Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **2**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

A Neumann problem for a system depending on the unknown boundary values of the solution

Pablo Amster and Alberto Déboli

Abstract

A semilinear system of second order ODEs under Neumann conditions is studied. The system has the particularity that its nonlinear term depends on the (unknown) Dirichlet values $\mathbf{y}(0)$ and $\mathbf{y}(1)$ of the solution. Asymptotic and non-asymptotic sufficient conditions of Landesman-Lazer type for existence of solutions are given. We generalize our previous results for a scalar equation, and a well known result by Nirenberg for a standard nonlinearity independent of $\mathbf{y}(0)$ and $\mathbf{y}(1)$.

Keywords: Two-ion electro-diffusion models; Landesman-Lazer conditions; nonlinear systems; topological degree.
2000 MSC: 34B15,34B99

1 Introduction

In [9], Leuchtag presented an m-ion electrodiffusion model consisting of the nonlinear coupled system

$$\begin{aligned} dn_i/dx &= \nu_i n_i p - c_i, \quad i = 1, \dots, m \\ dp/dx &= \sum_{i=1}^{m} \nu_i n_i \end{aligned} \tag{1}$$

where n_i is the number of ions with the same charge, p is the electric field, ν_i are non-zero integral signed valencies and c_i are real constants.

Different boundary value problems derived from these equations have been studied; for example, some particular cases of the two and three ions equations were solved in [5], [6]. The Painlevé structure of the equations has been described in [7].

An interesting case is studied in [14], for two ions with the same valency diffusing and migrating across a liquid junction under the influence of an electric field. Elimination of the ionic concentrations leads to the following problem for the unknown function y, which is proportional to the electric field in the rescaled interval $[0, 1]$:

$$y''(x) = y(x)\{\lambda - \frac{y(0)^2 - y(x)^2}{2} + [l\lambda + \frac{y(0)^2 - y(1)^2}{2}]x\} - [l\lambda + \frac{y(0)^2 - y(1)^2}{2}]D,$$

$$y'(0) = y'(1) = 0.$$

The constants $\lambda > 0$, $l > 0$ and $D \in (0, 1)$ depend on the physical parameters, such as the diffusion constant.

The problem is unconventional, since the equation depends on the yet to be determined values of the solution y at the boundary. Sufficient conditions for the existence of a positive solution are given in [14]: it is proven, essentially, that if λ is large enough with respect to the other parameters then the problem has a positive solution. Using a two-dimensional shooting argument, this restriction has been removed in [3]. A more general case with not necessarily equal valencies was studied in [4].

In the recent paper [2], an abstract version of this problem was considered. The right hand side of the equation was replaced by an arbitrary term $f(x, y(x), y(0), y(1))$, with $f : [0, 1]\times\mathbb{R}^3 \to \mathbb{R}$ continuous. Asymptotic conditions of Landesman-Lazer type [8], [10] have been obtained, more precisely:

Theorem 1.1 *[2] Assume that f is bounded, and that for every $x \in [0, 1]$ the limits*

$$\lim_{s\to\pm\infty} f(x, s + A, s, s + B) := f^{\pm}(x)$$

exist uniformly for $|A|, |B| \leq \|f\|_\infty$. Then the problem

$$y''(x) = f(x, y(x), y(0), y(1)), \qquad y'(0) = y'(1) = 0$$

admits a solution, provided that one of the following conditions holds:

$$\int_0^1 f^-(x)\, dx < 0 < \int_0^1 f^+(x)\, dx \tag{2}$$

or

$$\int_0^1 f^+(x)\, dx < 0 < \int_0^1 f^-(x)\, dx. \tag{3}$$

Furthermore, a stronger result under non-asymptotic conditions has been proved. Roughly speaking, if for $i = 1, 2$ there exist functions $\rho_i(x)$ and appropriate compact sets $K_i \subset \mathbb{R}^3$ such that

$$f(x, y, v, w) < \rho_1(x) \qquad \forall\ (y, v, w) \in K_1,$$

$$f(x, y, v, w) > \rho_2(x) \qquad \forall\ (y, v, w) \in K_2$$

and

$$\int_0^1 \rho_1(x)\, dx = \int_0^1 \rho_2(x)\, dx = 0,$$

then the problem has at least one solution.

It is observed that the nonlinearity f is not necessarily bounded, although some growth conditions are assumed. Also, the sets K_i cannot be arbitrarily small; their sizes depend on f (for details see [2, Thm 2]).

In this paper, we extend the results of [2] to a system of n equations, namely the problem

$$\begin{cases} \mathbf{y}''(x) = \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)), & x \in (0,1) \\ \mathbf{y}'(0) = \mathbf{y}'(1) = 0 \end{cases} \tag{4}$$

where $\mathbf{f} : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous.

Our first theorem can be regarded, in some sense, as an extension of a result proved by Nirenberg in [11].

Theorem 1.2 *Assume that* $\mathbf{f}$ *is bounded, and that for every* $x \in [0,1]$ *the limits*

$$\lim_{s\to+\infty} \mathbf{f}(x, s\mathbf{v} + A, s\mathbf{v}, s\mathbf{v} + B) := \mathbf{f}_{\mathbf{v}}(x)$$

exist uniformly for $|\mathbf{v}| = 1$ *and* $|A|, |B| \le \|\mathbf{f}\|_\infty$. *Further, assume that*

(N1) $\int_0^1 \mathbf{f}_{\mathbf{v}}(x)\, dx \ne 0$ *for every* $\mathbf{v} \in \mathbb{S}^{n-1} := \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}| = 1\}$.

(N2) $deg(\Phi) \ne 0$, *where* $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ *is defined by* $\Phi(\mathbf{v}) := \frac{\int_0^1 \mathbf{f}_{\mathbf{v}}(x)\, dx}{|\int_0^1 \mathbf{f}_{\mathbf{v}}(x)\, dx|}$.

Then problem (4) *admits a solution.*

As in the case $n = 1$, we shall also prove a non-asymptotic result. In first place, the boundedness condition on $\mathbf{f}$ will be replaced by the more general assumption that its range is contained in an 'angular sector' of $\mathbb{R}^n$. More precisely, we shall assume the existence of $\mathbf{c} \in \mathbb{R}^n$ and linearly independent hyperplanes $H_1, \dots, H_n$ such that

$$Im(\mathbf{f}) \subset \mathbb{R}^n \backslash \left(\mathbf{c} + \bigcup_{j=1}^{n} H_j \right). \tag{5}$$

Without loss of generality, we may suppose $H_j = \{\mathbf{z}_j\}^\perp$, with $\{\mathbf{z}_j\}_{1\le j\le n} \subset \mathbb{S}^{n-1}$ a basis of $\mathbb{R}^n$ and

$$\langle \mathbf{f}(x, \mathbf{y}, \mathbf{v}, \mathbf{w}) - \mathbf{c}, \mathbf{z}_j \rangle > 0$$

for every $(x, \mathbf{y}, \mathbf{v}, \mathbf{w}) \in [0,1] \times \mathbb{R}^{3n}$. In this case, an obvious necessary condition for the existence of solutions is that $\langle \mathbf{c}, \mathbf{z}_j \rangle < 0$.

In second place, the assumption on the existence of uniform limits will be removed. We shall assume, instead, that $\mathbf{f}$ does not rotate too fast, in a sense that will be specified below.

For convenience, let us define, for any $\mathbf{v} \in \mathbb{R}^n$, the neighborhood $Q(\mathbf{v})$ given by

$$Q(\mathbf{v}) := \{\mathbf{w} \in \mathbb{R}^n : |\langle \mathbf{w} - \mathbf{v}, \mathbf{z}_j \rangle| < 2|\langle \mathbf{c}, \mathbf{z}_j \rangle| \text{ for } 1 \le j \le n\}.$$

Moreover, consider the function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\phi(\mathbf{v}) := \int_0^1 \mathbf{f}(x, \mathbf{v}, \mathbf{v}, \mathbf{v})\, dx. \tag{6}$$

The Brouwer degree of ϕ at 0 over a bounded open set $D \subset \mathbb{R}^n$ shall be denoted by $deg_B(\phi, D, 0)$. Finally, the convex hull of a set $X \subset \mathbb{R}^n$ shall be denoted by $co(X)$.

Theorem 1.3 *Assume that* (5) *holds. If there exists a bounded domain* $D \subset \mathbb{R}^n$ *such that*

(H1)

$$0 \notin co(\mathbf{f}([0,1] \times Q(\mathbf{v}) \times \{\mathbf{v}\} \times Q(\mathbf{v}))) \tag{7}$$

for all $\mathbf{v} \in \partial D$*.*

(H2)

$$deg_B(\phi, D, 0) \neq 0.$$

Then (4) *has at least one solution.*

Remark 1.1 *Condition* (7) *forbids* $\mathbf{f}$ *to rotate too fast around zero near the boundary of* D*. It can be seen as an adaptation to this situation of an analogous condition introduced in [13] for a second order periodic problem. Rapid rotation is allowed in the main result of [1], although some 'largeness' condition on the nonlinearity is required to compensate this effect.*

The paper is organized as follows. In the next section, we introduce an abstract functional setting for problem (4) and prove the continuation theorem that will be used in the proof of our main theorems. In section 3, we apply the continuation theorem for proving Theorems 1.2 and 1.3. Finally, in section 4 we present some examples and final remarks.

2 The abstract setting

Inspired in [2], we convert our problem into a $4n$-dimensional system of first order equations

$$\begin{cases} \mathbf{y}'(x) = \mathbf{u}(x) \\ \mathbf{u}'(x) = \mathbf{f}(x, \mathbf{y}(x), \mathbf{v}(x), \mathbf{w}(x)) \\ \mathbf{v}'(x) = 0 \\ \mathbf{w}'(x) = 0, \end{cases} \tag{8}$$

with the following boundary conditions:

$$\begin{cases} \mathbf{u}(0) = \mathbf{u}(1) = 0 \\ \mathbf{y}(0) = \mathbf{v}(0) \\ \mathbf{y}(1) = \mathbf{w}(1). \end{cases} \tag{9}$$

Next, consider the Banach Space

$$\mathbb{E} := \{\mathbf{X} := (\mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in C([0,1], \mathbb{R}^n)^4 : \mathbf{X} \text{ satisfies } (9)\},$$

equipped with the standard norm

$$\|\mathbf{X}\| := \max\{\|\mathbf{y}\|_\infty, \|\mathbf{u}\|_\infty, \|\mathbf{v}\|_\infty, \|\mathbf{w}\|_\infty\}.$$

In this setting, the problem can be interpreted in the context of the so-called *resonant systems*. Indeed, the kernel of the linear operator $L(\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{w}) := (\mathbf{y}'-\mathbf{u},\mathbf{u}',\mathbf{v}',\mathbf{w}')$ over the subspace of C^1 elements of $\mathbb{E}$ is the n-dimensional subspace spanned by the vectors $\mathbf{X_c} = (\mathbf{c},\mathbf{0},\mathbf{c},\mathbf{c})$, where $\mathbf{c} \in \mathbb{R}^n$.

In order to apply the Leray-Schauder degree method to the problem, let us introduce an operator $K : [0,1]\times\mathbb{E} \to \mathbb{E}$ in the following way. For $\mathbf{X} \in \mathbb{E}$, define

$$F_{\mathbf{X}}(x) := \int_0^x \mathbf{f}(s,\mathbf{y}(s),\mathbf{v}(s),\mathbf{w}(s))\, ds$$

$$\mathbf{c} = \mathbf{c}(\mathbf{X}) := \mathbf{y}(0) + F_{\mathbf{X}}(1)$$

and

$$S(\mathbf{X})(x) := \left(\int_0^x F_{\mathbf{X}}(s)\, ds, F_{\mathbf{X}}(x) - xF_{\mathbf{X}}(1), \mathbf{0}, \int_0^1 F_{\mathbf{X}}(s)\, ds\right).$$

Finally, set

$$K(\sigma,\mathbf{X}) := \mathbf{X_c} + \sigma S(\mathbf{X}) \tag{10}$$

We claim that $\mathbf{X} \in \mathbb{E}$ is a solution of (8) if and only if $\mathbf{X}$ is a fixed point of $K(1,\cdot)$. More generally, we have:

Lemma 2.1 *Let $\mathbf{X} \in \mathbb{E}$ and $0 < \sigma \le 1$. Then $\mathbf{X}$ is a fixed point of $K(\sigma,\cdot)$ if and only if $\mathbf{X}$ satisfies:*

$$\begin{cases} \mathbf{y}'(x) = \mathbf{u}(x) \\ \mathbf{u}'(x) = \sigma\mathbf{f}(x,\mathbf{y}(x),\mathbf{y}(0),\mathbf{y}(1)) \\ \mathbf{v}'(x) = 0 \\ \mathbf{w}'(x) = 0. \end{cases} \tag{11}$$

<u>*Proof*</u>: If $\mathbf{X} = K(\sigma,\mathbf{X})$, then its first coordinate is given by

$$\mathbf{y}(x) = \mathbf{y}(0) + F_{\mathbf{X}}(1) + \sigma\int_0^x F_{\mathbf{X}}(s)\, ds.$$

It follows that $F_{\mathbf{X}}(1) = 0$, and $\mathbf{y}'(x) = \sigma F_{\mathbf{X}}(x) = \mathbf{u}(x)$. Moreover, $\mathbf{y}''(x) = \mathbf{u}'(x) = \sigma\mathbf{f}(x,\mathbf{y}(x),\mathbf{v}(x),\mathbf{w}(x))$, and using the last two coordinates in the fixed point equation, we deduce:

$$\mathbf{v} \equiv \mathbf{y}(0), \qquad \mathbf{w} \equiv \mathbf{y}(0) + \sigma\int_0^1 F_{\mathbf{X}}(s)\, ds = \mathbf{y}(1).$$

Conversely, if $\mathbf{X}$ satisfies (11), then $\mathbf{v} \equiv \mathbf{y}(0)$, $\mathbf{w} \equiv \mathbf{y}(1)$ and

$$\mathbf{u}' = \sigma\mathbf{f}(x,\mathbf{y}(x),\mathbf{v},\mathbf{w}).$$

As $\mathbf{u}(0) = \mathbf{u}(1) = 0$, it is seen that $F_{\mathbf{X}}(1) = 0$. Moreover,

$$\mathbf{u}(x) = \sigma \int_0^x \mathbf{f}(s, \mathbf{y}(s), \mathbf{v}, \mathbf{w})\, ds = \sigma F_{\mathbf{X}}(x),$$

and as $\mathbf{y}' = \mathbf{u}$ we deduce that $\mathbf{y}(x) = \mathbf{y}(0) + \sigma \int_0^x F_{\mathbf{X}}(s)\, ds$. Hence $\mathbf{w} = \mathbf{y}(0) + \sigma \int_0^1 F_{\mathbf{X}}(s)\, ds$, and the proof is complete. ∎

The preceding lemma induces us to define the homotopy $H : [0,1] \times \mathbb{E} \to \mathbb{E}$ given by

$$H(\sigma, \mathbf{X}) = \mathbf{X} - K(\sigma, \mathbf{X}) = \mathbf{X} - \mathbf{X_c} - \sigma S(\mathbf{X}),$$

with $\mathbf{c} = \mathbf{c}(\mathbf{X})$, $\mathbf{X_c}$ and $S(\mathbf{X})$ as before.

It is easy to see that $K_\sigma := K(\sigma, \cdot) : \mathbb{E} \to \mathbb{E}$ is compact for any $\sigma \in [0,1]$. Furthermore, the range of K_0 is contained in $Ker(L)$. Indeed, if $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{X} = (\mathbf{v}, 0, \mathbf{v}, \mathbf{v})$, then $\mathbf{c} = \mathbf{v} + \int_0^1 \mathbf{f}(x, \mathbf{v}, \mathbf{v}, \mathbf{v})\, dx = \mathbf{v} + \phi(\mathbf{v})$, and

$$H_0(\mathbf{X}) = \mathbf{X} - \mathbf{X_c} = -(\phi(\mathbf{v}), 0, \phi(\mathbf{v}), \phi(\mathbf{v})).$$

In other words, if Ω is an open subset of $\mathbb{E}$ such that H_σ does not vanish on $\partial\Omega$ for $\sigma \in [0,1]$, then its Leray-Schauder (LS) degree may be computed by

$$deg_{LS}(H_1, \Omega, \mathbf{0}) = deg_{LS}(H_0, \Omega, \mathbf{0}) = deg_B(H_0|_{Ker(L)}, \Omega \cap Ker(L), \mathbf{0}).$$

Moreover, as $\Omega \cap Ker(L) = \{(\mathbf{v}, 0, \mathbf{v}, \mathbf{v}) : \mathbf{v} \in G_\Omega\}$ for some open bounded $G_\Omega \subset \mathbb{R}^n$, we conclude that $deg_{LS}(H_1, \Omega, \mathbf{0}) = (-1)^n deg_B(\phi, G_\Omega, 0)$. Thus we have proved:

Theorem 2.1 *Let $\Omega \subset \mathbb{E}$ be open and bounded and let $G_\Omega \subset \mathbb{R}^n$ as before. Assume that*

1. *(11) has no solutions on $\partial\Omega$ for $\sigma \in (0,1)$.*
2. *$\phi(\mathbf{v}) \neq 0$ for $\mathbf{v} \in \partial G_\Omega$.*
3. *$deg(\phi, G_\Omega, 0) \neq 0$.*

Then (8) has at least one solution $\mathbf{X} \in \overline{\Omega}$.

3 Proof of the main results

Proof of Theorem 1.2:

According with the continuation theorem, we shall firstly prove that solutions of (11) with $0 < \sigma < 1$ are bounded. By contradiction, suppose that $\mathbf{X}_n$ satisfies (11) with $0 < \sigma_n < 1$ and $\|\mathbf{X}_n\| \to \infty$. Then

$$\mathbf{y}_n''(x) = \sigma_n \mathbf{f}(x, \mathbf{y}_n(x), \mathbf{y}_n(0), \mathbf{y}_n(0)), \qquad \mathbf{y}'(0) = \mathbf{y}'(1) = 0,$$

and hence

$$\|\mathbf{y}_n - \mathbf{y}_n(0)\|_\infty \leq \|\mathbf{y}_n'\|_\infty \leq \|\mathbf{y}_n''\|_\infty \leq \|\mathbf{f}\|_\infty.$$

This implies that $\mathbf{u}_n$, $\mathbf{y}_n-\mathbf{v}_n$ and $\mathbf{w}_n-\mathbf{v}_n$ are bounded and $|\mathbf{v}_n| = |\mathbf{y}_n(0)| \to \infty$. Moreover,

$$\int_0^1 \mathbf{f}(x, \mathbf{y}_n(x), \mathbf{y}_n(0), \mathbf{y}_n(1))dx = \mathbf{y}_n'(1) - \mathbf{y}_n'(0) = 0. \tag{12}$$

Passing to a subsequence if necessary, we may suppose that $\frac{\mathbf{v}_n}{|\mathbf{v}_n|} \to \mathbf{v} \in \mathbb{S}^{n-1}$ and by dominated convergence we deduce:

$$\int_0^1 \mathbf{f}(x, \mathbf{y}_n(x), \mathbf{y}_n(0), \mathbf{y}_n(1))dx \to \int_0^1 \mathbf{f}_{\mathbf{v}}(x)\, dx \neq 0,$$

a contradiction.

On the other hand, it is easy to see that if R is large enough then $deg(\Phi) = deg(\phi, B_R(0), 0)$ and taking $\Omega \subset \mathbb{E}$ as a large ball centered at 0 the proof follows. ■

<u>*Proof of Theorem 1.3*:</u>

For simplicity, let us introduce the following notation for $j = 1, \ldots, n$:

$$x_j := \langle \mathbf{x}, \mathbf{z}_j \rangle \qquad \text{for } \mathbf{x} \in \mathbb{R}^n$$

$$m_j := 2|c_j|.$$

We shall apply the continuation theorem over the set

$$\Omega := \{(\mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{E} : \mathbf{v} \in D, \|y_j - v_j\|_\infty, \|w_j - v_j\|_\infty, \|u_j\|_\infty < m_j \ \forall\, j\}.$$

If $\mathbf{X} = (\mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ solves (11) for some $\sigma \in (0, 1)$, then

$$\mathbf{y}''(x) = \sigma \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)), \quad \mathbf{y}'(0) = \mathbf{y}'(1) = 0$$

and hence

$$y_j''(x) = \sigma\langle \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)), \mathbf{z}_j\rangle = \sigma\langle \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)) - \mathbf{c}, \mathbf{z}_j\rangle + \sigma c_j.$$

This implies

$$|y_j''(x)| < \langle \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)) - \mathbf{c}, \mathbf{z}_j\rangle + |c_j|.$$

From the Neumann condition, integration in both terms of the preceding inequality yields

$$\textstyle\int_0^1 |y_j''(x)|dx < 2|c_j| = m_j,$$

for each $1 \leq j \leq n$.

Moreover, $y_j'(0) = \langle \mathbf{z}_j, \mathbf{u}(0)\rangle = 0$, then

$$|y_j'(x)| \leq \int_0^x |y_j''(t)|dt < m_j,$$

that is

$$\|y_j'\|_\infty < m_j.$$

Also,

$$|y_j(x) - y_j(0)| \leq \int_0^x |y_j'(t)|dt \leq \|y'\|_\infty < m_j$$

for every $x \in [0,1]$ and, in particular,

$$|y_j(1) - y_j(0)| \leq \|y_j - y_j(0)\|_\infty < m_j$$

for each $1 \leq j \leq n$.

Summarizing, $\|y_j - v_j\|_\infty, \|w_j - v_j\|_\infty, \|u_j\|_\infty < m_j$. Thus, if $\mathbf{X} \in \partial\Omega$ then $\mathbf{v} \in \partial D$, and

$$(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)) \in I \times Q(\mathbf{v}) \times \{\mathbf{v}\} \times Q(\mathbf{v})$$

for every $x \in [0,1]$. It follows that $\mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1))$ lies in a compact subset of $\mathbf{f}([0,1] \times Q(\mathbf{v}) \times \{\mathbf{v}\} \times Q(\mathbf{v}))$.

From (7) and the geometric version of the Hahn-Banach theorem, there exists a vector $\mathbf{m} = \mathbf{m}(\mathbf{v})$ such that

$$\langle \mathbf{m}\ ,\ \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)) \rangle > 0$$

for all $x \in [0,1]$ and we obtain a contradiction:

$$0 < \int_0^1 \langle \mathbf{m}, \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1)) \rangle\ dx = \left\langle \mathbf{m}, \int_0^1 \mathbf{f}(x, \mathbf{y}(x), \mathbf{y}(0), \mathbf{y}(1))\ dx \right\rangle = 0.$$

Finally, it is clear that $G_\Omega = D$ and the continuation theorem applies.

■

Remark 3.1 *If* $\mathbf{f}$ *is bounded, then the neighborhood* $Q(\mathbf{v})$ *may be replaced by* $B_r(\mathbf{v})$*, with* $r = \|\mathbf{f}\|_\infty$.

4 Examples and final remarks

The following example, inspired in [12], shows that Theorem 1.3 is not necessarily stronger than Theorem 1.2. Let $n = 2$, identify $\mathbb{R}^2$ with the complex plane and consider the function $\mathbf{f} : [0,1] \times \mathbb{C}^3 \to \mathbb{C}$ given by

$$\mathbf{f}(x, z, z_0, z_1) = \frac{e^{i\alpha x} z}{\sqrt{|z|^2 + 1}} + \gamma(z_0, z_1)$$

with $\alpha \in \mathbb{R}$ and $\lim_{|z_0|,|z_1| \to \infty} \gamma(z_0, z_1) = \gamma$, $|\gamma| < 1$. It is clear that the radial limits

$$\mathbf{f}_z(x) = \lim_{s \to +\infty} \mathbf{f}(x, sz + A, sz, sz + B) = e^{i\alpha x} z + \gamma$$

are uniform for $|z| = 1$, $|A|, |B| \leq 1 + \|\gamma\|_\infty$, and conditions of Theorem 1.2 are satisfied if $\alpha \neq 2k\pi$ for $k \in \mathbb{Z} \backslash \{0\}$. However, assumptions of Theorem 1.3 do not hold for example when $|\alpha| > \pi$ and $\|\gamma\|_\infty$ is small.

Beside this example, it is worth noticing that Theorem 1.3 improves Theorem 1.2 in a wide range of cases. With this aim, let us state the following result, which constitutes an extension, sharper than Theorem 1.3, of the main theorem in [2] for the case $n = 1$:

Theorem 4.1 *Assume that* (5) *holds. Furthermore, assume there exists a bounded domain $D \subset \mathbb{R}^n$ such that (H1') and (H2) are satisfied, with*

(H1') For every $\mathbf{v} \in \partial D$ there exists a continuous function $\rho : [0,1] \to \mathbb{R}^n$ such that $\int_0^1 \rho(x)\,dx = 0$ and

$$0 \notin co(\mathbf{f}_\rho([0,1] \times Q(\mathbf{v}) \times \{\mathbf{v}\} \times Q(\mathbf{v}))) \tag{13}$$

where $\mathbf{f}_\rho(x, \mathbf{y}, \mathbf{v}, \mathbf{w}) := \mathbf{f}(x, \mathbf{y}, \mathbf{v}, \mathbf{w}) - \rho(x)$.

Then (4) *has at least one solution.*

The proof is similar to the proof of Theorem 1.3 and thus omitted. It is easy to verify that the preceding result is stronger than Theorem 1.2 in the particular case $\mathbf{f}(x, \mathbf{y}, \mathbf{v}, \mathbf{w}) = \rho(x) + \mathbf{g}(\mathbf{y}, \mathbf{v}, \mathbf{w})$.

Indeed, let us prove in first place that the mapping $\mathbf{v} \mapsto \mathbf{g_v}$ is continuous. For $\varepsilon > 0$, fix s such that $|\mathbf{g}(s\mathbf{v}, s\mathbf{v}, s\mathbf{v}) - \mathbf{g_v}| < \frac{\varepsilon}{4}$ for every $\mathbf{v} \in \mathbb{S}^{n-1}$, then

$$|\mathbf{g_w} - \mathbf{g_v}| \leq |\mathbf{g}(s\mathbf{w}, s\mathbf{w}, s\mathbf{w}) - \mathbf{g}(s\mathbf{v}, s\mathbf{v}, s\mathbf{v})| + \frac{\varepsilon}{2} < \varepsilon$$

for $\mathbf{w}$ sufficiently close to $\mathbf{v}$. In particular, this implies that $|\mathbf{g_v}| \geq c$ for every $\mathbf{v} \in \mathbb{S}^{n-1}$, where c is a positive constant.

Now fix s_0 such that $|\mathbf{g}(s\mathbf{v} + A, s\mathbf{v}, s\mathbf{v} + B) - \mathbf{g_v}| < c$ for every $\mathbf{v} \in \mathbb{S}^{n-1}$, $s \geq s_0$ and $|A|, |B| \leq \|\mathbf{f}\|_\infty$. Taking $D = B_R(0)$ with $R > s_0$, for $\mathbf{w} = R\mathbf{v} \in \partial D$ and $|A|, |B| < \|\mathbf{f}\|_\infty$ we obtain:

$$\langle \mathbf{g}(\mathbf{w} + A, \mathbf{w}, \mathbf{w} + B), \mathbf{g_v} \rangle \geq |\mathbf{g_v}|^2 - |\mathbf{g}(\mathbf{w} + A, \mathbf{w}, \mathbf{w} + B) - \mathbf{g_v}| > 0.$$

This implies that the convex hull of $\mathbf{g}(B_{\|\mathbf{f}\|_\infty}(\mathbf{w}) \times \{\mathbf{w}\} \times B_{\|\mathbf{f}\|_\infty}(\mathbf{w}))$ lies at one side of the hyperplane $\{g_\mathbf{v}\}^\perp$ and, in particular, it does not contain the null vector. From Remark 3.1, we conclude that (13) is satisfied.

Remark 4.2 *In all the preceding results, it is clear that the role of* $\mathbf{y}(0)$ *and* $\mathbf{y}(1)$ *may be exchanged. For example,* (13) *may be replaced by*

$$0 \notin co(\mathbf{f}_\rho([0,1] \times Q(\mathbf{v}) \times Q(\mathbf{v}) \times \{\mathbf{v}\})).$$

5 Acknowledgements

This work has been supported by projects UBACyT 20020090100067 and PIP 11220090100637 CONICET.

References

[1] P. Amster and M. Clapp, Periodic solutions of resonant systems with rapidly rotating nonlinearities, Differential Equations and Dynamical Systems, Series A 31 No. 2 (2011), 373-383.

[2] P. Amster and A. Déboli, A nonlinear problem depending on the unknown Dirichlet values of the solution. Differential Equations and Dynamical Systems 18, No 4 (2010), 363-372.

[3] P. Amster, M. K. Kwong and C. Rogers, On a Neumann Boundary Value Problem for Painlevé II in Two Ion Electro-Diffusion. To appear in Nonlinear Analysis, TMA.

[4] P. Amster, M. K. Kwong and C. Rogers, A Neumann Boundary Value Problem in Two-Ion Electro-diffusion with $\nu_+ + \nu_- \neq 0$. Submitted.

[5] P. Amster, C. Rogers, On boundary value problems in three-ion electrodiffusion. J. Math. Anal. Appl. 333 (2007), 42-51.

[6] P. Amster, M. C. Mariani, C. Rogers and C. C. Tisdell, On two-point boundary value problems in muti-ion electrodiffusion. J. Math. Anal. Appl. 289 (2004), 712-721.

[7] R. Conte, W. K. Schief and C. Rogers, Painlevé structure of a multi-ion electrodiffusion system, J. Physics A: Math. Theor. 40 (2007).

[8] E. Landesman and A. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609-623.

[9] H. R. Leuchtag, A family of differential equations arising from multi-ion electrodiffusion, J. Math. Phys., 22, 1317-1320 (1981).

[10] J. Mawhin, Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance. Bol. de la Sociedad Española de Mat. Aplicada 16 (2000), 45-65.

[11] L. Nirenberg, Generalized degree and nonlinear problems, Contributions to nonlinear functional analysis, Ed. E. H. Zarantonello, Academic Press New York (1971), 1-9.

[12] R. Ortega and L. Sanchez, Periodic solutions of forced oscillators with several degrees of freedom, Bull. London Math. Soc. 34 (2002), 308-318

[13] D. Ruiz and J. R. Ward Jr., Some notes on periodic systems with linear part at resonance, Discrete and Continuous Dynamical Systems 11 (2004), 337-350.

[14] H. B. Thompson, Existence for two-point boundary value problems in two ion electrodiffusion, Journal of Mathematical Analysis and Applications 184, No. 1 (1994) 82-94.

(Received February 25, 2012)

Pablo Amster[1,2] and Alberto Déboli[1]

[1] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina.

[2] Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

E-mails: pamster@dm.uba.ar – adeboli@dm.uba.ar