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ON THE DULAC FUNCTIONS FOR MULTIPLY CONNECTED DOMAINS

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Abstract. We provide a method to find Dulac functions in multiply connected domains using partial differential equations. We also present some applications and examples in order to illustrate our results.

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1. Introduction

Many important problems of the qualitative theory of differential equations in the plane are related to closed orbits, and this is reason to study them. But, deciding whether an arbitrary differential equation has or does not have periodic orbits, and how many they might be, are difficult questions that remain open. Given an open set $\Omega \subset \mathbb{R}^2$ we consider

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \quad (x_1,\ x_2) \in \Omega, \end{cases} \tag{1}$$

where f_1, f_2 are C^1-functions.

The Bendixson–Dulac criterion is well-known on the non-existence of periodic orbits for planar differential system. We state this criterion (see [4, p. 137]), and throughout the paper we will use the Lebesgue measure.

Theorem 1. (Bendixson–Dulac criterion) *Let $h(x_1, x_2)$ be a C^1-function in a simply connected domain $\Omega \subset \mathbb{R}^2$ such that $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ does not change sign in Ω and vanishes at most on a set of measure zero. Then the system* (1) *does not have periodic orbits in Ω.*

The Bendixson–Dulac criterion gives a sufficient condition for the non-existence of periodic orbits in simply connected sets. However, the Bendixson–Dulac criterion requires an auxiliary function, a *Dulac function*, which is not easy to find. The Bendixson–Dulac criterion admits some important generalizations, for example to multiply connected domains, which provides bounds for the number of limit cycles. Thus the Bendixson–Dulac criterion and their extensions are a very useful tool for investigation of limit cycles, following the works [1, 3], we can see that is necessary to study expressions of the type

$$f_1\frac{\partial h}{\partial x_1}+f_2\frac{\partial h}{\partial x_2}+sh\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right),$$

where $s\in\mathbb{R}$ and h are unknowns to be determined, we continue calling h a *Dulac function*. As already mentioned, the downside of these results is that there is no algorithm for finding h, although there are some procedures (see [1], [2] and [3]) that apply to certain systems; frequently Dulac functions are obtained by intuition.

Our goal in this paper is to present a method to construct Dulac functions for multiply connected domains, which is based on solutions of certain partial differential equations. The method is systematic and has advantages of application for several systems. We also give some applications and examples in order to illustrate the applicability of the results.

2. Preliminaries and results

Consider the vector field $F(x_1,x_2)=(f_1(x_1,x_2),f_2(x_1,x_2))$, then the system (1) can be rewritten in the form

$$\dot{x}=F(x),\qquad x=(x_1,x_2)\in\Omega. \tag{2}$$

As usual the divergence of the vector field F is defined by

$$\operatorname{div}(F)=\operatorname{div}(f_1,f_2)=\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}.$$

We consider $C^0(\Omega,\mathbb{R})$ the set of continuous functions and define the sets

$$\mathcal{F}_\Omega^\pm:=\{f\in C^0(\Omega,\ \mathbb{R}^\pm\cup\{0\})\ :f\text{ vanishes only on a measure zero set}\},$$

and

$$\mathcal{F}_\Omega:=\mathcal{F}_\Omega^-\cup\mathcal{F}_\Omega^+.$$

Recall that an open subset $\Omega\subset\mathbb{R}^2$ intuitively is said to be l-*connected* if it has l-holes, that is, if its first fundamental group is a free group with l-generators, we denote $l(\Omega)=l$.

For $h:\Omega\to\mathbb{R}$ a continuous function, let $Z(h):=\{x\in\Omega\ :\ h(x)=0\}$ be the set of zeros of h.

Following [3] we denote by $l(\Omega,h)$ the sum of the quantities $l(U)$ over all the connected components U of $\Omega\setminus Z(h)$, also denote by $co(h)$ the numbers of closed ovals of $Z(h)$ contained in Ω.

The following proposition states a version of extended Bendixson–Dulac criterion:

Proposition 1. [3, cor. 1] *Let $\Omega \subset \mathbb{R}^2$ be an open set with a regular boundary. Suppose that for an analytic function $h : \Omega \to \mathbb{R}$ and a real number s we have*

$$M_s := f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} + sh \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = \langle \nabla h, F \rangle + sh \operatorname{div}(F), \tag{3}$$

does not change sign and vanishes only on a measure zero set. Then the limit cycles of system (1) *are either totally contained in $Z(h)$, or do not intersect $Z(h)$. Moreover, the number of limit cycles contained in $Z(h)$ is at most $co(h)$ and the number N of limit cycles that do not intersect $Z(h)$ satisfies*

$$N \leq \begin{cases} l(\Omega) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ l(\Omega, h) & \text{if } s < 0. \end{cases} \tag{4}$$

Furthermore, for any $s \neq 0$ the limit cycles of this second type are hyperbolic.

We call a function h in Proposition 1 a *Dulac function*. Note that this result also provides information on the cycles's hyperbolicity, which is of interest in relation to perturbation problems. This result gives a tool to delimit the number of limit cycles when a Dulac function h is known. Next, following results for simply connected sets in [5], we present a method for obtaining Dulac functions involving solutions of certain partial differential equation. We have the following:

Proposition 2. *Let $\Omega \subset \mathbb{R}^2$ be an open set with a regular boundary. Suppose that there are both $s \in \mathbb{R}$ and a function $c : \Omega \to \mathbb{R}$ such that*

$$\langle \nabla h, F \rangle + sh \operatorname{div}(F) = ch, \tag{5}$$

admits an analytic solution h, with ch defined on Ω and does not change sign and vanishes only on a null measure subset. Then h is a Dulac function and the conclusions of Proposition 1 are true. In particular, the number of limit cycles contained in $Z(h)$ is at most $co(h)$ and the number N of limit cycles that do not intersect $Z(h)$ satisfies

$$N \leq \begin{cases} l(\Omega) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ l(\Omega, h) & \text{if } s < 0. \end{cases} \tag{6}$$

Proof: The function ch is positive or negative and only vanishes on a zero measure set. We propose M_s with the specific form $M_s = ch$, now we substitute this relation into the equation (3); thus we get the expression

$$ch = \langle \nabla h, F \rangle + sh\,\mathrm{div}(F),$$

since ch does not change sign and vanishes only on a null measure subset, then the hypotheses of Proposition 1 are satisfied. Therefore the validity of the result follows. $\square$

Remark: Note that it is possible to take the function c finite almost everywhere on Ω, but ch defined and finite on all Ω.

Next we consider some examples and applications to illustrate the proposed technique.

Example 1. *Consider the system*

$$\begin{cases} \dot{x}_1 &= x_1 - x_2 - x_1^3, \\ \dot{x}_2 &= x_1 + x_2 - x_2^3. \end{cases}$$

The associated equation (5) *is*

$$(x_1 - x_2 - x_1^3)\frac{\partial h}{\partial x_1} + (x_1 + x_2 - x_2^3)\frac{\partial h}{\partial x_2} = h(c - s(2 - 3x_1^2 - 3x_2^2)), \tag{7}$$

taking

$$c = \frac{-4(x_1^4 + x_2^4) - \frac{5}{2}(x_1^2 + x_2^2) - 6x_1^2x_2^2 + 1}{x_1^2 + x_2^2 + \frac{1}{2}},$$

note that $c \in \mathcal{F}_\Omega$, *where* $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq \frac{2}{5}\}$, *besides, if* $s = 1$ *then*

$$h(x_1, x_2) = x_1^2 + x_2^2 + \frac{1}{2},$$

is a solution of the equation (7)*; furthermore, h is a Dulac function on* Ω.

Since $Z(h) = \emptyset$ *it contains no ovals. In particular,* $co(h) = 0$.

Since $l(\Omega) = 1$ *and* $s > 0$, *then by Proposition 2 the system has at most one limit cycle in* Ω.

Example 2. *The next system has exactly one limit cycle.*

$$\begin{cases} \dot{x}_1 &= x_1^3 + x_1x_2^2 - x_1 - x_2, \\ \dot{x}_2 &= x_2^3 + x_2x_1^2 + x_1 - x_2. \end{cases} \tag{8}$$

The associated equation (5) *is*

$$(x_1^3 + x_1x_2^2 - x_1 - x_2)\frac{\partial h}{\partial x_1} + (x_2^3 + x_2x_1^2 + x_1 - x_2)\frac{\partial h}{\partial x_2} = h[c - s(4x_1^2 + 4x_2^2 - 2)].$$

Assume that $h = h(z)$ *depends on a function* $z = z(x_1, x_2)$*, thus we get*

$$\left[(x_1^3 + x_1x_2^2 - x_1 - x_2)\frac{\partial z}{\partial x_1} + (x_2^3 + x_2x_1^2 + x_1 - x_2)\frac{\partial z}{\partial x_2}\right]\frac{d\ln h}{dz} = c - s(4x_1^2 + 4x_2^2 - 2). \tag{9}$$

To simplify the equation we take z *such that* $-x_2\frac{\partial z}{\partial x_1} + x_1\frac{\partial z}{\partial x_2} = 0$*; hence we obtain* $z := x_1^2 + x_2^2$ *and* (9) *reduces to*

$$\frac{d\ln h}{dz} = \frac{c - s(4x_1^2 + 4x_2^2 - 2)}{2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)}, \tag{10}$$

where the right side depends on z*, denoted by* η*; for simplicity we take* $\eta(z) = \frac{1}{z}$*. Rewriting we obtain*

$$c = \frac{2z(z-1) + s(4x_1^2 + 4x_2^2 - 2)z}{z}, \tag{11}$$

choosing $s = -1$*, we get* $c = -2(x_1^2 + x_2^2)$ *and equation* (10) *is written as*

$$\frac{\frac{\partial h}{\partial z}}{h} = \frac{1}{z},$$

whose solution is

$$h(x_1, x_2) = z = x_1^2 + x_2^2.$$

We note that $ch = -2z^2$*, which is continuous, does not change sign, and vanishes only on a measure zero set; therefore,* h *is a Dulac function.*

Note that $Z(h) = \{0\}$ *contains no ovals, and in particular* $co(h) = 0$*.*

Since $s < 0$ *and* $l(\mathbb{R}^2, h) = 1$*, then by Proposition 2, the system has at most one limit cycle in* $\mathbb{R}^2$*.*

Furthermore, it is easy to check that $z - 1 = 0$ *is a periodic solution for the system.*

It is well known that gradient systems never possess periodic orbits. In this case, Proposition 2 yields the following:

Example 3. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a C^2-function, which vanishes at most only on a zero measure set, then the system $\dot{x} = -\operatorname{grad} V(x)$ admits a Dulac function and does not support periodic orbits.*

Indeed, the associated equation (5) *is*

$$-\frac{\partial V}{\partial x_1}\frac{\partial h}{\partial x_1} - \frac{\partial V}{\partial x_2}\frac{\partial h}{\partial x_2} = h(c + s(\operatorname{div}(\operatorname{grad} V))). \tag{12}$$

Taking $c(x) = -||\operatorname{grad} V(x)||$ and $s = 0$, then $h(x) = \exp(V(x))$ is a solution of the equation (12)*; furthermore, h is a Dulac function on $\mathbb{R}^2$.*

Since $Z(h) = \emptyset$ and $s = 0$, then by Proposition 2 the gradient system admits no periodic curves.

Example 4. *Consider the system*

$$\begin{cases} \dot{x}_1 &= x_2^3, \\ \dot{x}_2 &= (5x_1^2 - 1)x_2^3 - x_1^3. \end{cases}$$

The associated equation (5) *is*

$$x_2^3\frac{\partial h}{\partial x_1} + \left((5x_1^2 - 1)x_2^3 - x_1^3\right)\frac{\partial h}{\partial x_2} = h\left(c - s3x_2^2(5x_1^2 - 1)\right), \tag{13}$$

taking z such that $x_2^3\dfrac{\partial z}{\partial x_1} - x_1^3\dfrac{\partial z}{\partial x_2} = 0$, then $z = \frac{x_2^4}{4} + \frac{x_1^4}{4} + a$ for some constant a and (13) *becomes*

$$\frac{d \ln h}{dz} = \frac{c - s3x_2^2(5x_1^2 - 1)}{x_2^6(5x_1^2 - 1)}, \tag{14}$$

since the right side depends on z, denoted by η, thus

$$c = -\eta(z)\left(x_2^6(5x_1^2 - 1)\right) + s3x_2^2(5x_1^2 - 1), \tag{15}$$

taking $s = -\frac{4}{3}$, $\eta(z) = \frac{1}{z}$ and simplifying we have

$$c = -\frac{x_2^2(5x_1^2 - 1)(x_1^4 + 4a)}{z},$$

taking $a = -\frac{1}{100}$, then

$$c = \frac{x_2^2(5x_1^2 - 1)^2(5x_1^2 + 1)}{25z}, \tag{16}$$

choosing c in this way, then (14) *becomes* $\frac{d\ln h}{dz} = z$ *whose solution is*

$$h(x_1, x_2) = z = \frac{x_2^4}{4} + \frac{x_1^4}{4} - \frac{1}{100}.$$

Since $ch \in \mathcal{F}_{\mathbb{R}^2}$*, then* h *is a Dulac function.*

Note that $Z(h)$ *is an oval, then* $co(h) = 1$*; nevertheless* $Z(h)$ *contains no limit cycles, because the vector field is not tangent to* $Z(h)$*. Also* $l(\mathbb{R}^2, h) = 1$*, thus by Proposition 2 the system has at most* 1 *limit cycle in* $\mathbb{R}^2$*.*

The next application gives us information about the widely known uniqueness of limit cycles for *Van der Pol's equation.* However we want to point out the technique to produce a Dulac function given by Proposition 2.

Example 5. *For* $r \in \mathbb{R}$*, the Van der Pol's system*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\epsilon(x_1^2 + r)x_2 - x_1, \end{cases} \tag{17}$$

has at most one periodic orbit. In effect, using the associated equation (5) *we have*

$$x_2 \frac{\partial h}{\partial x_1} + (-\epsilon(x_1^2 + r)x_2 - x_1)\frac{\partial h}{\partial x_2} = h(c + s(\epsilon(x_1^2 + r))). \tag{18}$$

We can assume that $r \neq 0$*, otherwise the result follows from Theorem 1. Now we try to solve* (18) *for some* c *satisfying the conditions of Proposition 2. First assume that* $h = h(z)$ *depends on a function* $z = z(x_1, x_2)$*, thus we get*

$$\left[x_2 \frac{\partial z}{\partial x_1} + \left(-\epsilon(x_1^2 + r)x_2 - x_1\right)\frac{\partial z}{\partial x_2}\right]\frac{\partial h}{\partial z} = h\left(c + s\epsilon(x_1^2 + r)\right), \tag{19}$$

to simplify the equation we take z *such that* $x_2 \dfrac{\partial z}{\partial x_1} - x_1 \dfrac{\partial z}{\partial x_2} = 0$*; hence we define* $z := x_1^2 + x_2^2 + a$ *for some constant* a *and* (19) *reduces to*

$$\frac{\frac{\partial h}{\partial z}}{h} = \frac{c + s\epsilon(x_1^2 + r)}{-2\epsilon(x_1^2 + r)x_2^2}, \tag{20}$$

where the right side depends on z*, denoted by* η*. Rewriting we obtain*

$$c = -2\epsilon\eta(z)(x_1^2 + r)x_2^2 - s\epsilon(x_1^2 + r) = -(x_1^2 + r)\epsilon[2\eta(z)x_2^2 + s], \tag{21}$$

for simplicity we take $\eta(z) = \frac{1}{z}$*, and note that*

$$[2\eta(z)x_2^2 + s] = \frac{2x_2^2 + s(x_1^2 + x_2^2 + a)}{(x_1^2 + x_2^2 + a)},$$

from here we choose $s = -2$ *and* $a = r$*, then* $c = \frac{-2\epsilon(x_1^2+r)^2}{x_1^2+x_2^2+r}$ *and equation* (19) *is written as*

$$\frac{\frac{\partial h}{\partial z}}{h} = \frac{1}{z},$$

whose solution is

$$h(x_1, x_2) = z = x_1^2 + x_2^2 + r.$$

We note that $ch = -2\epsilon(x_1^2 + r)^2$*, which is continuous, does not change sign, and vanishes only on a measure zero set; therefore,* h *is a Dulac function.*

If $r > 0$*,* $Z(h)$ *contains no ovals, then* $co(h) = 0$*. Also* $l(\mathbb{R}^2, h) = 0$*; therefore by Proposition 2, the system admits no periodic orbits.*

On the other hand, if $r < 0$*,* $Z(h)$ *is a circle, and therefore* $co(h) = 1$*; nevertheless,* $Z(h)$ *contains no limit cycles, because the vector field is not tangent to* $Z(h)$*. Also* $l(\mathbb{R}^2, h) = 1$*, thus by Proposition 2 the system has at most* 1 *limit cycle in* $\mathbb{R}^2$*.*

We note that in particular, we have recovered the celebrated Dulac function due to Cherkas.

The next result deals with the non-existence of periodic orbits for the system (1).

Proposition 3. *If* $f_1 + f_2$*, does not change sign, and vanishes only on a measure zero set, then the system* (1) *admits no periodic orbits in* $\mathbb{R}^2$*.*

Proof: The associated equation (5) is

$$f_1(x_1, x_2)\frac{\partial h}{\partial x_1} + f_2(x_1, x_2)\frac{\partial h}{\partial x_2} = h[c - s(\mathrm{div}(f_1, f_2))]. \tag{22}$$

taking $z = x_1 + x_2$, the equation (22) becomes

$$(f_1(x_1, x_2) + f_2(x_1, x_2))\frac{d \ln h(z)}{dz} = c - s(\mathrm{div}(f_1, f_2)). \tag{23}$$

Taking $c = f_1 + f_2$ and $s = 0$, we have that $h(x_1, x_2) = \exp(x_1)\exp(x_2)$ is a solution of the equation (23); therefore, h is a Dulac function.

Since $Z(h) = \emptyset$ and $s = 0$, then by Proposition 2 the system (1) admits no periodic orbits in $\mathbb{R}^2$. □

Example 6. *Consider the system*

$$\begin{cases} \dot{x}_1 = x_1^2 + ax_2 - x_1x_2^3, \\ \dot{x}_2 = x_1^4 - ax_2 + x_1x_2^3. \end{cases}$$

Since $f_1(x_1,x_2)+f_2(x_1,x_2)=x_1^2+x_1^4$, then by Proposition 3, the system supports no limit cycles.

Now we consider the system

$$\begin{cases} \dot{x}_1 = k(x_2), \\ \dot{x}_2 = l(x_1) + m(x_1)x_2^r, \; r \geq 1. \end{cases} \tag{24}$$

The following proposition gives conditions that do not admit limit cycles for this system.

Proposition 4. *If any of the following conditions holds*

(1) $\frac{k(x_2)l(x_1)}{x_2^r} \in \mathcal{F}_{\mathbb{R}^2}$,

(2) $m(x_1)x_2^r k(x_2) \in \mathcal{F}_{\mathbb{R}^2}$,

then the system (24) *admits no periodic orbits.*

Proof: In the first case the associated equation (5) is

$$k(x_2)\frac{\partial h}{\partial x_1} + (l(x_1) + m(x_1)x_2^r)\frac{\partial h}{\partial x_2} = h[c - s(m(x_1)rx_2^{r-1})]. \tag{25}$$

Now assume that $h = h(z)$ depends on $z = z(x_1, x_2)$, thus (25) is written as

$$\left[k(x_2)\frac{\partial z}{\partial x_1} + (l(x_1) + m(x_1)x_2^r)\frac{\partial z}{\partial x_2}\right]\frac{d\ln h}{dz} = c - sm(x_1)rx_2^{r-1},$$

taking z such that $k(x_2)\dfrac{\partial z}{\partial x_1} + m(x_1)x_2^r\dfrac{\partial z}{\partial x_2} = 0$, i.e., $z = \int^{x_2} \frac{k(\tau)}{(\tau)^r}d\tau - \int^{x_1} m(\tau)d\tau$. This yields

$$\frac{d\ln h}{dz} = \frac{x_2^r(c - s(m(x_1)rx_2^{r-1}))}{k(x_2)l(x_1)}, \tag{26}$$

so taking $c = \frac{k(x_2)l(x_1)}{x_2^r}$ and $s = 0$, the equation (26) is written as $\dfrac{d\ln h}{dz} = 1$, whose solution is $h(x_1,x_2) = e^z = \exp(\int^{x_2}\frac{k(\tau)}{(\tau)^r}d\tau - \int^{x_1} m(\tau)d\tau)$ which is a Dulac function; note that $Z(h)$ contains no ovals, and in particular $co(h) = 0$. By Proposition 2 the system (24) admits no periodic orbits.

For the second case: we take $z = z(x_1, x_2)$ such that

$$k(x_2)\frac{\partial z}{\partial x_1} + l(x_1)\frac{\partial z}{\partial x_2} = 0.$$

The proof proceeds in an analogous way. □

Example 7. *Consider the system*

$$\begin{cases} \dot{x}_1 = x_2^3, \\ \dot{x}_2 = -(x_1 + 3x_1^2)^4 + (1 + 2x_1 + 4x_1^5)x_2^3. \end{cases}$$

By (1) in Proposition 4, the system supports no limit cycles.

The next result deals with the uniqueness of periodic orbits for special cases (24), we have the following:

Proposition 5. *If* $k(x_2) = x_2$ *and* $m(x_1)L(x_1)x_2^{r-1} \in \mathcal{F}_{\mathbb{R}^2}$, *where* $L(x_1) = \int^{x_1} l(s)ds$. *Then the system* (24) *has at most one limit cycle.*

Proof: The associated equation (5) is

$$k(x_2)\frac{\partial h}{\partial x_1} + (l(x_1) + m(x_1)x_2^r)\frac{\partial h}{\partial x_2} = h[c - s(m(x_1)rx_2^{r-1})]. \tag{27}$$

Now assume that $h = h(z)$ depends on $z = z(x_1, x_2)$, thus (27) is written as

$$\left[k(x_2)\frac{\partial z}{\partial x_1} + (l(x_1) + m(x_1)x_2^r)\frac{\partial z}{\partial x_2}\right]\frac{d\ln h}{dz} = c - sm(x_1)rx_2^{r-1},$$

taking z such that $k(x_2)\dfrac{\partial z}{\partial x_1} + l(x_1)\dfrac{\partial z}{\partial x_2} = 0$, i.e., $z = \int^{x_2} k(\tau)d\tau - \int^{x_1} l(\tau)d\tau$. This yields

$$\frac{d\ln h}{dz} = \frac{c - s(m(x_1)rx_2^{r-1})}{m(x_1)k(x_2)x_2^r}, \tag{28}$$

where the right side depends on z, denoted by η. Rewriting we obtain

$$c = \eta(z)m(x_1)k(x_2)x_2^r + s(m(x_1)rx_2^{r-1}), \tag{29}$$

taking $\eta(z) = \frac{1}{z}$, then

$$c = \frac{m(x_1)k(x_2)x_2^r + s(m(x_1)rx_2^{r-1})(\frac{x_2^2}{2} - L(x_1))}{z}, \tag{30}$$

from here we choose $s = -\frac{2}{r}$, thus $c = -\frac{2m(x_1)L(x_1)x_2^{r-1}}{z}$ and equation (30) is written as

$$\frac{d\ln h}{dz} = \frac{1}{z},$$

whose solution is

$$h(x_1, x_2) = z = x_2^2 - L(x_1).$$

Since $ch = 2m(x_1)L(x_1)x_2^{r-1}$ which is continuous, does not change sign, and vanishes only on a measure zero set; therefore, h is a Dulac function. therefore, h is a Dulac function.

Note that $Z(h)$ has at most one oval, and therefore $co(h) = 1$; nevertheless, $Z(h)$ contains no limit cycles. Indeed, the condition $k(x_2)\dfrac{\partial z}{\partial x_1} + l(x_1)\dfrac{\partial z}{\partial x_2} = 0$, implies that the vector field $(x_2, l(x_1) + m(x_1)x_2^r)$ is not tangent to $z = 0$.

Since $s < 0$ and $l(\mathbb{R}^2, h) = 1$, then by Proposition 2 the system (24) has at most one limit cycle in $\mathbb{R}^2$. $\square$

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