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# Existence of periodic solutions for a class of functional integral equations*

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**Abstract**

In this paper, we investigate the existence of periodic solution for a class of nonlinear functional integral equation. We prove a fixed point theorem in a Banach algebra. As an application, an existence theorem about periodic solutions to the addressed functional integral equation is presented. In addition, an example is given to illustrate our result.

**Keywords:** functional integral equation; periodic solution.

**2000 Mathematics Subject Classification:** 45G10, 34K13.

## 1 Introduction

This paper has four main motivations. The first motivation is that recently, the study on the existence of solutions to various kinds of functional integral equations has became one of the most attractive topics in the theory of integral equations. Many authors have made a lot of interesting contributions on this topic. For example, we refer the readers to [1–7, 9–15, 17, 18, 20] and references therein. The second motivation is that in recent years, some authors have focused on the resolution of the operator equation $x = AxBx+Cx$ in Banach algebras, and obtained many valuable results (see, e.g., [2–5, 7, 9–13, 18] and references

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therein). Moreover, in these papers, the authors applied successfully their abstract results to the study on the existence of solutions to functional integral equations. The third motivation is that the authors of [19] studied the existence of periodic solutions for the following Fredholm integral equation:

$$y(t) = h(t) + \int_{\mathbb{R}} k(t,s) f(s, y(s)) ds, \quad t \in \mathbb{R},$$

by using nonlinear alternative of Leray-Schauder type. The fourth motivation is that in [16], the authors investigated the existence of almost periodic type solutions to the following functional integral equation:

$$y(t) = e(t, y(\alpha(t))) + g(t, y(\beta(t))) \left[ h(t) + \int_{\mathbb{R}} k(t,s) f(s, y(\gamma(s))) ds \right], \quad t \in \mathbb{R}.$$

Motivated by all the above works, in this paper, we first establish a fixed point theorem in a Banach algebra, and then, with its help, we discuss the existence of periodic solution for the following general functional integral equation:

$$x(t) = \sum_{i=1}^{n} f_i(t, x(a_i(t))) \cdot \int_{\mathbb{R}} k_i(t,s) g_i(s, x(b_i(s))) ds, \quad t \in \mathbb{R}, \tag{1.1}$$

where $n$ is a fixed positive integer, and $f_i$, $a_i$, $k_i$, $g_i$ and $b_i$ $(i = 1, \ldots, n)$ satisfy some conditions recalled in Section 2.

Throughout the rest of this paper, we denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, by $\mathbb{N}$ the set of positive integers, by $\mathfrak{C}(\mathbb{R}^+, \mathbb{R}^+)$ the set of all continuous and nondecreasing functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$, and by $\mathcal{P}_T(\mathbb{R})$ the Banach algebra of all $T$-periodic continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \max_{t \in [0,T]} |x(t)|, \quad x \in \mathcal{P}_T(\mathbb{R})$$

and the multiplication defined by

$$(x \cdot y)(t) = x(t) \cdot y(t), \quad x, y \in \mathcal{P}_T(\mathbb{R}),\ t \in \mathbb{R}.$$

**Definition 1.1.** *Let $X$ be a Banach space. A mapping $A : X \to X$ is called $\mathcal{D}$-Lipschitzian if there exists a function $\phi \in \mathfrak{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\|Ax - Ay\| \leq \phi(\|x - y\|)$$

*for all $x, y \in X$. In addition, the function $\phi$ is called a $\mathcal{D}$-function of $A$.*

## 2 Main results

**Theorem 2.1.** *Let $n$ be a positive integer, and $C$ be a nonempty, closed, convex and bounded subset of a Banach algebra $X$. Assume that the operators $A_i : X \to X$ and $B_i : C \to X$, $i = 1, 2, \dots, n$, satisfy*

(a) *for each $i \in \{1, 2, \dots, n\}$, $A_i$ is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\phi_i$;*

(b) *for each $i \in \{1, 2, \dots, n\}$, $B_i$ is continuous and $B_i(C)$ is precompact;*

(c) *for each $y \in C$, $x = \sum\limits_{i=1}^{n} A_i x \cdot B_i y$ implies that $x \in C$;*

*Then, the operator equation $x = \sum\limits_{i=1}^{n} A_i x \cdot B_i x$ has a solution provided that*

$$\sum_{i=1}^{n} M_i \phi_i(r) < r, \quad \forall r > 0,$$

*where $M_i = \sup\limits_{x \in C} \|B_i x\|$, $i = 1, 2, \dots, n$.*

*Proof.* For each $y \in C$, define an operator on $X$ by

$$\mathcal{S}_y x = \sum_{i=1}^{n} A_i x \cdot B_i y, \quad x \in X.$$

Denote

$$\psi(r) := \sum_{i=1}^{n} M_i \phi_i(r), \quad r > 0.$$

Then $\psi$ is continuous and nondecreasing. Moreover, $\psi(r) < r$ for all $r > 0$. For all $x_1, x_2 \in X$, we have

$$\begin{aligned}
& \|\mathcal{S}_y x_1 - \mathcal{S}_y x_2\| \\
= & \left\| \sum_{i=1}^{n} A_i x_1 \cdot B_i y - \sum_{i=1}^{n} A_i x_2 \cdot B_i y \right\| \\
\leq & \sum_{i=1}^{n} \|A_i x_1 - A_i x_2\| \cdot \|B_i y\| \\
\leq & \sum_{i=1}^{n} M_i \phi_i(\|x_1 - x_2\|) \\
= & \psi(\|x_1 - x_2\|).
\end{aligned}$$

Then, by using the well-known results in [8], we know that $\mathcal{S}_y$ has a unique fixed point $x_y$ in $X$.

Now, define an operator $\mathcal{S}$ on $C$ by

$$\mathcal{S}y = x_y, \quad y \in C,$$

where $x_y$ is the unique fixed point of $\mathcal{S}_y$ in $X$. Then,

$$\mathcal{S}y = x_y = \mathcal{S}_y x_y = \sum_{i=1}^{n} A_i x_y \cdot B_i y, \quad y \in C.$$

By the assumption (c), we know that $\mathcal{S}y = x_y \in C$ for all $y \in C$. In addition, for all $y, z \in C$, we have

$$\begin{aligned}
&\|\mathcal{S}y - \mathcal{S}z\| \\
= &\left\| \sum_{i=1}^{n} A_i x_y \cdot B_i y - \sum_{i=1}^{n} A_i x_z \cdot B_i z \right\| \\
\leq &\sum_{i=1}^{n} \|A_i x_y \cdot B_i y - A_i x_z \cdot B_i y + A_i x_z \cdot B_i y - A_i x_z \cdot B_i z\| \\
\leq &\sum_{i=1}^{n} M_i \phi_i(\|x_y - x_z\|) + \sum_{i=1}^{n} \|A_i x_z\| \cdot \|B_i y - B_i z\| \\
= &\psi(\|\mathcal{S}y - \mathcal{S}z\|) + \sum_{i=1}^{n} \|A_i x_z\| \cdot \|B_i y - B_i z\| \\
\leq &\psi(\|\mathcal{S}y - \mathcal{S}z\|) + \mathcal{M} \cdot \sum_{i=1}^{n} \|B_i y - B_i z\|, \qquad (2.1)
\end{aligned}$$

where

$$\begin{aligned}
\|A_i x_z\| &\leq \|A_i e\| + \|A_i x_z - A_i e\| \\
&\leq \|A_i e\| + \phi_i(\|x_z - e\|) \\
&\leq \max_{1 \leq i \leq n} \|A_i e\| + \phi_i\left( \|e\| + \sup_{y \in C} \|y\| \right) \\
&\leq \max_{1 \leq i \leq n} \|A_i e\| + \max_{1 \leq i \leq n} \left[ \phi_i\left( \|e\| + \sup_{y \in C} \|y\| \right) \right] := \mathcal{M}
\end{aligned}$$

for a fixed element $e \in C$.

Next, let us show that $\mathcal{S}(C)$ is precompact and $\mathcal{S} : C \to C$ is continuous. Let $\{y_m\}$ be a sequence in $C$. Noting that every $B_i(C)$ is precompact, there exists a subsequence $\{y_k\}$ of $\{y_m\}$ such that every $\{B_i y_k\}$ is convergent for each $i = 1, 2 \ldots, n$. For all $k_1, k_2 \in \mathbb{N}$, by (2.1), we have

$$\|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\| \leq \psi(\|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|) + \mathcal{M} \cdot \sum_{i=1}^{n} \|B_i y_{k_1} - B_i y_{k_2}\|. \qquad (2.2)$$

Since $\psi$ is continuous and nondecreasing, we have

$$\begin{aligned}
& \limsup_{k_1,k_2\to\infty} \psi(\|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|) \\
:= \;& \inf_{k\in\mathbb{N}} \sup_{k_1,k_2\geq k} \psi(\|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|) \\
= \;& \psi\left(\inf_{k\in\mathbb{N}} \sup_{k_1,k_2\geq k} \|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|\right) \\
:= \;& \psi\left(\limsup_{k_1,k_2\to\infty} \|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|\right),
\end{aligned}$$

which together with (2.2) yield that

$$\limsup_{k_1,k_2\to\infty} \|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\| \leq \psi\left(\limsup_{k_1,k_2\to\infty} \|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\|\right)$$

since every $\{B_iy_k\}$ is convergent. Noting that $\psi(r) < r$ for all $r > 0$, we conclude that

$$\limsup_{k_1,k_2\to\infty} \|\mathcal{S}y_{k_1} - \mathcal{S}y_{k_2}\| = 0,$$

which means that $\{\mathcal{S}y_k\}$ is a Cauchy sequence, and thus $\{\mathcal{S}y_k\}$ is convergent. So $\mathcal{S}(C)$ is precompact. In addition, letting $y_k \to y$ in $C$, it follows from (2.1) that

$$\|\mathcal{S}y_k - \mathcal{S}y\| \leq \psi(\|\mathcal{S}y_k - \mathcal{S}y\|) + \mathcal{M} \cdot \sum_{i=1}^{n} \|B_iy_k - B_iy\|.$$

Noting that $B_iy_k \to B_iy$, $i = 1, 2, \ldots, n$, we conclude

$$\limsup_{k\to\infty} \|\mathcal{S}y_k - \mathcal{S}y\| \leq \psi\left(\limsup_{k\to\infty} \|\mathcal{S}y_k - \mathcal{S}y\|\right),$$

which yields that

$$\lim_{k\to\infty} \|\mathcal{S}y_k - \mathcal{S}y\| = 0,$$

i.e., $\mathcal{S}y_k \to \mathcal{S}y$. Thus, $\mathcal{S} : C \to C$ is continuous.

Now, by using Schauder's fixed point theorem, we know that $\mathcal{S}$ has a fixed point $y_0 \in C$. Then, we have

$$y_0 = \mathcal{S}y_0 = x_{y_0} = \sum_{i=1}^{n} A_ix_{y_0} \cdot B_iy_0 = \sum_{i=1}^{n} A_iy_0 \cdot B_iy_0,$$

i.e., $y_0$ is a solution of the operator equation $x = \sum\limits_{i=1}^{n} A_ix \cdot B_ix$. □

**Remark 2.2.** In the case of $n = 1$, Theorem 2.1 is due to [12, Theorem 2.1]. However, due to some misprints, [12, Theorem 2.1] is essentially proved in the case of $n = 1$ and $\phi_1(r) = \alpha r$ for some constant $\alpha > 0$.

Next, we consider the existence of periodic solution for Eq. (1.1).

**Theorem 2.3.** *Let $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that the following assumptions hold:*

*(H1) For each $i \in \{1, 2, \dots, n\}$, $a_i, b_i : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $x(a_i(\cdot)) \in \mathcal{P}_T(\mathbb{R})$ for all $x \in \mathcal{P}_T(\mathbb{R})$.*

*(H2) For each $i \in \{1, 2, \dots, n\}$, $f_i(\cdot, x) \in \mathcal{P}_T(\mathbb{R})$ for any fixed $x \in \mathbb{R}$ and there exists a function $\phi_i \in \mathfrak{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$|f_i(t, x) - f_i(t, y)| \leq \phi_i(|x - y|), \quad \forall t \in \mathbb{R},\ \forall x, y \in \mathbb{R}.$$

*(H3) For each $i \in \{1, 2, \dots, n\}$, $g_i(\cdot, x)$ is measurable for all $x \in \mathbb{R}$, $g_i(t, \cdot)$ is continuous for almost all $t \in \mathbb{R}$, and for each $r > 0$, there exists a function $\mu_i^r \in L^p(\mathbb{R})$ such that $|g_i(t, x)| \leq \mu_i^r(t)$ for all $|x| \leq r$ and almost all $t \in \mathbb{R}$.*

*(H4) For each $i \in \{1, 2, \dots, n\}$, $k_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies that the map $t \to \widetilde{k}_i(t)$ is a continuous $T$-periodic function from $\mathbb{R}$ to $L^q(\mathbb{R})$, where $[\widetilde{k}_i(t)](s) = k_i(t, s)$, $\forall t, s \in \mathbb{R}$.*

*(H5) There exists a constant $M > 0$ such that*

$$\sum_{i=1}^{n} K_i \|\mu_i^M\|_p \cdot \phi_i(r) < r, \quad \forall r > 0,$$

*where $K_i = \max_{t \in [0,T]} \|\widetilde{k}_i(t)\|_q$; and*

$$\sum_{i=1}^{n} \left[ \sup_{t \in \mathbb{R}, |x| \leq \lambda} |f_i(t, x)| \cdot K_i \cdot \|\mu_i^M\|_p \right] < \lambda, \quad \forall \lambda > M.$$

*Then Eq. (1.1) has a continuous $T$-periodic solution.*

*Proof.* Let

$$(A_i x)(t) = f_i(t, x(a_i(t))), \quad x \in \mathcal{P}_T(\mathbb{R}),\ t \in \mathbb{R},$$

and

$$(B_i x)(t) = \int_{\mathbb{R}} k_i(t, s) g_i(s, x(b_i(s))) ds, \quad x \in \mathcal{P}_T(\mathbb{R}),\ t \in \mathbb{R}.$$

For each $x \in \mathcal{P}_T(\mathbb{R})$, it follows from (H1) and the periodicity of $f_i$ and $k_i$ that $A_i x$ and $B_i x$ are both $T$-periodic; in addition, it is not difficult to verify that $A_i x$ and $B_i x$ are both continuous. Thus, both $A_i$ and $B_i$ map $\mathcal{P}_T(\mathbb{R})$ into $\mathcal{P}_T(\mathbb{R})$.

We will use Theorem 2.1 to prove that Eq. (1.1) has a $T$-periodic solution. Next, let us verify all the assumptions of Theorem 2.1. Denote

$$C = \{x \in \mathcal{P}_T(\mathbb{R}) : \|x\| \leq M\}.$$

First, by (H2), for all $x, y \in \mathcal{P}_T(\mathbb{R})$, we have

$$\begin{aligned} \|A_i x - A_i y\| &= \max_{t\in\mathbb{R}} |f_i(t, x(a_i(t))) - f_i(t, y(a_i(t)))| \\ &\leq \max_{t\in\mathbb{R}} \phi_i\Big(|x(a_i(t)) - y(a_i(t))|\Big) \\ &\leq \phi_i(\|x - y\|), \end{aligned}$$

which means that $A_i$ is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\phi_i$, i.e., the assumption (a) of Theorem 2.1 holds.

Next, let us show that for each $i \in \{1, 2, \ldots, n\}$, $B_i$ is continuous. Let $x_k \to x$ in $\mathcal{P}_T(\mathbb{R})$. We have

$$\begin{aligned} |(B_i x_k)(t) - (B_i x)(t)| &\leq \int_{\mathbb{R}} |k_i(t, s)| \cdot |g_i(s, x_k(b_i(s))) - g_i(s, x(b_i(s)))| ds \\ &\leq \left(\int_{\mathbb{R}} |k_i(t, s)|^q ds\right)^{1/q} \cdot \left(\int_{\mathbb{R}} |g_i(s, x_k(b_i(s))) - g_i(s, x(b_i(s)))|^p ds\right)^{1/p} \\ &\leq \sup_{t\in\mathbb{R}} \|\widetilde{k}_i(t)\|_q \cdot \left(\int_{\mathbb{R}} |g_i(s, x_k(b_i(s))) - g_i(s, x(b_i(s)))|^p ds\right)^{1/p} \\ &\leq K_i \cdot \left(\int_{\mathbb{R}} |g_i(s, x_k(b_i(s))) - g_i(s, x(b_i(s)))|^p ds\right)^{1/p}. \end{aligned} \tag{2.3}$$

On the other hand, Let $r' = \sup_k \|x_k\| + 1$. Then $r' < +\infty$. By (H3), for almost all $t \in \mathbb{R}$, we have

$$|g_i(t, x_k(b_i(t))) - g_i(t, x(b_i(t)))| \leq 2\mu_i^{r'}(t)$$

and

$$\lim_{k\to\infty} g_i(t, x_k(b_i(t))) = g_i(t, x(b_i(t))).$$

Thus, by using the Lebesgue's dominated convergence theorem, we get

$$\lim_{k\to\infty} \int_{\mathbb{R}} |g_i(s, x_k(b_i(s))) - g_i(s, x(b_i(s)))|^p ds = 0,$$

which and (2.3) yield that $B_i x_k \to B_i x$ in $\mathcal{P}_T(\mathbb{R})$.

Now, let us prove that every $B_i(C)$ is precompact. Since for all $t \in \mathbb{R}$ and $x \in C$,

$$
\begin{aligned}
|(B_i x)(t)| &\leq \int_{\mathbb{R}} |k_i(t,s)| \cdot |g_i(s, x(b_i(s)))| ds \\
&\leq \int_{\mathbb{R}} |k_i(t,s)| \cdot |\mu_i^M(s)| ds \\
&\leq \left( \int_{\mathbb{R}} |k_i(t,s)|^q ds \right)^{1/q} \cdot \left( \int_{\mathbb{R}} |\mu_i^M(s)|^p ds \right)^{1/p} \\
&\leq K_i \cdot \|\mu_i^M\|_p < +\infty,
\end{aligned}
$$

$B_i(C)$ is uniformly bounded. In addition, for all $t_1, t_2 \in \mathbb{R}$ and $x \in C$, we have

$$
\begin{aligned}
|(B_i x)(t_1) - (B_i x)(t_2)| &\leq \int_{\mathbb{R}} |k_i(t_1,s) - k_i(t_2,s)| \cdot |g_i(s, x(b_i(s)))| ds \\
&\leq \left( \int_{\mathbb{R}} |k_i(t_1,s) - k_i(t_2,s)|^q ds \right)^{1/q} \cdot \left( \int_{\mathbb{R}} |\mu_i^M(s)|^p ds \right)^{1/p} \\
&= \|\widetilde{k}_i(t_1) - \widetilde{k}_i(t_2)\|_q \cdot \|\mu_i^M\|_p. \qquad (2.4)
\end{aligned}
$$

Since $t \to \widetilde{k}_i(t)$ is a continuous $T$-periodic function from $\mathbb{R}$ to $L^q(\mathbb{R})$, $t \to \widetilde{k}_i(t)$ is uniformly continuous on $\mathbb{R}$. Combining this with (2.4), we know that $B_i(C)$ is equicontinuous. Then, by using the well-known Arzéla-Ascoli Theorem, $B_i(C)$ is precompact. Thus, the assumption (b) of Theorem 2.1 holds.

Next, we show that the assumption (c) of Theorem 2.1 holds. Let $y \in C$ and $x = \sum_{i=1}^{n} A_i x \cdot B_i y$. Denote $\|x\| = \lambda$. We claim that $\lambda \leq M$. In fact, if $\lambda > M$, by (H5), we have

$$
\begin{aligned}
\lambda = \|x\| &= \left\| \sum_{i=1}^{n} A_i x \cdot B_i y \right\| \\
&\leq \sup_{t \in \mathbb{R}} \sum_{i=1}^{n} |f_i(t, x(a_i(t)))| \cdot \left| \int_{\mathbb{R}} k_i(t,s) g_i(s, y(b_i(s))) \right| ds \\
&\leq \sum_{i=1}^{n} \left[ \sup_{t \in \mathbb{R}, |x| \leq \lambda} |f_i(t,x)| \cdot K_i \cdot \|\mu_i^M\|_p \right] \\
&< \lambda,
\end{aligned}
$$

which is a contradiction. So $\lambda \leq M$, and thus $x \in C$.

At last, it follows from

$$
\sum_{i=1}^{n} K_i \|\mu_i^M\|_p \cdot \phi_i(r) < r, \quad \forall r > 0
$$

and

$$
\sup_{x \in C} \|B_i x\| \leq K_i \|\mu_i^M\|_p
$$

that

$$\sum_{i=1}^{n}\left[\sup_{x\in C}\|B_i x\|\cdot\phi_i(r)\right] < r, \quad \forall r>0.$$

Now, by Theorem 2.1, there exists $x_0\in C$ such that

$$x_0=\sum_{i=1}^{n}A_i x_0\cdot B_i x_0,$$

which means that $x_0(t)$ is a continuous $T$-periodic solution of Eq. (1.1). □

To complete this paper, we give an example to illustrate how Theorem 2.3 can be used.

**Example 2.4.** Let $n=2$, $p=1$, $q=\infty$,

$$a_1(t)=t-1,\quad b_1(t)=t^2,\quad a_2(t)=2t,\quad b_2(t)=|t|,$$

$$f_1(t,x)=\frac{x}{10}\sin t,\quad g_1(t,x)=\frac{\sin(xe^{t^2})}{2(1+t^2)},\quad k_1(t,s)=\frac{\cos t}{1+s^2},$$

and

$$f_2(t,x)=\frac{\cos t\sin x}{20},\quad g_2(t,x)=\frac{\arctan(tx)}{1+t^2},\quad k_2(t,s)=e^{-s^2}\sin t.$$

It is easy to see that (H1) and (H2) hold with $T=2\pi$, $\phi_1(r)=\frac{r}{10}$ and $\phi_2(r)=\frac{r}{20}$. In addition, we have

$$|g_1(t,x)|\le\frac{1}{2(1+t^2)},\quad |g_2(t,x)|\le\frac{\pi}{2}\cdot\frac{1}{1+t^2}.$$

Thus (H3) holds with $\mu_1^r(t)\equiv\frac{1}{2(1+t^2)}$ and $\mu_2^r(t)\equiv\frac{\pi}{2}\cdot\frac{1}{1+t^2}$. By a direct calculation, we can get (H4) holds and

$$K_1=\pi,\quad K_2=\sqrt{\pi}.$$

Letting $M=1$, we have

$$\sum_{i=1}^{2}K_i\|\mu_i^M\|_1\cdot\phi_i(r)\le\frac{\pi^2 r}{20}+\frac{\pi^2\sqrt{\pi}\cdot r}{40}<r,\quad \forall r>0,$$

and

$$\sum_{i=1}^{2}\left[\sup_{t\in\mathbb{R},|x|\le\lambda}|f_i(t,x)|\cdot K_i\cdot\|\mu_i^M\|_1\right]\le\frac{\pi^2\lambda}{20}+\frac{\pi^2\sqrt{\pi}}{40}<\lambda,\quad\forall\lambda>1.$$

Thus, (H5) holds.

By using Theorem 2.3, we know that the following functional integral equation

$$x(t)=\frac{\sin t\cos t\cdot x(t-1)}{20}\cdot\int_{\mathbb{R}}\frac{\sin[x(s^2)e^{s^2}]}{(1+s^2)^2}ds+\frac{\sin t\cos t\sin[x(2t)]}{20}\cdot\int_{\mathbb{R}}\frac{\arctan[sx(|s|)]}{1+s^2}e^{-s^2}ds$$

has a continuous $2\pi$-periodic solution.

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