Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **89**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF HIGHER ORDER QUASILINEAR DELAY DIFFERENTIAL EQUATIONS

B. BACULÍKOVÁ AND J. DŽURINA[1]

Abstract. In the paper, we offer such generalization of a lemma due to Philos (and partially Staikos), that yields many applications in the oscillation theory. We present its disposal in the comparison theory and we establish new oscillation criteria for $n-$th order delay differential equation

$$(E)\qquad \left(r(t)\left[x'(t)\right]^{\gamma}\right)^{(n-1)}+q(t)x^{\gamma}(\tau(t))=0.$$

The presented technique essentially simplifies the examination of the higher order differential equations.

1. Introduction

In this paper, we shall study the asymptotic and oscillation behavior of the solutions of the higher order delay differential equations

$$(E)\qquad \left(r(t)\left[x'(t)\right]^{\gamma}\right)^{(n-1)}+q(t)x^{\gamma}(\tau(t))=0.$$

Throughout the paper, we will assume $q,\tau,r\in C([t_0,\infty))$, and

- (H_1) $n\geq 3$, γ is the ratio of two positive odd integers,
- (H_2) $r(t)>0$, $q(t)>0$, $\tau(t)\leq t$, $\lim\limits_{t\to\infty}\tau(t)=\infty$.

Whenever, it is assumed

$$(1.1)\qquad R(t)=\int_{t_0}^{t}r^{-1/\gamma}(s)\,\mathrm{d}s\to\infty\quad\text{as } t\to\infty.$$

By a solution of Eq. (E) we mean a function $x(t)\in C^1([T_x,\infty))$, with $T_x\geq t_0$, which has the property $r(t)(x'(t))^{\gamma}\in C^{n-1}([T_x,\infty))$ and satisfies Eq. (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)|:t\geq T\}>0$ for all $T\geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

The problem of the oscillation of higher order differential equations has been widely studied by many authors, who have provided many techniques for obtaining oscillatory criteria for studied equations (see e.g. [1] - [19]).

Philos in [16] and [17] presented the following lemma.

1991 *Mathematics Subject Classification.* 34K11, 34C10.
Key words and phrases. n-th order differential equations, oscillation.
This work was supported by S.G.A. KEGA Grant No. 020TUKE-4/2012.
[1] Corresponding author.

Lemma A. *Assume that $z^{(i)}(t)$, $i = 1, 2, \dots, \ell$ are of constant signs such that $z^{(\ell-1)}(t)z^{(\ell)}(t) \le 0$ and $\lim_{t\to\infty} z(t) \ne 0$. Then for any $\lambda \in (0,1)$*

$$z(t) \ge \frac{\lambda}{(\ell-1)!} t^{\ell-1} z^{(\ell)}(t),$$

eventually.

This lemma essentially simplifies the examination of $n-th$ order differential equations of the form

$$y^{(n)}(t) + q(t)y^{\gamma}(\tau(t)) = 0 \tag{1.2}$$

since it provides needed relationship between $y(t)$ and $y^{(n-1)}(t)$ and this fact permit us to establish just one condition for oscillation of (1.2). This lemma is not applicable to differential equation (E). In this paper we offer a generalization of Lemma A, which works for (E) and permits to establish new oscillation criteria for it.

2. Main Results

The following result is a well-known lemma of Kiguradze see e.g. [6] or [14].

Lemma 1. *Let $z(t) \in C^1([t_0,\infty))$, and $r(t)(z'(t))^{\gamma} \in C^{k-1}([t_0,\infty))$ with $z(t) > 0$, $(r(t)(z'(t))^{\gamma})^{(k-1)} \le 0$ and not identically zero on a subray of $[t_0,\infty)$. Then there exist a $t_1 \ge t_0$ and an integer ℓ, $0 \le \ell \le k-1$, with $k+\ell$ odd so that*

$$\begin{aligned} \left(r(t)(z'(t))^{\gamma}\right)^{(i)}(t) &> 0, \quad i = 0,\dots,\ell-1, \quad \text{when } \ell \ge 1,\\ (-1)^{\ell+j-1}\left(r(t)(z'(t))^{\gamma}\right)^{(j)}(t) &> 0, \quad j = \ell,\dots,k-2, \end{aligned} \tag{2.1}$$

on $[t_1,\infty)$.

Now we are prepared to provide a generalization of Lemma A.

Lemma 2. *Let $z(t)$ be as in Lemma 1 and numbers t_1 and ℓ be assigned to $z(t)$ by Lemma 1. Then for $2 \le \ell \le k-1$*

$$z(t) \ge \frac{\left[\left(r(t)(z'(t))^{\gamma}\right)^{(k-2)}\right]^{1/\gamma}}{\left((k-2)!\right)^{1/\gamma}} \int_{t_1}^{t} r^{-1/\gamma}(s)(s-t_1)^{(k-2)/\gamma}\,\mathrm{d}s, \tag{2.2}$$

for $\ell = 1$

$$z(t) \ge \frac{\left[\left(r(t)(z'(t))^{\gamma}\right)^{(k-2)}\right]^{1/\gamma}}{\left((k-2)!\right)^{1/\gamma}} \int_{t_1}^{t} r^{-1/\gamma}(s)(t-s)^{(k-2)/\gamma}\,\mathrm{d}s, \tag{2.3}$$

for $t \ge t_1$.

Proof. Let ℓ be the integer assigned to function $z(t)$ as in Lemma 1. Assume that $\ell < k-1$, then for any s,t with $t \geq s \geq t_1$, we have

$$-\left(r(s)(z'(s))^\gamma\right)^{(k-3)} \geq \int_s^t \left(r(u)(z'(u))^\gamma\right)^{(k-2)} \mathrm{d}u \geq \left(r(t)(z'(t))^\gamma\right)^{(k-2)} (t-s).$$

Repeated integration in s from s to t yields

$$\left(r(s)(z'(s))^\gamma\right)^{(\ell-1)} \geq \left(r(t)(z'(t))^\gamma\right)^{(k-2)} \frac{(t-s)^{k-\ell-1}}{(k-\ell-1)!}. \tag{2.4}$$

It is easy to see that (2.4) holds also for $\ell = k-1$.

On the other hand, if $\ell \geq 2$, then for every $t \geq t_1$, we have

$$\left(r(t)(z'(t))^\gamma\right)^{(\ell-2)} \geq \int_{t_1}^t \left(r(s)(z'(s))^\gamma\right)^{(\ell-1)} \mathrm{d}s.$$

Repeated integration from t_1 to t leads to

$$r(t)(z'(t))^\gamma \geq \frac{1}{(\ell-2)!} \int_{t_1}^t \left(r(s)(z'(s))^\gamma\right)^{(\ell-1)} (t-s)^{\ell-2}\,\mathrm{d}s. \tag{2.5}$$

Setting (2.4) into (2.5), one gets

$$\begin{aligned}\left(r(t)(z'(t))^\gamma\right) &\geq \frac{\left(r(t)(z'(t))^\gamma\right)^{(k-2)}}{(\ell-2)!(k-\ell-1)!} \int_{t_1}^t (t-s)^{k-3}\,\mathrm{d}s \\ &\geq \frac{\left(r(t)(z'(t))^\gamma\right)^{(k-2)}}{(k-2)!} (t-t_1)^{k-2}.\end{aligned}$$

or simply

$$z'(t) \;\geq\; \frac{\left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma}}{\left((k-2)!\right)^{1/\gamma}} r^{-1/\gamma}(t)(t-t_1)^{(k-2)/\gamma}.$$

Integrating the last inequality from t_1 to t, we get (2.2). We have verified the first part of the lemma.

Now assume that $\ell = 1$. It follows from (2.4) that

$$r(s)(z'(s))^\gamma \geq \left(r(t)(z'(t))^\gamma\right)^{(k-2)} \frac{(t-s)^{k-2}}{(k-2)!}. \tag{2.6}$$

On the other hand,

$$z(t) \geq \int_{t_1}^t z'(s)\,\mathrm{d}s = \int_{t_1}^t r^{1/\gamma}(s) z'(s) r^{-1/\gamma}(s)\,\mathrm{d}s. \tag{2.7}$$

Combining (2.6) together with (2.7), we get (2.3). The proof is complete now. □

Imposing additional condition, we are able to joint (2.4) and (2.5) to just one estimate.

Lemma 3. *Let $z(t)$ be as in Lemma 1 and $\lim_{t\to\infty} z(t) \neq 0$. Let $r'(t) \geq 0$. Then for any $\lambda \in (0,1)$ there exists some $t_\lambda \geq t_1$ such that*

$$z(t) \geq \frac{\gamma\lambda t^{(k-2+\gamma)/\gamma}}{\left((k-2)!\right)^{1/\gamma}(k-2+\gamma)} r^{-1/\gamma}(t)\left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma} \tag{2.8}$$

for $t \geq t_\lambda$.

Proof. Note that $r'(t) \geq 0$ implies that $r^{-1/\gamma}(t)$ is nonincreasing. Assume that ℓ is the integer associated with $z(t)$ in Lemma 1. If $2 \leq \ell \leq k-2$, then using (2.2), we have

$$z(t) \geq \frac{\left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma}}{\left((k-2)!\right)^{1/\gamma}} r^{-1/\gamma}(t)\gamma\frac{(t-t_1)^{(k-2+\gamma)/\gamma}}{k-2+\gamma} \tag{2.9}$$

It is easy to see that for any $\lambda \in (0,1)$ there exists a $t_\lambda \geq t_1$ such that $t - t_1 \geq \lambda^{\gamma/(k-2+\gamma)}t$ for $t \geq t_\lambda$, which in view of (2.9) yields (2.8).

If $\ell = 1$, then proceeding similarly as above it can be shown that (2.3) implies (2.8).

If $\ell = 0$, then for any s, t with $t \geq s \geq t_1$

$$-\left(r(s)(z'(s))^\gamma\right)^{(k-3)} \geq \left(r(t)(z'(t))^\gamma\right)^{(k-2)}(t-s).$$

Repeated integration in s from s to t yields

$$-r(s)(z'(s))^\gamma \geq \left(r(t)(z'(t))^\gamma\right)^{(k-2)}\frac{(t-s)^{k-2}}{(k-2)!}$$

or

$$-z'(s) \geq \left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma} r^{-1/\gamma}(s)\frac{(t-s)^{(k-2)/\gamma}}{\left((k-2)!\right)^{1/\gamma}}.$$

An integration from s to t, yields

$$\begin{aligned} z(s) &\geq \left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma}\int_s^t r^{-1/\gamma}(s)\frac{(t-s)^{(k-2)/\gamma}}{\left((k-2)!\right)^{1/\gamma}}\,\mathrm{d}s \\ &\geq \left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma} r^{-1/\gamma}(t)\frac{\gamma\,(t-s)^{(k-2+\gamma)/\gamma}}{\left((k-2)!\right)^{1/\gamma}(k-2+\gamma)}. \end{aligned}$$

Setting $s = \left(1 - \lambda^{\gamma/2(k-2-\gamma)}\right)t$, we have

$$z\left(\left(1-\lambda^{\gamma/2(k-2-\gamma)}\right)t\right) \geq \left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma}\frac{\gamma\lambda^{1/2}\,r^{-1/\gamma}(t)\,t^{(k-2+\gamma)/\gamma}}{\left((k-2)!\right)^{1/\gamma}(k-2+\gamma)}.$$

Moreover,

$$\lim_{t\to\infty}\frac{z(t)}{z\left(\left(1-\lambda^{\gamma/2(k-2-\gamma)}\right)t\right)} = 1 > \lambda^{1/2}.$$

Therefore,

$$z(t) \geq \lambda^{1/2} z\left(\left(1-\lambda^{\gamma/2(k-2-\gamma)}\right)t\right),$$

and consequently,

$$z(t) \geq \frac{\gamma\lambda t^{(k-2+\gamma)/\gamma}}{\left((k-2)!\right)^{1/\gamma}(k-2+\gamma)} r^{-1/\gamma}(t) \left[\left(r(t)(z'(t))^\gamma\right)^{(k-2)}\right]^{1/\gamma}$$

The proof is complete now. $\square$

Remark 1. *For $r(t) \equiv 1$ and $\gamma = 1$, Lemma 3 reduces to Lemma A.*

3. Applications

To present usefulness of Lemma 2 and Lemma 3, we apply both to establish new oscillatory results for (E), based also on comparison principles.

Theorem 1. *Assume that the first order delay differential equation*

$$(E_1) \qquad y'(t) + \frac{q(t)}{(n-2)!}\left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(s-t_1)^{(n-2)/\gamma}\,\mathrm{d}s\right)^\gamma y(\tau(t)) = 0$$

is oscillatory. Moreover, for n-even the first order delay differential equation

$$(E_2) \qquad y'(t) + \frac{q(t)}{(n-2)!}\left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(\tau(t)-s)^{(n-2)/\gamma}\,\mathrm{d}s\right)^\gamma y(\tau(t)) = 0$$

is oscillatory and for n-odd condition

$$(P_0) \qquad \int_{t_0}^{\infty} r^{-1/\gamma}(u)\left(\int_u^\infty q(s)(s-u)^{n-2}\,\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u = \infty.$$

holds. Then

(i) *for n even, (E) is oscillatory;*
(ii) *for n odd, each nonoscillatory solution of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (E), let say positive. Then $\left(r(t)\left[x'(t)\right]^\gamma\right)^{(n-1)} < 0$ and there exist a $t_1 \geq t_0$ and an integer ℓ with $n+\ell$ odd such that (2.1) holds.

If $2 \leq \ell \leq n-1$, Then by Lemma 2

$$x(t) \geq \frac{\left[\left(r(t)(x'(t))^\gamma\right)^{(n-2)}\right]^{1/\gamma}}{\left((n-2)!\right)^{1/\gamma}} \int_{t_1}^{t} r^{-1/\gamma}(s)(s-t_1)^{(n-2)/\gamma}\,\mathrm{d}s,$$

Then $y(t) = \left(r(t)(x'(t))^\gamma\right)^{(n-2)}$ is positive and

$$x^\gamma(\tau(t)) \geq \frac{y(\tau(t))}{(n-2)!}\left(\int_{t_1}^{t} r^{-1/\gamma}(s)(s-t_1)^{(n-2)/\gamma}\,\mathrm{d}s\right)^\gamma,$$

Setting to (E), we see that $y(t)$ is a positive solution of the delay differential inequality

$$y'(t) + \frac{q(t)}{(n-2)!}\left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(s-t_1)^{n-2}\,\mathrm{d}s\right)^\gamma y(\tau(t)) \leq 0.$$

By Theorem 1 in [15] the corresponding equation (E_1) has also a positive solution. A contradiction.

If $\ell = 1$, which is possible only when n is even, Lemma 2 implies

$$x(t) \geq \frac{\Big[\big(r(t)(x'(t))^\gamma\big)^{(n-2)}\Big]^{1/\gamma}}{\big((n-2)!\big)^{1/\gamma}} \int_{t_1}^t r^{-1/\gamma}(s)(t-s)^{(n-2)/\gamma}\,\mathrm{d}s,$$

and proceeding as above, we find out that (E_2) has a positive solution. A contradiction and the proof is finished for n even.

Assume that $\ell = 0$, note that it is possible only of n is odd. Since $x'(t) < 0$, then there exists a finite $\lim_{t\to\infty} x(t) = c \geq 0$. We claim that $c = 0$. If not, that $x(\tau(t)) \geq c > 0$, eventually, let us say for $t \geq t_2$. An integration of (E) from t to ∞ yields

$$\big(r(t)\big(x'(t)\big)^\gamma\big)^{(n-2)} \geq \int_t^\infty q(s)x^\gamma(\tau(s))\,\mathrm{d}s$$

Integrating $n-2$ times from t to ∞, we get

$$-r(t)\big(x'(t)\big)^\gamma \geq \int_t^\infty q(s)x^\gamma(\tau(s))\frac{(s-t)^{n-2}}{(n-2)!}\,\mathrm{d}s$$

or equivalently

$$-x'(t) \geq r^{-1/\gamma}(t)\left(\int_t^\infty q(s)x^\gamma(\tau(s))\frac{(s-t)^{n-2}}{(n-2)!}\,\mathrm{d}s\right)^{1/\gamma} \tag{3.1}$$

Integrating again from t_2 to ∞, we get

$$x(t_2) \geq c\int_{t_2}^\infty r^{-1/\gamma}(u)\left(\int_u^\infty q(s)\frac{(s-u)^{n-2}}{(n-2)!}\,\mathrm{d}s\right)^{1/\gamma}\,\mathrm{d}u$$

which contradicts (P_0). The proof is complete. □

Employing any result (e.g. Theorem 2.1.1 in [14]) for the oscillation of (E_1) and (E_2), we immediately obtain criteria for studied properties of (E).

Corollary 1. *Assume that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^t q(u)\left(\int_{t_1}^{\tau(u)} r^{-1/\gamma}(s)(s-t_1)^{(n-2)/\gamma}\,\mathrm{d}s\right)^\gamma \mathrm{d}u > \frac{(n-2)!}{\mathrm{e}}$$

and

$$\liminf_{t\to\infty} \int_{\tau(t)}^t q(u)\left(\int_{t_1}^{\tau(u)} r^{-1/\gamma}(s)(\tau(s)-t)^{(n-2)/\gamma}\,\mathrm{d}s\right)^\gamma \mathrm{d}u > \frac{(n-2)!}{\mathrm{e}},$$

Moreover, for n-odd assume that (P_0) hold. Then

(i) *for n even, (E) is oscillatory;*
(ii) *for n odd, each nonoscillatory solution of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.*

The results of Theorem 1 and Corollary 1 can be simplified provided that we impose additional condition on the function $r(t)$.

Theorem 2. *Let $r'(t) \geq 0$. Assume that for some $\lambda \in (0,1)$ the first order delay differential equation*

$$(E_3) \qquad y'(t) + \frac{\gamma^\gamma \lambda^\gamma}{(n-2)!(n-2+\gamma)^\gamma} \frac{q(t)\tau^{n-2+\gamma}(t)}{r(\tau(t))} y(\tau(t)) = 0$$

is oscillatory. Then

(i) *for n even, (E) is oscillatory;*
(ii) *for n odd, each nonoscillatory solution of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Assume that $x(t)$ is an eventually positive solution of (E). Then $(r(t)\,[x'(t)]^\gamma)^{(n-1)} < 0$ and there exist a $t_1 \geq t_0$ and an integer ℓ with $n+\ell$ odd such that (2.1) holds. If n is odd suppose that $\lim_{t\to\infty} x(t) \neq 0$ (for n is even this is obvious). Then it follows from Lemma 3 that

$$x(t) \geq \frac{\gamma\lambda t^{(n-2+\gamma)/\gamma}}{\big((n-2)!\big)^{1/\gamma}(n-2+\gamma)} r^{-1/\gamma}(t) \left[\big(r(t)(x'(t))^\gamma\big)^{(n-2)}\right]^{1/\gamma},$$

that is, $y(t) = (r(t)(x'(t))^\gamma)^{(n-2)}$ satisfies

$$x^\gamma(\tau(t)) \geq \frac{\gamma^\gamma\lambda^\gamma\tau^{n-2+\gamma}(t)}{(n-2)!(n-2+\gamma)^\gamma} \frac{y(\tau(t))}{r(\tau(t))}.$$

Setting to (E), we see that $y(t)$ is a positive solution of the differential inequality

$$y'(t) + \frac{\gamma^\gamma \lambda^\gamma}{(n-2)!(n-2+\gamma)^\gamma} \frac{q(t)\tau^{n-2+\gamma}(t)}{r(\tau(t))} y(\tau(t)) \leq 0.$$

By Theorem 1 in [15] the corresponding equation (E_3) has also a positive solution. A contradiction. □

Corollary 2. *Let $r'(t) \geq 0$. If*

$$(P_1) \qquad \liminf_{t\to\infty} \int_{\tau(t)}^t \frac{q(s)\tau^{n-2+\gamma}(s)}{r(\tau(s))} \,\mathrm{d}s > \frac{(n-2)!}{\mathrm{e}} \left(\frac{n-2+\gamma}{\gamma}\right)^\gamma.$$

Then

(i) *for n even, (E) is oscillatory;*
(ii) *for n odd, every nonoscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. It is easy to see from (P_1) that there exist some $\lambda \in (0,1)$ such that

$$\liminf_{t\to\infty} \int_{\tau(t)}^t \frac{\gamma^\gamma\lambda^\gamma}{(n-2)!(n-2+\gamma)^\gamma} \frac{q(s)\tau^{n-2+\gamma}(s)}{r(\tau(s))} \,\mathrm{d}s > \frac{1}{\mathrm{e}},$$

But according to Theorem 2.1.1 in [14] this condition guarantees oscillation of (E_3). the assertion now follows from Theorem 2. □

Example 1. *We consider the fourth order delay differential equation*

$$(E_3) \qquad \left(t\left(x'(t)\right)^3\right)''' + \frac{a}{t^5}\,x^3(\lambda t) = 0, \quad a > 0, \quad 0 < \lambda < 1, \quad t \geq 1.$$

Condition (P_1) *reduces to*

$$a\lambda^4 \ln\left(\frac{1}{\lambda}\right) > \frac{2}{\mathrm{e}}\left(\frac{5}{3}\right)^3, \tag{3.2}$$

which by Corollary 2 guarantees oscillation of (E_3)*. On the other hand, it is easy to see that for* $a\lambda^{3/2} = 15/2^6$ *condition* (3.2) *fails and* (E_3) *has a nonoscillatory solution* $x(t) = t^{1/2}$*.*

If we enforce condition (P_0), we can obtain oscillation of (E) even if n is odd.

Theorem 3. *Let* $\tau'(t) \geq 0$*. Assume that both first order delay differential equations* (E_1) *and* (E_2) *are oscillatory. Moreover, for* n*-odd assume that*

$$(P_2) \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} r^{-1/\gamma}(u)\left(\int_u^t q(s)(s-u)^{n-2}\,\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u > \left((n-2)!\right)^{1/\gamma}.$$

Then (E) *is oscillatory.*

Proof. Assume that $x(t)$ is a positive solution of (E). Then there exist a $t_1 \geq t_0$ and an integer ℓ with $n+\ell$ odd such that (2.1) holds. Taking into account the proof of Theorem 1, it is sufficient to eliminate the case $\ell = 0$. If we admit that $\ell = 0$, then we are led to (3.1). Integrating it from t to ∞, we get

$$x(t) \geq \int_t^\infty r^{-1/\gamma}(u)\left(\int_u^\infty x^\gamma(\tau(s))q(s)\frac{(s-u)^{n-2}}{(n-2)!}\,\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u,$$

which implies

$$\begin{aligned} x(\tau(t)) &\geq \int_{\tau(t)}^{t} r^{-1/\gamma}(u)\left(\int_u^t x^\gamma(\tau(s))q(s)\frac{(s-u)^{n-2}}{(n-2)!}\,\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u \\ &\geq x(\tau(t))\int_{\tau(t)}^{t} r^{-1/\gamma}(u)\left(\int_u^t q(s)\frac{(s-u)^{n-2}}{(n-2)!}\,\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u, \end{aligned}$$

which contradicts (P_2). □

Corollary 3. *Let* $\tau'(t) \geq 0$ *and* $r'(t) \geq 0$*. If* (P_1) *and* (P_2) *hold, then* (E) *is oscillatory.*

Proof. Assume that $x(t)$ is a positive solution of (E). Then there exist a $t_1 \geq t_0$ and an integer ℓ with $n+\ell$ odd such that (2.1) holds. It follows from Theorem 2 and Corollary 2 that $\ell = 0$, but this case is eliminated by (P_1). □

Example 2. *We consider the third order delay differential equation*

$$(E_4) \qquad \Big(t\big(x'(t)\big)^3\Big)'' + \frac{a}{t^4}\, x^3(\lambda t) = 0, \quad a > 0, \quad 0 < \lambda < 1, \quad t \geq 1.$$

Condition (P_1) *simplifies to*

$$a\lambda^3 \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}\left(\frac{4}{3}\right)^3,$$

which by Corollary 2 guarantees that every nonoscillatory solution $x(t)$ *of* (E_4) *tends to zero. Note that for* $a = \alpha^3(3\alpha+2)(3\alpha+3)\lambda^{3\alpha}$*, with* $\alpha > 0$ *one such solution is* $x(t) = t^{-\alpha}$*. On the other hand,* (P_2) *takes the form*

$$a\left(\ln\frac{1}{\lambda}\right)^3 > 6,$$

which according to Corollary 3 yields oscillation of (E_4)*.*

References

[1] B. Baculíková, J. Džurina, *Oscillation of third-order neutral differential equations*, Math. Comput. Modelling 52 (2010), 215–226.

[2] B. Baculíková, J. Graef, J. Džurina, *On the oscillation of higher order delay differential equations*, Nonlinear oscillations, 15 (2012), 13–24.

[3] B. Baculíková, J. Džurina, *Oscillation of third-order nonlinear differential equations*, Applied Math. Letters, 24 (2011), 466–470.

[4] B. Baculíková, J. Džurina, *Oscillation of third-order functional differential equations* , EJQTDE, 43 (2010), 1–10.

[5] Džurina J., Stavroulakis I.P.: *Oscillation criteria for second-order delay differential equations*, Appl. Math. Comp. 140 (2003), No2, 445–453.

[6] J. Džurina, *Comparison theorems for nonlinear ODE's* Math. Slovaca 42 (1992), 299–315.

[7] L. H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1994.

[8] S. R. Grace, R. P. Agarwal, R. Pavani and E. Thandapani, *On the oscillation of certain third order nonlinear functional differential equations*, Appl. Math. Comp. 202 (2008),102–112.

[9] S. R. Grace, B. S. Lalli, *Oscillation of even order differential equations with deviating arguments* J. Math. Appl. Anal. 147 (1990) 569–579.

[10] T. Li, Z. Han, P. Zhao, S. Sun, *Oscillation of even order nonlinear neutral delay differential equations*, Adv. Difference Equ.(2010), Article ID 184180,

[11] I. T. Kiguradze and T. A. Chaturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Acad. Publ., Dordrecht 1993.

[12] R. G. Koplatadze, G. Kvinkadze, I Stavroulakis, *Properties A and B of n-th order linear differential equations with deviating argument*, Georgian Math. J. 6 (1999) 553–566.

[13] T. Kusano and M. Naito, *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan 3 (1981), 509–533.

[14] G. S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.

[15] Ch. G. Philos, *On the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delay*, Arch. Math. 36 (1981), 168–178.

[16] Ch. G. Philos, *Oscillation and asymptotic behavior of linear retarded differential equations of arbitrary order*, Univ. Ioannina, Tech. Report No. 57, 1981

[17] Ch. G. Philos, *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay*, J. Austral. Math. Soc. 36 (1984), 176–186.
[18] Ch. Zhang, T. Li, B. Sun, E. Thandapani, *On the Oscillation of Higher-Order Half-Linear Delay Differential Equations*, Appl. Math. Letters (2011),
[19] Q. Zhang, J. Yan, L. Gao, *Oscillation behavior of even order nonlinear neutral differential equations with variable coefficients*, Comput. Math. Appl. 59(2010), 426–430.

(Received July 6, 2012)

(B. Baculíková, J. Džurina) Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia

E-mail address: {blanka.baculikova,jozef.dzurina}@tuke.sk