Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **6**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

Multiple positive solutions for second order impulsive differential equation*

Weihua Jiang†, Qiang Zhang, Weiwei Guo
College of Sciences, Hebei University of Science and Technology
Shijiazhuang, 050018, Hebei, P. R. China

Abstract: We investigate the existence of positive solutions to a three-point boundary value problem of second order impulsive differential equation. Our analysis rely on the Avery-Peterson fixed point theorem in a cone. An example is given to illustrate our result.

Keywords: impulsive differential equation; fixed point theorem; positive solution; completely continuous operator

1. Introduction

Impulsive differential equations have very good applications in economics, biology, ecology and other fields(see[1-3]). Many authors are interested in the boundary value problem of impulsive differential equations (see [4-23]). For example, in [6,7], R. P. Agarwal and D. O'Regan studied the existence of solutions for the boundary value problems

$$y''(t) + \phi(t)f(t, y(t)) = 0, \quad t \in (0, 1) \setminus \{t_1, t_2, \cdots, t_m\},$$
$$\Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, 2, \cdots, m,$$
$$\Delta y'(t_k) = J_k(y(t_k^-)), \quad k = 1, 2, \cdots, m,$$
$$y(0) = y(1) = 0,$$

by using Krasnoselskii's fixed point theorem and the Leggett Williams fixed point theorem, respectively. Using the fixed point index theory, T. Jankowski ([23]) obtained the existence of solutions for the boundary value problem

$$x''(t) + \alpha(t)f(x(\alpha(t))) = 0, \quad t \in (0, 1) \setminus \{t_1, t_2, \cdots, t_m\},$$

*This work is supported by the Natural Science Foundation of China (11171088), the Doctoral Program Foundation of Hebei University of Science and Technology (QD201020) and the Foundation of Hebei University of Science and Technology (XL201136).

†Corresponding author. E-mail address: weihuajiang@hebust.edu.cn (Weihua Jiang).

$$\Delta y'(t_k) = Q_k(x(t_k)), \quad k = 1, 2, \cdots, m,$$
$$x(0) = 0, \quad \beta x(\eta) = x(1).$$

In paper [26], quite general impulsive boundary value problems

$$u''(t) + p(t)u'(t) + q(t)u(t) + g(t)f(t, u(t)) = 0, \quad t \in (0,1),\ t \neq \tau,$$
$$\Delta u_{(t=\tau)}) = I(u(\tau)),$$
$$\Delta u'_{(t=\tau)}) = N(u(\tau)),$$
$$a_1u(0) - b_1u'(0) = \alpha[u],\ a_2u(1) - b_2u'(1) = \beta[u].$$

are treated.

Motivated by the excellent results mentioned above and the methods used in [24], in this paper, we examine the second order impulsive equation

$$\begin{cases} u''(t) + \phi(t)f(t, u(t)) = 0, \quad t \in (0,1) \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = \alpha u(\xi), \quad u'(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha, \xi \in (0,1)$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $\xi \neq t_k,\ k = 1, 2, \cdots, m$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ (respectively $u(t_k^-)$) denotes the right limit (respectively left limit) of $u(t)$ at $t = t_k$. Also $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$. Our result complements the results of [6,7,23] and it can solve the problems which cannot be solved by the results of [26](see example 3.1).

We define the Banach space:

$$PC[0,1] = \{u : [0,1] \to R, \text{ there exists } u_k \in C[t_k, t_{k+1}] \text{ such that } u(t) = u_k(t)$$
$$\text{for } t \in (t_k, t_{k+1}],\ k = 0, 1, \cdots, m,\ u(0) = u(0+0)\},$$

with the norm

$$\|u\| = \sup\{|u(t)| : t \in [0,1] \setminus \{t_1, \cdots, t_m\}\},$$

where $t_0 = 0,\ t_{m+1} = 1$.

A positive solution of the problem (1.1) means a function $u \in PC[0,1]$ which satisfies (1.1) with $u(t) > 0,\ t \in [0,1]$.

In this paper, we will always suppose that the following conditions hold:

(C_1) $\phi \in C(0,1)$ with $\phi > 0$ on $(0,1)$ and $\phi \in L^1[0,1]$.

(C_2) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

(C_3) I_k , J_k :$[0,\infty) \to R$ are continuous for $k = 1, 2, \cdots, m$.

(C_4) There exists a function $\Omega: \ \{u : u \in PC[0,1], \ u \geq 0\} \to [0,+\infty)$ and a constant $0 < c_0 < 1$ such that

$$c_0\Omega(u) \leq \omega_0(t,u) \leq \Omega(u), \quad (t,u) \in [0,1] \times \{u : u \in PC[0,1], \ u \geq 0\},$$

where

$$\omega_0(t,u) = \frac{\alpha}{1-\alpha}\sum_{t_k<\xi}[I_k(u(t_k)) + (\xi - t_k)J_k(u(t_k))] \\ + \sum_{t_k<t}\left[I_k(u(t_k)) - \frac{\alpha\xi + (1-\alpha)t_k}{1-\alpha}J_k(u(t_k))\right] - \sum_{t\leq t_k}\frac{\alpha\xi + (1-\alpha)t}{1-\alpha}J_k(u(t_k)).$$

2. Preliminaries

For $y \in L[0,1]$, let's consider the following problem:

$$\begin{cases} u''(t) + y(t) = 0, \quad t \in (0,1) \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\cdots,m, \\ \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1,2,\cdots,m, \\ u(0) = \alpha u(\xi), \quad u'(1) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 Let $u \geq 0$. Then u is a solution of the problem (2.1) if and only if it satisfies

$$u(t) = \int_0^1 G(t,s)y(s)ds + \omega_0(t,u), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{1-\alpha}\begin{cases} s, & s < \xi, s < t, \\ \alpha s + (1-\alpha)t, & t \leq s \leq \xi, \\ \alpha\xi + (1-\alpha)s, & \xi \leq s \leq t, \\ \alpha\xi + (1-\alpha)t, & \xi < s, t < s, \end{cases}$$

$\omega_0(t,u)$ is the same as in condition (C_4).

Proof. Let u be a solution of the problem (2.1), then

$$u''(t) = -y(t). \tag{2.3}$$

For $t \in (0,t_1]$, integrating (2.3) from 0 to t, we have

$$u'(t) = c_1 - \int_0^t y(s)ds,$$

$$u(t) = c_2 + c_1 t - \int_0^t (t-s)y(s)ds.$$

So, we have

$$u(t_1^-) = c_1 t_1 - \int_0^{t_1}(t_1 - s)y(s)ds + c_2, \tag{2.4}$$

$$u'(t_1^-) = c_1 - \int_0^{t_1} y(s))ds. \tag{2.5}$$

For $t \in (t_1, t_2]$, integrating (2.3) from t_1 to t, we have

$$u(t) = b_2 + b_1(t - t_1) - \int_{t_1}^{t}(t - s)y(s)ds. \tag{2.6}$$

By (2.1), (2.4), (2.5) and (2.6), we have

$$b_2 = I_1(u(t_1)) + c_1 t_1 - \int_0^{t_1}(t_1 - s)y(s)ds + c_2,$$

$$b_1 = J_1(u(t_1)) + c_1 - \int_0^{t_1} y(s)ds.$$

Thus,

$$u(t) = I_1(u(t_1)) + c_1 t - \int_0^{t}(t - s)y(s)ds + J_1(u(t_1))(t - t_1) + c_2.$$

For $t \in (t_k, t_{k+1}]$, by the same way, we can get

$$u(t) = c_1 t + c_2 - \int_0^{t}(t - s)y(s)ds + \sum_{i=1}^{k}(t - t_i)J_i(u(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)). \tag{2.7}$$

By $u'(1) = 0$ and (2.7), we have

$$c_1 = \int_0^1 y(s)ds - \sum_{i=1}^{m} J_i(u(t_i)).$$

It follows from (2.7) and $u(0) = \alpha u(\xi)$ that

$$\begin{aligned} c_2 = {} & \frac{\alpha}{1-\alpha}[\xi \int_0^1 y(s)ds - \int_0^{\xi}(\xi - s)y(s)ds - \sum_{k=1}^{m} \xi J_k(u(t_k)) + \sum_{t_k<\xi}(\xi - t_k)J_k(u(t_k)) \\ & + \sum_{t_k<\xi} I_k(u(t_k))]. \end{aligned}$$

So, we get

$$\begin{aligned} u(t) = {} & \int_0^1 ty(s)ds + \frac{\alpha\xi}{1-\alpha}\int_0^1 y(s)ds - \frac{\alpha}{1-\alpha}\int_0^{\xi}(\xi - s)y(s)ds - \int_0^{t}(t - s)y(s)ds \\ & + \frac{\alpha}{1-\alpha}\sum_{t_k<\xi}[I_k(u(t_k)) + (\xi - t_k)J_k(u(t_k))] + \sum_{t_k<t}\left[I_k(u(t_k)) - \frac{\alpha\xi + (1-\alpha)t_k}{1-\alpha}J_k(u(t_k))\right] \\ & - \sum_{t\le t_k}\frac{\alpha\xi + (1-\alpha)t}{1-\alpha}J_k(u(t_k)) \\ = {} & \int_0^1 ty(s)ds + \frac{\alpha\xi}{1-\alpha}\int_0^1 y(s)ds - \frac{\alpha}{1-\alpha}\int_0^{\xi}(\xi - s)y(s)ds - \int_0^{t}(t - s)y(s)ds + \omega_0(t, u). \end{aligned}$$

For $t \le \xi$, we obtain

$$u(t) = \int_0^{t}\frac{s}{1-\alpha}y(s)ds + \int_t^{\xi}\frac{\alpha s + (1-\alpha)t}{1-\alpha}y(s)ds + \int_{\xi}^{1}\frac{\alpha\xi + (1-\alpha)t}{1-\alpha}y(s)ds + \omega_0(t, u).$$

For $t \geq \xi$, we have

$$u(t) = \int_0^\xi \frac{s}{1-\alpha} y(s)ds + \int_\xi^t \frac{\alpha\xi + (1-\alpha)s}{1-\alpha} y(s)ds + \int_t^1 \frac{\alpha\xi + (1-\alpha)t}{1-\alpha} y(s)ds + \omega_0(t,u).$$

So, we get

$$u(t) = \int_0^1 G(t,s)y(s)ds + \omega_0(t,u).$$

Conversely, if $u(t)$ satisfies (2.2), it's easy to get that $u(t)$ is a solution of (2.1). □

Lemma 2.2. The function $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and it satisfies

$$\rho_0 g(s) \leq G(t,s) \leq g(s), \quad t,s \in [0,1],$$

where $g(s) = \dfrac{s}{1-\alpha}, \quad \rho_0 = \alpha\xi.$

Proof. The proof of this lemma is easy. So, we omit it. □

Now we define a cone P on $PC[0,1]$ and an operator $T : P \to PC[0,1]$ as follows:

$$P = \{u \in PC[0,1] : u(t) \geq 0, \inf_{t\in[0,1]} u(t) \geq \rho\|u\|\}, \text{ where } \rho = \min\{c_0,\ \rho_0\}.$$

$$Tu(t) = \int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u).$$

Obviously, if $u \in P$ is a fixed point of T, it is a solution of the problem (1.1).

Lemma 2.3. Assume $(C_1) - (C_4)$ hold. Then $T : P \to P$ is a completely continuous operator.

Proof. By (C_1), (C_2) and (C_4), we have $Tu(t) \geq 0,\ u \in P$. By (C_4) and Lemma 2.2, we can get

$$\begin{aligned} |Tu(t)| &= |\int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u)| \\ &\leq \int_0^1 g(s)\phi(s)f(s,u(s))ds + \Omega(u), \end{aligned}$$

and

$$\begin{aligned} \inf_{t\in[0,1]} Tu(t) &= \inf_{t\in[0,1]} \left[\int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u)\right] \\ &\geq \rho_0 \int_0^1 g(s)\phi(s)f(s,u(s))ds + c_0\Omega(u) \\ &\geq \rho\|Tu\|. \end{aligned}$$

This shows that $T : P \to P$. By the continuity of $f,\ I_k,\ J_k,\ k = 1,2,\cdots,m$, we can easily obtain that $T : P \to P$ is continuous. Let $S \subset P$ be bounded. Obviously, $T(S) \subset P$ is bounded. For $u \in S,\ t,t' \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &|Tu(t) - Tu(t')| \leq \textstyle\int_0^1 |G(t,s) - G(t',s)|\phi(s)f(s,u(s))ds + |\omega_0(t,u) - \omega_0(t',u)| \\ &\leq \textstyle\int_0^1 |G(t,s) - G(t',s)|\phi(s)f(s,u(s))ds + |t-t'| \sum\limits_{k=1}^m |J_k(u(t_k))|. \end{aligned}$$

By (C_1), the uniform continuity of G on $[0,1]\times[0,1]$, the boundedness of f on $[0,1]\times S$ and the boundedness of J_k on S, we obtain that $T(S)$ is quasi-equicontinuous on [0,1]. By [1], T is a compact map. So, $T: P \to P$ is completely continuous. $\square$

In order to obtain our main results, we need the following definitions and theorem.

Definition 2.1. A map ϕ is said to be a non-negative, continuous and concave functional on a cone P of a real Banach space E iff $\phi: P \to R_+$ is continuous and

$$\phi(tx + (1-t)y) \geq t\phi(x) + (1-t)\phi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.2. A map Φ is said to be a non-negative, continuous and convex functional on a cone P of a real Banach space E iff $\Phi: P \to R_+$is continuous and

$$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Let φ and Θ be non-negative, continuous and convex functional on P, Φ be a non-negative, continuous and concave functional on P, and Ψ be a non-negative continuous functional on P. Then, for positive numbers a, b, c and d, we define the following sets:

$$P(\varphi, d) = \{x \in P : \varphi(x) < d\},$$

$$P(\varphi, \Phi, b, d) = \{x \in P : b \leq \Phi(x), \varphi(x) \leq d\},$$

$$P(\varphi, \Theta, \Phi, b, c, d) = \{x \in P : b \leq \Phi(x), \Theta(x) \leq c, \varphi(x) \leq d\},$$

$$R(\varphi, \Psi, a, d) = \{x \in P : a \leq \Psi(x), \varphi(x) \leq d\}.$$

We will use the following fixed point theorem of Avery and Peterson to study the problem (1.1), (2.1).

Theorem 2.1[25]. Let P be a cone in a real Banach space E. Let φ and Θ be non-negative, continuous and convex functionals on P, Φ be a non-negative, continuous and concave functional on P, and Ψ be a non-negative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d,

$$\Phi(x) \leq \Psi(x) \text{ and } \|x\| \leq M\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Suppose that

$$T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that the following conditions are satisfied:

(S_1) $\{x \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(x) > b\} \neq \emptyset$ and $\Phi(Tx) > b$ for $x \in P(\varphi, \Theta, \Phi, b, c, d)$;
(S_2) $\Phi(Tx) > b$ for $x \in P(\varphi, \Phi, b, d)$ with $\Theta(Tx) > c$;
(S_3) $0 \notin R(\varphi, \Psi, a, d)$ and $\Psi(Tx) < a$ for $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = a$.

Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\varphi, d)}$, such that

$$\varphi(x_i) \leq d, \text{ for } i = 1, 2, 3,$$

and

$$b < \Phi(x_1),\ a < \Psi(x_2),\ \Phi(x_2) < b,$$
$$\Psi(x_3) < a.$$

3. Main results

We define a concave function $\Phi(x) = \inf\limits_{t\in[0,1]} |x(t)|$ and convex functions $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Theorem 3.1. Suppose $(C_1) - (C_4)$ hold. In additions, we assume that there exist positive constants $\mu,\ L,\ a,\ b,\ c,\ d$ with $a < b < \dfrac{b}{\rho} = c < d$, $\mu > D_1 + D_2,\ 0 < L < \rho(D_1 + D_3)$, where $D_1 = \int_0^1 g(s)\phi(s)ds,\ D_2,\ D_3 \geq 0$, such that the following conditions hold:

(A_1) $f(t, u) \leq \dfrac{d}{\mu}$, for $(t, u) \in [0, 1]\times[0, d]$, and $\omega_0(t, u) \leq \dfrac{D_2}{\mu}d$, for $u \in P,\ \|u\| \leq d$;

(A_2) $f(t, u) \geq \dfrac{b}{L}$, for $(t, u) \in [0, 1] \times \left[b, \dfrac{b}{\rho}\right]$, and $\omega_0(t, u) \geq \dfrac{D_3}{L}b$, for $u \in P,\ b \leq u(t) \leq \dfrac{b}{\rho},\ t \in [0, 1]$;

(A_3) $f(t, u) \leq \dfrac{a}{\mu}$, for $(t, u) \in [0, 1]\times[0, a]$, and $\omega_0(t, u) \leq \dfrac{D_2}{\mu}a$, for $u \in P,\ \|u\| \leq a$.

Then the problem (1.1) has at least two positive solutions when $f(t, 0) \equiv 0,\ t \in [0, 1]$ and at least three positive solutions when $f(t, 0) \not\equiv 0,\ t \in [0, 1]$.

Proof. Take $u \in \overline{P(\varphi, d)}$. By assumption (A_1), we have

$$\begin{aligned}\varphi(Tu) = \|Tu\| &\leq \int_0^1 g(s)\phi(s)f(s, u(s))ds + \frac{D_2}{\mu}d \\ &\leq \frac{d}{\mu}\int_0^1 g(s)\phi(s)ds + \frac{D_2}{\mu}d = \frac{D_1}{\mu}d + \frac{D_2}{\mu}d < d.\end{aligned}$$

Thus, $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

Let's prove that condition S_1 holds.

Take $u(t) = \dfrac{b(\rho + 1)}{2\rho}$, $t \in [0, 1]$. By simple calculation, we can get that

$$\|u\| = \frac{b(\rho + 1)}{2\rho} < \frac{b}{\rho} = c,$$

and

$$\Phi(u) = \inf_{t\in[0,1]} |u(t)| = \frac{b(\rho+1)}{2\rho} > b.$$

Therefore,

$$\{u \in P(\varphi, \Theta, \Phi, b, c, d) : b < \Phi(u)\} \neq \emptyset.$$

$u \in P(\varphi, \Theta, \Phi, b, c, d)$ means that $b \leq u(t) \leq \dfrac{b}{\rho},\ t \in [0,1]$. By (A_2), we get

$$\Phi(Tu) = \inf_{t\in[0,1]} |Tu(t)| \geq \rho\left[\int_0^1 g(s)\phi(s)f(s,u(s))ds + \frac{b}{L}D_3\right] \geq \rho\frac{b}{L}(D_1 + D_3) > b.$$

So, condition S_1 holds.

Now we will show that condition S_2 holds.

Take $u \in P(\varphi, \Phi, b, d)$ and $\|Tu\| > \dfrac{b}{\rho} = c$. Considering $Tu \in P$, we get

$$\Phi(Tu) = \inf_{t\in[0,1]} |Tu(t)| \geq \rho\|Tu\| > \rho \cdot \frac{b}{\rho} = b,$$

This shows that condition S_2 is satisfied.

In the following we will show that the condition S_3 is satisfied. Since $\Psi(0) = 0,\ 0 < a,\ 0 \notin R(\varphi, \Psi, a, d)$. Assume that $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = \|u\| = a$. Then, by (A_3), we have

$$\Psi(Tu) = \|Tu(t)\| \leq \int_0^1 g(s)\phi(s)f(s,u(s))ds + \frac{a}{\mu}D_2 \leq \frac{a}{\mu}(D_1 + D_2) < a.$$

Thus, condition S_3 is satisfied. By Theorem 2.1, we get that the problem (1.1) has at least three solutions $u_1, u_2, u_3 \in P$ satisfying

$$\|u_i\| \leq d,\ i = 1,2,3\ ,\ \text{and } b < \inf_{t\in[0,1]} |u_1(t)|,$$

$$a \leq \|u_2\|,\ \inf_{t\in[0,1]} |u_2(t)| < b,\ \|u_3\| < a.$$

Obviously, $u_1(t) > 0,\ u_2(t) > 0,\ t \in [0,1]$. If $f(t,0) \not\equiv 0,\ t \in [0,1]$, then $u = 0$ is not a solution of (1.1). So, $u_3 \neq 0$. This, together with $u_3 \in P$, means that $u_3(t) > 0,\ t \in [0,1]$. $\square$

Example 3.1. Consider the following boundary value problem

$$\begin{cases} u''(t) + f(t,u(t)) = 0,\ t \in (0,1) \setminus \{\frac{1}{8}\}, \\ \Delta u(\frac{1}{8}) = I_1(u(\frac{1}{8})), \\ \Delta u'(\frac{1}{8}) = J_1(u(\frac{1}{8})), \\ u(0) = \frac{1}{4}u(\frac{1}{4}),\quad u'(1) = 0, \end{cases} \tag{3.1}$$

where

$$f(t,u)=\begin{cases}\frac{1}{4}u^2t, & t\in[0,1],\ u\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}u^2t(1-u)+(60+2\sqrt{u}t)(u-\frac{1}{2}), & t\in[0,1],\ u\in[\frac{1}{2},1],\\ 30+\sqrt{u}t, & t\in[0,1],\ u\in[1,16],\\ 30+4t, & t\in[0,1],\ u\in[16,\infty).\end{cases}$$

Corresponding to Theorem 3.1, we take $\alpha=\xi=\frac{1}{4}, c_0=\frac{1}{6}, \rho=\frac{1}{16}, \mu=2, D_1=\int_0^1 g(s)ds=\frac{2}{3}, D_2=\frac{1}{3}, D_3=0, L=\frac{1}{30}, I_1(\omega)=\frac{1}{64}\sqrt{\omega}, J_1(\omega)=\frac{-\sqrt{\omega}}{64}, \Omega(u)=\frac{3\sqrt{u(\frac{1}{8})}}{128}$, and

$$\omega_0(t,u)=\begin{cases}\frac{3\sqrt{u(\frac{1}{8})}}{128}, & t>\frac{1}{8},\\ (\frac{3}{8}+t)\frac{1}{64}\sqrt{u(\frac{1}{8})}, & t\le\frac{1}{8}.\end{cases}$$

It is easy to check that $\frac{1}{6}\Omega(u)\le\omega_0(t,u)\le\Omega(u)$. Let $a=\frac{1}{2}, b=1, d=68$. By simple calculation, we can get that the conditions of Theorem 3.1 are satisfied. So, the problem (3.1) has at least three solutions $u_1,u_2,u_3\in P$ satisfying

$$\|u_i\|\le 68,\ i=1,2,3\ ,$$

and

$$1<\Phi(u_1),\ \frac{1}{2}<\|u_2\|,\ \Phi(u_2)<1,\ \|u_3\|<\frac{1}{2},$$

where u_1,u_2 are positive solutions of (3.1).

Remark. Corresponding to the condition (C_3) in [26], we get $(d_1I+e_1N)(\omega)=\frac{9}{512}\sqrt{\omega},\ (d_2I+e_2N)(\omega)=\frac{1}{64}\sqrt{\omega}$. The problem (3.1) cannot be solved by the Theorems in [26] because the condition (C_3) in [26] is not satisfied. So, our result may be considered as a complementary result of [26].

Acknowledgments. The authors are grateful to editor and anonymous referees for their constructive comments and suggestions which led to improvement of the original manuscript.

References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of impulsive ordinary differential equations, World Scientific, Singapore, 1989.

[2] D. Guo, J. Sun, Z. Liu, Functional Methods of Ordinary Differential equation[M], Ji'nan: Shandong Science and Technology Press, 1995.

[3] L. Hu, L. Liu, Y. Wu, Positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations, Appl. Math. Comput. 196(2008) 550-562.

[4] D. Guo, Multiple positive solutions of a boundary value problem for nth-order impulsive integro-differential equations in Banach spaces, Nonlinear Anal. 63(2005) 618-641.

[5] L.H. Erbe, W. Krawcewicz, Existence of solutions to boundary value problems for impulsive second order differential inclusions, Rocky Mountain J. Math. 22(1992) 1-20.

[6] R.P. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput. 114(2000) 51-59.

[7] R.P. Agarwal, D. O'Regan, A Multiplicity results for second order impulsive differential equations via the Leggett Williams fixed point theorem, Appl. Math. Comput. 161(2005) 433-439.

[8] R.P. Agarwal, D. O'Regan, Existence of triple solutions to integral and discrete equations via the Leggett Williams fixed point theorem, Rocky Mountain J. Math. 31(2001) 23-35.

[9] D. Anderson, R.I. Avery, A.C. Peterson, Three positive solutions to a discrete focal boundary value problem, J. Comput. Appl. Math. 88(1998) 103-118.

[10] P.W. Eloe, J. Henderson, Positive solutions of boundary value problems for ordinary differential equations with impulsive, Dyn. Contin. Discrete Impuls. Syst. 4(1998) 285-294.

[11] D. Guo, X. Liu, Multiple positive solutions of boundary-value problems for impulsive differential equations, Nonlinear Anal. 25(1995) 327-337.

[12] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press Inc., New York, 1998.

[13] T. Jankowski, First-order impulsive ordinary differential equations with advanced arguments, J. Math. Anal. Appl. 331(2007) 1-12.

[14] T. Jankowski, Existence of solutions for second order impulsive differential equations with deviating arguments, Nonlinear Anal. 67(2007) 1764-1774.

[15] D. Jiang, Multiple positive solutions for boundary value problems of second order delay differential equations, Appl. Math. Lett. 15(2002) 575-583.

[16] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone iterative techniques for nonlinear differential equations, Pitman, Boston, 1985.

[17] D. Guo, A class of second-order impulsive integuo-differential equations on unbounded domain in a Banach space, Appl. Math. Comput. 125(2002) 59-77.

[18] A.M. Samoilenko, N.A. Perestyuk, Impulsive differential equations, World Scientific, Singapore, 1995.

[19] E. Lee, Y. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equations, Appl. Math. Comput. 158(2004) 745-759.

[20] R. Ma, Positive solutions for nonlinear three-point boundary-value problem, Electron. J. Differential. Equations. 34(1999) 1-8.

[21] R. Ma, Multiplicity of positive solutions for second-order three-point boundary-value problems, Comput. Math. Appl. 40(2000) 193-204.

[22] T. Jankowski, Positive solutions to second order four-point boundary value problems for impulsive differential equations, Appl. Math. Comput. 202(2008) 550-561.

[23] T. Jankowski, Positive solutions of three-point boundary value problems for second order impulsive differential equations with advanced arguments, Appl. Math. Comput. 197(2008) 179-189.

[24] T. Jankowski, Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. Nonlinear Anal. 74(2011) 3775-3785.

[25] R.I. Avery, A.C. Peterson, Three positive fixed points of nonlinear operators on order Banach spaces, Comput. Math. Appl. 42(2001) 313-322.

[26] G. Infante, P. Pietramala and M. Zima, Positive solutions for a class of nonlinear impulsive BVPs via fixed point index, Topological Methods in Nonlinear Anal. 36 (2010), 263-284.

(Received March 30, 2012)