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ON A SYSTEM OF HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. We investigate the existence and nonexistence of positive solutions for a system of nonlinear higher-order ordinary differential equations subject to some multi-point boundary conditions.

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1 Introduction

We consider the system of nonlinear higher-order ordinary differential equations

$$(S) \qquad \begin{cases} u^{(n)}(t) + c(t)f(v(t)) = 0, \;\; t \in (0,T), \\ v^{(m)}(t) + d(t)g(u(t)) = 0, \;\; t \in (0,T), \end{cases}$$

with the multi-point boundary conditions

$$(BC) \qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i) + a_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(T) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i) + b_0, \end{cases}$$

where n, m, p, $q \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $p \geq 3$, $q \geq 3$ and $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$.

By using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem $(S)-(BC)$. By a positive solution of $(S)-(BC)$ we mean a pair of functions $(u,v) \in C^n([0,T];\mathbb{R}_+)\times C^m([0,T];\mathbb{R}_+)$ satisfying (S) and (BC) with $u(t) > 0$, $v(t) > 0$ for all $t \in (0,T]$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem.

Multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by J. Henderson, R. Luca, S. K. Ntouyas and I. K. Purnaras, by using the Guo-Krasnosel'skii fixed point theorem. Namely, in [2], the authors give sufficient conditions for λ, μ, f and g such that the system of differential equations

$$(S_1) \qquad \begin{cases} u^{(n)}(t) + \lambda c(t) f(u(t),v(t)) = 0, & t \in (0,T), \\ v^{(m)}(t) + \mu d(t) g(u(t),v(t)) = 0, & t \in (0,T), \end{cases}$$

with the boundary conditions (BC) with $a_0 = b_0 = 0$ (denoted by (BC_1)) has positive solutions. The system (S_1) with $f(u,v) = \widetilde{f}(v)$, $g(u,v) = \widetilde{g}(u)$ and $n = m$ (denoted by $(\widetilde{S}_1)$) with the boundary conditions (BC_1) where $n = m$, $p = q$, $a_i = b_i$, $\xi_i = \eta_i$ for $i = 1,\ldots,p-2$, has been studied in [19]. In [6], the authors studied the system $(\widetilde{S}_1)$ with $T = 1$ and the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$, $v(1) = \alpha v(\eta)$, where $0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$. We also mention the paper [23], where the authors used fixed point index theory to prove the existence of positive solutions for the system (S_1) with $\lambda = \mu = 1$ and (BC_1), where $\frac{1}{2} \le \xi_1 < \cdots < \xi_{p-2} < 1$, $\frac{1}{2} \le \eta_1 < \cdots < \eta_{q-2} < 1$. For multi-point boundary value problems for nonlinear higher-order ordinary differential equations we mention the papers [1], [15].

The systems (S) and (S_1) with $n = m = 2$ subject to various boundary conditions were studied in [3], [7], [8], [10], [11], [17], [20]. Some discrete versions of these nonlinear second-order boundary value problems have been investigated in [4], [5], [9], [12], [18], [21].

Our results obtained in this paper were inspired by the paper [16], where the authors studied the existence and nonexistence of positive solutions for the m-point boundary value problem on time scales

$$\begin{cases} u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \ \ t \in (0,T), \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \ \ u(T) - \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) = b, \ \ m \ge 3, \ \ b > 0, \end{cases}$$

where in this case $(0, T)$ denotes a time scale interval.

Multi-point boundary value problems for ordinary differential equations or finite difference equations have applications in a variety of different areas of applied mathematics and physics. For example the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [22]); also many problems in the theory of elastic stability can be handled as multi-point problems (see [24]). The study of multi-point boundary value problems for second order differential equations was initiated by Il'in and Moiseev (see [13], [14]). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors, by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for a n-th order differential equation (problem $(1)-(2)$ below). In Section 3, we shall prove our main results, and in Section 4, we shall present a simple example which illustrate the obtained results.

2 Auxiliary results

In this section, we shall present some auxiliary results from [15] and [19] related to the following n-th order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \;\; t \in (0, T), \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i). \tag{2}$$

Lemma 2.1 *([15], [19]) If $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \neq 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $y \in C([0, T])$, then the solution of (1)-(2) is given by*

$$u(t) = \frac{t^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1} y(s)\,ds - \frac{t^{n-1}}{d(n-1)!}\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\,ds$$
$$- \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y(s)\,ds, \;\; 0 \le t \le T.$$

Lemma 2.2 *([15], [19]) Under the assumptions of Lemma 2.1, Green's function for the boundary value problem (1)-(2) is given by*

$$G_1(t,s)$$
$$= \begin{cases} \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{p-2} a_i(\xi_i - s)^{n-1}\right] - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \\ \qquad if \ \ \xi_j \le s < \xi_{j+1}, \ \ s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{p-2} a_i(\xi_i - s)^{n-1}\right], \\ \qquad if \ \ \xi_j \le s < \xi_{j+1}, \ \ s \ge t, \ \ j = 0, \ldots, p-3, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1} - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \ \ if \ \ \xi_{p-2} \le s \le T, \ \ s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}, \ \ if \ \ \xi_{p-2} \le s \le T, \ \ s \ge t, \ \ (\xi_0 = 0). \end{cases}$$

Using the above Green's function the solution of problem (1)-(2) is expressed as $u(t) = \displaystyle\int_0^T G_1(t,s)y(s)\,ds$.

Lemma 2.3 *([15], [19]) If $a_i > 0$ for all $i = 1, \ldots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$ and $y \in C([0,T])$, $y(t) \ge 0$ for all $t \in [0,T]$, then the solution u of problem (1)-(2) satisfies $u(t) \ge 0$ for all $t \in [0,T]$.*

Lemma 2.4 *([19]) If $a_i > 0$ for all $i = 1, \ldots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$, $y(t) \ge 0$ for all $t \in [0,T]$, then the solution of problem (1)-(2) satisfies*

$$\begin{cases} u(t) \le \dfrac{T^{n-1}}{d(n-1)!}\displaystyle\int_0^T (T-s)^{n-1}y(s)\,ds, \ \ \forall t \in [0,T], \\ u(\xi_j) \ge \dfrac{\xi_j^{n-1}}{d(n-1)!}\displaystyle\int_{\xi_{p-2}}^T (T-s)^{n-1}y(s)\,ds, \ \ \forall\, j = \overline{1, p-2}. \end{cases}$$

Lemma 2.5 *([15]) Assume that $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i > 0$ for all $i = 1, \ldots, p-2$, $d > 0$ and $y \in C([0,T])$, $y(t) \ge 0$ for all $t \in [0,T]$. Then the solution of problem (1)-(2) satisfies $\inf_{t \in [\xi_{p-2}, T]} u(t) \ge \gamma_1 \|u\|$, where*

$$\gamma_1 = \begin{cases} \min\left\{\dfrac{a_{p-2}(T-\xi_{p-2})}{T-a_{p-2}\xi_{p-2}}, \dfrac{a_{p-2}\xi_{p-2}^{n-1}}{T^{n-1}}\right\}, \ \ if \ \ \displaystyle\sum_{i=1}^{p-2} a_i < 1, \\ \min\left\{\dfrac{a_1\xi_1^{n-1}}{T^{n-1}}, \dfrac{\xi_{p-2}^{n-1}}{T^{n-1}}\right\}, \ \ if \ \ \displaystyle\sum_{i=1}^{p-2} a_i \geq 1. \end{cases}$$

We can also formulate similar results as Lemma 2.1 - Lemma 2.5 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \ \ t \in (0,T), \tag{3}$$

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i). \tag{4}$$

If $e = T^{m-1} - \displaystyle\sum_{i=1}^{q-2} b_i\eta_i^{m-1} \neq 0$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$ and $h \in C([0,T])$, we denote by G_2 Green's function corresponding to problem (3)-(4), that is

$$G_2(t,s)$$
$$= \begin{cases} \dfrac{t^{m-1}}{e(m-1)!}\left[(T-s)^{m-1} - \displaystyle\sum_{i=j+1}^{q-2} b_i(\eta_i-s)^{m-1}\right] - \dfrac{1}{(m-1)!}(t-s)^{m-1}, \\ \qquad\qquad if \ \ \eta_j \leq s < \eta_{j+1}, \ \ s \leq t, \\ \dfrac{t^{m-1}}{e(m-1)!}\left[(T-s)^{m-1} - \displaystyle\sum_{i=j+1}^{q-2} b_i(\eta_i-s)^{m-1}\right], \\ \qquad\qquad if \ \ \eta_j \leq s < \eta_{j+1}, \ \ s \geq t, \ \ j = 0, \ldots q-3, \\ \dfrac{t^{m-1}}{e(m-1)!}(T-s)^{m-1} - \dfrac{1}{(m-1)!}(t-s)^{m-1}, \ \ if \ \ \eta_{q-2} \leq s \leq T, \ \ s \leq t, \\ \dfrac{t^{m-1}}{e(m-1)!}(T-s)^{m-1}, \ \ if \ \ \eta_{q-2} \leq s \leq T, \ \ s \geq t, \ \ (\eta_0 = 0). \end{cases}$$

Under similar assumptions as those from Lemma 2.5, we have the inequality $\inf\limits_{t\in[\eta_{q-2},T]} v(t) \geq \gamma_2\|v\|$, where v is the solution of problem (3)-(4) and γ_2 is given by

$$\gamma_2 = \begin{cases} \min\left\{\dfrac{b_{q-2}(T-\eta_{q-2})}{T-b_{q-2}\eta_{q-2}}, \dfrac{b_{q-2}\eta_{q-2}^{m-1}}{T^{m-1}}\right\}, & if \ \sum_{i=1}^{q-2} b_i < 1, \\ \min\left\{\dfrac{b_1\eta_1^{m-1}}{T^{m-1}}, \dfrac{\eta_{q-2}^{m-1}}{T^{m-1}}\right\}, & if \ \sum_{i=1}^{q-2} b_i \geq 1. \end{cases}$$

3 Main results

We present the assumptions that we shall use in the sequel:

$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $a_i > 0$, $i = 1, \ldots, p-2$, $b_i > 0$, $i = 1, \ldots, q-2$, $d = T^{n-1} - \sum_{i=1}^{p-2} a_i\xi_i^{n-1} > 0$, $e = T^{m-1} - \sum_{i=1}^{q-2} b_i\eta_i^{m-1} > 0$.

$(H2)$ The functions c, $d : [0,T] \to [0,\infty)$ are continuous and there exist t_0, $\widetilde{t}_0 \in [\theta_0, T]$ such that $c(t_0) > 0$, $d(\widetilde{t}_0) > 0$, where $\theta_0 = \max\{\xi_{p-2}, \eta_{q-2}\}$.

$(H3)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and there exists $c_0 > 0$ such that $f(u) < \dfrac{c_0}{L}$, $g(u) < \dfrac{c_0}{L}$ for all $u \in [0, c_0]$, where

$$L = \max\left\{\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)\,ds,\ \frac{T^{m-1}}{e(m-1)!}\int_0^T (T-s)^{m-1}d(s)\,ds\right\}.$$

$(H4)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and satisfy the conditions $\lim\limits_{u\to\infty} \dfrac{f(u)}{u} = \infty$, $\lim\limits_{u\to\infty} \dfrac{g(u)}{u} = \infty$.

First we present an existence result for the positive solutions of $(S) - (BC)$.

Theorem 3.1 *Assume that the assumptions* $(H1), (H2)$ *and* $(H3)$ *hold. Then the problem* $(S)-(BC)$ *has at least one positive solution for* $a_0 > 0$ *and* $b_0 > 0$ *sufficiently small.*

Proof. We consider the problems

$$\begin{cases} h^{(n)}(t) = 0, \quad t \in (0,T), \\ h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0, \quad h(T) = \sum_{i=1}^{p-2} a_i h(\xi_i) + 1, \end{cases} \tag{5}$$

$$\begin{cases} w^{(m)}(t) = 0, \ \ t \in (0,T), \\ w(0) = w'(0) = \cdots = w^{(m-2)}(0) = 0, \ \ w(T) = \displaystyle\sum_{i=1}^{q-2} b_i w(\eta_i) + 1. \end{cases} \tag{6}$$

The above problems (5) and (6) have the solutions

$$h(t) = \frac{t^{n-1}}{d}, \quad w(t) = \frac{t^{m-1}}{e}, \ \ t \in [0,T]. \tag{7}$$

We define the functions $x(t)$ and $y(t), \ t \in [0,T]$ by

$$x(t) = u(t) - a_0 h(t), \ \ y(t) = v(t) - b_0 w(t), \ \ t \in [0,T],$$

where (u,v) is solution of $(S)-(BC)$. Then $(S)-(BC)$ can be equivalently written as

$$\begin{cases} x^{(n)}(t) + c(t) f(y(t) + b_0 w(t)) = 0, \ \ t \in (0,T), \\ y^{(m)}(t) + d(t) g(x(t) + a_0 h(t)) = 0, \ \ t \in (0,T), \end{cases} \tag{8}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ x(T) = \displaystyle\sum_{i=1}^{p-2} a_i x(\xi_i), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0), \ \ y(T) = \displaystyle\sum_{i=1}^{q-2} b_i y(\eta_i). \end{cases} \tag{9}$$

Using the Green's functions given in Section 2, a pair (x,y) is a solution of the problem (8)-(9) if and only if (x,y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \displaystyle\int_0^T G_1(t,s) c(s) f\left(\int_0^T G_2(s,\tau) d(\tau) g(x(\tau) + a_0 h(\tau))\, d\tau + b_0 w(s) \right) ds, \\ y(t) = \displaystyle\int_0^T G_2(t,s) d(s) g(x(s) + a_0 h(s))\, ds, \ \ 0 \le t \le T, \end{cases} \tag{10}$$

where $h(t), \ w(t), \ t \in [0,T]$ are given by (7).

We consider the Banach space $X = C([0,T])$ with the supremum norm $\|\cdot\|$ and define the set

$$K = \{x \in C([0,T]), \ \ 0 \le x(t) \le c_0, \ \ \forall\, t \in [0,T]\} \subset X.$$

We also define the operator $\mathcal{A}: K \to X$ by

$$\mathcal{A}(x)(t) = \int_0^T G_1(t,s)c(s)f\left(\int_0^T G_2(s,\tau)d(\tau)g(x(\tau)+a_0h(\tau))d\tau \right.$$
$$\left. +b_0w(s)\right) ds, \;\; 0 \le t \le T, \;\; x \in K.$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(H3)$, we deduce

$$f(y(t)+b_0w(t)) \le \frac{c_0}{L}, \;\; g(x(t)+a_0h(t)) \le \frac{c_0}{L}, \;\; \forall\, t \in [0,T], \;\; \forall\, x,\, y \in K.$$

Then, by using Lemma 2.3, we obtain $\mathcal{A}(x)(t) \ge 0$ for all $t \in [0,T]$ and $x \in K$. By Lemma 2.4, for all $x \in K$, we have

$$\int_0^T G_2(s,\tau)d(\tau)g(x(\tau)+a_0h(\tau))\, d\tau \le \frac{T^{m-1}}{e(m-1)!}\int_0^T (T-\tau)^{m-1}d(\tau)g(x(\tau)$$
$$+a_0h(\tau))\, d\tau \le \frac{c_0T^{m-1}}{eL(m-1)!}\int_0^T (T-\tau)^{m-1}d(\tau)\, d\tau \le c_0, \;\; \forall\, s \in [0,T],$$

and

$$\mathcal{A}(x)(t) \le \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)f\left(\int_0^T G_2(s,\tau)d(\tau)g(x(\tau)+a_0h(\tau))d\tau \right.$$
$$\left. +b_0w(s)\right) ds \le \frac{c_0T^{n-1}}{dL(n-1)!}\int_0^T (T-s)^{n-1}c(s)ds \le c_0, \;\; \forall\, t \in [0,T].$$

Therefore $\mathcal{A}(K) \subset K$.

Using standard arguments, we deduce that $\mathcal{A}$ is completely continuous ($\mathcal{A}$ is compact, that is for any bounded set $B \subset K$, $\mathcal{A}(B) \subset K$ is relatively compact by Arzèla-Ascoli theorem, and $\mathcal{A}$ is continuous). By the Schauder fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $x \in K$. This element together with y given by $y(t) = \int_0^T G_2(t,s)d(s)g(x(s)+a_0h(s))\, ds, \;\; t \in [0,T]$ represents a solution for (8)-(9). This shows that our problem $(S)-(BC)$ has a positive solution $u = x + a_0h, \;\; v = y + b_0w$ for sufficiently small a_0 and b_0. □

In what follows, we present sufficient conditions for the nonexistence of the positive solutions of $(S)-(BC)$.

Theorem 3.2 *Let the assumptions* $(H1)$, $(H2)$ *and* $(H4)$ *be satisfied. Then the problem* $(S)-(BC)$ *has no positive solution for* a_0 *and* b_0 *sufficiently large.*

Proof. We suppose that (u, v) is a positive solution of $(S)-(BC)$. Then $x = u - a_0 h$, $y = v - b_0 w$ is a solution for (8)-(9), where h and w are the solutions of problems (5) and (6) (given by (7)). By Lemma 2.3, we have $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in [0, T]$, and by $(H2)$ we deduce that $\|x\| > 0$, $\|y\| > 0$. Using Lemma 2.5, we also have $\inf_{t\in[\xi_{p-2},T]} x(t) \geq \gamma_1 \|x\|$ and $\inf_{t\in[\eta_{q-2},T]} y(t) \geq \gamma_2 \|y\|$, where γ_1, γ_2 are defined in Section 2.

Using now (7), we deduce that $\inf_{t\in[\xi_{p-2},T]} h(t) = \xi_{p-2}^{n-1}/d$. Therefore

$$\inf_{t\in[\xi_{p-2},T]} h(t) = \xi_{p-2}^{n-1}\|h\|/T^{n-1} \geq \gamma_1 \|h\|.$$

In a similar manner we obtain $\inf_{t\in[\eta_{q-2},T]} w(t) \geq \gamma_2 \|w\|$.

Therefore, we obtain

$$\begin{aligned}
\inf_{t\in[\xi_{p-2},T]}(x(t) + a_0 h(t)) &\geq \inf_{t\in[\xi_{p-2},T]} x(t) + a_0 \inf_{t\in[\xi_{p-2},T]} h(t) \geq \gamma_1 \|x + a_0 h\|,\\
\inf_{t\in[\eta_{q-2},T]}(y(t) + b_0 w(t)) &\geq \inf_{t\in[\eta_{q-2},T]} y(t) + b_0 \inf_{t\in[\eta_{q-2},T]} w(t) \geq \gamma_2 \|y + b_0 w\|.
\end{aligned}$$

We now consider

$$\begin{aligned}
R = \Bigg(\min\Bigg\{&\frac{\gamma_2 \xi_{p-2}^{n-1}}{d(n-1)!}\int_{\theta_0}^{T}(T-s)^{n-1}c(s)\,ds,\\
&\frac{\gamma_1 \eta_{q-2}^{m-1}}{e(m-1)!}\int_{\theta_0}^{T}(T-s)^{m-1}d(s)\,ds\Bigg\}\Bigg)^{-1} > 0.
\end{aligned}$$

By $(H4)$, for R defined above, we deduce that there exists $M > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for all $u \geq M$.

We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that

$$\inf_{t\in[\theta_0,T]}(x(t) + a_0 h(t)) \geq M \quad \text{and} \quad \inf_{t\in[\theta_0,T]}(y(t) + b_0 w(t)) \geq M.$$

By using Lemma 2.4 and the above considerations, we have

$$
\begin{aligned}
& y(\eta_{q-2}) \geq \frac{\eta_{q-2}^{m-1}}{e(m-1)!} \int_{\eta_{q-2}}^{T} (T-s)^{m-1} d(s) g(x(s)+a_0 h(s))\, ds \\
& \geq \frac{\eta_{q-2}^{m-1}}{e(m-1)!} \int_{\theta_0}^{T} (T-s)^{m-1} d(s) g(x(s)+a_0 h(s))\, ds \\
& \geq \frac{2R\eta_{q-2}^{m-1}}{e(m-1)!} \int_{\theta_0}^{T} (T-s)^{m-1} d(s)(x(s)+a_0 h(s))\, ds \\
& \geq \frac{2R\eta_{q-2}^{m-1}}{e(m-1)!} \inf_{\tau\in[\theta_0,T]} (x(\tau)+a_0 h(\tau)) \int_{\theta_0}^{T} (T-s)^{m-1} d(s)\, ds \\
& \geq \frac{2R\eta_{q-2}^{m-1}}{e(m-1)!} \inf_{\tau\in[\xi_{p-2},T]} (x(\tau)+a_0 h(\tau)) \int_{\theta_0}^{T} (T-s)^{m-1} d(s)\, ds \\
& \geq \frac{2R\gamma_1\eta_{q-2}^{m-1}}{e(m-1)!} \|x+a_0 h\| \int_{\theta_0}^{T} (T-s)^{m-1} d(s)\, ds \geq 2\|x+a_0 h\| \geq 2\|x\|.
\end{aligned}
$$

Therefore, we obtain

$$
\|x\| \leq y(\eta_{q-2})/2 \leq \|y\|/2. \tag{11}
$$

In a similar manner, we deduce

$$
\begin{aligned}
& x(\xi_{p-2}) \geq \frac{\xi_{p-2}^{n-1}}{d(n-1)!} \int_{\xi_{p-2}}^{T} (T-s)^{n-1} c(s) f(y(s)+b_0 w(s))\, ds \\
& \geq \frac{\xi_{p-2}^{n-1}}{d(n-1)!} \int_{\theta_0}^{T} (T-s)^{n-1} c(s) f(y(s)+b_0 w(s))\, ds \\
& \geq \frac{2R\xi_{p-2}^{n-1}}{d(n-1)!} \int_{\theta_0}^{T} (T-s)^{n-1} c(s)(y(s)+b_0 w(s))\, ds \\
& \geq \frac{2R\xi_{p-2}^{n-1}}{d(n-1)!} \inf_{\tau\in[\theta_0,T]} (y(\tau)+b_0 w(\tau)) \int_{\theta_0}^{T} (T-s)^{n-1} c(s)\, ds \\
& \geq \frac{2R\xi_{p-2}^{n-1}}{d(n-1)!} \inf_{\tau\in[\eta_{q-2},T]} (y(\tau)+b_0 w(\tau)) \int_{\theta_0}^{T} (T-s)^{n-1} c(s)\, ds \\
& \geq \frac{2R\gamma_2\xi_{p-2}^{n-1}}{d(n-1)!} \|y+b_0 w\| \int_{\theta_0}^{T} (T-s)^{n-1} c(s)\, ds \geq 2\|y+b_0 w\| \geq 2\|y\|.
\end{aligned}
$$

So, we obtain

$$
\|y\| \leq x(\xi_{p-2})/2 \leq \|x\|/2. \tag{12}
$$

By (11) and (12), we obtain $\|x\| \leq \|y\|/2 \leq \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution. $\square$

4 An example

We consider $T = 1$, $c(t) = ct$, $d(t) = dt$, $t \in [0, 1]$, c, $d > 0$; $n = 4$, $m = 3$, $p = 4$, $q = 5$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $a_1 = 2$, $a_2 = \frac{1}{2}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = \frac{3}{4}$, $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{3}$. Then $d = \frac{7}{9} > 0$, $e = \frac{5}{8} > 0$.

We also consider the functions f, $g : [0,\infty) \to [0,\infty)$, $f(x) = \dfrac{\widetilde{a}x^\alpha}{x^\beta + \widetilde{c}}$, $g(x) = \dfrac{\widetilde{b}x^\gamma}{x^\delta + \widetilde{d}}$ with $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d} > 0$, α, β, γ, $\delta > 0$, $\alpha > \beta + 1$, $\gamma > \delta + 1$. We have $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{g(x)}{x} = \infty$. The constant L from $(H3)$ is in this case

$$L = \max\left\{\frac{1}{d(n-1)!}\int_0^1 cs(1-s)^{n-1}\,ds, \frac{1}{e(m-1)!}\int_0^1 ds(1-s)^{m-1}\,ds\right\}$$
$$= \max\left\{\frac{3c}{280}, \frac{d}{15}\right\}.$$

We choose $c_0 = 1$ and if we select $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ satisfying the conditions

$$\widetilde{a} < \frac{1+\widetilde{c}}{L} = (1+\widetilde{c})\min\left\{\frac{280}{3c}, \frac{15}{d}\right\},\ \ \widetilde{b} < \frac{1+\widetilde{d}}{L} = (1+\widetilde{d})\min\left\{\frac{280}{3c}, \frac{15}{d}\right\},$$

then we obtain $f(x) \le \frac{\widetilde{a}}{1+\widetilde{c}} < \frac{1}{L}$, $g(x) \le \frac{\widetilde{b}}{1+\widetilde{d}} < \frac{1}{L}$ for all $x \in [0, 1]$.

Thus all the assumptions $(H1) - (H4)$ are satisfied. By Theorem 3.1 and Theorem 3.2 we deduce that the nonlinear second-order differential system

$$\begin{cases} u^{(4)}(t) + ct\dfrac{\widetilde{a}v^\alpha(t)}{v^\beta(t)+\widetilde{c}} = 0, & t \in (0,1), \\ v^{(3)}(t) + dt\dfrac{\widetilde{b}u^\gamma(t)}{u^\delta(t)+\widetilde{d}} = 0, & t \in (0,1), \end{cases}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = u''(0) = 0, \ \ u(1) = 2u(\frac{1}{3}) + \frac{1}{2}u(\frac{2}{3}) + a_0, \\ v(0) = v'(0) = 0, \ \ v(1) = v(\frac{1}{4}) + \frac{1}{2}v(\frac{1}{2}) + \frac{1}{3}v(\frac{3}{4}) + b_0, \end{cases}$$

has at least one positive solution for sufficiently small $a_0 > 0$ and $b_0 > 0$ and no positive solution for sufficiently large a_0 and b_0.

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