Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **42**, 1–14; http://www.math.u-szeged.hu/ejqtde/

Application of the bifurcation method to the modified Boussinesq equation

Shaoyong Li ✉

College of Mathematics and Information Sciences, Shaoguan University, Shaoguan 512005, China

Received 24 February 2013, appeared 20 August 2014
Communicated by Gabriele Villari

Abstract. In this paper, we investigate the modified Boussinesq equation

$$u_{tt} - u_{xx} - \varepsilon u_{xxxx} - 3(u^2)_{xx} + 3(u^2 u_x)_x = 0.$$

Firstly, we give a property of the solutions of the equation, that is, if $1 + u(x,t)$ is a solution, so is $1 - u(x,t)$. Secondly, by using the bifurcation method of dynamical systems we obtain some explicit expressions of solutions for the equation, which include kink-shaped solutions, blow-up solutions, periodic blow-up solutions and solitary wave solutions. Some previous results are extended.

Keywords: modified Boussinesq equation, bifurcation method, exact solutions.

2010 Mathematics Subject Classification: 34C23, 76B25.

1 Introduction

In recent years, nonlinear phenomena have been studied in all fields of science and engineering, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. Many nonlinear evolution equations play an important role in the analysis of these phenomena.

In order to find the traveling wave solutions of these nonlinear evolution equations, there have been many methods, such as inverse scattering method [6], the Bäcklund transformation method [14], Jacobi elliptic function method [10], F-expansion and extended F-expansion method [18, 19], $(\frac{G'}{G})$-expansion method [16, 20], the bifurcation method of dynamical systems [8, 9, 11, 12, 17], and so on.

The bad and good Boussinesq equations [13] are as follows

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0, \tag{1.1}$$

and

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} = 0, \tag{1.2}$$

✉ Email: lishaoyongok@sohu.com

which were introduced by the French scientist Joseph Boussinesq (1842–1929) to describe the 1870s model equation for the propagation of long waves on the surface of water with a small amplitude. Equation (1.1) is used to describe the two-dimensional flow of shallow-water waves having small amplitudes. There is a dense connection to the so-called Fermi–Pasta–Ulam (FPU) problem. The existence of Lax pair, Bäcklund transformation and some soliton-type solutions is known [13, 21]. Equation (1.2) describes the two-dimensional irrotational flow of an inviscid liquid in a uniform rectangular channel. There are known results due to local well-posedness, global existence and blow-up of some solutions [3]. The bad and good Boussinesq equations have been studied by using the VIM, HPM, ADM, Exp-function method, and F-expansion method [1, 2, 7, 15].

Dai et al. [4] studied the explicit homoclinic orbits solutions for the bad Boussinesq equation with periodic boundary condition and even constraint, and periodic soliton solutions for the good Boussinesq equation with even constraint. G. Forozani and M. Ghorveei Nosrat [5], by adding a nonlinear term of the form $3(u^2u_x)_x$ to the bad and good Boussinesq equations, studied the following modified bad and good Boussinesq equations

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0, \tag{1.3}$$

and

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0. \tag{1.4}$$

They obtained variant solutions such as kink, anti-kink, compacton and periodic solutions for these equations by using the standard tanh, the extended tanh method and a mathematical method based on the reduction of order.

In the present paper, combining the modified bad and good Boussinesq equations, we consider the following modified Boussinesq equation

$$u_{tt} - u_{xx} - \varepsilon u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0, \tag{1.5}$$

where ε is a nonzero constant. When $\varepsilon = 1$, equation (1.5) reduces to the modified bad Boussinesq equation (1.3). When $\varepsilon = -1$, equation (1.5) reduces to the modified good Boussinesq equation (1.4). In order to search for the traveling wave solutions of equation (1.5), here we study equation (1.5) by using the bifurcation method mentioned above. Firstly, we give a property of the solutions of equation (1.5), that is, if $1 + u(x,t)$ is a solution, so is $1 - u(x,t)$. Secondly, we obtain some explicit expressions of solutions for equation (1.5), which include kink-shaped solutions, blow-up solutions, periodic blow-up solutions and solitary wave solutions. After checking over these solutions carefully, we find that some solutions are, in fact, exactly the same as those solutions given in [5]. To our knowledge, many other solutions are new.

This paper is organized as follows. In Section 2, we state our main results which are included in three propositions. In Section 3, we give the theoretical derivations for the propositions respectively. A brief conclusion is given in Section 4.

2 Main results

In this section we list our main results. To relate conveniently, for given constant wave speed c, let

$$\xi = x - ct, \tag{2.1}$$

$$g_0 = \frac{2}{3\sqrt{3}}(4-c^2)^{3/2}. \tag{2.2}$$

Using the notations above, our main results are stated in Proposition 2.1 (the property of the solutions of equation (1.5)) and Propositions 2.2, 2.3 (the exact explicit expressions of solutions for equation (1.5)).

Proposition 2.1. *There exists a property of the solutions of equation* (1.5)*, that is, if* $1+u(x,t)$ *is a solution of equation* (1.5)*, so is* $1-u(x,t)$*.*

Proposition 2.2. *When* $\varepsilon > 0$*, equation* (1.5) *has the following exact solutions.*

(1) Let g *denote the integral constant in equation* (3.3)*. If* $g=0$*, we obtain two kink-shaped solutions*

$$u_{1\pm}(x,t) = 1 \pm \sqrt{4-c^2}\tanh\left(\sqrt{\frac{4-c^2}{2\varepsilon}}\xi\right), \tag{2.3}$$

two blow-up solutions

$$u_{2\pm}(x,t) = 1 \pm \sqrt{4-c^2}\coth\left(\sqrt{\frac{4-c^2}{2\varepsilon}}\xi\right), \tag{2.4}$$

and four periodic blow-up solutions

$$u_{3\pm}(x,t) = 1 \pm \sqrt{2(4-c^2)}\sec\left(\sqrt{\frac{4-c^2}{2\varepsilon}}\xi\right), \tag{2.5}$$

$$u_{4\pm}(x,t) = 1 \pm \sqrt{2(4-c^2)}\csc\left(\sqrt{\frac{4-c^2}{2\varepsilon}}\xi\right). \tag{2.6}$$

(2) If $0<g<g_0$*, we obtain two solitary wave solutions*

$$u_{5\pm}(x,t) = 1 \pm \frac{2(1-2\alpha)\sqrt{4-c^2}+\sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha}+\sqrt{2(1-\alpha)}\cosh(\eta_1\xi)}, \tag{2.7}$$

two blow-up solutions

$$u_{6\pm}(x,t) = 1 \pm \frac{\sqrt{4-c^2}(\sqrt{2\alpha(3\alpha-1)}\sinh(\eta_1\xi)+2\alpha\cosh(\eta_1\xi)+2(2\alpha-1))}{\sqrt{2(3\alpha-1)}\sinh(\eta_1\xi)+2\sqrt{\alpha}\cosh(\eta_1\xi)-2\sqrt{\alpha}}, \tag{2.8}$$

and two periodic blow-up solutions

$$u_{7\pm}(x,t) = 1 \pm \frac{\sqrt{2}(4-c^2-2\varphi_2^2)-\varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2-\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}, \tag{2.9}$$

where α *depends on* g*, it is such that* $\sqrt{\alpha(4-c^2)}^3+(c^2-4)\sqrt{\alpha(4-c^2)}+g=0$*, and* $\frac{1}{3}<\alpha<1$

$$\eta_1 = \sqrt{\frac{(4-c^2)(3\alpha-1)}{\varepsilon}}, \tag{2.10}$$

$$\eta_2 = \sqrt{\frac{4-c^2-3\varphi_2^2}{\varepsilon}}, \tag{2.11}$$

$$\varphi_2 = \frac{1}{2}\left(\sqrt{(4-c^2)(4-3\alpha)}-\sqrt{\alpha(4-c^2)}\right).$$

(3) If $g = g_0$, we obtain six blow-up solutions

$$u_{8\pm}(x,t) = 1 \pm \frac{(4-c^2)\xi^2 - \sqrt{6(4-c^2)}\varepsilon\xi + 6\varepsilon}{\sqrt{3(4-c^2)}\xi^2 - 3\sqrt{2}\varepsilon\xi}, \tag{2.12}$$

$$u_{9\pm}(x,t) = 1 \pm \frac{(4-c^2)\xi^2 + \sqrt{6(4-c^2)}\varepsilon\xi + 6\varepsilon}{\sqrt{3(4-c^2)}\xi^2 + 3\sqrt{2}\varepsilon\xi}, \tag{2.13}$$

and

$$u_{10\pm}(x,t) = 1 \pm \sqrt{\frac{4-c^2}{3}}\frac{2(4-c^2)\xi^2 + 9\varepsilon}{2(4-c^2)\xi^2 - 3\varepsilon}. \tag{2.14}$$

Proposition 2.3. *When $\varepsilon < 0$, equation (1.5) has the following exact solutions*

(1°) If $g = 0$, we obtain two solitary wave solutions

$$u_{11\pm}(x,t) = 1 \pm \sqrt{2(4-c^2)}\,\mathrm{sech}\left(\sqrt{\frac{c^2-4}{\varepsilon}}\xi\right). \tag{2.15}$$

(2°) If $0 < g < g_0$, we obtain four solitary wave solutions

$$u_{12\pm}(x,t) = 1 \pm \frac{2(1-2\alpha)\sqrt{4-c^2} - \sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha} - \sqrt{2(1-\alpha)}\cosh(\eta_1\xi)}, \tag{2.16}$$

$$u_{13\pm}(x,t) = 1 \pm \frac{2(1-2\alpha)\sqrt{4-c^2} + \sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha} + \sqrt{2(1-\alpha)}\cosh(\eta_1\xi)}, \tag{2.17}$$

and four periodic wave solutions

$$u_{14\pm}(x,t) = 1 \pm \frac{\sqrt{2}(4-c^2-2\varphi_2^2) + \varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2 + \sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}, \tag{2.18}$$

$$u_{15\pm}(x,t) = 1 \pm \frac{\sqrt{2}(4-c^2-2\varphi_2^2) - \varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2 - \sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}, \tag{2.19}$$

where $0 < \alpha < \frac{1}{3}$.

(3°) If $g = g_0$, we obtain two solitary wave solutions

$$u_{16\pm}(x,t) = 1 \pm \sqrt{\frac{4-c^2}{3}}\frac{2(4-c^2)\xi^2 + 9\varepsilon}{2(4-c^2)\xi^2 - 3\varepsilon}. \tag{2.20}$$

Remark 2.4. If we check the above solutions carefully, we can discover an interesting fact, that is, (2.9) and (2.19) have the same expressions, so do (2.14) and (2.20). However, they are different kinds of solutions under corresponding parametric conditions. In fact, (2.9) are periodic blow-up wave solutions, while (2.19) are periodic solutions. Meanwhile, (2.14) are blow-up solutions, while (2.20) are solitary wave solutions. On the other hand, (2.7) and (2.17), which have the same expressions, are both solitary wave solutions.

3 The theoretic derivations for main results

In this section, we will give the derivations for our main results. Firstly we derive Proposition 2.1, the property of the solutions of equation (1.5). If $1+u(x,t)$ is a solution of equation (1.5), that is $1+u(x,t)$ satisfies equation (1.5), then we have

$$\begin{aligned}
&(1+u)_{tt}-(1+u)_{xx}-\varepsilon(1+u)_{xxxx}-3((1+u)^2)_{xx}+3((1+u)^2(1+u)_x)_x\\
&\quad=u_{tt}-u_{xx}-\varepsilon u_{xxxx}-3(1+2u+u^2)_{xx}+3((1+u)^2u_x)_x\\
&\quad=u_{tt}-u_{xx}-\varepsilon u_{xxxx}-6((1+u)u_{xx}+(u_x)^2)+6(1+u)(u_x)^2+3(1+u)^2u_{xx}\\
&\quad=u_{tt}-\varepsilon u_{xxxx}+(3u^2-4)u_{xx}+6u(u_x)^2=0.
\end{aligned}\tag{3.1}$$

On the other hand, substituting $1-u(x,t)$ into the left side of equation (1.5), we have

$$\begin{aligned}
&(1-u)_{tt}-(1-u)_{xx}-\varepsilon(1-u)_{xxxx}-3((1-u)^2)_{xx}+3((1-u)^2(1-u)_x)_x\\
&\quad=-u_{tt}+u_{xx}+\varepsilon u_{xxxx}-3(1-2u+u^2)_{xx}-3((1-u)^2u_x)_x\\
&\quad=-u_{tt}+u_{xx}+\varepsilon u_{xxxx}-6((u-1)u_{xx}+(u_x)^2)+6(1-u)(u_x)^2-3(1-u)^2u_{xx}\\
&\quad=-u_{tt}+\varepsilon u_{xxxx}-(3u^2-4)u_{xx}-6u(u_x)^2\\
&\quad=-(u_{tt}-\varepsilon u_{xxxx}+(3u^2-4)u_{xx}+6u(u_x)^2)=0,\qquad\text{(according to (3.1))}
\end{aligned}$$

thus $1-u(x,t)$ is a solution of equation (1.5).

Secondly we derive Propositions 2.2 and 2.3, the explicit expressions of solutions for equation (1.5). We look for the traveling wave solutions of equation (1.5) in the form of

$$u(x,t)=1+\varphi(\xi),\tag{3.2}$$

where ξ was given in (2.1). Substituting (3.2) into equation (1.5) and integrating twice with respect to ξ, we get

$$\varphi''=\frac{1}{\varepsilon}(\varphi^3+(c^2-4)\varphi+g+g_*\xi),\tag{3.3}$$

where g and g_* are two integral constants. In order to use the bifurcation method of dynamical systems, we consider the case $g_*=0$.

Letting $y=\varphi'$, we obtain the following planar system

$$\begin{cases}\dfrac{d\varphi}{d\xi}=y,\\[2mm] \dfrac{dy}{d\xi}=\dfrac{1}{\varepsilon}(\varphi^3+(c^2-4)\varphi+g),\end{cases}\tag{3.4}$$

which has the first integral

$$H(\varphi,y)=y^2-\frac{1}{\varepsilon}\left(\frac{1}{2}\varphi^4+(c^2-4)\varphi^2+2g\varphi\right)=h,$$

where h is an integral constant.

Now we consider the phase portraits of system (3.4). Set

$$f_0(\varphi)=\varphi^3+(c^2-4)\varphi,$$

and

$$f(\varphi)=\varphi^3+(c^2-4)\varphi+g.$$

It is easy to obtain the two extreme points of $f_0(\varphi)$ as follows

$$\varphi_{\pm}^{*} = \pm\sqrt{\frac{4-c^2}{3}},$$

where $|c| < 2$. Let $g_0 = |f_0(\varphi_{\pm}^{*})| = \frac{2}{3\sqrt{3}}(4-c^2)^{3/2}$, which is in (2.2).

Let $(\varphi_*, 0)$ be one of the singular points of system (3.4). Then the characteristic values of the linearized system of system (3.4) at the singular point $(\varphi_*, 0)$ are

$$\lambda_{\pm} = \pm\sqrt{\frac{f'(\varphi_*)}{\varepsilon}}.$$

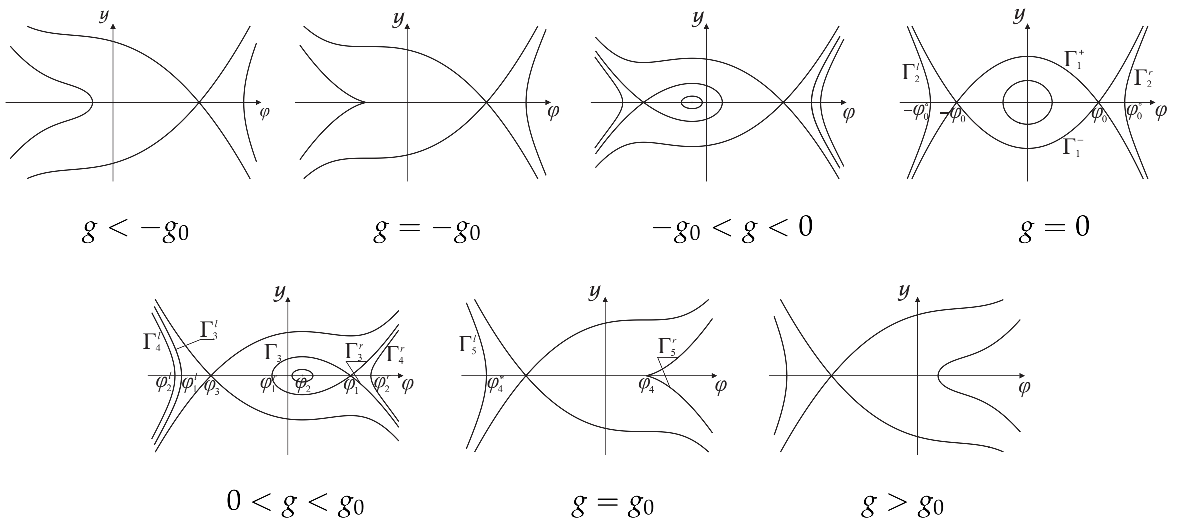

Figure 3.1: The phase portraits of system (3.4) when $\varepsilon > 0$.

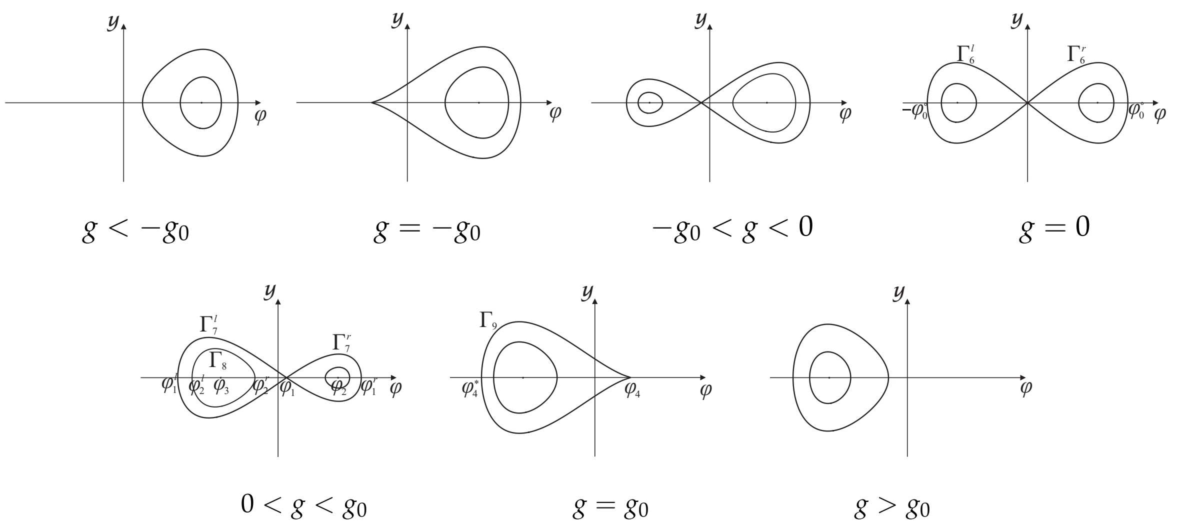

Figure 3.2: The phase portraits of system (3.4) when $\varepsilon < 0$.

According to the qualitative theory of dynamical systems, we therefore know the following

(i) If $\frac{f'(\varphi_*)}{\varepsilon} > 0$, $(\varphi_*, 0)$ is a saddle point.

(ii) If $\frac{f'(\varphi_*)}{\varepsilon} < 0$, $(\varphi_*, 0)$ is a center point.

(iii) If $\frac{f'(\varphi_*)}{\varepsilon} = 0$, $(\varphi_*, 0)$ is a degenerate saddle point.

From the analysis above, we obtain the phase portraits of system (3.4) in Figure 3.1 (when $\varepsilon > 0$) and Figure 3.2 (when $\varepsilon < 0$).

Now we will obtain the explicit expressions of solutions for equation (1.5) when $\varepsilon > 0$.

(1) If $g = 0$, we consider two kinds of orbits.

(i) Firstly, we see that there are two heteroclinic orbits Γ_1^+ and Γ_1^- connected at saddle points $(-\varphi_0, 0)$ and $(\varphi_0, 0)$. On the (φ, y)-plane the expressions of the heteroclinic orbits are given as

$$y = \pm\frac{1}{\sqrt{2\varepsilon}}(\varphi_0^2 - \varphi^2), \tag{3.5}$$

where $\varphi_0 = \sqrt{4 - c^2}$. Substituting (3.5) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the heteroclinic orbits Γ_1^+ and Γ_1^-. Meanwhile for simplicity, we assume that $\varphi(\xi) \to 0$ and ∞ respectively as $\xi \to 0$, then it follows that

$$\pm\int_0^{\varphi} \frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{\varphi_0^2 - s^2} = \int_0^{\xi} \mathrm{d}s,$$

and

$$\pm\int_{\varphi}^{\infty} \frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{s^2 - \varphi_0^2} = \int_{\xi}^{0} \mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\sqrt{4 - c^2}\tanh\left(\sqrt{\frac{4 - c^2}{2\varepsilon}}\xi\right),$$

and

$$\varphi(\xi) = \pm\sqrt{4 - c^2}\coth\left(\sqrt{\frac{4 - c^2}{2\varepsilon}}\xi\right).$$

Noting that $u(x,t) = 1 + \varphi(\xi)$ with $\xi = x - ct$, we get two kink-shaped solutions $u_{1\pm}(x,t)$ and two blow-up solutions $u_{2\pm}(x,t)$ as (2.3) and (2.4).

(ii) Secondly, from the phase portrait, we note that there are two special orbits Γ_2^l and Γ_2^r, which have the same Hamiltonian as that of the center point $(0,0)$. On the (φ, y)-plane the expressions of the two orbits are given as

$$y = \pm\frac{1}{\sqrt{2\varepsilon}}\varphi\sqrt{\varphi^2 - 2(4 - c^2)}. \tag{3.6}$$

Substituting (3.6) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the two orbits Γ_2^l and Γ_2^r, it follows that

$$\pm\int_{\varphi}^{\infty} \frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{s\sqrt{s^2 - 2(4 - c^2)}} = \int_{\xi}^{0} \mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\sqrt{2(4 - c^2)}\sec\left(\sqrt{\frac{4 - c^2}{\varepsilon}}\xi\right).$$

At the same time, we note that if $\varphi(\xi)$ is a solution of system (3.4), then $\varphi(\xi+\kappa)$ is also a solution of system (3.4). Specially, taking $\kappa=\frac{\pi}{2}$, we get another two solutions

$$\varphi(\xi)=\pm\sqrt{2(4-c^2)}\csc\left(\sqrt{\frac{4-c^2}{\varepsilon}}\xi\right).$$

Noting that $u(x,t)=1+\varphi(\xi)$ with $\xi=x-ct$, we get four periodic blow-up solutions $u_{3\pm}(x,t)$ and $u_{4\pm}(x,t)$ as (2.5) and (2.6).

(2) If $0<g<g_0$, we set the largest root of $f(\varphi)=0$ be $\varphi_1=\sqrt{\alpha(4-c^2)}$ $(\frac{1}{3}<\alpha<1)$, then we can get another two roots as follows

$$\varphi_2=\frac{1}{2}\left(-\sqrt{\alpha(4-c^2)}+\sqrt{(4-c^2)(4-3\alpha)}\right), \tag{3.7}$$

$$\varphi_3=\frac{1}{2}\left(-\sqrt{\alpha(4-c^2)}-\sqrt{(4-c^2)(4-3\alpha)}\right). \tag{3.8}$$

(i) Firstly, from the phase portrait, we note that there are a heteroclinic orbit Γ_3 and two special orbits Γ_3^l, Γ_3^r, which have the same Hamiltonian as that of the saddle point $(\varphi_1,0)$. On the (φ,y)-plane the expressions of these orbits are given as

$$y=\pm\frac{1}{\sqrt{2\varepsilon}}\sqrt{(\varphi-\varphi_1)^2(\varphi-\varphi_1^l)(\varphi-\varphi_1^r)}, \tag{3.9}$$

where

$$\varphi_1^l=-\sqrt{\alpha(4-c^2)}-\sqrt{2(1-\alpha)(4-c^2)}, \tag{3.10}$$

$$\varphi_1^r=-\sqrt{\alpha(4-c^2)}+\sqrt{2(1-\alpha)(4-c^2)}. \tag{3.11}$$

Substituting (3.9) into $\mathrm{d}\varphi/\mathrm{d}\xi=y$ and integrating them along the orbits Γ_3, Γ_3^l and Γ_3^r, it follows that

$$\pm\int_{\varphi_1^r}^{\varphi}\frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{(\varphi_1-s)\sqrt{(s-\varphi_1^l)(s-\varphi_1^r)}}=\int_0^{\xi}\mathrm{d}s.$$

and

$$\pm\int_{\varphi}^{\infty}\frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{(s-\varphi_1)\sqrt{(s-\varphi_1^l)(s-\varphi_1^r)}}=\int_{\xi}^{0}\mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi)=\pm\frac{2(1-2\alpha)\sqrt{4-c^2}+\sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha}+\sqrt{2(1-\alpha)}\cosh(\eta_1\xi)},$$

and

$$\varphi(\xi)=\pm\frac{\sqrt{4-c^2}(\sqrt{2\alpha(3\alpha-1)}\sinh(\eta_1\xi)+2\alpha\cosh(\eta_1\xi)+2(2\alpha-1))}{\sqrt{2(3\alpha-1)}\sinh(\eta_1\xi)+2\sqrt{\alpha}\cosh(\eta_1\xi)-2\sqrt{\alpha}},$$

where η_1 is given in (2.10). Noting that $u(x,t)=1+\varphi(\xi)$ with $\xi=x-ct$, we get two solitary wave solutions $u_{5\pm}(x,t)$ and two blow-up solutions $u_{6\pm}(x,t)$ as (2.7) and (2.8).

(ii) Secondly, from the phase portrait, we note that there are two special orbits Γ_4^l and Γ_4^r, which have the same Hamiltonian as that of the center point $(\varphi_2, 0)$. On the (φ, y)-plane the expressions of these orbits are given as

$$y = \pm\frac{1}{\sqrt{2\varepsilon}}\sqrt{(\varphi-\varphi_2)^2(\varphi-\varphi_2^l)(\varphi-\varphi_2^r)}, \tag{3.12}$$

where

$$\varphi_2^l = -\varphi_2 - \sqrt{2(4-c^2-\varphi_2^2)}, \tag{3.13}$$

$$\varphi_2^r = -\varphi_2 + \sqrt{2(4-c^2-\varphi_2^2)}. \tag{3.14}$$

Substituting (3.12) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbits Γ_4^l and Γ_4^r, it follows that

$$\pm\int_{\varphi}^{\infty}\frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{(s-\varphi_2)\sqrt{(s-\varphi_2^l)(s-\varphi_2^r)}} = \int_{\xi}^{0}\mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\frac{\sqrt{2}(4-c^2-2\varphi_2^2) - \varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2 - \sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)},$$

where η_2 is given in (2.11). Noting that $u(x,t) = 1+\varphi(\xi)$ with $\xi = x-ct$, we get two periodic blow-up solutions $u_{7\pm}(x,t)$ as (2.9).

(3) If $g = g_0$, from the phase portrait, we see that there are two orbits Γ_5^l and Γ_5^r, which have the same Hamiltonian with the degenerate saddle point $(\varphi_4, 0)$. On the (φ, y)-plane the expressions of these orbits are given as

$$y = \pm\frac{1}{\sqrt{2\varepsilon}}\sqrt{(\varphi-\varphi_4)^3(\varphi-\varphi_4^*)}, \tag{3.15}$$

where

$$\varphi_4 = \sqrt{\frac{1}{3}(4-c^2)}, \tag{3.16}$$

$$\varphi_4^* = -3\varphi_4 = -\sqrt{3(4-c^2)}. \tag{3.17}$$

Substituting (3.15) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbits Γ_5^l and Γ_5^r, it follows that

$$\pm\int_{\varphi}^{+\infty}\frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{(s-\varphi_4)\sqrt{(s-\varphi_4)(s-\varphi_4^*)}} = \int_{\xi}^{0}\mathrm{d}s,$$

and

$$\pm\int_{\varphi}^{\varphi_4^*}\frac{\sqrt{2\varepsilon}\,\mathrm{d}s}{(\varphi_4-s)\sqrt{(s-\varphi_4)(s-\varphi_4^*)}} = \int_{\xi}^{0}\mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\frac{(4-c^2)\xi^2 - \sqrt{6(4-c^2)\varepsilon}\xi + 6\varepsilon}{\sqrt{3(4-c^2)}\xi^2 - 3\sqrt{2\varepsilon}\xi},$$

$$\varphi(\xi) = \pm \frac{(4-c^2)\xi^2 + \sqrt{6(4-c^2)\varepsilon}\xi + 6\varepsilon}{\sqrt{3(4-c^2)}\xi^2 + 3\sqrt{2\varepsilon}\xi},$$

and

$$\varphi(\xi) = \pm\sqrt{\frac{4-c^2}{3}\frac{2(4-c^2)\xi^2 + 9\varepsilon}{2(4-c^2)\xi^2 - 3\varepsilon}}.$$

Noting that $u(x,t) = 1 + \varphi(\xi)$ with $\xi = x - ct$, we get six blow-up solutions $u_{8\pm}(x,t)$, $u_{9\pm}(x,t)$ and $u_{10\pm}(x,t)$ as (2.12)–(2.14).

Heretofore, we have completed the derivations for the Proposition 2.2. Now we will obtain the explicit expressions of solutions for equation (1.5) when $\varepsilon < 0$.

(1°) If $g = 0$, from the phase portrait, we see that there are two symmetric homoclinic orbits Γ_6^l and Γ_6^r connected at the saddle point $(0,0)$. On the (φ, y)-plane the expressions of these orbits are given as

$$y = \pm\frac{1}{\sqrt{-2\varepsilon}}\sqrt{\varphi^2(2(4-c^2) - \varphi^2)}. \tag{3.18}$$

Substituting (3.18) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbits Γ_6^l and Γ_6^r, it follows that

$$\pm\int_{-\varphi_0^\circ}^{\varphi} \frac{\sqrt{-2\varepsilon}\mathrm{d}s}{s\sqrt{2(4-c^2) - s^2}} = \int_0^{\xi} \mathrm{d}s,$$

and

$$\pm\int_{\varphi}^{\varphi_0^\circ} \frac{\sqrt{-2\varepsilon}\mathrm{d}s}{s\sqrt{2(4-c^2) - s^2}} = \int_{\xi}^{0} \mathrm{d}s,$$

where $\varphi_0^\circ = \sqrt{2(4-c^2)}$. Computing the integrals above, we have

$$\varphi(\xi) = \pm\sqrt{2(4-c^2)}\,\mathrm{sech}\left(\sqrt{\frac{c^2-4}{\varepsilon}}\xi\right).$$

Noting that $u(x,t) = 1 + \varphi(\xi)$ with $\xi = x - ct$, we get two solitary wave solutions $u_{11\pm}(x,t)$ as (2.15).

(2°) If $0 < g < g_0$, we set the middle root of $f(\varphi) = 0$ be $\varphi_1 = \sqrt{\alpha(4-c^2)}$ $(0 < \alpha < \frac{1}{3})$, then we can get another two roots φ_2 and φ_3 as (3.7) and (3.8).

(i) Firstly, from the phase portrait, we note that there are two homoclinic orbits Γ_7^l and Γ_7^r connected at the saddle point $(\varphi_1, 0)$. On the (φ, y)-plane the expressions of these orbits are given as

$$y = \pm\frac{1}{\sqrt{-2\varepsilon}}\sqrt{(\varphi-\varphi_1)^2(\varphi-\varphi_1^l)(\varphi-\varphi_1^r)}, \tag{3.19}$$

where φ_1^l and φ_1^r are given in (3.10) and (3.11). Substituting (3.19) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbits Γ_7^l and Γ_7^r, it follows that

$$\pm\int_{\varphi_1^l}^{\varphi} \frac{\sqrt{-2\varepsilon}\,\mathrm{d}s}{(\varphi_1 - s)\sqrt{(s-\varphi_1^l)(\varphi_1^r - s)}} = \int_0^{\xi} \mathrm{d}s,$$

and

$$\pm\int_{\varphi}^{\varphi_1^r} \frac{\sqrt{-2\varepsilon}\,\mathrm{d}s}{(s-\varphi_1)\sqrt{(s-\varphi_1^l)(\varphi_1^r - s)}} = \int_{\xi}^{0} \mathrm{d}s.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\frac{2(1-2\alpha)\sqrt{4-c^2} - \sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha} - \sqrt{2(1-\alpha)}\cosh(\eta_1\xi)},$$

and

$$\varphi(\xi) = \pm\frac{2(1-2\alpha)\sqrt{4-c^2} + \sqrt{2\alpha(1-\alpha)(4-c^2)}\cosh(\eta_1\xi)}{2\sqrt{\alpha} + \sqrt{2(1-\alpha)}\cosh(\eta_1\xi)},$$

where η_1 is given in (2.10). Noting that $u(x,t) = 1 + \varphi(\xi)$ with $\xi = x - ct$, we get four solitary wave solutions $u_{12\pm}(x,t)$ and $u_{13\pm}(x,t)$ as (2.16) and (2.17).

(ii) Secondly, from the phase portrait, we note that there is a special periodic orbit Γ_8, which has the same Hamiltonian as that of the center point $(\varphi_2, 0)$. On the (φ, y)-plane the expressions of this orbit are given as

$$y = \pm\frac{1}{\sqrt{-2\varepsilon}}\sqrt{(\varphi_2-\varphi)^2(\varphi-\varphi_2^l)(\varphi_2^r-\varphi)}, \tag{3.20}$$

where φ_2^l and φ_2^r are given in (3.13) and (3.14). Substituting (3.20) into $d\varphi/d\xi = y$ and integrating them along the orbit Γ_8, it follows that

$$\pm\int_{\varphi}^{\varphi_2^r}\frac{\sqrt{-2\varepsilon}\,ds}{(\varphi_2-s)\sqrt{(s-\varphi_2^l)(\varphi_2^r-s)}} = \int_{\xi}^{0} ds,$$

and

$$\pm\int_{\varphi_2^l}^{\varphi}\frac{\sqrt{-2\varepsilon}\,ds}{(\varphi_2-s)\sqrt{(s-\varphi_2^l)(\varphi_2^r-s)}} = \int_{0}^{\xi} ds.$$

Computing the integrals above, we have

$$\varphi(\xi) = \pm\frac{\sqrt{2}(4-c^2-2\varphi_2^2) + \varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2 + \sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)},$$

and

$$\varphi(\xi) = \pm\frac{\sqrt{2}(4-c^2-2\varphi_2^2) - \varphi_2\sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)}{\sqrt{2}\varphi_2 - \sqrt{4-c^2-\varphi_2^2}\cos(\eta_2\xi)},$$

where η_2 is given in (2.11). Noting that $u(x,t) = 1 + \varphi(\xi)$ with $\xi = x - ct$, we get four periodic wave solutions $u_{14\pm}(x,t)$ and $u_{15\pm}(x,t)$ as (2.18) and (2.19).

(3°) If $g = g_0$, from the phase portrait, we see that there is a homoclinic orbit Γ_9, which passes the degenerate saddle point $(\varphi_4, 0)$. On the (φ, y)-plane the expressions of the homoclinic orbit are given as

$$y = \pm\frac{1}{\sqrt{-2\varepsilon}}\sqrt{(\varphi_4-\varphi)^3(\varphi_4^*-\varphi)}, \tag{3.21}$$

where φ_4 and φ_4^* are given in (3.16) and (3.17). Substituting (3.21) into $d\varphi/d\xi = y$ and integrating them along the orbit Γ_9, it follows that

$$\pm\int_{\varphi_4^*}^{\varphi}\frac{\sqrt{-2\varepsilon}\,ds}{(\varphi_4-s)\sqrt{(\varphi_4-s)(s-\varphi_4^*)}} = \int_{0}^{\xi} ds.$$

ing the integrals above, we have

$$\varphi(\xi)=\pm\sqrt{\frac{4-c^2}{3}\frac{2(4-c^2)\xi^2+9\varepsilon}{2(4-c^2)\xi^2-3\varepsilon}}.$$

Noting that $u(x,t)=1+\varphi(\xi)$ with $\xi=x-ct$, we get two solitary wave solutions $u_{16\pm}(x,t)$ as (2.20).

Heretofore, we have completed the derivations for the Proposition 2.3.

Remark 3.1. One may find that we only consider the case when $g\geq 0$ in Propositions 2.2 and 2.3. In fact, we can get exactly the same solutions in the opposite case. Meanwhile, we assume that $4-c^2>0$ in our studies. For $4-c^2<0$, these solutions also satisfy equation (1.5) but are complex forms.

Remark 3.2. We have also tested the correctness of these solutions by using the software Mathematica. Here, we list a testing order, the other testing orders are similar. For instance, the orders for testing $u_{16}(x,t)$ are as follows:

$$\begin{aligned}
&\xi=x-ct;\\
&u=1+\sqrt{\tfrac{4-c^2}{3}\tfrac{2(4-c^2)\xi^2+9\varepsilon}{2(4-c^2)\xi^2-3\varepsilon}};\\
&u_{tt}=\mathtt{D}[u,\{t,2\}];\\
&u_{xx}=\mathtt{D}[u,\{x,2\}];\\
&u_{xxx}=\mathtt{D}[u,\{x,3\}];\\
&(u^2)_{xx}=\mathtt{D}[u^2,\{x,2\}];\\
&(u^2u_x)_x=\mathtt{D}[u^2\mathtt{D}[u,x],x];\\
&\mathtt{Simplify}[u_{tt}-u_{xx}-\varepsilon u_{xxx}-3(u^2)_{xx}+3(u^2u_x)_x]\\
&0
\end{aligned}$$

4 Conclusion

In this paper, we investigated the modified Boussinesq equation (1.5) by using the bifurcation method of dynamical systems. We gave a property of the solutions of the equation (see Proposition 2.1). We obtained some precise explicit expressions of traveling wave solutions $u_i(x,t)$ ($i=8$–14, 18–26) (see Propositions 2.2 and 2.3), which include kink-shaped solutions, blow-up solutions, periodic blow-up solutions and solitary wave solutions. Our work extended some previous results [2, 5, 7]. The method can be applied to many other nonlinear evolution equations and we believe that many new results wait for further discovery by this method.

Acknowledgements

The author wishes to thank the editor and the anonymous referee for many valuable suggestions leading to an improvement of this paper. This work is supported by the Science Foundation of Shaoguan University (201320501), Shaoguan Science and Technology Foundation (313140546) and Guangdong Provincial culture of seedling of China (2013LYM0081).

References

[1] T. A. Abassy, M. A. El-Tawil, H. El-Zoheiry, Modified variational iteration method for Boussinesq equation, *Comput. Math. Appl.* **54**(2007), 955–965. MR2395633; url

[2] M. A. Abdou, A. A. Soliman, S. T. El-Basyony, New application of Exp-function method for improved Boussinesq equation, *Phys. Lett. A.* **369**(2007), 469–475. url

[3] J. L. Bona, R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, *Commun. Math. Phys.* **118**(1988), 15–29. MR0954673; url

[4] Z. D. Dai, J. Huang, M. R. Jiang, S. H. Wang, Homoclinic orbits and periodic solitons for Boussinesq equation with even constraint, *Chaos. Soliton. Fract.* **26**(2005), 1189–1194. MR2143667; url

[5] G. Forozani, M. G. Nosrat, Solitary solution of modified bad and good Boussinesq equation by using of tanh and extended tanh methods, *J. Pure Appl. Math* **44**(2013), 497–510. MR3142556; url

[6] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Method for solving the Korteweg-deVries equation, *Phys. Rev. Lett.* **19**(1967), 1095–1097. url

[7] M. Javidi, Y. Jalilian, Exact solitary wave solution of Boussinesq equation by VIM, *Chaos. Soliton. Fract.* **36**(2008), 1256–1260. MR2388970; url

[8] S. Y. Li, R. Liu, Some explicit expressions and interesting bifurcation phenomena for nonlinear waves in generalized Zakharov equations, *Abstr. Appl. Anal.* **2013**, Art. ID 869438, 19 pp. MR3055940; url

[9] J. B. Li, Z. R. Liu, Smooth and non-smooth traveling waves in a nonlinearly dispersive equation, *Appl. Math. Model.* **25**(2000), 41–56. url

[10] S. K. Liu, Z. T. Fu, S. D. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, *Phys. Lett. A.* **289**(2001), 69–74. MR1862082; url

[11] Z. R. Liu, T. P. Jiang, P. Qing, Q.F. Xu, Trigonometric function periodic wave solutions and their limit forms for the KdV-like and the PC-like equations, *Math. Probl. Eng.* **2011**, Art. ID 810217, 23 pp. MR2823612; url

[12] Z. R. Liu, Y. Liang, The explicit nonlinear wave solutions and their bifurcations of the generalized Camassa–Holm equation, *Int. J. Bifur. Chaos.* **21**(2011), 3119–3136. MR2871965; url

[13] H. P. McKean, Boussinesq's equation on the circle, *Comm. Pure Appl. Math.* **34**(1981), 599–691. MR0622617; url

[14] R. M. Miura (editor), *Bäcklund transformation, the inverse scattering method, solitons, and their applications.*, Lecture Notes in Mathematics, Vol. 515, Springer-Verlag, Berlin, 1976. MR0497409

[15] M. Rafei, D. D. Ganji, H. R. Mohammadi Daniali, H. Pashaei, Application of homotopy perturbation method to the RLW and generalized modified Boussinesq equations, *Phys. Lett. A.* **364**(2007), 1–6. MR2288007; url

[16] M. Song, Y. L. Ge, Application of the $(\frac{G'}{G})$-expansion method to (3+1)-dimensional nonlinear evolution equations, *Comput. Math. Appl.* **60**(2010), 1220–1227. MR2672922; url

[17] M. Song, S. Y. Li, J. Cao, New exact solutions for the (2+1)-dimensional Broer–Kaup–Kupershmidt equations, *Abstr. Appl. Anal.* **2010**, Art. ID 652649, 9 pp. MR2746008; url

[18] M. L. Wang, X. Z. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, *Chaos. Soliton. Fract.* **24**(2005), 1257–1268. MR2123272; url

[19] M. L. Wang, X. Z. Li, Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations, *Phys. Lett. A.* **343**(2005), 48–54. MR2149694; url

[20] M. L. Wang, X. Z. Li, J.L. Zhang, The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A.* **372**(2008), 417–423. MR2381823; url

[21] J. Weiss, The Painlevé property and Bäcklund transformations for the sequence of Boussinesq equations, *J. Math. Phys.* **26**(1985), 258–269. MR0776493; url