



Existence and multiplicity results for a coupled system of Kirchhoff type equations

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Abstract. This paper deals with a coupled system of Kirchhoff type equations in \mathbb{R}^3 . Under suitable assumptions on the potential functions $V(x)$ and $W(x)$, we obtain the existence and multiplicity of nontrivial solutions when the parameter λ is sufficiently large. The method combines the Nehari manifold and the mountain-pass theorem.

Keywords: Kirchhoff type equation, ground state solution, mountain-pass theorem.

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1 Introduction

In this paper, we consider the coupled system of Kirchhoff type equations

$$\begin{cases} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + \lambda V(x)u = \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^3, \\ - \left(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx \right) \Delta v + \lambda W(x)v = \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^3, \\ u(x) \rightarrow 0, v(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty, \end{cases} \quad (\mathcal{K})_\lambda$$


where $a > 0, b > 0$ are constants, $\lambda > 0$ is a parameter, $\alpha > 2, \beta > 2$ satisfy $\alpha + \beta < 2^* = 6$, and $V(x), W(x)$ are nonnegative continuous potential functions on \mathbb{R}^3 .

In recent years, many papers have extensively considered the scalar Kirchhoff equation

$$\begin{cases} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, one can see [1, 4, 6, 11, 12, 15] and the references therein. Problem (1.1) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u), \quad (1.2)$$

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which was proposed by Kirchhoff in [8] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model considers the changes in length of the string produced by transverse vibrations.

There are also many works on the existence and multiplicity results for the scalar case of $(\mathcal{K})_1$

$$\begin{cases} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), u > 0 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.3)$$

where f is a subcritical function and satisfies certain conditions. We would mention the recent paper [14], by applying symmetric mountain-pass theorem, the author obtained the existence results for nontrivial solutions and a sequence of high energy solutions for problem (1.3). Subsequently, Liu and He [9] proved the existence of infinitely many high energy solutions for (1.3) when f is a subcritical nonlinearity which does not need to satisfy the usual Ambrosetti–Rabinowitz conditions. Further related results can be seen in [7, 10, 13] and the references therein.

The purpose of this paper is to study the existence and multiplicity results for a coupled system of Kirchhoff type equations in \mathbb{R}^3 . To the best of our knowledge, problem $(\mathcal{K})_\lambda$ has not been considered before, the main difficulties lie in the appearance of the non-local term and the lack of compactness due to the unboundedness of the domain \mathbb{R}^3 . Motivated by the work mentioned above, we will get the existence and multiplicity results of nontrivial solutions for λ large enough by exploiting the Nehari manifold method and the mountain-pass theorem.

Before stating our main results, we need to introduce some assumptions and notations:

(A₁) $V(x), W(x) \in \mathcal{C}(\mathbb{R}^3, [0, +\infty))$ and $\Omega := \text{int}(V^{-1}(0)) = \text{int}(W^{-1}(0))$ is nonempty with smooth boundary and $\bar{\Omega} = V^{-1}(0) = W^{-1}(0)$;

(A₂) there exist $M_1, M_2 > 0$ such that

$$\mathcal{L}(\{x \in \mathbb{R}^3 \mid V(x) \leq M_1\}) < \infty, \quad \mathcal{L}(\{x \in \mathbb{R}^3 \mid W(x) \leq M_2\}) < \infty,$$

where \mathcal{L} denotes the Lebesgue measure in \mathbb{R}^3 .

The hypothesis (A₂) was first introduced by Bartsch and Wang [3] in the study of a non-linear Schrödinger equation. Let $E_V = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\}$ and $E_W = \{v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} W(x)v^2 dx < +\infty\}$ with the norms $\|u\|_{\lambda, V}^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda V(x)u^2) dx$ and $\|v\|_{\lambda, W}^2 = \int_{\mathbb{R}^3} (a|\nabla v|^2 + \lambda W(x)v^2) dx$ respectively. For any given $\lambda > 0$, we consider the Hilbert space $E := E_V \times E_W$ endowed with the norm

$$\|(u, v)\|_\lambda = \sqrt{\|u\|_{\lambda, V}^2 + \|v\|_{\lambda, W}^2}.$$

The energy functional associated with $(\mathcal{K})_\lambda$ is defined on E by

$$\mathcal{I}_\lambda(u, v) = \frac{1}{2} \|(u, v)\|_\lambda^2 + \frac{b}{4} (Y^2(u) + Y^2(v)) - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx,$$

where $Y(w) = \int_{\mathbb{R}^3} |\nabla w|^2 dx$. In view of the assumptions (A₁) and (A₂), the energy functional $\mathcal{I}_\lambda(u, v)$ is well defined and belongs to $\mathcal{C}^1(E, \mathbb{R})$. It is well known that the weak solutions of problem $(\mathcal{K})_\lambda$ are the critical points of the energy functional $\mathcal{I}_\lambda(u, v)$.

The main results we get are the following:

Theorem 1.1. *Suppose that (A_1) and (A_2) hold. Then there is $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the system $(\mathcal{K})_\lambda$ has a ground state solution.*

Theorem 1.2. *Suppose that (A_1) and (A_2) hold. Then for any given $k \in \mathbb{N}$, there exists $\Lambda_k > 0$ such that for each $\lambda \geq \Lambda_k$, the system $(\mathcal{K})_\lambda$ possesses at least k pairs of nontrivial solutions.*

This paper is organized as follows. In Section 2, we will prove some important lemmas that will be used for the proofs of the main results. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2 Some preliminary lemmas

In this paper, C, C_1, C_2, \dots denote positive (possibly different) constants. \rightarrow (respectively \rightharpoonup) denotes strong (respectively weak) convergence. $o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. B_r denotes a ball centered at the origin with radius $r > 0$. For a given set $K \subset \mathbb{R}^3$, we set $K^c = \mathbb{R}^3 \setminus K$. We define the minimax c_λ as

$$c_\lambda = \inf_{(u,v) \in \mathcal{N}_\lambda} \mathcal{I}_\lambda(u, v), \quad (2.1)$$

where \mathcal{N}_λ denotes the Nehari manifold associated with \mathcal{I}_λ given by

$$\mathcal{N}_\lambda = \{(u, v) \in E \setminus \{(0, 0)\} : \langle \mathcal{I}'_\lambda(u, v), (u, v) \rangle = 0\},$$

and $\langle \cdot, \cdot \rangle$ is the duality product between E and its dual space E^{-1} . A ground state solution of $(\mathcal{K})_\lambda$ means a solution (u, v) of $(\mathcal{K})_\lambda$ with $\mathcal{I}_\lambda(u, v) = c_\lambda$. Note that \mathcal{N}_λ contains every nonzero solution of problem $(\mathcal{K})_\lambda$. Hereafter, we suppose that (A_1) and (A_2) are satisfied.

Lemma 2.1. *Let $(u, v) \in \mathcal{N}_\lambda$, then there exists $\sigma > 0$ which is independent of λ such that $\|(u, v)\|_\lambda \geq \sigma$.*

Proof. First, by Young's inequality, we get

$$|u|^\alpha |v|^\beta \leq \frac{\alpha}{\alpha + \beta} |u|^{\alpha + \beta} + \frac{\beta}{\alpha + \beta} |v|^{\alpha + \beta},$$

then by the continuity of the Sobolev embedding $E_V \hookrightarrow L^s(\mathbb{R}^3)$ and $E_W \hookrightarrow L^s(\mathbb{R}^3)$ for $2 \leq s \leq 6$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx &\leq \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^{\alpha + \beta} dx + \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^3} |v|^{\alpha + \beta} dx \\ &\leq C_1 \|u\|_V^{\alpha + \beta} + C_2 \|v\|_W^{\alpha + \beta} \leq C \|(u, v)\|_\lambda^{\alpha + \beta}, \end{aligned} \quad (2.2)$$

where $C > 0$ is independent of λ . So, by (2.2), for any $(u, v) \in \mathcal{N}_\lambda$ we have

$$\begin{aligned} 0 = \langle \mathcal{I}'_\lambda(u, v), (u, v) \rangle &= \|(u, v)\|_\lambda^2 + b(Y^2(u) + Y^2(v)) - 2 \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\ &\geq \|(u, v)\|_\lambda^2 - 2C \|(u, v)\|_\lambda^{\alpha + \beta}. \end{aligned}$$

Note that $\alpha + \beta > 2$, thus there exists $\sigma > 0$ such that $\|(u, v)\|_\lambda \geq \sigma$. □

Lemma 2.2. *Suppose that $\{(u_n, v_n)\}$ is a $(PS)_c$ -sequence for $\mathcal{I}_\lambda(u, v)$. Then we have*

- (i) $\{(u_n, v_n)\}$ is bounded in E ;

(ii) if $c \neq 0$, then $c \geq c_0$, for some $c_0 > 0$ is independent of λ .

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$ sequence for $\mathcal{I}_\lambda(u, v)$, that is, $\mathcal{I}_\lambda(u_n, v_n) = c + o_n(1)$ and $\mathcal{I}'_\lambda(u_n, v_n) = o_n(1)$. Then we have that

$$\begin{aligned} c + o_n(1) - \frac{1}{4}o_n(\|(u_n, v_n)\|_\lambda) &= \mathcal{I}_\lambda(u_n, v_n) - \frac{1}{4}\langle \mathcal{I}'_\lambda(u_n, v_n), (u_n, v_n) \rangle \\ &= \frac{1}{2}\|(u_n, v_n)\|_\lambda^2 + \left(\frac{1}{2} - \frac{2}{\alpha + \beta}\right) \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx \\ &\geq \frac{1}{2}\|(u_n, v_n)\|_\lambda^2, \end{aligned} \quad (2.3)$$

which implies that $\{(u_n, v_n)\}$ is bounded in E .

On the other hand, we have

$$\begin{aligned} o_n(\|(u_n, v_n)\|_\lambda) &= \langle \mathcal{I}'_\lambda(u_n, v_n), (u_n, v_n) \rangle \\ &= \|(u_n, v_n)\|_\lambda^2 + b(Y^2(u_n) + Y^2(v_n)) - 2 \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx \\ &\geq \|(u_n, v_n)\|_\lambda^2 - 2C\|(u_n, v_n)\|_\lambda^{\alpha+\beta} \text{ (by (2.2))}, \end{aligned}$$

since $\alpha + \beta > 2$, there exists $0 < \sigma_1 < 1$ such that

$$\langle \mathcal{I}'_\lambda(u_n, v_n), (u_n, v_n) \rangle \geq \frac{1}{4}\|(u_n, v_n)\|_\lambda^2, \text{ for } \|(u_n, v_n)\|_\lambda < \sigma_1. \quad (2.4)$$

Now, if $c < \frac{\sigma_1^2}{2}$ and $\{(u_n, v_n)\}$ is a $(PS)_c$ -sequence of \mathcal{I}_λ , then by (2.3)

$$\lim_{n \rightarrow \infty} \|(u_n, v_n)\|_\lambda^2 \leq 2c < \sigma_1^2.$$

Hence, $\|(u_n, v_n)\|_\lambda < \sigma_1$ for n large, then by (2.4)

$$\frac{1}{4}\|(u_n, v_n)\|_\lambda^2 \leq \langle \mathcal{I}'_\lambda(u_n, v_n), (u_n, v_n) \rangle = o_n(\|(u_n, v_n)\|_\lambda),$$

which implies $\|(u_n, v_n)\|_\lambda \rightarrow 0$ as $n \rightarrow \infty$ and $c = 0$, it follows that (ii) holds for $c_0 = \sigma_1^2/2$. \square

Lemma 2.3. Suppose that (A_1) – (A_2) hold and let C^* be fixed. Given $\varepsilon > 0$ there exist $\Lambda_\varepsilon = \Lambda(\varepsilon, C^*) > 0$ and $\rho_\varepsilon = \rho(\varepsilon, C^*) > 0$ such that, if $\{(u_n, v_n)\}$ is a $(PS)_c$ -sequence of $\mathcal{I}_\lambda(u, v)$ with $c \leq C^*, \lambda \geq \Lambda_\varepsilon$, then

$$\limsup_{n \rightarrow \infty} \int_{B_{\rho_\varepsilon}^c} |u_n|^\alpha |v_n|^\beta dx \leq \varepsilon. \quad (2.5)$$

Proof. For $\rho > 0$, we set

$$A(\rho) := \{x \in \mathbb{R}^3 : |x| \geq \rho, V(x) \geq M_1\}, \quad B(\rho) := \{x \in \mathbb{R}^3 : |x| \geq \rho, V(x) < M_1\},$$

then

$$\begin{aligned} \int_{A(\rho)} |u_n|^2 dx &\leq \frac{1}{\lambda M_1} \int_{\mathbb{R}^3} \lambda V(x) u_n^2 dx \\ &\leq \frac{1}{\lambda M_1} \int_{\mathbb{R}^3} (a|\nabla u_n|^2 + \lambda V(x) u_n^2) dx \\ &\leq \frac{1}{\lambda M_1} (2c + o_n(\|(u_n, v_n)\|_\lambda)) \\ &\leq \frac{1}{\lambda M_1} (2C^* + o_n(\|(u_n, v_n)\|_\lambda)) \\ &\rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned} \quad (2.6)$$

Using the Hölder inequality and (2.3), for $1 < q < 3$ we obtain

$$\begin{aligned} \int_{B(\rho)} |u_n|^2 dx &\leq \left(\int_{\mathbb{R}^3} |u_n|^{2q} dx \right)^{\frac{1}{q}} \cdot \mathcal{L}(B(\rho))^{\frac{q-1}{q}} \\ &\leq C_4 \|u_n\|_{H^1(\mathbb{R}^3)}^2 \cdot \mathcal{L}(B(\rho))^{\frac{q-1}{q}} \\ &\leq C_4 \cdot 2C^* \cdot \mathcal{L}(B(\rho))^{\frac{q-1}{q}} \rightarrow 0 \text{ as } \rho \rightarrow \infty, \end{aligned} \quad (2.7)$$

where $C_4 = C_4(q)$ is a positive constant. Setting $\theta = \frac{3(\alpha+\beta-2)}{2(\alpha+\beta)}$ and using the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} \int_{B_\rho^c} |u_n|^{\alpha+\beta} dx &\leq C \left(\int_{B_\rho^c} |\nabla u_n|^2 dx \right)^{\frac{(\alpha+\beta)\theta}{2}} \cdot \left(\int_{B_\rho^c} |u_n|^2 dx \right)^{\frac{(\alpha+\beta)(1-\theta)}{2}} \\ &\leq C_5 \|(u_n, v_n)\|_\lambda^{(\alpha+\beta)\theta} \cdot \left(\int_{A(\rho)} |u_n|^2 dx + \int_{B(\rho)} |u_n|^2 dx \right)^{\frac{(\alpha+\beta)(1-\theta)}{2}} \\ &\leq C_6 \left(\int_{A(\rho)} |u_n|^2 dx + \int_{B(\rho)} |u_n|^2 dx \right)^{\frac{(\alpha+\beta)(1-\theta)}{2}} \\ &\rightarrow 0 \text{ as } \lambda, \rho \rightarrow \infty \text{ (by (2.6) and (2.7)).} \end{aligned} \quad (2.8)$$

Similarly,

$$\int_{B_\rho^c} |v_n|^{\alpha+\beta} dx \leq \varepsilon \text{ for } \lambda, \rho \text{ large.} \quad (2.9)$$

At last, using the Hölder inequality, (2.8) and (2.9) we have that

$$\limsup_{n \rightarrow \infty} \int_{B_{\rho_\varepsilon}^c} |u_n|^\alpha |v_n|^\beta dx \leq \limsup_{n \rightarrow \infty} \left(\int_{B_{\rho_\varepsilon}^c} |u_n|^{\alpha+\beta} dx \right)^{\frac{\alpha}{\alpha+\beta}} \left(\int_{B_{\rho_\varepsilon}^c} |v_n|^{\alpha+\beta} dx \right)^{\frac{\beta}{\alpha+\beta}} \leq \varepsilon.$$

This concludes the proof of Lemma 2.3. \square

The following Brézis–Lieb type lemma is proved in [5, Lemma 4.2].

Lemma 2.4. *Let $\{(u_n, v_n)\} \subset E$ be a sequence such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E . Then we have*

$$\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^3} |u_n - u|^\alpha |v_n - v|^\beta dx = \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx + o_n(1).$$

Lemma 2.5. *Let $\lambda > 0$ be fixed and $\{(u_n, v_n)\}$ is a $(PS)_c$ -sequence of \mathcal{I}_λ . Then*

- (i) *up to a subsequence $(u_n, v_n) \rightharpoonup (u, v)$ in E with (u, v) being a weak solution of $(\mathcal{K})_\lambda$;*
- (ii) *$\{(u_n - u, v_n - v)\}$ is a $(PS)_d$ -sequence for \mathcal{I}_λ with $d = c - \mathcal{I}_\lambda(u, v)$.*

Proof. (i) Since $\{(u_n, v_n)\}$ is bounded in E (see Lemma 2.2(i)), then there is a subsequence of $\{(u_n, v_n)\}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in E as $n \rightarrow \infty$. In order to see that (u, v) is a critical point of \mathcal{I}_λ , we recall that $(u_n, v_n) \rightharpoonup (u, v)$ in E , $(u_n, v_n) \rightarrow (u, v)$ for almost every $x \in \mathbb{R}^3$, $(u_n, v_n) \rightarrow (u, v)$ in $L_{loc}^{s_1}(\mathbb{R}^3) \times L_{loc}^{s_2}(\mathbb{R}^3)$, $2 \leq s_1, s_2 < 6$. It is easy to see that for any $(\varphi, \psi) \in E$, we have

$$\langle \mathcal{I}'_\lambda(u, v), (\varphi, \psi) \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{I}'_\lambda(u_n, v_n), (\varphi, \psi) \rangle = 0.$$

Therefore (u, v) is a critical point of \mathcal{I}_λ .

(ii) Let $(\tilde{u}_n, \tilde{v}_n) = (u_n - u, v_n - v)$. Now we verify that

$$\mathcal{I}_\lambda(\tilde{u}_n, \tilde{v}_n) = c - \mathcal{I}_\lambda(u, v) \text{ as } n \rightarrow \infty \quad (2.10)$$

and

$$\mathcal{I}'_\lambda(\tilde{u}_n, \tilde{v}_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.11)$$

By the Brézis–Lieb lemma, we have that

$$\begin{aligned} \|(\tilde{u}_n, \tilde{v}_n)\|_\lambda^2 &= \|(u_n, v_n)\|_\lambda^2 - \|(u, v)\|_\lambda^2 + o_n(1), \\ \left(\int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 dx\right)^2 &= \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + o_n(1), \\ \left(\int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 dx\right)^2 &= \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 - \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 + o_n(1). \end{aligned}$$

To show (2.10) we observe

$$\begin{aligned} \mathcal{I}_\lambda(\tilde{u}_n, \tilde{v}_n) &= \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla \tilde{u}_n|^2 + \lambda V(x)|\tilde{u}_n|^2) dx + \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla \tilde{v}_n|^2 + \lambda W(x)|\tilde{v}_n|^2) dx \\ &\quad + \frac{b}{4} \left(\left(\int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 dx\right)^2 + \left(\int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 dx\right)^2 \right) - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \\ &= \mathcal{I}_\lambda(u_n, v_n) - \mathcal{I}_\lambda(u, v) + o_n(1) \\ &\quad + \frac{2}{\alpha + \beta} \left(\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx - \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \right). \end{aligned} \quad (2.12)$$

From Lemma 2.4, $\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx - \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \rightarrow 0$ as $n \rightarrow \infty$. Thus from (2.12) we obtain (2.10).

In order to show (2.11), let $(\varphi, \psi) \in E$. We note that

$$\begin{aligned} \langle \mathcal{I}'_\lambda(\tilde{u}_n, \tilde{v}_n), (\varphi, \psi) \rangle &= \langle \mathcal{I}'_\lambda(u_n, v_n), (\varphi, \psi) \rangle - \langle \mathcal{I}'_\lambda(u, v), (\varphi, \psi) \rangle - \frac{2\alpha}{\alpha + \beta} \int_{\mathbb{R}^3} |\tilde{u}_n|^{\alpha-2} |\tilde{v}_n|^\beta \tilde{u}_n \varphi dx \\ &\quad - \frac{2\beta}{\alpha + \beta} \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^{\beta-2} \tilde{v}_n \psi dx + \frac{2\alpha}{\alpha + \beta} \int_{\mathbb{R}^3} |u_n|^{\alpha-2} |v_n|^\beta u_n \varphi dx \\ &\quad + \frac{2\beta}{\alpha + \beta} \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^{\beta-2} v_n \psi dx - \frac{2\alpha}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^{\alpha-2} |v|^\beta u \varphi dx \\ &\quad - \frac{2\beta}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^{\beta-2} v \psi dx. \end{aligned} \quad (2.13)$$

Since $\mathcal{I}'_\lambda(u_n, v_n) \rightarrow 0$ and $u_n \rightarrow u, v_n \rightarrow v$ in $L^s(\mathbb{R}^3)$ ($2 \leq s < 6$), we have

$$\lim_{n \rightarrow \infty} \sup_{\|\varphi\|_{\lambda, V} \leq 1} \int_{\mathbb{R}^3} \left(|\tilde{u}_n|^{\alpha-2} |\tilde{v}_n|^\beta \tilde{u}_n - |u_n|^{\alpha-2} |v_n|^\beta u_n + |u|^{\alpha-2} |v|^\beta u \right) \varphi dx = 0, \quad (2.14)$$

$$\lim_{n \rightarrow \infty} \sup_{\|\psi\|_{\lambda, W} \leq 1} \int_{\mathbb{R}^3} \left(|\tilde{u}_n|^\alpha |\tilde{v}_n|^{\beta-2} \tilde{v}_n - |u_n|^\alpha |v_n|^{\beta-2} v_n + |u|^\alpha |v|^{\beta-2} v \right) \psi dx = 0. \quad (2.15)$$

Thus combining (2.13)–(2.15) we obtain that

$$\lim_{n \rightarrow \infty} \langle \mathcal{I}'_\lambda(\tilde{u}_n, \tilde{v}_n), (\varphi, \psi) \rangle = 0, \quad \forall (\varphi, \psi) \in E,$$

which implies (2.12) and this completes the proof of Lemma 2.5. \square

3 Proof of the main results

We begin with the following lemma.

Lemma 3.1. *Suppose that (A_1) and (A_2) hold. Then for any $C_0 > 0$, there exists $\Lambda_0 > 0$ such that \mathcal{I}_λ satisfies the $(PS)_c$ -condition for all $c \leq C_0$ and $\lambda \geq \Lambda_0$.*

Proof. Let $c_0 > 0$ be given by Lemma 2.2 (ii) and choose $\varepsilon > 0$ such that $\varepsilon < \frac{c_0(\alpha+\beta)}{\alpha+\beta-2}$. Then for given $C_0 > 0$, we choose $\Lambda_\varepsilon > 0$ and $\rho_\varepsilon > 0$ as in Lemma 2.3. We claim that $\Lambda_0 = \Lambda_\varepsilon$ is just required in Lemma 3.1. Let $\{(u_n, v_n)\} \subset E$ be a $(PS)_c$ -sequence of $\mathcal{I}_\lambda(u, v)$ with $c \leq C_0$ and $\lambda \geq \Lambda_0$. By Lemma 2.5, we may suppose that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E and then $\{(\tilde{u}_n, \tilde{v}_n)\} = \{(u_n - u, v_n - v)\}$ is a $(PS)_d$ -sequence of \mathcal{I}_λ with $d = c - \mathcal{I}_\lambda(u, v)$. We claim that $d = 0$. Arguing by contradiction, assume that $d \neq 0$. Lemma 2.2 (ii) implies that $d \geq c_0 > 0$. Since $(\tilde{u}_n, \tilde{v}_n)$ is a $(PS)_d$ -sequence of \mathcal{I}_λ , we have

$$\mathcal{I}_\lambda(\tilde{u}_n, \tilde{v}_n) = d + o_n(1), \quad \mathcal{I}'_\lambda(\tilde{u}_n, \tilde{v}_n) = o_n(1).$$

Then we get

$$\begin{aligned} d + o_n(1) - \frac{1}{2}o_n(\|(u_n, v_n)\|_\lambda) &= \mathcal{I}_\lambda(\tilde{u}_n, \tilde{v}_n) - \frac{1}{2}\langle \mathcal{I}'_\lambda(\tilde{u}_n, \tilde{v}_n), (\tilde{u}_n, \tilde{v}_n) \rangle \\ &= -\frac{b}{4}(Y^2(\tilde{u}_n) + Y^2(\tilde{v}_n)) + \left(1 - \frac{2}{\alpha + \beta}\right) \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \\ &\leq \left(1 - \frac{2}{\alpha + \beta}\right) \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx, \end{aligned} \quad (3.1)$$

from which we deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \geq d \left(1 - \frac{2}{\alpha + \beta}\right)^{-1} \geq \frac{\alpha + \beta}{\alpha + \beta - 2} c_0. \quad (3.2)$$

On the other hand, Lemma 2.3 implies

$$\limsup_{n \rightarrow \infty} \int_{B_{\rho_\varepsilon}^c} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \leq \varepsilon < \frac{c_0(\alpha + \beta)}{\alpha + \beta - 2}.$$

This implies $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v)$ in E with $(u, v) \neq (0, 0)$, which is a contradiction. Therefore $d = 0$ and it follows from (2.3) that

$$\lim_{n \rightarrow \infty} \|(\tilde{u}_n, \tilde{v}_n)\|_\lambda^2 \leq 2d = 0,$$

hence $(\tilde{u}_n, \tilde{v}_n) \rightarrow (0, 0)$ in E , that is, $(u_n, v_n) \rightarrow (u, v)$ in E . This completes the proof of Lemma 3.1. \square

The following lemma implies that \mathcal{I}_λ possesses the mountain-pass geometry.

Lemma 3.2. *The functional \mathcal{I}_λ satisfies the following conditions.*

- (i) *There exist $\rho, \eta > 0$ such that $\mathcal{I}_\lambda(u, v) \geq \eta$ for all $\|(u, v)\|_\lambda = \rho$.*
- (ii) *There exists $(u_0, v_0) \in B_\rho^c(0)$ such that $\mathcal{I}_\lambda(u_0, v_0) < 0$.*

Proof. (i) By (2.2) we have that

$$\begin{aligned} \mathcal{I}_\lambda(u, v) &= \frac{1}{2} \|(u, v)\|_\lambda^2 + \frac{b}{4} (Y^2(u) + Y^2(v)) - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\ &\geq \frac{1}{2} \|(u, v)\|_\lambda^2 - C \|(u, v)\|_\lambda^{\alpha+\beta}, \end{aligned} \quad (3.3)$$

since $\alpha + \beta > 2$, we can choose some $\eta > 0$, $\rho > 0$ such that $\mathcal{I}_\lambda(u, v) \geq \eta$ for $\|(u, v)\|_\lambda = \rho$.

(ii) We note that for each $\lambda > 0$, $\mathcal{I}_\lambda(0, 0) = 0$. Furthermore, for $(u, v) \in E \setminus \{0, 0\}$, since $\alpha + \beta > 4$, we get that

$$\mathcal{I}_\lambda(t(u, v)) = \left(\frac{t^2}{2} \|(u, v)\|_\lambda^2 + \frac{bt^4}{4} (Y^2(u) + Y^2(v)) - \frac{2t^{\alpha+\beta}}{\alpha + \beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \right) \rightarrow -\infty$$

as $t \rightarrow +\infty$. Hence, we can choose $t_0 > 0$ large enough such that $\|t_0(u, v)\|_\lambda > \rho$ and $\mathcal{I}_\lambda(t_0(u, v)) < 0$. Let $(u_0, v_0) = t_0(u, v)$, then (ii) holds. \square

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2, the functional \mathcal{I}_λ satisfies the mountain-pass geometry, then using a version of the mountain-pass theorem without (PS) condition, there exists a (PS)-sequence $\{(u_n, v_n)\} \subset E$ satisfying

$$\mathcal{I}_\lambda(u_n, v_n) \rightarrow c_\lambda \text{ and } \mathcal{I}'_\lambda(u_n, v_n) \rightarrow 0.$$

Moreover, by Lemma 2.2 (i), $\{(u_n, v_n)\}$ is bounded in E . Then, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E and $(u_n, v_n) \rightarrow (u, v)$ for almost every $x \in \mathbb{R}^3$. By Lemma 3.1, there exists $\lambda^* > 0$, such that $(u_n, v_n) \rightarrow (u, v)$ in E for $\lambda \geq \lambda^*$. Furthermore, by Lemma 2.5 we have that $\mathcal{I}'_\lambda(u, v) = 0$. By Lemma 2.1, we know that $(u, v) \neq (0, 0)$, then $(u, v) \in \mathcal{N}_\lambda$, and using Fatou's lemma we get

$$\begin{aligned} \mathcal{I}_\lambda(u, v) &= \mathcal{I}_\lambda(u, v) - \frac{1}{4} \langle \mathcal{I}'_\lambda(u, v), (u, v) \rangle \\ &= \frac{1}{4} \|(u, v)\|_\lambda^2 + \left(\frac{1}{2} - \frac{2}{\alpha + \beta} \right) \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{1}{4} \|(u_n, v_n)\|_\lambda^2 + \left(\frac{1}{2} - \frac{2}{\alpha + \beta} \right) \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx \right) \\ &= \liminf_{n \rightarrow \infty} \left(\mathcal{I}_\lambda(u_n, v_n) - \frac{1}{4} \langle \mathcal{I}'_\lambda(u_n, v_n), (u_n, v_n) \rangle \right) \\ &= c_\lambda. \end{aligned}$$

Hence, $\mathcal{I}_\lambda(u, v) \leq c_\lambda$. On the other hand, from the definition of c_λ , we have $c_\lambda \leq \mathcal{I}_\lambda(u, v)$. So, $\mathcal{I}_\lambda(u, v) = c_\lambda$, that is (u, v) is a ground state solution of problem $(\mathcal{K})_\lambda$. \square

To prove Theorem 1.2 we need the following version of the symmetric mountain-pass theorem [2].

Theorem 3.3. *Let X be a real Banach space and $W \subset X$ a finite dimensional subspace. Suppose that $J \in \mathcal{C}^1(X, \mathbb{R})$ is an even functional satisfying $J(0) = 0$ and*

(a) *there exists a constant $\rho > 0$ such that $J|_{\partial B_\rho(0)} \geq 0$;*

(b) *there exists $M_0 > 0$ such that $\sup_{z \in W} J(z) < M_0$.*

If J satisfies $(PS)_c$ for any $0 < c < M_0$, then J possesses at least $\dim W$ pairs of nontrivial critical points.

Proof of Theorem 1.2. Obviously, $\mathcal{I}_\lambda(u, v)$ is an even functional. Given $k \in \mathbb{N}$, we set

$$W = \text{span}\{(\phi_1, \phi_1), \dots, (\phi_k, \phi_k)\},$$

where ϕ_i is the eigenfunction corresponding to the i -th eigenvalue of $(-\Delta, H_0^1(\Omega))$ and Ω is defined in assumption (A_1) , then $\dim W = k$. Since all norms in a finite dimensional space are equivalent, for each $i = 1, \dots, k$, we have that

$$\lim_{t \rightarrow +\infty} \mathcal{I}_\lambda(t(\phi_i, \phi_i)) = \lim_{t \rightarrow +\infty} \left(at^2 \int_\Omega |\nabla \phi_i|^2 dx + \frac{bt^4}{2} \left(\int_\Omega |\nabla \phi_i|^2 dx \right)^2 - \frac{2t^{\alpha+\beta}}{\alpha + \beta} \int_\Omega |\phi_i|^{\alpha+\beta} dx \right) = -\infty$$

uniformly in λ . Since W has finite dimension we obtain $M_k > 0$, independent of $\lambda > 0$, such that

$$\sup_{(u,v) \in W} \mathcal{I}_\lambda(u, v) < M_k.$$

Moreover, similar to the proof of Lemma 3.2 (i) we may obtain $\rho > 0$, independent of $\lambda > 0$, such that

$$\mathcal{I}_\lambda(u, v) \geq 0 \text{ for } \|(u, v)\|_\lambda = \rho.$$

In view of Lemma 3.1, there exists $\Lambda_k > 0$ such that \mathcal{I}_λ satisfies $(PS)_c$ for any $c \leq M_k$ and $\lambda \geq \Lambda_k$. Thus, for any fixed $\lambda \geq \Lambda_k$ we may apply Theorem 3.3 to obtain k pairs of nontrivial solutions. Theorem 1.2 is proved. \square

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