solutions of the so-called *symplectic dynamic systems* which cover both linear Hamiltonian differential systems (1.2) and symplectic difference systems. We recall some results of [15] in the next section.

The paper is organized as follows. In the next section we recall properties of principal/recessive and nonprincipal/dominant solutions of differential and difference systems and we present basic facts of the so-called time scale calculus. The third section contains the main results of this paper, sufficient conditions for the existence of principal and nonprincipal solutions of symplectic dynamic systems and some of their properties. The last section is devoted to remarks concerning the results of the paper and contains also some suggestions for the further investigation.

2. Auxiliary results.

We start this section with a brief survey of the basic properties of principal (recessive) and nonprincipal (dominant) solutions of differential and difference equations. Suppose that (1.1) is nonoscillatory, i.e. any nontrivial solution is eventually positive or negative. Then among all solutions one can distinguish a unique (up to a multiple by a nonzero constant) solution $\tilde{x}$ which is less than any other solution in the sense that

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \tag{2.1}$$

for any solution x of (1.1) which is linearly independent of $\tilde{x}$. This solution is said to be the *principal* solution. Differentiating the ratio $x/\tilde{x}$ and using the Wronskian identity $r(x'\tilde{x} - x\tilde{x}') = \text{const}$, it is not difficult to verify that (2.1) is equivalent to

$$\int^{\infty} \frac{dt}{r(t)\tilde{x}^2(t)} = \infty. \tag{2.2}$$

Another (equivalent) characterization of the principal solution is based on the fact that if x is a solution of (1.1) then $w := \frac{r(t)x'}{x}$ is a solution of the associated Riccati equation

$$w' + c(t) + \frac{w^2}{r(t)} = 0. \tag{2.3}$$

A solution $\tilde{x}$ of (1.1) is principal if and only if $\tilde{w} = \frac{r(t)\tilde{x}'}{\tilde{x}}$ is the *eventually minimal* solution of (2.3) in the sense that any other solution w of (2.3) satisfies eventually the inequality $w(t) > \tilde{w}(t)$. A *nonprincipal solution* of (1.1) is any solution which is linearly independent of the principal solution $\tilde{x}$ and it is characterized by $\int^{\infty} r^{-1}(t)x^{-2}(t)\,dt < \infty$. If x is a nonprincipal solution, then

$$\tilde{x}(t) = x(t)\int_t^{\infty} \frac{ds}{r(s)x^2(s)}$$

is the principal solution.

The Sturm-Liouville difference equation (1.3) is said to be *nonoscillatory* if any nontrivial solution satisfies

$$r_k x_k x_{k+1} > 0 \quad \text{eventually.} \tag{2.4}$$

It is known that the Sturmian comparison and separation theory extends to (1.3), in particular, if there exists a solution x of (1.3) satisfying (2.4) then any other solution has also this property. A nonoscillatory solution $\tilde{x}$ of (1.3) is said to be *recessive* if

$$\lim_{k\to\infty} \frac{\tilde{x}_k}{x_k} = 0$$

for any linearly independent solution x. The last limit relation is equivalent to

$$\sum_{k=N}^{\infty} \frac{1}{r_k \tilde{x}_k \tilde{x}_{k+1}} = \infty$$

and this is equivalent to the fact that $w_k = \frac{r_k \Delta x_k}{x_k}$ is the eventually minimal solution of the discrete Riccati equation

$$\Delta w_k + c_k + \frac{w_k^2}{r_k + w_k} = 0.$$

There exist more equivalent characterizations of the principal resp. recessive solutions of (1.1) and (1.3), e.g. as the so-called *zero maximal* solution [26] or as solutions of a certain boundary value problem, see [11, Chap. II]. However, to present them here in a consistent form exceeds the scope of this contribution.

Next we turn our attention to the extension of the concepts of principal and recessive solution to linear Hamiltonian differential systems and symplectic difference systems. Together with (1.2) consider its matrix version (referred again as (1.2))

$$X' = A(t)X + B(t)U, \quad U' = C(t)X - A^T(t)U,$$

where X, U are $n \times n$ matrices. We suppose that the matrices B, C are symmetric and B is nonnegative definite. Recall that this system is said to be *nonoscillatory* if there exists a conjoined basis $\binom{X}{U}$ (i.e. a $2n \times n$ matrix solution such that $X^T U$ is symmetric and rank $\binom{X}{U} \equiv n$) such that $X(t)$ is nonsingular for large t. System (1.2) is said to be *eventually controllable* if there exists $T \in \mathbb{R}$ such that the trivial solution $\binom{x}{u} = \binom{0}{0}$ is the only solution for which $x(t) = 0$ on a nondegenerate subinterval of $[T, \infty)$. A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of a nonoscillatory system (1.2) is said to be the *principal solution* if

$$\lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) = 0$$

for any conjoined basis $\binom{X}{U}$ such that the (constant) matrix

$$X^T(t)\tilde{U}(t) - U^T(t)\tilde{X}(t) \quad \text{is nonsingular.} \tag{2.5}$$

Any conjoined basis satisfying (2.5) is said to be the *nonprincipal* solution of (1.2). Principal and nonprincipal solutions of eventually controllable systems can be characterized equivalently as conjoined bases whose first component satisfies

$$\lim_{t\to\infty} \lambda_1 \left(\int_T^t X^{-1}(s)B(s)X^{T-1}(s)\, ds \right) = \infty$$

resp.

$$\lim_{t\to\infty} \lambda_n \left(\int_T^t X^{-1}(s)B(s)X^{T-1}(s)\, ds \right) < \infty,$$

where λ_1, λ_n denote the least and largest eigenvalue of the matrix indicated. Another equivalent characterization of the principal solution of (1.2) is via the associated Riccati matrix equation

$$W' + A^T(t)W + WA(t) + WB(t)W - C(t) = 0 \tag{2.6}$$

related to (1.2) by the substitution $W = UX^{-1}$. A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ is principal if and only if $\tilde{W} = \tilde{U}\tilde{X}^{-1}$ is the eventually minimal solution of (2.6) in the sense that for any other symmetric solution W of this equation the matrix $W(t) - \tilde{W}(t)$ is nonnegative definite eventually.

A *symplectic difference system* is the first order recurrence system of the form

$$z_{k+1} = \mathcal{S}_k z, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \tag{2.7}$$

where $x, u \in \mathbb{R}^n$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices such that the matrix $\mathcal{S}$ is symplectic, i.e. $\mathcal{S}^T \mathcal{J} \mathcal{S} = \mathcal{J}$ with $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Symplectic difference systems cover a large variety of difference equations and systems, among them also as a very special case Sturm-Liouville difference equation (1.3). Indeed, using the substitution $u = r\Delta x$ this equation can be written as the first order system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r_k} \\ -p_k & 1 - \frac{p_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

and it is easy to see that the matrix in this system is really symplectic.

A $2n \times n$ matrix solution $\binom{X}{U}$ of (2.7) is said to be a *conjoined basis* if $X^T U$ is symmetric and $\operatorname{rank} \binom{X}{U} = n$. System (2.7) is said to be *disconjugate* in a discrete interval $[l, m]$, $l, m \in \mathbb{N}$, if the $2n \times n$ matrix solution $\binom{X}{U}$ given by the initial condition $X_l = 0$, $U_l = I$ satisfies

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0 \tag{2.8}$$

for $k = l, \dots, m$. Here Ker, † and $\geq$ stand for the kernel, Moore-Penrose generalized inverse and nonnegative definiteness of a matrix indicated, respectively. System (2.7) is said to be *nonoscillatory* if there exists $N \in \mathbb{N}$ such that this system is disconjugate on $[N,\infty)$ and it is said to be *oscillatory* in the opposite case. System (2.7) is said to be *eventually controllable* if there exist $N,\kappa\in\mathbb{N}$ such that for any $m\ge N$ the trivial solution $\binom{x}{u}=\binom{0}{0}$ is the only solution for which $x_m=x_{m+1}=\dots=x_{m+\kappa}=0$. A conjoined basis $\binom{\tilde X}{\tilde U}$ of (2.7) is said to be a *principal solution* if $\tilde X_k$ are nonsingular, $X_kX_{k+1}^{-1}\mathcal{B}_k\ge 0$, both for large k, and for any other conjoined basis $\binom{X}{U}$ for which the (constant) matrix $X^T\tilde U-U^T\tilde X$ is nonsingular we have

$$\lim_{k\to\infty} X_k^{-1}\tilde X_k=0. \tag{2.9}$$

Note that the existence of a conjoined basis $\binom{X}{U}$ such that its first component X is nonsingular and the second condition in (2.8) holds for large k implies that the first component of any other conjoined basis has the same property, see [9]. Using the Wronskian-type identity for solutions of (2.7) it is not difficult to show that (2.9) is for eventually controllable systems equivalent to

$$\lim_{k\to\infty}\lambda_1\left(\sum^{k} X_{j+1}^{-1}\mathcal{B}_j(X_j^T)^{-1}\right)=\infty.$$

Nonprincipal solutions of (2.7) can be defined similarly as for linear Hamiltonian differential systems (1.2).

Now we recall some basic facts of the time scale calculus, see [6,20,21], unifying the differential and difference calculus. A *time scale* $\mathbb{T}$ is any closed subset of the set of real numbers $\mathbb{R}$, an alternative terminology for the time scale is *measure chain*. On any time scale $\mathbb{T}$ we define the following operators and concepts:

$$\sigma(t):=\inf\{s\in\mathbb{T},\ s>t\},\quad \rho(t):=\sup\{s\in\mathbb{T},\ s<t\}$$

are the *forward* and *backward shift operators*. A point $t\in\mathbb{T}$ is said to be *left-dense* (l-d) if $\rho(t)=t$, *right-dense* (r-d) if $\sigma(t)=t$, *left-scattered* (l-s) if $\rho(t)<t$, *right-scattered* (r-s) if $\sigma(t)>t$ and it is said to be *dense* if it is r-d or l-d. The *graininess* μ of a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$. For a function $f:\mathbb{T}\to\mathbb{R}$ (the range $\mathbb{R}$ of f may be replaced by any Banach space) it is defined the *generalized derivative* $f^\Delta(t)$ as follows. For every $\varepsilon>0$ there exists a neighborhood U of t such that

$$|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s|\quad\text{for all } s\in U$$

If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, $\mu(t)=0$ and $f^\Delta=f'$ is the usual derivative. In case $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$, $\mu(t)=1$ and $f^\Delta=\Delta f$ is the forward difference operator.

Directly one can verify the following basic rules of the differential calculus on time scales

$$\begin{gathered}
[f(t)\pm g(t)]^\Delta=f^\Delta(t)\pm g^\Delta(t),\quad f(\sigma(t))=f(t)+\mu(t)f^\Delta(t),\\
[f(t)g(t)]^\Delta=f^\Delta(t)g(\sigma(t))+f(t)g^\Delta(t)=f^\Delta(t)g(t)+f(\sigma(t))g^\Delta(t),\\
\left\{\frac{f(t)}{g(t)}\right\}^\Delta=\frac{f^\Delta(t)g(t)-f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.
\end{gathered}$$

For the investigation of solvability of *dynamic equations* on time scales (dynamic equation means an equation involving an unknown function together with its generalized derivatives) we need also the following concepts. Here the usual notation for an interval $[a,b]$ actually means the set $\{t \in \mathbb{T},\, t \in [a,b]\}$, open and half open intervals are defined in the same way.

A function $f : [a,b] \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each r-d point and there exists a finite left limit in all l-d points, and this function is said to be *rd-continuously differentiable* if its generalized derivative exists and it is rd-continuous. To every rd-continuous function f there exists its *generalized antiderivative* – a function F such that $F^\Delta = f$. Using the antiderivative we define $\int_a^b f(t)\Delta t := F(b) - F(a)$. A function f is said to be *regressive* if $1 + \mu(t)f(t) \neq 0$ (the mapping $x \longmapsto (\mathrm{id} + \mu(t)f(t))x$ is invertible if the range of f is a Banach space). The initial value problem for the linear dynamic equation

$$z^\Delta = g(t)z, \quad z(t_0) = z_0$$

with a regressive and rd-continuous function g has the unique solution which depends continuously on the initial condition.

We finish this section with definition and basic properties of solutions of the so-called symplectic dynamic systems. A *symplectic dynamic system* on a time scale $\mathbb{T}$ is the first order linear dynamic system

$$z^\Delta = \mathcal{S}(t)z, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \tag{2.10}$$

where $x, u : \mathbb{T} \to \mathbb{R}^n$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $\mathcal{S}$ satisfies

$$\mathcal{J}\mathcal{S}(t) + \mathcal{S}^T(t)\mathcal{J} + \mu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) \equiv 0, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.11}$$

If $\mathbb{T} = \mathbb{R}$ then we get the first order differential system

$$z' = \begin{pmatrix} x \\ u \end{pmatrix}' = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

and (2.11) (with $\mu \equiv 0$) implies $\mathcal{B} = \mathcal{B}^T$, $\mathcal{C} = \mathcal{C}^T, \mathcal{D} = -\mathcal{A}^T$, i. e. (2.10) is really a linear Hamiltonian differential system (1.2). In case $\mathbb{T} = \mathbb{N}$ we have $z_k^\Delta = \Delta z_k = z_{k+1} - z_k$ and substituting this into (2.10) we get the system

$$z_{k+1} = (I + \mathcal{S}_k)z_k. \tag{2.12}$$

From (2.11) with $\mu \equiv 1$ immediately follows that the matrix $(I + \mathcal{S})$ is symplectic and hence (2.12) is a symplectic difference system.

Condition (2.11) implies that the matrix-valued function S is regressive (since it implies that the matrix $(I + \mu S)$ is symplectic and hence invertible). Hence, if $\mathcal{S}$ is rd-continuous, an initial condition determines the unique solution of (2.10).

System (2.10) is said to be *dense normal* on an interval $[a,b]$ if for any dense point $s \in (a,b]$ the trivial solution $\binom{x}{u} \equiv \binom{0}{0}$ is the only solution of (2.10) for which $x(t) \equiv 0$ on $[a,s]$. System (2.10) is said to be *eventually dense normal* if there exists $T \in \mathbb{T}$ and $\kappa \in \mathbb{N}$ such that this system is dense normal on $[T,\infty)$ and if there is no dense point in (T,∞) then for any $t_1 \geq T$ $x^{\sigma^k}(t_1) = 0$, $k = 0,\dots,\kappa$ implies $\binom{x}{u} \equiv \binom{0}{0}$ on (t_1,∞). Here $\sigma^k = \underbrace{\sigma \circ \cdots \circ \sigma}_{k-\text{times}}$, $\sigma^0(t) = t$.

III. Principal and nonprincipal solutions.

In this section we suppose that the time scale $\mathbb{T}$ is not bounded above, i.e. $\sup\{t \in \mathbb{T}\} = \infty$. In the sequel we adopt the usual "time scale" notation. We write $f^\sigma(t)$ instead of $f(\sigma(t))$ and $t \to \infty$ means that t attains arbitrarily large values from $\mathbb{T}$. The inequality $Q_1 \geq Q_2$ $(\leq)$ between two symmetric matrices of the same dimension means that $Q_1 - Q_2$ is nonnegative (nonpositive) definite.

We start with basic results of the transformation theory of symplectic dynamic systems (further SDS) (2.10), for details we refer to [15] and [22]. The transformation

$$z = \mathcal{R}(t)w, \quad \mathcal{R}(t) = \begin{pmatrix} H(t) & 0 \\ K(t) & (H^T(t))^{-1} \end{pmatrix}, \quad w = \begin{pmatrix} y \\ v \end{pmatrix} \tag{3.1}$$

where H, K are $n \times n$ matrices of rd-continuously differentiable functions such that H is nonsingular and $H^T K$ is symmetric, transforms (2.10) into another SDS

$$\bar{z}^\Delta = \bar{\mathcal{S}}(t)\bar{z}, \quad \bar{\mathcal{S}}(t) = \begin{pmatrix} \bar{\mathcal{A}}(t) & \bar{\mathcal{B}}(t) \\ \bar{\mathcal{C}}(t) & \bar{\mathcal{D}}(t) \end{pmatrix} \tag{3.2}$$

with

$$\begin{aligned} \bar{\mathcal{A}} &= -(H^\sigma)^{-1}(H^\Delta - \mathcal{A}H - \mathcal{B}K), \\ \bar{\mathcal{B}} &= (H^\sigma)^{-1}\mathcal{B}(H^T)^{-1}, \\ \bar{\mathcal{C}} &= (K^\sigma)^T(H^\Delta - \mathcal{A}H - \mathcal{B}K) - (H^\sigma)^T(K^\Delta - \mathcal{C}H - \mathcal{D}K), \\ \bar{\mathcal{D}} &= (H^\Delta + \mathcal{D}^T H^\sigma - \mathcal{B}^T K^\sigma)^T (H^T)^{-1}. \end{aligned} \tag{3.3}$$

This transformation preserves oscillatory properties of the transformed system which means that (3.2) is nonoscillatory if and only if (2.10) is nonoscillatory. If $\binom{X}{U}$ is a conjoined basis such that $X(t)$ is nonsingular then setting $H = X$, $K = U$ in (3.1) we have $\bar{\mathcal{A}} = 0$, $\bar{\mathcal{B}} = 0$, $\bar{\mathcal{C}} = 0$ in (3.2). In particular,

$$\begin{aligned} \bar{X}(t) &= X(t) \int_{t_1}^{t} (X^\sigma(s))^{-1}\mathcal{B}(s)(X^T(s))^{-1}\,\Delta s, \\ \bar{U}(t) &= U(t) \int_{t_1}^{t} (X^\sigma(s))^{-1}\mathcal{B}(s)(X^T(s))^{-1}\,\Delta s + (X^T(t))^{-1} \end{aligned}$$

is a conjoined basis of (2.10) for which $X^T\bar{U} - U^T\bar{X} = I$.

The definition of the concept of the principal solution of symplectic dynamic system reads as follows.

Definition. A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (2.10) is said to be a *principal solution* of SDS (2.10) if $\tilde{X}(t)$ is nonsingular,

$$(\tilde{X}^\sigma(t))^{-1}\mathcal{B}(t)(\tilde{X}^T(t))^{-1} \geq 0,$$

both for large t, and

$$\lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) = 0 \tag{3.4}$$

for any conjoined basis $\binom{X}{U}$ for which the (constant) matrix

$$L := X^T \tilde{U} - U^T \tilde{X} \quad \text{is nonsingular.} \tag{3.5}$$

Any conjoined basis $\binom{X}{U}$ for which (3.4) and (3.5) hold is said to be a *nonprincipal* solution.

The following theorem concerns the existence of the principal and nonprincipal solution of (2.10) and unifies statements concerning the existence of principal and recessive solutions of (1.2) and (2.7), respectively.

Theorem 3.1. *Suppose that* (2.10) *is nonoscillatory and eventually dense normal. Then this system possesses the principal solution* $\binom{\tilde{X}}{\tilde{U}}$. *This solution is equivalently characterized by*

$$\lim_{t\to\infty} \lambda_1 \left(\int^t (\tilde{X}^\sigma(s))^{-1} \mathcal{B}(s) (\tilde{X}^T(s))^{-1} \,\Delta s \right) = \infty. \tag{3.6}$$

Any conjoined basis $\binom{X}{U}$ *for which* (3.5) *holds is nonprincipal and this solution is characterized by the relation*

$$\lim_{t\to\infty} \lambda_n \left(\int^t (X^\sigma(s))^{-1} \mathcal{B}(s) (X^T(s))^{-1} \,\Delta s \right) < \infty. \tag{3.7}$$

Proof. Let $t_0 \in \mathbb{T}$ be sufficiently large and consider the solution $\binom{X}{U}$ given by the initial condition $X(t_0) = 0$, $U(t_0) = I$. Nonoscillation and eventual dense normality of (2.10) imply that there exists $t_1 > t_0$ such that $X(t)$ is nonsingular and

$$(X^\sigma(t))^{-1} \mathcal{B}(t) \left(X^T(t)\right)^{-1} = X^{-1}(t) \left[X(t)(X^\sigma(t))^{-1}\mathcal{B}(t)\right] (X^T(t))^{-1}$$

is nonnegative definite for $t \ge t_1$. Denote

$$\tilde{\mathcal{B}}(t) := (X^\sigma(t))^{-1} \mathcal{B}(t) \left(X^T(t)\right)^{-1}, \quad \mathcal{G}(t; X) := \int_{t_1}^t \tilde{\mathcal{B}}(s) \,\Delta s$$

and let

$$\bar{X}(t) = X(t)[I + \mathcal{G}(t;X)], \quad \bar{U}(t) = U(t)[I + \mathcal{G}(t;X)] + (X^T(t))^{-1}. \tag{3.8}$$

Then $\binom{\bar{X}}{\bar{U}}$ is a conjoined basis for which $\bar{X}^T U - \bar{U}^T X = -I$.

Since $\mathcal{G}(t;X)$ is nonnegative definite, $\bar{X}$ is nonsingular for $t \ge t_1$. Hence any conjoined basis $\binom{\hat{X}}{\hat{U}}$ of (2.10) can be expressed in the form

$$\hat{X} = \bar{X}[M + \mathcal{G}(t;\bar{X})N], \quad \hat{U} = \bar{U}[M + \mathcal{G}(t;\bar{X})N] + (\bar{X}^T)^{-1}, \tag{3.9}$$

where M, N are constant $n \times n$ matrices such that $M^T N$ is symmetric and

$$\mathcal{G}(t;\bar{X}) = \int_{t_1}^t (\bar{X}^\sigma(s))^{-1} \mathcal{B}(s) (\bar{X}^T(s))^{-1} \,\Delta s.$$

In particular, the solution $\binom{X}{U}$ is also of this form and substituting $t = t_1$ into (3.9) we get $M = I, N = -I$, hence

$$X = \bar{X}[I - \mathcal{G}(t; \bar{X})], \quad U = \bar{U}[I - \mathcal{G}(t; \bar{X})] - (\bar{X}^T)^{-1} \tag{3.10}$$

The first equalities in (3.9) and (3.10) imply that

$$I = [I - \mathcal{G}(t; \bar{X})][I + \mathcal{G}(t; X)].$$

Since the second factor in the last equality is a nondecreasing matrix-valued function, the first factor is nonincreasing and $0 \leq \mathcal{G}(t; \bar{X}) < I$, hence there exists a nonnegative definite matrix limit $\mathcal{G}_\infty = \lim_{t\to\infty} \mathcal{G}(t; \bar{X})$. Denote

$$\tilde{X} = \bar{X}[\mathcal{G}_\infty - \mathcal{G}(t; \bar{X})], \quad \tilde{U} = \bar{U}[\mathcal{G}_\infty - \mathcal{G}(t; \bar{X})] - (\bar{X}^T)^{-1}.$$

Then $\tilde{X}^T \bar{U} - \tilde{U}^T \bar{X} = I$ and

$$\lim_{t\to\infty} \bar{X}^{-1}(t)\tilde{X}(t) = \lim_{t\to\infty} [\mathcal{G}_\infty - \mathcal{G}(t; \bar{X})] = 0,$$

i.e. $\binom{\tilde{X}}{\tilde{U}}$ is a principal solution of (2.10). Concerning the equivalent characterization of the principal solution (3.6), if this limit relation holds, then for

$$X(t) = \tilde{X}(t)\mathcal{G}(t; \tilde{X}), \quad U(t) = \tilde{U}(t)\mathcal{G}(t; \tilde{X}) + (\tilde{X}^T(t))^{-1}$$

with

$$\mathcal{G}(t; \bar{X}) = \int_{t_1}^{t} (\tilde{X}^\sigma(s))^{-1}\mathcal{B}(s)(\tilde{X}^T(s))^{-1}\,\Delta s,$$

we have $\tilde{X}^T U - \tilde{U}^T X = I$ and

$$\begin{aligned}\lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) &= \lim_{t\to\infty}\left(\int_{t_1}^{t} \tilde{\mathcal{B}}(s)\,\Delta s\right)^{-1} \leq \lim_{t\to\infty} \lambda_n \left(\int_{t_1}^{t} \mathcal{B}(s)\,\Delta s\right)^{-1} \\ &= \lim_{t\to\infty} \frac{1}{\lambda_1\left(\int_{t_1}^{t} \mathcal{B}(s)\,\Delta s\right)} = 0.\end{aligned}$$

In the last computation the inequality between a symmetric matrix and a scalar quantity actually means the matrix inequality between a matrix and the identity matrix multiplied by the scalar quantity.

Conversely, suppose that (3.4), (3.5) hold. Without loss of generality we can suppose that $L = I$ in (3.4). Then $\binom{X}{U}$ can be expressed in the form

$$X = \tilde{X}[M + \mathcal{G}(t; \tilde{X})], \quad U = \tilde{U}[M + \mathcal{G}(t; \tilde{X})] + (\tilde{X}^T)^{-1}.$$

Moreover, we can suppose that $M = 0$ since if X satisfies (3.4) then $X - \tilde{X}M$ satisfies this limit relation as well since

$$\lim_{t\to\infty} [X(t) - \tilde{X}(t)M]^{-1}\tilde{X}(t) = \lim_{t\to\infty} [I - X^{-1}(t)\tilde{X}(t)M]^{-1}X^{-1}(t)\tilde{X}(t) = 0$$

for any constant matrix M. Consequently,

$$0 = \lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) = \lim_{t\to\infty} \left[\mathcal{G}(t;\tilde{X})\right]^{-1}$$

and hence

$$0 = \lim_{t\to\infty} \lambda_n \left[(\mathcal{G}(t;\tilde{X}))^{-1}\right] = \lim_{t\to\infty} \frac{1}{\lambda_1(\mathcal{G}(t;\tilde{X}))}$$

which implies (3.6) since $\mathcal{G}(\cdot;\tilde{X})$ is the nondecreasing matrix-valued function.

Finally, the equivalent characterization of the nonprincipal solution follows from the fact that if $L = I$ in (3.5) (what we can again suppose without loss of generality), then

$$(X^{-1}\tilde{X})^{\Delta} = -(X^{\sigma})^{-1}\mathcal{B}(X^T)^{-1}$$

and remaining arguments are the same as in the proof of equivalence of (3.4) and (3.6). □

The next theorem shows that the principal solution of (2.10) defines a solution of the associated Riccati matrix equation

$$Q^{\Delta} = [\mathcal{C} + \mathcal{D}Q - Q\mathcal{A} - Q\mathcal{B}Q][I + \mu(\mathcal{A} + \mathcal{B}Q)]^{-1} \tag{3.11}$$

related to (2.10) by the Riccati substitution $Q = UX^{-1}$, which has the same extremal property as in the continuous and discrete case.

Theorem 3.2. *Suppose that* (2.10) *is nonoscillatory and let* $\binom{\tilde{X}}{\tilde{U}}$ *be its principal solution. Then the solution* $\tilde{Q} = \tilde{U}\tilde{X}^{-1}$ *of the associated Riccati equation* (3.11) *is eventually minimal in the sense that if* Q *is any solution of this equation which exists on some interval* $[T,\infty)$ *and* $[I + \mu(\mathcal{A} + \mathcal{B}Q)^{-1}]\mathcal{B} \geq 0$ *in this interval, then* $Q(t) \geq \tilde{Q}(t)$ *for* $t \in [T,\infty)$.

Proof. Let $\binom{X}{U}$ be a conjoined basis of (2.10) which defines Q, i.e. $Q = UX^{-1}$ and denote $M = \tilde{X}^T U - \tilde{U}^T X$. Then we have

$$Q - \tilde{Q} = UX^{-1} - \tilde{U}\tilde{X}^{-1} = (\tilde{X}^T)^{-1}[\tilde{X}^T U - \tilde{U}^T X]X^{-1} = (\tilde{X}^T)^{-1}[MX^{-1}\tilde{X}]\tilde{X}^{-1}$$

and

$$\begin{aligned}(MX^{-1}\tilde{X})^{\Delta} &= -M(X^{\sigma})^{-1}(\mathcal{A}X + \mathcal{B}U)X^{-1}\tilde{X} + M(X^{\sigma})^{-1}(\mathcal{A}\tilde{X} + \mathcal{B}\tilde{U}) \\ &= M(X^{\sigma})^{-1}\mathcal{B}(X^T)^{-1}(-U^T\tilde{X} + X^T\tilde{U}) = -M(X^{\sigma})^{-1}\mathcal{B}(X^T)^{-1}M^T.\end{aligned}$$

Now,

$$\begin{aligned}[I + \mu(\mathcal{A} + \mathcal{B}Q)]^{-1}\mathcal{B} &= X[X + \mu(\mathcal{A}X + \mathcal{B}U)]^{-1} = X[X + \mu X^{\Delta}]^{-1}\mathcal{B} \\ &= X(X^{\sigma})^{-1}\mathcal{B}(X^T)^{-1}X^T \geq 0\end{aligned}$$

for $t \geq T$, hence also $(X^{\sigma})^{-1}\mathcal{B}(X^T)^{-1} \geq 0$, i.e. $MX^{-1}\tilde{X}$ is the nonincreasing matrix-valued sequence. Since $MX^{-1}\tilde{X} \to 0$ as $t \to \infty$, this matrix-valued function must be nonnegative definite which implies that $Q - \tilde{Q} \geq 0$ for $t \geq T$. □

The following theorem proves the essential uniqueness of the principal solution of (2.10).

Theorem 3.3. *Let* $\binom{\tilde X}{\tilde U}$, $\binom{\bar X}{\bar U}$ *be two principal solutions of* (2.10)*. Then there exists a constant nonsingular* $n \times n$ *matrix* M *such that* $\binom{\tilde X}{\tilde U} = \binom{\bar X}{\bar U} M$.

Proof. Let $\tilde Q = \tilde U \tilde X^{-1}$, $\bar Q = \bar U \bar X^{-1}$ be the solutions of (3.11) generated by $\binom{\tilde X}{\tilde U}$ and $\binom{\bar X}{\bar U}$, respectively, i.e. $\tilde X, \bar X$ are solutions of the first order liner dynamic systems

$$\tilde X^{\Delta} = (\mathcal{A} + \mathcal{B}\tilde Q)\tilde X, \quad \bar X^{\Delta} = (\mathcal{A} + \mathcal{B}\bar Q)\bar X.$$

By the previous theorem we have $\tilde Q \le \bar Q$ and $\bar Q \le \tilde Q$ eventually, so $\tilde Q = \bar Q$. This means that $\tilde X$, $\bar X$ are fundamental matrices of the same first order linear dynamic systems, hence there exists a constant nonsingular $n \times n$ matrix M such that $\tilde X = \bar X M$ and then $\tilde U = \tilde Q \tilde X = \bar Q \bar X M = \bar U M$. $\square$

4. Remarks.

In this last section we present some remarks concerning our previous results and also some suggestions for the further investigation.

(i) Observe that transformation (3.1) of SDS's is “principal solutions invariant”, i.e. $\binom{X}{U}$ is a principal solution of (2.10) if and only if $\binom{Y}{V} = \mathcal{R}^{-1}\binom{X}{U}$ is a principal solution of (3.2). This follows immediately from the identity $(X^\sigma)^{-1}\mathcal{B}(X^T)^{-1} = (Y^\sigma)^{-1}\bar{\mathcal{B}}(Y^T)^{-1}$.

(ii) Consider a general transformation

$$z = \mathcal{R}(t)w, \quad \mathcal{R} = \begin{pmatrix} H & M \\ K & N \end{pmatrix}, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad w = \begin{pmatrix} y \\ v \end{pmatrix} \tag{4.1}$$

with a symplectic matrix $\mathcal{R}$. This transformation transforms (2.10) again into a symplectic system, see [15]. It is a natural question when this transformation preserves oscillatory nature of transformed systems and converts the principal solution into the principal solution. In the theory of linear Hamiltonian systems (1.2) and symplectic difference systems (2.7) this problem is closely related to the so-called *reciprocity principle* and its generalization.

Transformation (4.1) with $\mathcal{R} = \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ transforms (1.2) into the so-called *reciprocal system*

$$y' = -A^T(t)y - C(t)z, \quad z' = -B(t)y + A(t)z. \tag{4.2}$$

It is known ([2,30]) that if $B(t) \ge 0$, $C(t) \le 0$ and both (1.2) and (4.2) are eventually controllable, then (1.2) is nonoscillatory if and only if (4.2) is nonoscillatory. Moreover, if

$$\lim_{t\to\infty} \lambda_1 \left(\int^t H^{-1}(s)B(s)(H^T(s))^{-1}\, ds \right) = \infty$$

and $\binom{X}{U}$ is a principal solution of (1.2) then $\binom{Y}{Z} = \binom{U}{-X} = \mathcal{J}\binom{X}{U}$ is a principal solution of (4.2), see [3]. These results were extended to general transformation (4.1) in [12,13] and papers [8,14] contain a discrete version of these statements. However, as pointed out in [6], there are some discrepancies between “differential” and “difference” results which have not been explained yet. The time scale point of view where (2.1) and (2.7) are special cases of symplectic dynamic system (2.10) could perhaps give an explanation of these discrepancies.

(iii) Suppose that (2.10) is nonoscillatory and eventually dense normal, and let

$$\binom{X}{U} = \binom{X(t;a,b)}{U(t;a,b)}$$

be its $2n\times n$ matrix solution satisfying the boundary condition $X(a)=I$, $X(b)=0$ (such solution exists and is unique if $a<b$ are sufficiently large as we will show later). Let $\binom{\tilde X}{\tilde U}$, $\binom{\bar X}{\bar U}$ be the principal and nonprincipal solutions of (2.10) such that $\tilde X(a)=I$, $\bar X(a)=I$. Any $2n\times n$ matrix solution $\binom{X}{U}$ is of the form $X=\tilde XM+\bar XN$, $U=\tilde UM+\bar UN$ with constant $n\times n$ matrices M,N. Substituting there the conditions at $t=a$ and $t=b$ we get $I=M+N$, $\quad 0=\tilde X(b)M+\bar X(b)N$, i.e.

$$M=\left[I-\bar X^{-1}(b)\tilde X^{-1}(b)\right]^{-1}, \quad N=\left[I-\tilde X^{-1}(b)\bar X(b)\right]^{-1}.$$

Since $\bar X^{-1}(b)\tilde X(b)\to 0$ as $b\to\infty$, the matrices in brackets are really nonsingular if b is sufficiently large, hence our boundary value has the unique solution and $M\to I$, $N\to 0$ as $b\to\infty$ (and a is fixed), which means that

$$\binom{X(t;a,b)}{U(t;a,b)} \to \binom{\tilde X}{\tilde U}, \quad \text{as } t\to\infty,$$

uniformly on every compact interval $[a,\tau]$, $\tau>a$. Consequently, we have shown another construction of the principal solution of (2.10), as the limit of solutions of a certain boundary value problem associated with (2.10).

(iv) Together with (1.1) consider another Sturm-Liouville equation

$$(R(t)y')'+C(t)y=0 \tag{4.3}$$

and suppose that this equation is the majorant of (1.1), i.e. $C(t)\ge c(t)$ and $r(t)\le R(t)$. If (4.3) is nonoscillatory then (1.1) is nonoscillatory as well by the Sturm comparison theorem. Denote by $\tilde x$, $\tilde y$ principal solutions of (1.1) and (4.3) and by $\tilde w$, $\tilde v$ the corresponding minimal solutions of the associated Riccati equations. Then $\tilde v(t)\ge \tilde w(t)$ for large t, see [18]. A similar inequality between minimal solutions of Riccati matrix differential equations (2.6) can be found in [23,32] and a discrete version of this inequality is given in [9]. Since the assumptions under which inequality between eventually minimal solutions holds differ in the discrete and continuous case (at least optically), it would be interesting to find their unification in the scope of the theory of SDS (2.10) and associated Riccati equation (3.11).

(v) This last remark deals with the so-called *zero maximality* of the principal solution. To explain the idea, consider the solution of (1.1) given by the initial condition $x(b)=0$, $r(b)x'(b)=-1$ and let a be its firs zero point left of b. Then by the Sturm separation theorem any solution of (1.1) which is not proportional to x has exactly one zero in (a,b). Now, let $\tilde x$ be the principal solution of (1.1) and let a be its largest zero point (if any). Then every solution which is linearly independent of $\tilde x$ (i.e. nonprincipal) has exactly one zero in (a,∞), see [24,26]. Consequently, the principal solution behaves like a solution given by the "initial condition" $\tilde x(\infty)=0$. This property of the principal solution was extended to linear Hamiltonian systems e.g. in [32] and to difference Hamiltonian systems in [10]. Similarly as in the case of the reciprocity principle, the continuous and discrete results are not quite analogical and the qualitative theory of SDS's (2.10) may explain this discrepancy.

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Mathematical Institute of the
Academy of Sciences of the
Czech Republic,
Zizkova 22, CZ-616 62 Brno
Czech Republic
E-mail: dosly@math.muni.cz