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Ulam-Hyers stability for partial differential inclusions

Vasile L. Lazăr

"Vasile Goldiş" Western University Arad, Romania.

M.Viteazul str., no.26, 440030, Satu Mare, Romania

E-mail: vasilazar@yahoo.com

Abstract

Using the weakly Picard operator technique, we will present Ulam-Hyers stability results for integral inclusions of Fredholm and Volterra type and for the Darboux problem associated to a partial differential inclusion.

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1 Introduction

The Ulam stability (Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin,...) of various functional equations has been investigated by many authors (see [14], [15], [6], [8], [3], [9], [13], [25], [30], [31]). There are some results for differential equations ([16], [18], [19], [23], [36]), integral equations ([5], [17], [35]), for difference equations [4], [28], [29], [44]), etc. ([10], [11], [32]). For other results in the case of fixed point problems and coincidence point problems see [2], [26], [34], [37], [39].

The aim of this paper is to present existence and Ulam-Hyers stability results for some problems associated with integral inclusions and partial differential inclusions.

2 Ulam-Hyers stability via weakly Picard operators

Let (X, d) be a metric space and consider the following families of subsets of X:

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\},\ P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded}\},$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\},\ P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\}.$$

We will denote by $\bar{B}(x_0, r)$ the closure of $B(x_0, r)$ in (X, d), where $B(x_0, r) := \{x \in X | d(x_0, x) < r\}$ is the open ball centered at $x_0 \in X$ with radius $r > 0$ and by $\tilde{B}(x_0, r)$ the closed ball centered at $x_0 \in X$ with radius $r > 0$, i.e., $\tilde{B}(x_0, r) := \{x \in X | d(x_0, x) \leq r\}$.

If (X, d) is a metric space, then the gap functional in $P(X)$ is defined as

$$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}.$$

In particular, if $x_0 \in X$ then $D_d(x_0, B) := D_d(\{x_0\}, B)$.

We will denote by H the generalized Pompeiu-Hausdorff functional on $P(X)$, defined as

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) = \max\{\sup_{a \in A} D_d(a, B), \sup_{b \in B} D_d(b, A)\}.$$

Let (X, d) be a metric space. If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called a fixed point for F if and only if $x \in F(x)$. The set $Fix(F) := \{x \in X |\ x \in F(x)\}$ is called the fixed point set of F, while $SFix(F) = \{x \in X |\ \{x\} = F(x)\}$ is called the strict fixed point set of F.

For a multivalued operator $F : X \to P(Y)$ the graph of F will be denoted by

$$Graph(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Notice that $f : X \to Y$ is a selection for $F : X \to P(Y)$ if $f(x) \in F(x)$, for each $x \in X$.

In particular, when F is a singlevalued operator, we obtain the similar well-known concepts in fixed point theory.

For the following notions see I.A. Rus [33] and [37], I.A. Rus, A. Petruşel, A. Sîntămărian [40] and A. Petruşel [27].

Definition 2.1. *Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, f is a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from $x \in X$ converges, for all $x \in X$ and its limit is a fixed point of f.*

If f is a WPO, then we consider the operator

$$f^\infty : X \to X \text{ defined by } f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that $f^\infty(X) = Fix(f)$.

Definition 2.2. *Let (X, d) be a metric space, $f : X \to X$ be a WPO and $c > 0$ be a real number. By definition, the operator f is a c-weakly Picard operator (briefly c-WPO) if and only if*

$$d(x, f^\infty(x)) \le c\ d(x, f(x)),\ \textit{for all } x \in X.$$

In the multivalued case we have the following concepts.

Definition 2.3. *Let (X, d) be a metric space, and $F : X \to P_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard (briefly MWP) operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence*

$(x_n)_{n\in\mathbb{N}}$ *such that:*

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, *for each* $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a fixed point of* F.

Remark 2.1. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *satisfying condition* (i) *and* (ii) *in the Definition* 2.3 *is called a sequence of successive approximations of* F *starting from* $(x, y) \in Graph(F)$.

If $F : X \to P(X)$ is a MWP operator, then we define $F^{\infty} : Graph(F) \to P(FixF)$ by the formula $F^{\infty}(x, y) := \{ \ z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z \ \}$.

Definition 2.4. Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous at 0 and $\psi(0) = 0$. Then $F : X \to P(X)$ is said to be a multivalued ψ-weakly Picard operator if it is a multivalued weakly Picard operator and there exists a selection $f^{\infty} : Graph(F) \to Fix(F)$ of F^{∞} such that

$$d(x, f^{\infty}(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in Graph(F).$$

If there exists $c > 0$ such that $\psi(t) = ct$, for each $t \in \mathbb{R}_+$, then F is called a multivalued c-weakly Picard operator.

Recall that, if (X, d) is a metric space, then $F : X \to P_{cl}(X)$ is said to be a multivalued α-contraction if $\alpha \in [0, 1)$ and

$$H_d(F(x), F(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X,$$

Example 2.1. *Let* (X, d) *be a complete metric space and* $F : X \to P_{cl}(X)$ *be a multivalued* α*-contraction. Then* F *is a c-MWP operator, where* $c = (1 - \alpha)^{-1}$.

For the theory of weakly Picard operators, see [33] for the singlevalued case and [40] and [27] for the multivalued one.

We present now some Ulam-Hyers stability concepts for the fixed point problem associated with a multivalued operator.

Definition 2.5. Let (X, d) be a metric space and $F : X \to P(X)$ be a multivalued operator. The fixed point inclusion

$$x \in F(x),\ x \in X \tag{2.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$D_d(y, F(y)) \le \varepsilon \tag{2.2}$$

there exists a solution x^* of the fixed point inclusion (2.1) such that

$$d(y^*, x^*) \le \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (2.1) is said to be Ulam-Hyers stable.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the fixed point inclusion (2.1) for multivalued operators with compact values.

Theorem 2.1. *(I.A. Rus [37]) Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be a multivalued ψ-weakly Picard operator. Then, the fixed point inclusion (2.1) is generalized Ulam-Hyers stable.*

3 Existence and Ulam-Hyers stability for integral inclusions

We consider here some integral inclusion of Fredholm and Volterra type. Throughout this section we will denote by $\|\cdot\|$ the supremum norm in $C([a, b], \mathbb{R}^n)$ and by $|\cdot|$ a norm in $\mathbb{R}^n$.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [38]) if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.

Recall also the notion of strict comparison function. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [38]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$.

The mappings $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(t) = at$ (where $a \in [0, 1[$) and respectively $\varphi(t) = \frac{t}{1+t}$, for each $t \in \mathbb{R}_+$ are examples of strict comparison functions.

The following result, a generalization of Covitz-Nadler fixed point principle (see [24], [7]) is known in the literature as Węgrzyk's fixed point theorem.

Theorem 3.2. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function and*

$$H(F(x_1), F(x_2)) \leq \varphi(d(x_1, x_2)), \text{ for all } x_1, x_2 \in X.$$

Then $Fix(F)$ is nonempty and for any $x_0 \in X$ there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

Remark 3.2. *It is worth noting that, in the conditions of above result, if additionally $SFix(F) \neq \emptyset$, then $Fix(F) = SFix(F) = \{x^*\}$, see Sîntămărian [42]. Moreover, in this case, if the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(t) := t - \varphi(t)$ is strictly increasing and onto, then, since*

$$d(x, x^*) \leq D(x, F(x)) + H(F(x), F(x^*)) \leq D(x, F(x)) + \varphi(d(x, x^*)), \text{ for all } x \in X,$$

we get that

$$d(x, x^*) \leq \beta^{-1}(D(x, F(x)), \text{ for all } x \in X,$$

This immediately implies that the fixed point problem $x \in F(x)$, $x \in X$ is generalized Ulam-Hyers stable with function β^{-1}.

Another Ulam-Hyers stability result, more efficient for applications, was proved in [21].

Theorem 3.3. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then:*

(i) (existence of the fixed point) F is a MWP operator;

(ii) (Ulam-Hyers stability for the fixed point inclusion) If additonally $\varphi(qt) \le q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, then F is a ψ-MWP operator, with $\psi(t) := t + s(t)$, for each $t \in \mathbb{R}_+$ (where $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$);

(iii) (data dependence of the fixed point set) Let $S : X \to P_{cl}(X)$ be a multivalued φ-contraction and $\eta > 0$ be such that $H(S(x), F(x)) \le \eta$, for each $x \in X$. Suppose that $\varphi(qt) \le q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$. Then $H(Fix(S), Fix(F)) \le \psi(\eta)$.

We will present now, using the above mentioned results, some existence and Ulam-Hyers stability theorems for multivalued operatorial inclusions.

Consider first the following Fredholm type integral inclusion.

$$x(t) \in \int_a^b K(t, s, x(s))ds + g(t), \; t \in [a, b]. \tag{3.3}$$

The main result concerning the stability of the Fredholm integral inclusion (3.3) is the following.

Theorem 3.4. *Let $K : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ and $g : [a, b] \to \mathbb{R}^n$ such that:*

(a) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ such that for each $t \in [a, b]$ and $u \in \mathbb{R}^n$ we have $K(t, s, u) \subset M(s)B(0; 1)$, a.e. $s \in [a, b]$;

(b) for each $u \in \mathbb{R}^n$ $K(\cdot, \cdot, u) : [a, b] \times [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is jointly measurable;

(c) for each $(s, u) \in [a, b] \times \mathbb{R}^n$ $K(\cdot, s, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is lower semi-continuous;

(d) there exists a continuous function $p : [a,b] \times [a,b] \to \mathbb{R}_+$ *with* $\sup_{t\in[a,b]} \int_a^b p(t,s)ds \leq 1$ *and a strict comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for each* $(t,s) \in [a,b] \times [a,b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t,s,u), K(t,s,v)) \leq p(t,s) \cdot \varphi(|u-v|); \tag{3.4}$$

(e) g *is continuous.*

Then the following conclusions hold:

(a) the integral inclusion (3.3) has least one solution, i.e., there exists $x^* \in C([a,b], \mathbb{R}^n)$ *which satisfies (3.3), for each* $t \in [a,b]$*.*

(b) If additionally $\varphi(qt) \leq q\varphi(t)$ *for every* $t \in \mathbb{R}_+$ *(where* $q > 1$*) and* $t = 0$ *is a point of uniform convergence for the series* $\sum_{n=1}^{\infty} \varphi^n(t)$*, then the integral inclusion (3.3) is generalized Ulam-Hyers stable with function* ψ *(where* $\psi(t) := t + s(t)$*, for each* $t \in \mathbb{R}_+$ *and* $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$*), i.e., for each* $\varepsilon > 0$ *and for any* ε*-solution* y *of (3.3), that is any* $y \in C([a,b], \mathbb{R}^n)$ *for which there exists* $u \in C([a,b], \mathbb{R}^n)$ *such that*

$$u(t) \in \int_a^b K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon,\ \text{for each } t \in [a,b]),$$

there exists a solution x^* *of the integral inclusion (3.3) such that*

$$|y(t) - x^*(t)| \leq \psi(\varepsilon),\ \text{for each } t \in [a,b].$$

Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (3.4) holds.

Proof. (a) Define the multivalued operator $T : C([a,b], \mathbb{R}^n) \to \mathcal{P}(C([a,b], \mathbb{R}^n))$ by

$$T(x) := \left\{ v \in C([a,b], \mathbb{R}^n) |\ v(t) \in \int_a^b K(t,s,x(s))ds + g(t),\ t \in [a,b] \right\}.$$

Then, (3.3) is equivalent to the fixed point inclusion

$$x \in T(x),\ x \in C([a,b],\mathbb{R}^n). \tag{3.5}$$

The proof is organized in several steps. We successively prove:

1. $T(x) \in P_{cp}(C([a,b],\mathbb{R}^n))$.

From (e) and Theorem 2 in Rybiński [41] we have that for each $x \in C([a,b],\mathbb{R}^n)$ there exists $k(t,s) \in K(t,s,x(s))$, for all $(t,s) \in [a,b]$, such that $k(t,s)$ is integrable with respect to s and continuous with respect to t. Then $v(t) := \int_a^b k(t,s)ds + g(t)$, has the property $v \in T(x)$. Moreover, from (a) and (b), via Theorem 8.6.3. in Aubin and Frankowska [1], we get that $T(x)$ is a compact set, for each $x \in C([a,b],\mathbb{R}^n)$.

2. $H(T(x_1),T(x_2)) \le \varphi(\|x_1 - x_2\|)$, for each $x_1, x_2 \in C([a,b],\mathbb{R}^n)$.

Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality (3.4) is strict. Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \int\limits_a^b K(t,s,x_1(s))ds + g(t)$, $t \in [a,b]$. It follows that $v_1(t) = \int\limits_a^b k_1(t,s)ds + g(t)$, $t \in [a,b]$, for some $k_1(t,s) \in K(t,s,x_1(s))$, $(t,s) \in [a,b] \times [a,b]$.

From (d) we have $H(K(t,s,x_1(s)),K(t,s,x_2(s)) < p(t,s)\varphi(|x_1(s) - x_2(s)|) \le p(t,s)\varphi(\|x_1 - x_2\|)$. Thus, there exists $w \in K(t,s,x_2(s))$ such that $|k_1(t,s) - w| \le p(t,s)\varphi(\|x_1 - x_2\|)$, for $t,s \in [a,b]$.

Let us define $U : [a,b] \times [a,b] \to P(\mathbb{R}^n)$, by $U(t,s) = \{w|\ |k_1(t,s) - w| \le p(t,s)\varphi(\|x_1 - x_2\|)\}$. Since the multi-valued operator $V(t,s) := U(t,s) \cap K(t,s,x_2(s))$ is jointly measurable and lower semi-continuous in t there exists $k_2(t,s)$ a selection for V, jointly measurable (and, hence, integrable in s) and continuous in t. Hence, $k_2(t,s) \in K(t,s,x_2(s))$ and $|k_1(t,s) - k_2(t,s)| \le p(t,s)\varphi(\|x_1 - x_2\|)$, for each $t,s \in [a,b]$.

Consider $v_2(t) = \int\limits_a^b k_2(t,s)ds + g(t)$, $t \in [a,b]$. Then, we have:

$|v_1(t)-v_2(t)| \leq \int_a^b |k_1(t,s)-k_2(t,s)|ds \leq \int_a^b p(t,s)\varphi(\|x_1-x_2\|)ds \leq \varphi(\|x_1-x_2\|)$.

A similar relation can be obtained by interchanging the roles of x_1 and x_2. Thus the second step follows.

The first conclusion follows by the above mentioned Węgrzyk's fixed point theorem, see Theorem 3.3 (i) (see also [43]).

(b) We will prove that the fixed point inclusion problem (3.5) is generalized Ulam-Hyers stable. Indeed, let $\varepsilon > 0$ and $y \in C([a,b],\mathbb{R}^n)$ for which there exists $u \in C([a,b],\mathbb{R}^n)$ such that

$$u(t) \in \int_a^b K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

$$\text{and } \|u-y\| \leq \varepsilon.$$

Then $D_{\|\cdot\|}(y,T(y)) \leq \varepsilon$. Moreover, by the above proof we have that T is a multivalued φ-contraction and using Theorem 3.3(i)-(ii), we obtain that T is a multivalued ψ-weakly Picard operator. Then, by Theorem 2.1 we obtain that the fixed point problem (3.5) is generalized Ulam-Hyers stable. Thus, the integral inclusion (3.4) is generalized Ulam-Hyers stable.

Concerning the last conclusion of the theorem, we apply Theorem 3.3 (iii). □

A second application concerns an integral inclusion of Volterra type.

$$x(t) \in \int_a^t K(t,s,x(s))ds + g(t),\ t \in [a,b]. \tag{3.6}$$

By a similar method, we can prove the following.

Theorem 3.5. *Let $K : [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ and $g : [a,b] \to \mathbb{R}^n$ such that:*

(a) there exists an integrable function $M : [a,b] \to \mathbb{R}_+$ such that for each $t \in [a,b]$ and $u \in \mathbb{R}^n$ we have $K(t,s,u) \subset M(s)B(0;1)$, a.e. $s \in [a,b]$;

(b) for each $u \in \mathbb{R}^n$ $K(\cdot,\cdot,u) : [a,b] \times [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ is jointly measurable;

(c) for each $(s,u) \in [a,b] \times \mathbb{R}^n$ $K(\cdot,s,u) : [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ is lower semi-continuous;

(d) there exists a continuous function $p : [a,b] \to \mathbb{R}_+^*$ and a strict comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(\lambda t) \le \lambda\varphi(t)$, for each $t \in \mathbb{R}_+$ and each $\lambda \ge 1$, such that for each $(t,s) \in [a,b] \times [a,b]$ and each $u,v \in \mathbb{R}^n$ we have that

$$H(K(t,s,u),K(t,s,v)) \le p(s) \cdot \varphi(|u-v|); \tag{3.7}$$

(e) g is continuous.

Then the following conclusions hold:

(a) the integral inclusion (3.6) has at least one solution, i.e., there exists $x^* \in C([a,b],\mathbb{R}^n)$ which satisfies (3.6) for each $t \in [a,b]$;

(b) If additionally $\varphi(qt) \le q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, then the integral inclusion (3.3) is generalized Ulam-Hyers stable with function ψ (where $\psi(t) := t + s(t)$, for each $t \in \mathbb{R}_+$ and $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$), i.e., for each $\varepsilon > 0$ and for any ε-solution y of (3.6), that is, any $y \in C([a,b],\mathbb{R}^n)$ for which there exists $u \in C([a,b],\mathbb{R}^n)$ such that

$$u(t) \in \int_a^t K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$|u(t) - y(t)| \le \varepsilon,\ \textit{for each } t \in [a,b]),$$

there exists a solution x^* of the integral inclusion (3.6) such that

$$|y(t) - x^*(t)| \le \psi(c\varepsilon),\ \textit{for each } t \in [a,b] \textit{ and some } c > 0.$$

Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (3.7) holds.

Proof. We consider the multi-valued operator $T : C([a,b],\mathbb{R}^n) \to \mathcal{P}(C([a,b],\mathbb{R}^n))$

$$T(x) := \left\{ v \in C([a,b],\mathbb{R}^n) |\ v(t) \in \int_a^t K(t,s,x(s))ds + g(t),\ t \in [a,b] \right\}.$$

Then, (3.6) is equivalent to the fixed point inclusion

$$x \in T(x),\ x \in C([a,b],\mathbb{R}^n). \tag{3.8}$$

As in the proof of Theorem 3.4 we obtain $T(x) \in P_{cp}(C([a,b],\mathbb{R}^n))$. Next, we will prove that T is a multivalued φ-contraction on $C([a,b],\mathbb{R}^n)$.

Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality (3.7) is strict. Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \int_a^t K(t,s,x_1(s))ds + g(t)$, $t \in [a,b]$. It follows that $v_1(t) = \int_a^b k_1(t,s)ds + g(t)$, $t \in [a,b]$, for some $k_1(t,s) \in K(t,s,x_1(s))$, $(t,s) \in [a,b] \times [a,b]$.

From (d) we have $H(K(t,s,x_1(s)), K(t,s,x_2(s))) < p(s)\varphi(|x_1(s) - x_2(s)|)$. Thus, there exists $w \in K(t,s,x_2(s))$ such that $|k_1(t,s) - w| \leq p(s)\varphi(|x_1(s) - x_2(s)|)$, for $t, s \in [a,b]$.

Let us define $U : [a,b] \times [a,b] \to P(\mathbb{R}^n)$, by $U(t,s) = \{w |\ |k_1(t,s) - w| \leq p(t,s)\varphi(|x_1(s) - x_2(s)|)\}$. Since the multivalued operator $V(t,s) := U(t,s) \cap K(t,s,x_2(s))$ is jointly measurable and lower semi-continuous in t there exists $k_2(t,s)$ a selection for V, jointly measurable (hence, integrable in s) and continuous in t. Hence, $k_2(t,s) \in K(t,s,x_2(s))$ and $|k_1(t,s) - k_2(t,s)| \leq p(s)\varphi(|x_1(s) - x_2(s)|)$, for each $t, s \in [a,b]$.

Consider $v_2(t) = \int_a^t k_2(t,s)ds + g(t)$, $t \in [a,b]$. We denote by $\|\cdot\|_B$ a Bielecki-type norm in $C([a,b],\mathbb{R}^n)$, given by $\|x\|_B := \sup_{t\in[a,b]} (|x(t)|e^{-q(t)}))$, where $q(t) := \int_a^t p(s)ds$.

Then, for each $t \in [a,b]$, we have:

$|v_1(t) - v_2(t)| \leq \int_a^t |k_1(t,s) - k_2(t,s)|ds \leq \int_a^t p(s)\varphi(|x_1(s) - x_2(s)|)ds = \int_a^t p(s)\varphi(e^{q(s)}|x_1(s) - x_2(s)|e^{q(s)})ds \leq \int_a^t p(s)e^{q(s)}\varphi(\|x_1 - x_2\|_B)ds = \varphi(\|x_1 - x_2\|_B)(e^{q(t)} - e^{q(a)}) \leq \varphi(\|x_1 - x_2\|_B)e^{q(t)}$. Thus, we immediately get

$$\|v_1 - v_2\|_B \leq \varphi(\|x_1 - x_2\|_B).$$

A similar relation can be obtained by interchanging the roles of x_1 and x_2. Thus, we have that

$$H_{\|\cdot\|_B}(T(x_1), T(x_2)) \leq \varphi(\|x_1 - x_2\|_B), \text{ for each } x_1, x_2 \in C([a,b], \mathbb{R}^n),$$

which proves that T is a multivalued φ-contraction. The conclusion (a) follows by the above mentioned Węgrzyk's fixed point theorem, see Theorem 3.3 (i) (see also [43]).

(b) We will prove that the fixed point inclusion problem (3.6) is generalized Ulam-Hyers stable. For this purpose, it is enough to prove that the fixed point inclusion problem (3.8) is generalized Ulam-Hyers stable. For this purpose, let $\varepsilon > 0$ and $y \in C([a,b], \mathbb{R}^n)$ for which there exists $u \in C([a,b], \mathbb{R}^n)$ such that

$$u(t) \in \int_a^t K(t,s,y(s))ds + g(t), \ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon, \text{ for each } t \in [a,b].$$

Notice that

$$\|\cdot\|_B \leq \|\cdot\| \leq \|\cdot\|_B \, e^{\tau q(b)}.$$

Then, we obtain that $\|u - y\|_B \leq \|u - y\| \leq \varepsilon$. Thus, $D_{\|\cdot\|_B}(y, T(y)) \leq \varepsilon$. Moreover, by the above proof, T is a multivalued φ-contraction with respect to $\|\cdot\|_B$ and, thus, T is a MWP operator. Using Theorem 3.3(i)-(ii), we obtain that T is a multivalued ψ-MWP operator. Thus, conclusion (b) is a consequence of Theorem 2.1. Hence, there exists a solution x^* of the integral inclusion (3.6) such that

$$\|y - x^*\|_B \leq \psi(\varepsilon).$$

Hence,

$$|y(t) - x^*(t)| \leq \psi(e^{\tau q(b)}\varepsilon), \text{ for each } t \in [a, b].$$

Concerning the last conclusion of the theorem, we apply Theorem 3.3 (iii).

□

4 Existence and Ulam-Hyers stability for partial differential inclusions

Let us consider the following Darboux problem for a second order differential inclusion

$$(4.9) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} \in F(x, y, u(x, y)) \\ u(x, 0) = \lambda(x, 0), \; u(0, y) = \lambda(0, y), \end{cases}$$

where $F : I_1 \times I_2 \times \mathbb{R}^m \to P_{cl}(\mathbb{R}^m)$ (with $I_i = [0, T_i]$, $i \in \{1, 2\}$) and $\lambda(x, y) = \alpha(x) + \beta(y) - \alpha(0)$ (with α, β continuous functions on I_1 respectively I_2 and $\alpha(0) = \beta(0)$).

Denote by $\Pi = I_1 \times I_2$ and let $a > 0$. By L^1 we will denote the Banach space of all measurable Lebesgue functions $\eta : \Pi \to \mathbb{R}^m$, endowed with the norm

$$\|\eta\|_1 = \int\int_{\Pi} e^{-a(x+y)} |\eta(x, y)| dx dy.$$

Let C be the Banach space of continuous functions $u : \Pi \to \mathbb{R}^m$, with the norm $\|u\|_C = \sup_{(x,y)\in\Pi} |u(x, y)|$ and let $\tilde{C}$ be the linear subspace of C consisting of all $\lambda \in C$ such that there exist continuous functions $\alpha \in C(I_1, \mathbb{R}^m)$ and $\beta \in C(I_2, \mathbb{R}^m)$ with $\alpha(0) = \beta(0)$ satisfying $\lambda(x, y) = \alpha(x) + \beta(y) - \alpha(0)$, for all $x, y \in I_1 \times I_2$. Obviously, $\tilde{C}$ with the norm of C is a separable Banach space.

By definition, the Darboux problem (4.9) is called Ulam-Hyers stable if for each $\varepsilon > 0$ and for any ε-solution w of (4.9), there exists a solution u^* of (4.9) such that $|w(x, y) - u^*(x, y)| \leq c\varepsilon$, for each $(x, y) \in \Pi$ and for some $c > 0$.

We have the following existence and Ulam-Hyers stability result.

Theorem 4.6. *Consider the Darboux Problem (4.9) and suppose that the above mentioned conditions hold. Suppose also that the following assumptions hold:*

i) for each $u \in \mathbb{R}^m$, $F(\cdot, \cdot, u)$ *is measurable;*

ii) there exists $k > 0$ *such that a.e.* $(x, y) \in I_1 \times I_2$ *the multifunction* $F(x, y, \cdot)$ *is k-Lipschitz;*

iii) $a > \sqrt{k}$.

Then, the Darboux Problem (4.9) has at least one solution and it is Ulam-Hyers stable.

Proof. For $\lambda \in \tilde{C}, \eta \in L^1$ define

$$T_\lambda(\eta) := \{\mu \in L^1 : \mu(x, y) \in M_{\lambda,\eta}(x, y), \text{ a. e. on } \Pi\},$$

where

$$M_{\lambda,\eta}(x, y) = F(x, y, \lambda(x, y) + \int_0^x \int_0^y \eta(s, t)dsdt), (s, t) \in \Pi.$$

Notice that F_{T_λ} coincides with the solution set of the considered problem. Moreover, we have that $T_\lambda : L^1 \to P_{cl}(L^1)$ and it is a MWP operator. Indeed, we have

$$H_1(T_\lambda(\eta_1), T_\lambda(\eta_2)) \leq \frac{k}{a^2} \cdot \|\eta_1 - \eta_2\|_1, \text{ for all } \lambda \in \tilde{C} \text{ and } \eta_1, \eta_2 \in L^1.$$

Thus, T_λ is a $\frac{k}{a^2}$-multivalued contraction on L^1 and hence is a MWP operator. Thus, there exists $u^* \in L^1$ a fixed point for T_λ, which is also a solution for the Darboux Problem (4.9). For the second part of our theorem it is enough to prove that T_λ is a multivalued c-weakly Picard operator. Since T_λ is a $\frac{k}{a^2}$-multivalued contraction on L^1, we immediately get (see Example 2.1) that T_λ is a multivalued c-weakly Picard operator with $c := \frac{1}{1-ka^{-2}}$. Thus, the second conclusion follows by Theorem 2.1. □

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