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A general Lipschitz uniqueness criterion for scalar ordinary differential equations

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Abstract. The classical Lipschitz-type criteria guarantee unique solvability of the scalar initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, by putting restrictions on $|f(t,x) - f(t,y)|$ in dependence of $|x - y|$. Geometrically it means that the field differences are estimated in the direction of the x-axis. In 1989, Stettner and the second author could establish a generalized Lipschitz condition in both arguments by showing that the field differences can be measured in a suitably chosen direction $v = (d_t, d_x)$, provided that it does not coincide with the directional vector $(1, f(t_0, x_0))$.

Considering the vector v depending on t, a new general uniqueness result is derived and a short proof based on the implicit function theorem is developed. The advantage of the new criterion is shown by an example. A comparison with known results is given as well.

Keywords: fundamental theory of ordinary differential equations, initial value problems, uniqueness, Lipschitz type conditions.

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1 Introduction

We consider the scalar initial value problem

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0, \tag{1.1}$$

and assume throughout the paper that $f\colon D \to \mathbb{R}$ is a continuous function on an open neighborhood D of the point $(t_0, x_0) \in \mathbb{R}^2$. Problem (1.1) is called *locally uniquely solvable* if there exists an open interval I containing t_0 such that (1.1) has exactly one solution on I.

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The uniqueness problem of (1.1) attracts permanent attention because it is not really solved up to now as simple examples show. The classical Lipschitz condition and its generalizations [1], including the results by Nagumo, Osgood, Perron and Kamke, consider $|f(t,x)-f(t,y)|$ in dependence of $|x-y|$ and thus measure the field differences in the direction of the x-axis. In 1989, Stettner and Nowak [9] could establish a generalized Lipschitz condition in both arguments. The field differences can be measured in a suitably chosen direction $v=(d_t,d_x)$, provided that it does not coincide with the directional vector $(1,f(t_0,x_0))$. The particular case with the t-axis as direction, thus requiring a Lipschitz condition with respect to the first argument of f, if $f(t_0,x_0)\neq 0$, was independently published first by Mortici [6] and then by Cid and López Pouso [2, 4]. Stettner and Nowak's paper is written in German, and therefore it is maybe non-accessible by not German-speaking colleagues as it is also remarked by Cid and López Pouso [3]. Hoag [5] extends the approach of a Lipschitz condition in the first argument including cases when $f(t_0,x_0)=0$.

In Section 2, considering the vector v depending on t, a new general uniqueness result is derived. We give a rather short proof based on the implicit function theorem. In Section 3 we compare our criterion with known results and show the advantage by an example.

2 A general Lipschitz uniqueness criterion

Theorem 2.1. *Let $v(t)=(\varphi(t),\psi(t))$ be a continuously differentiable vector on an open neighborhood of t_0 with real entries φ and ψ such that*

(i) $\psi(t_0)\neq f(t_0,x_0)\varphi(t_0)$,

(ii) for a constant $L\geq 0$ and every $k\in\mathbb{R}$

$$|f(t,x)-f(t+k\varphi(t),x+k\psi(t))|\leq L|k| \tag{2.1}$$

whenever the arguments of f are well-defined and belong to D.

Then (1.1) *is locally uniquely solvable.*

Proof. Peano's theorem guarantees that (1.1) has at least one solution $x\colon [t_0-\alpha_0,t_0+\alpha_0]\to\mathbb{R}$ for some $\alpha_0>0$. By assumption (i) there exists $\alpha\in(0,\alpha_0]$ with $\psi(t)\neq f(t,x(t))\varphi(t)$ for all $t\in(t_0-\alpha,t_0+\alpha)$. To prove that (1.1) is locally uniquely solvable with solution x on $I:=(t_0-\alpha,t_0+\alpha)$ assume to the contrary that there exists a solution $y\colon I\to\mathbb{R}$ of (1.1) and $x\not\equiv y$ on $[t_0,t_0+\alpha)$ (the case $x\not\equiv y$ on $(t_0-\alpha,t_0]$ is treated similarly). For $t_1:=\sup\{t\in[t_0,t_0+\alpha):x(s)=y(s)\text{ for }s\in[t_0,t]\}$ we have $t_1\in[t_0,t_0+\alpha)$, $x(t_1)=y(t_1)=:x_1$ by continuity and also

$$\psi(t_1)\neq f(t_1,x_1)\varphi(t_1). \tag{2.2}$$

We show that the equation

$$y(t+k(t)\varphi(t))=x(t)+k(t)\psi(t) \tag{2.3}$$

is uniquely solvable with respect to $k=k(t)$ on a subinterval of I. The problem suggests to apply the implicit function theorem. Let

$$F(t,k):=y(t+k\varphi(t))-x(t)-k\psi(t).$$

This function is defined in an open set containing $(t_1, 0)$ with the property

$$F(t_1, 0) = y(t_1) - x(t_1) = 0.$$

As

$$\frac{\partial F}{\partial k}(t, k) = f(t + k\varphi(t), y(t + k\varphi(t)))\varphi(t) - \psi(t),$$

we get with assumption (2.2)

$$\frac{\partial F}{\partial k}(t_1, 0) = f(t_1, x_1)\varphi(t_1) - \psi(t_1) \neq 0\,.$$

The implicit function theorem (cf., e.g., [8, Theorem 9.28]) now yields that there exists a unique continuously differentiable function $k = k(t)$ on an open interval $I_1 \subset I$ containing t_1 such that $k(t_1) = 0$ and $F(t, k(t)) = 0$ for all $t \in I_1$.

We show that $k(t) \equiv 0$ on a subinterval of I_1 with $t_1 \in I_1$. Due to (2.2), there exist a constant $\eta > 0$ and an open interval $I_2 \subset I_1$ containing t_1 such that

$$|f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)| \geq \eta \quad \text{for} \quad t \in I_2.$$

Moreover, there exists a constant M such that

$$|f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))| \leq M, \quad |\varphi'(t)| \leq M, \quad |\psi'(t)| \leq M, \quad t \in I_2.$$

Now we consider $u(t) := k^2(t)$ on I_2. Using the derivative of the function $k(t)$, relation (2.3) and inequality (2.1) we get for $t \in I_2$

$$\begin{aligned}
\dot{u}(t) = 2k(t)\dot{k}(t) &= 2k(t)\frac{\dot{x}(t) - \dot{y}(t + k(t)\varphi(t))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{\dot{y}(t + k(t)\varphi(t))\varphi(t) - \psi(t)} \\
&= 2k(t)\frac{f(t, x(t)) - f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)} \\
&= 2k(t)\frac{f(t, x(t)) - f(t + k(t)\varphi(t), x(t) + k(t)\psi(t))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)} \\
&\leq \frac{2(L + M^2 + M)}{\eta}k^2(t) \;=\; \frac{2(L + M^2 + M)}{\eta}u(t)
\end{aligned}$$

which is equivalent to

$$\frac{d}{dt}\left[u(t)\exp\left(-\frac{2(L + M^2 + M)}{\eta}(t - t_1)\right)\right] \leq 0\,.$$

Since $u(t_1) = k^2(t_1) = 0$, we get $u(t) = k^2(t) \equiv 0$ and hence from (2.3), $x(t) \equiv y(t)$ on I_2, which contradicts the definition of t_1. $\square$

3 Concluding remarks and comparison with known results

The function $k(t)$ in the proof of Theorem 2.1 measures in the case when $v(t)$ is a unit vector the distance between the points $(t, x(t))$ and $(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))$ on the graphs of the solutions x and y because

$$\operatorname{dist}\big((t, x(t)), (t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\big) = |k(t)|\sqrt{\varphi^2(t) + \psi^2(t)} = |k(t)|.$$

By the specification $v(t) = (\varphi(t), \psi(t)) = (0, 1)$ we get the well-known Lipschitz condition. The specification $v(t) = (\varphi(t), \psi(t)) = (1, 0)$ yields the result by Mortici cited above. The latter case contains the following special uniqueness criterion which is given in [7]. It was already known by Peano.

Corollary 3.1. *If $f\colon \mathbb{R} \to \mathbb{R}^+$ is continuous and positive then the equation $\dot{x} = f(x)$ has uniqueness, i.e. exactly one solution passes through every point of $\mathbb{R}^2$.*

Finally, the choice $v(t) = (\varphi(t), \psi(t)) = (d_t, d_x)$ turns our result into the following criterion published in German by Stettner and Nowak [9].

Theorem 3.2. *Let D be an open neighborhood of the point (t_0, x_0) and $f\colon D \to \mathbb{R}$ be continuous on D. Let d_t, d_x be real numbers such that*

i) $d_t^2 + d_x^2 > 0$,

ii) $d_x \neq f(t, x)d_t$ on D,

iii) for a constant $L \geq 0$ and every $k \in \mathbb{R}$ the inequality

$$|f(t, x) - f(t + kd_t, x + kd_x)| \leq L|k|$$

is satisfied whenever the arguments of f are in D.

Then (1.1) has at most one solution.

Now we illustrate the advantage of Theorem 2.1.

Example 1. Consider the initial value problem

$$\frac{dx}{dt} = f(t, x), \quad x(0) = 0, \tag{3.1}$$

where

$$f(t, x) := \begin{cases} 1 + x, & x < t^2, \\ 1 + x + \sqrt{x - t^2}, & x \geq t^2. \end{cases}$$

It is easy to check that f is not Lipschitz continuous with respect to x in any neighborhood of $(0, 0)$, and the problem cannot be treated by Theorem 3.2 using a constant vector $v = (d_t, d_x)$. Nevertheless, problem (3.1) is locally unique which can be shown by Theorem 2.1 using the vector $v(t) = (\varphi(t), \psi(t)) = (1, 2t)$. As $0 = \psi(0) \neq f(0, 0)\varphi(0) = 1$, assumption (i) is fulfilled. We briefly explain that assumption (ii) also holds on an arbitrary open and bounded neighbourhood $D \subset \mathbb{R} \times \mathbb{R}$ of $(0, 0)$. Let $M_1 := \sup\{|t| : (t, x) \in D\} < \infty$ and $L := 2M_1 + 1$. Consider the theoretically possible cases

$$\begin{aligned} &\alpha) \quad x < t^2 \wedge x + 2tk < (t + k)^2, \\ &\beta) \quad x < t^2 \wedge x + 2tk \geq (t + k)^2, \\ &\gamma) \quad x \geq t^2 \wedge x + 2tk < (t + k)^2, \\ &\delta) \quad x \geq t^2 \wedge x + 2tk \geq (t + k)^2, \end{aligned}$$

and note that β) is impossible. Then condition (2.1) of the form

$$|f(t, x) - f(t + k, x + 2tk)| \leq L|k|$$

is also fulfilled, since in the case α)

$$|f(t,x) - f(t+k, x+2tk)| = |1 + x - (1 + x + 2tk)| = 2|t||k| \leq 2M_1|k| \leq L|k|,$$

in the case γ), regarding that $\sqrt{x - t^2} < |k|$,

$$\begin{aligned} |f(t,x) - f(t+k, x+2tk)| &= |1 + x + \sqrt{x-t^2} - (1 + x + 2tk)| \\ &\leq |k| + 2|t||k| \leq |k| + 2M_1|k| = L|k| \end{aligned}$$

and in the case δ), regarding that $\sqrt{x - t^2} \geq |k|$,

$$\begin{aligned} &|f(t,x) - f(t+k, x+2tk)| \\ &\quad = \left|1 + x + \sqrt{x-t^2} - \left(1 + x + 2tk + \sqrt{x + 2tk - (t+k)^2}\right)\right| \\ &\quad \leq 2|t||k| + \left|\sqrt{x-t^2} - \sqrt{x - t^2 - k^2}\right| \leq 2M_1|k| + \left|\frac{k^2}{\sqrt{x-t^2} + \sqrt{x-t^2-k^2}}\right| \\ &\quad \leq 2M_1|k| + \left|\frac{k^2}{\sqrt{x-t^2}}\right| \leq 2M_1|k| + \left|\frac{k^2}{k}\right| = 2M_1|k| + |k| = L|k|, \end{aligned}$$

where without loss of generality we can assume $k \neq 0$.

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References

[1] R. P. Agarwal, V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, World Scientific Publishing, 1993. MR1336820; url

[2] J. Á. Cid, R. López Pouso, Does Lipschitz with respect to x imply uniqueness for the differential equation $y' = f(x,y)$?, *Amer. Math. Monthly* **116**(2009), No. 1, 61–66. MR2478753; url

[3] J. Á. Cid, R. López Pouso, Addendum to [2], *Amer. Math. Monthly* **117**(2010), Editor's Endnotes, p. 754.

[4] J. Á. Cid, R. López Pouso, On first-order ordinary differential equations with nonnegative right-hand sides, *Nonlinear Anal.* **52**(2003), 1961–1977. MR1954592; url

[5] J. T. Hoag, Existence and uniqueness of a local solution for $x' = f(t,x)$ using inverse functions, *Electron. J. Differential Equations* **2013**, No. 124, 1–3. MR3065077

[6] C. Mortici, On the solvability of the Cauchy problem, *Nieuw Arch. Wiskd. IV. Ser.* **17**(1999), 21–23. MR1702578

[7] W. Rudin, Nonuniqueness and growth in first-order differential equations, *Amer. Math. Monthly* **89**(1982), 241–244. MR0650671; url

[8] W. Rudin, *Principles of mathematical analysis*, third edition, McGraw-Hill Book Co., 1976. MR0385023

[9] H. Stettner, Chr. Nowak, Eine verallgemeinerte Lipschitzbedingung als Eindeutigkeitskriterium bei gewöhnlichen Differentialgleichungen (in German) [A generalized Lipschitz condition as criterion of uniqueness in ordinary differential equations], *Math. Nachr.* **141**(1989), 33–35. MR1014412; url