

BIFURCATION OF A NONLINEAR ELLIPTIC SYSTEM FROM THE FIRST EIGENVALUE

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ABSTRACT. We study the following bifurcation problem in a bounded domain Ω in \mathbb{R}^N :

$$\begin{cases} -\Delta_p u = \lambda |u|^\alpha |v|^\beta v + f(x, u, v, \lambda) & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^\alpha |v|^\beta u + g(x, u, v, \lambda) & \text{in } \Omega \\ (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

We prove that the principal eigenvalue λ_1 of the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^\alpha |v|^\beta v & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^\alpha |v|^\beta u & \text{in } \Omega \\ (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \end{cases}$$

is simple and isolated and we prove that $(\lambda_1, 0, 0)$ is a bifurcation point of the system mentioned above.

1. INTRODUCTION

The purpose of this paper is to illustrate a global bifurcation phenomenon for the nonlinear elliptic system

$$(BS) \quad \begin{cases} -\Delta_p u = \lambda |u|^\alpha |v|^\beta v + f(x, u, v, \lambda) & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^\alpha |v|^\beta u + g(x, u, v, \lambda) & \text{in } \Omega \\ (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases}$$

where Ω is a bounded domain not necessary regular in \mathbb{R}^N , $N \geq 1$, α, β, p and q are real numbers satisfying suitable conditions which ensure the results. The system (BS) is weakly coupled in the sense that the interaction is present only in the "source terms", while the differential terms have only one dependent variable each. The differential operator involved is the well-known p-Laplacian $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ which reduces to the ordinary Laplacian Δu , when $p = 2$. The nonlinearities f and g satisfy some hypotheses to be specified later. To

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system (BS) we associate the eigenvalue problem system

$$(ES) \quad \begin{cases} -\Delta_p u = \lambda |u|^\alpha |v|^\beta v & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^\alpha |v|^\beta u & \text{in } \Omega \\ (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

We say that λ is an eigenvalue of (ES) if there exists a nontrivial pair $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ that satisfies (ES) in the following sense

$$(S) \quad \begin{cases} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int_{\Omega} [\lambda |u|^\alpha |v|^\beta v \phi] \, dx \\ \int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \psi \, dx = \int_{\Omega} [\lambda |u|^\alpha |v|^\beta u \psi] \, dx, \end{cases}$$

for any $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

The goal of this work is the study of the main properties (simplicity, isolation) of the least positive eigenvalue denote λ_1 of (ES) in theorem 3.1. These properties are well-known in the scalar case of one equation, we refer the reader to the bibliography contained in [14] where several results are cited.

We prove that the eigenvectors (u, v) associated to λ_1 have definite sign in Ω . So, we show that this property implies that λ_1 is simple and isolated in the spectrum. Note that our result of isolation follows by adaptation a technique used by Anane [1] for scalar p -Laplacian in a smooth bounded domain. Concerning the bifurcation problem, we prove the existence of bifurcation branch of nontrivial solutions of (BS) from λ_1 , (see Theorem 3.2.). Here we use abstract methods of nonlinear functional analysis based on the generalized topological degree in order to get a new result.

System (BS) has been studied by Chaïb [6] in the case $\Omega = \mathbb{R}^N$. The author extended Diaz-Saás inequality in \mathbb{R}^N and claimed that λ_1 is simple. In [12], the authors obtained similar result by considering for a quite different system. In the case of a bounded domain sufficiently regular the following eigenvalue system

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} |v|^{\beta+1} u & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v & \text{in } \Omega \\ (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \end{cases}$$

was studied in [5], where the author proved only the simplicity result of the first eigenvalue.

Here, we address to (ES) and (BS) without any regularity assumption on the boundary $\partial\Omega$.

Bifurcation problems in the case of one equation in a smooth bounded domain involving p -Laplacian operator were studied by [2],[9] and [10] under some restrictive conditions on the nonlinearity sources. In any bounded domain, we cite recent results of [7] and [8].

A global bifurcation result from the first eigenvalue of a particular system is considered in [13] under some restrictive hypotheses on the nonlinearities f and g to be specified at the end of this paper.

In this work, we consider a bifurcation system for which we investigate the system improving the conditions on the nonlinearities f and g in any bounded domain with some condition of homogeneity type connecting p , q , α and β . For this end we use a generalized degree of Leray-Schauder.

This paper is organized as follows: in Section 2 we introduce some assumptions, definitions and we prove some auxiliary results that are the key point for proving our results; Section 3 contains a results and proofs of simplicity and isolation of λ_1 , finally we verify that the topological degree has a jump when λ crosses λ_1 , which implies the bifurcation results.

2. ASSUMPTIONS AND PRELIMINARIES

Through this paper Ω will be a bounded domain of \mathbb{R}^N . $W_0^{1,p}(\Omega)$ will denote the usual Sobolev space endowed with norm $\|\nabla u\|_p = (\int_{\Omega} |\nabla u|^p dx)^{\frac{1}{p}}$. We will write $\|\cdot\|_p$ for the L^p -norm. We will also denote for $p > 1$, $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of p and p^* the critical exponent, that is $p^* = \infty$ if $p \geq N$ and $p^* = \frac{Np}{N-p}$ if $1 < p < N$.

Here we use the following norm in product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

$$\|(u, v)\| = \|\nabla u\|_p + \|\nabla v\|_q.$$

and $\|\cdot\|_*$ is the dual norm.

2.1. Assumptions. We assume that $p, q > 1$, $\alpha > 0$, $\beta > 0$ and

$$\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1 \tag{2.1}$$

(f) $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory's conditions in the first three variables and

$$\lim_{|(r,s)| \rightarrow +\infty} \frac{f(x, r, s, \lambda)}{|r|^{\gamma-1} r |s|^{\delta+1}} = 0; \tag{2.2}$$

$$f(x, r, s, \lambda) = o(|r|^{p-1}) \text{ as } |(r, s)| \rightarrow 0 \tag{2.3}$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded intervals of \mathbb{R} .

(g) $g : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory's conditions in the first three variables and

$$\lim_{|(r,s)| \rightarrow +\infty} \frac{g(x, r, s, \lambda)}{|r|^{\gamma+1}|s|^{\delta-1}} = 0; \quad (2.4)$$

$$g(x, r, s, \lambda) = o(|s|^{q-1}) \text{ as } |(r, s)| \rightarrow 0 \quad (2.5)$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded intervals of \mathbb{R} ; with

$$\gamma + 1 > p \text{ or } \delta + 1 > q$$

and

$$\delta + \gamma + 2 < \min(p^*, q^*),$$

for some nonnegative reals γ and δ .

Remark 2.1. *The homogeneity assumption (2.1) justifies the notion of eigenvalues.*

2.2. Definitions. 1. We say that $(\lambda, (u, v))$ in $\mathbb{R} \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is a solution of (BS) if for any $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$\begin{cases} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int_{\Omega} [\lambda |u|^{\alpha} |v|^{\beta} v + f(x, u, v, \lambda)] \phi \, dx \\ \int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \psi \, dx = \int_{\Omega} [\lambda |u|^{\alpha} |v|^{\beta} u + g(x, u, v, \lambda)] \psi \, dx, \end{cases} \quad (2.6)$$

for all $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We note that the pair $(\lambda, (0, 0))$ is a solution of (BS) for every $\lambda \in \mathbb{R}$. The pairs of this form will be called the trivial solutions of (BS) . We say that $P = (\bar{\lambda}, (0, 0))$ is a bifurcation point of (BS) if in any neighbourhood of P in $\mathbb{R} \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ there exists a nontrivial solution of (BS) .

2. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle \cdot, \cdot \rangle$. We shall deal with mapping T acting from X into X^* . T is demicontinuous at u in X , if $u_n \rightarrow u$ strongly in X , implies that $Tu_n \rightharpoonup Tu$ weakly in X^* . T is said to belong to the class (S_+) , if for any sequence u_n weakly convergent to u in X and $\limsup_{n \rightarrow +\infty} \langle Tu_n, u_n - u \rangle \leq 0$, it follows that $u_n \rightarrow u$ strongly in X .

2.3. Degree theory. If $T \in (S_+)$ and T is demicontinuous, then it is possible to define the degree

$$Deg[T; D; 0]$$

where $D \subset X$ is a bounded open set such that $Tu \neq 0$ for any $u \in \partial D$. Its properties are analogous to the ones of the Leray-Schauder degree (cf.[3]).

A point $u_0 \in X$ will be called a critical point of T if $Tu_0 = 0$. We say that u_0 is an isolated critical point of T if there exists $\epsilon > 0$ such that for any $u \in B_\epsilon(u_0)$ (open ball in X centered to u_0 and the radius ϵ), $Tu \neq 0$ if $u \neq u_0$. Then the limit

$$Ind(T, u_0) = \lim_{\epsilon \rightarrow 0^+} Deg[T; B_\epsilon(u_0); 0]$$

exists and is called the index of the isolated critical point u_0 .

Assume, furthermore, that T is a potential operator, i.e., for some continuously differentiable functional $\Phi : X \rightarrow \mathbb{R}$, $\Phi'(u) = Tu$, $u \in X$. Then we have the following two lemmas which we can find in [9] or [10]. These lemmas are crucial results in our bifurcation argument.

Lemma 2.1. *Let u_0 be a local minimum of Φ and an isolated critical point of T . Then*

$$Ind(T, u_0) = 1.$$

Lemma 2.2. *Assume that $\langle Tu, u \rangle > 0$ for all $u \in X$, $\|u\|_X = \rho$. Then*

$$Deg[T; B_\rho(0); 0] = 1.$$

2.4. Preliminaries. This subsection establishes an abstract framework used to prove our main results.

Let define, for $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, the operators

$$\langle A_u(u, v), (\phi, \psi) \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx,$$

$$\langle A_v(u, v), (\phi, \psi) \rangle = \int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \psi \, dx,$$

$$\langle B_u(u, v), (\phi, \psi) \rangle = \int_{\Omega} |u|^\alpha |v|^\beta v \phi \, dx,$$

$$\langle B_v(u, v), (\phi, \psi) \rangle = \int_{\Omega} |u|^\alpha |v|^\beta u \psi \, dx,$$

with

$$A(u, v) = \frac{\alpha + 1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{\beta + 1}{q} \int_{\Omega} |\nabla v|^q dx \text{ and } B(u, v) = \int_{\Omega} |u|^\alpha |v|^\beta uv dx$$

are functionals of class C^1 and

$$A_u = \frac{1}{\alpha + 1} \frac{\partial A}{\partial u}, \quad A_v = \frac{1}{\beta + 1} \frac{\partial A}{\partial v}, \quad B_u = \frac{1}{\alpha + 1} \frac{\partial B}{\partial u}, \quad B_v = \frac{1}{\beta + 1} \frac{\partial B}{\partial v}.$$

Define also

$$\langle F^\lambda(u, v), (\phi, \psi) \rangle = \int_{\Omega} f(x, u, v, \lambda) \phi dx,$$

$$\langle G^\lambda(u, v), (\phi, \psi) \rangle = \int_{\Omega} g(x, u, v, \lambda) \psi dx.$$

Remarks 2.1. (i) Due to (2.6) a pair (u, v) is a weak solution of (BS) if and only if

$$A_u(u, v) - \lambda B_u(u, v) = F^\lambda(u, v) \text{ in } (W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))'$$

and

$$A_v(u, v) - \lambda B_v(u, v) = G^\lambda(u, v) \text{ in } (W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))'$$

(ii) The operators A_u and A_v are odd and satisfy (S_+) .

Lemma 2.3. B_u and B_v are well defined, completely continuous and odd functionals.

Proof.

Step 1. Definition of B_u and B_v .

By Hölder's inequality, we have

$$|\langle B_u(u, v), (\phi, \psi) \rangle| = \left| \int_{\Omega} |u|^\alpha |v|^\beta v \phi dx \right| \leq \|u\|_p^\alpha \|v\|_q^{\beta+1} \|\phi\|_p < +\infty$$

for any $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Thus B_u is well defined. Similarly, we prove that B_v is also well defined.

Step 2. Compactness of B_u and B_v .

Let $\{(u_n, v_n)\}_{n \geq 0} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be a sequence such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We claim that $B_u(u_n, v_n) \rightarrow B_u(u, v)$ strongly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, i.e. for all $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$\sup_{\|\nabla \phi\|_p + \|\nabla \psi\|_q \leq 1} |\langle B_u(u_n, v_n) - B_u(u, v), (\phi, \psi) \rangle| = o(1) \text{ as } n \rightarrow +\infty.$$

Indeed, Hölder's inequality implies

$$\begin{aligned}
 |\langle B_u(u_n, v_n) - B_u(u, v), (\phi, \psi) \rangle| &= \left| \int_{\Omega} (|u_n|^\alpha |v_n|^\beta v_n - |u|^\alpha |v|^\beta v) \phi \, dx \right| \\
 &\leq \int_{\Omega} |v_n|^{\beta+1} (|u_n|^\alpha - |u|^\alpha) |\phi| \, dx \\
 &\quad + \int_{\Omega} |u|^\alpha (|v_n|^\beta v_n - |v|^\beta v) |\phi| \, dx \\
 &\leq \| |u_n|^\alpha - |u|^\alpha \|_{\frac{p}{\alpha}} \| \phi \|_p \| v_n \|_q^{\beta+1} \\
 &\quad + \| |u|^\alpha \|_p \| \phi \|_p \| |v_n|^\beta v_n - |v|^\beta v \|_{\frac{q}{\beta+1}}.
 \end{aligned}$$

Due to the continuity of Nemytskii operator $u \rightarrow |u|^\alpha$ (resp. $v \rightarrow |v|^\beta v$)

from $L^p(\Omega)$ into $L^{\frac{p}{\alpha}}(\Omega)$ (resp. from $L^q(\Omega)$ into $L^{\frac{q}{\beta+1}}(\Omega)$) and Rellich's Theorem, there exists $n_0 \geq 0$ such that for all $n \geq n_0$ we have

$$\| |u_n|^\alpha - |u|^\alpha \|_{\frac{p}{\alpha}} = o(1), \tag{2.7}$$

$$\| |v_n|^\beta v_n - |v|^\beta v \|_{\frac{q}{\beta+1}} = o(1). \tag{2.8}$$

Thus (v_n) is bounded in $L^p(\Omega)$. Finally from (2.7) and (2.8) we have the claim.

The compactness of B_v can be proved by the same argument.

The oddness of B_u and B_v is obvious. The proof is completed. \blacksquare

Lemma 2.4. $F^\lambda(.,.)$ and $G^\lambda(.,.)$ are well defined, completely continuous and $F^\lambda(0,0) = G^\lambda(0,0) = 0$ and we have

$$\lim_{\substack{\|\nabla u\|_p \rightarrow 0 \\ \|\nabla v\|_q \rightarrow 0}} \frac{F^\lambda(u, v)}{\|\nabla u\|_p^{p-1} + \|\nabla v\|_q^{q-1}} = 0 \text{ in } (W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))', \tag{2.9}$$

and

$$\lim_{\substack{\|\nabla u\|_p \rightarrow 0 \\ \|\nabla v\|_q \rightarrow 0}} \frac{G^\lambda(u, v)}{\|\nabla u\|_p^{p-1} + \|\nabla v\|_q^{q-1}} = 0 \text{ in } (W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))', \tag{2.10}$$

uniformly with respect to λ in bounded subsets of \mathbb{R} .

Proof.

(2.2) and (2.3) imply that for any $\epsilon > 0$ there are two real $\xi = \xi(\epsilon)$ and $M = M(\xi) > 0$ such that for a.e. $x \in \Omega$

$$|f(x, u, v, \lambda)| \leq \epsilon |u|^{p-1} + M |u|^\gamma |v|^{\delta+1}.$$

• Compactness:

Here, we show the compactness of F^λ .

If $\delta + 1 > p$, let $\theta \in \mathbb{R}^+$ such that:

$$\begin{cases} \delta + \gamma + 2 < \theta < \min(p^*, q^*) & \text{if } \gamma + 1 > p \\ \max(\delta + \gamma + 2, p) < \theta < \min(p^*, q^*) & \text{if } \gamma + 1 \leq p. \end{cases}$$

By using Hölder's and Young's inequalities we have

$$\begin{aligned} \int_{\Omega} |f(x, \lambda, u, v)|^{\theta'} dx &\leq \epsilon \int_{|u(x)|+|v(x)| \leq \xi} |u|^{\theta'(p-1)} dx + M \int_{|u(x)|+|v(x)| \geq \xi} |u|^{\theta'\delta} |v|^{\theta'(\gamma+1)} dx \\ &\leq \epsilon \int_{\Omega} |u|^{\theta'(p-1)} dx + M \int_{\Omega} |u|^{\theta'\gamma} |v|^{\theta'(\delta+1)} dx \\ &\leq M_1 \int_{\Omega} |u|^\theta dx + M_2 \left(\int_{\Omega} |u|^\theta dx \right)^{\frac{\gamma}{\theta-1}} \left(\int_{\Omega} |v|^\theta dx \right)^{\frac{\delta+1}{\theta-1}} \end{aligned}$$

Thus F^λ is the Nemytskii continuous operator from $L^\theta(\Omega) \times L^\theta(\Omega)$ into $L^{\theta'}(\Omega)$.

On the other hand, let $\{(u_n, v_n)\}_n \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and strongly in $L^\theta(\Omega) \times L^\theta(\Omega)$.

So,

$$\begin{aligned} \|F^\lambda(u_n, v_n) - F^\lambda(u, v)\|_* &= \sup_{\|(\phi, \psi)\| \leq 1} \left| \int_{\Omega} (f(x, \lambda, u_n, v_n) - f(x, \lambda, u, v)) \phi dx \right| \\ &\leq \|f(x, \lambda, u_n, v_n) - f(x, \lambda, u, v)\|_{\theta'} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence F^λ is compact.

By a similar argument, we can show that G^λ is compact.

• For the limit (2.9) and (2.10), we argue as follows:

Hölder's inequality yields for all $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$\begin{aligned} \|F^\lambda(u, v)\|_* &\leq \sup_{\|(\phi, \psi)\| \leq 1} \left| \left(\epsilon \int_\Omega |u|^{p-1} \phi \, dx + M \int_\Omega |u|^\gamma |v|^{\delta+1} \phi \, dx \right) \right| \\ &\leq \sup_{\|(\phi, \psi)\| \leq 1} (\epsilon \|u\|_p^{p-1} \|\phi\|_p + M |\Omega|^{\omega_0} \|u\|_{p^*}^\gamma \|v\|_{q^*}^{\delta+1} \|\phi\|_{p^*}) \\ &\leq c \sup_{\|(\phi, \psi)\| \leq 1} (\epsilon \|\nabla u\|_p^{p-1} + M \|\nabla u\|_p^\gamma \|\nabla v\|_q^{\delta+1}) \|\nabla \phi\|_p \end{aligned}$$

where $\omega_0 = \frac{p^* q^*}{p^* q^* - (\gamma+1)p^* - (\delta+1)q^*}$ and $c \in \mathbb{R}^+$ is a constant.

So, we distinguish two cases:

If $\gamma + 1 > p$ then,

$$\frac{\|F^\lambda(u, v)\|_*}{\|\nabla u\|_p^{p-1} + \|\nabla v\|_q^{q-1}} \leq c(\epsilon + M \|\nabla u\|_p^{\gamma-(p-1)} \|\nabla v\|_q^{\delta+1}). \quad (2.11)$$

If $\delta + 1 > q$ then,

$$\frac{\|F^\lambda(u, v)\|_*}{\|\nabla u\|_p^{p-1} + \|\nabla v\|_q^{q-1}} \leq c\epsilon + cM \|\nabla u\|_p^\gamma \|\nabla v\|_q^{(\delta+1)-(q-1)}. \quad (2.12)$$

Therefore, by passing to the limit in (2.11) or (2.12), (2.9) follows.

There is the same proof of (2.10). ■

Remark 2.2. *Note that every continuous map $T : X \rightarrow X^*$ is also demicontinuous. Note also, that if $T \in (S_+)$ then $(T + K) \in (S_+)$ for any compact operator $K : X \rightarrow X^*$.*

Remark 2.3. *λ is an eigenvalue of (ES) if and only if the system*

$$\begin{cases} A_u(u, v) - \lambda B_u(u, v) = 0 \\ A_v(u, v) - \lambda B_v(u, v) = 0 \end{cases}$$

has a nontrivial solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Proposition 2.1. *Under the assumption (2.1), (ES) has a principal eigenvalue λ_1 characterized variationally as follows*

$$\lambda_1 = \inf \{ A(u, v); (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), B(u, v) = 1 \}.$$

The proof of this proposition is more or less the same as F. De Thélin [5] for the system case (ES) modulo a suitable modification.

Now, let

$$\begin{aligned} \Gamma_p(u, \phi) &= \int_{\Omega} |\nabla u|^p dx + (p-1) \int_{\Omega} |\nabla \phi|^p \left(\frac{|u|}{\phi}\right)^p dx \\ &\quad - p \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \nabla u \left(\frac{|u|^{p-2} u}{\phi^{p-1}}\right) dx \\ &= \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} \frac{\Delta_p \phi}{\phi^{p-1}} |u|^p dx \end{aligned}$$

for all $(u, \phi) \in (W_0^{1,p}(\Omega) \cap C^1(\Omega))^2$ with $\phi > 0$ in Ω .

The following lemma is the heart on the proof of the simplicity.

Lemma 2.5. *For all $(u, \phi) \in (W_0^{1,p}(\Omega) \cap C^{1,\nu}(\Omega))^2$ with $\phi > 0$ in Ω and $\nu \in (0, 1)$, we have $\Gamma_p(u, \phi) \geq 0$ i.e*

$$\int_{\Omega} |\nabla u|^p dx \geq \int_{\Omega} \frac{-\Delta_p \phi}{\phi^{p-1}} |u|^p dx, \quad (2.13)$$

and if $\Gamma_p(u, \phi) = 0$ there is $c \in \mathbb{R}$ such that $u = c\phi$.

Proof.

By Young's inequality we have for $\epsilon > 0$,

$$\begin{aligned} \int_{\Omega} \nabla u |\nabla \phi|^{p-2} \nabla \phi \frac{u|u|^{p-2}}{\phi^{p-1}} dx &\leq \int_{\Omega} |\nabla u| |\nabla \phi|^{p-1} \left(\frac{|u|}{\phi}\right)^{p-1} dx \\ &\leq \int_{\Omega} \left(\frac{\epsilon^p}{p} |\nabla u|^p + \frac{p-1}{p\epsilon^p} |u|^p |\nabla \phi|^p\right) dx. \end{aligned} \quad (2.14)$$

For $\epsilon = 1$ we have by integration over Ω ,

$$p \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \nabla u \left(\frac{|u|^{p-2} u}{\phi^{p-1}}\right) dx \leq \int_{\Omega} |\nabla u|^p dx + (p-1) \int_{\Omega} \left|\frac{u}{\phi}\right|^p |\nabla \phi|^p dx.$$

Thus

$$\Gamma_p(u, \phi) \geq 0.$$

On the other hand, if $\Gamma_p(u, \phi) = 0$ by (2.14), we obtain for $\epsilon = 1$,

$$p \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \nabla u \left(\frac{|u|^{p-2} u}{\phi^{p-1}}\right) dx - \int_{\Omega} |\nabla u|^p dx - (p-1) \int_{\Omega} \left|\frac{u}{\phi}\right|^p |\nabla \phi|^p dx = 0 \quad (2.15)$$

and

$$\int_{\Omega} \left\{ \nabla u \nabla \phi |\nabla \phi|^{p-2} \frac{u|u|^{p-2}}{\phi^{p-1}} - |\nabla u| |\nabla \phi|^{p-1} \left(\frac{|u|}{\phi}\right)^{p-1} \right\} dx = 0. \quad (2.16)$$

Thanks to (2.15) we find $|\nabla u| = |\frac{u}{\phi}\nabla\phi|$. Thus from (2.16), it follows that $\nabla u = \epsilon\frac{u}{\phi}\nabla\phi$, where $|\epsilon| = 1$. Consequently, since $\Gamma_p(u, \phi) = 0$, we obtain $\epsilon = 1$ and $\nabla(\frac{u}{\phi}) = 0$. Therefore, there is $c \in \mathbb{R}$ such that $u = c\phi$. ■

- Remarks 2.2.**
- (i) *By adapting the argument of [5] we can show that the eigenvectors associated to λ_1 without any additional smoothness assumption on $\partial\Omega$, are in $L^\infty \times L^\infty$.*
 - (ii) *Thanks to an advanced result in regularity theory of [4], we deduce with (i) that the positive solution of (ES) associated to λ_1 is in $C_{loc}^1(\Omega) \times C_{loc}^1(\Omega)$.*
 - (iii) *According to the Maximum principle of [17] applied to each equation we conclude that (ES) has a priori a positive eigenvector associated to λ_1 .*

3. MAIN RESULTS

3.1. Simplicity and isolation results.

- Theorem 3.1.**
- (i) *λ_1 is simple.*
 - (ii) *λ_1 is the unique eigenvalue of (ES) having an eigenvector (u, v) not changing its sign, i.e., $uv > 0$ in Ω .*
 - (iii) *There is $c > 0$ such that*

$$|\Omega^-| \geq (|\lambda|c)^\omega,$$

*where $(\lambda, (u, v))$ is an eigenpair of (ES), $\Omega^- = \{x \in \Omega, u(x) < 0 \text{ and } v(x) < 0\}$ and $\omega = \frac{p^*q^*}{p^*q^* - (\alpha+1)q^* - (\beta+1)p^*}$.*

- (iv) *λ_1 is isolated.*

Proof.

- (i) Let (u, v) and (ϕ, ψ) be two eigenvectors of (ES) associated to λ_1 with (u, v) is positive ($u \geq 0, v \geq 0$). Thanks to definition of λ_1 and

Hölder's inequality we have in view of remark 2.4 that

$$\begin{aligned}
 A(\phi, \psi) &= \lambda_1 B(\phi, \psi) \\
 &\leq \lambda_1 \int_{\Omega} u^{\alpha+1} v^{\beta+1} \frac{|\phi|^{\alpha+1} |\psi|^{\beta+1}}{u^{\alpha+1} v^{\beta+1}} dx \\
 &\leq \lambda_1 \int_{\Omega} u^{\alpha+1} v^{\beta+1} \left[\frac{\alpha+1}{p} \frac{|\phi|^p}{u^p} + \frac{\beta+1}{q} \frac{|\psi|^q}{v^q} \right] dx \\
 &\leq \lambda_1 \int_{\Omega} \left[\frac{\alpha+1}{p} \frac{u^{\alpha} v^{\beta+1}}{u^{p-1}} |\phi|^p + \frac{\beta+1}{q} \frac{u^{\alpha+1} v^{\beta}}{v^{q-1}} |\psi|^q \right] dx \\
 &\leq \frac{\alpha+1}{p} \int_{\Omega} \frac{-\Delta_p u}{u^{p-1}} |\phi|^p dx + \frac{\beta+1}{q} \int_{\Omega} \frac{-\Delta_q v}{v^{q-1}} |\psi|^q dx.
 \end{aligned}$$

Thanks to lemma 2.5 we have

$$A(\phi, \psi) = \frac{\alpha+1}{p} \int_{\Omega} \frac{-\Delta_p u}{u^{p-1}} |\phi|^p dx + \frac{\beta+1}{q} \int_{\Omega} \frac{-\Delta_q v}{v^{q-1}} |\psi|^q dx.$$

Thus

$$\int_{\Omega} |\nabla \phi|^p dx = \int_{\Omega} \frac{-\Delta_p u}{u^{p-1}} |\phi|^p dx \text{ and } \int_{\Omega} |\nabla \psi|^q dx = \int_{\Omega} \frac{-\Delta_q v}{v^{q-1}} |\psi|^q dx.$$

Hence by Lemma 3.2 again, there exist k_1 and k_2 in \mathbb{R} such that $u = k_1 \phi$ and $v = k_2 \psi$.

(ii) Let (u, v) be a positive eigenvector of (ES) associated to λ and (ϕ, ψ) a positive solution of (ES) associated to λ_1 . It is clear that $\lambda_1 \leq \lambda$ and by Hölder's inequality we have

$$\begin{aligned}
 A(\phi, \psi) &= \lambda_1 B(\phi, \psi) \\
 &\leq \lambda \int_{\Omega} u^{\alpha+1} v^{\beta+1} \frac{\phi^{\alpha+1} \psi^{\beta+1}}{u^{\alpha+1} v^{\beta+1}} dx \\
 &\leq \lambda \int_{\Omega} u^{\alpha+1} v^{\beta+1} \left[\frac{\alpha+1}{p} \frac{\phi^p}{u^p} + \frac{\beta+1}{q} \frac{\psi^q}{v^q} \right] dx \\
 &\leq \lambda \int_{\Omega} \left[\frac{\alpha+1}{p} \frac{u^{\alpha} v^{\beta+1}}{u^{p-1}} \phi^p + \frac{\beta+1}{q} \frac{u^{\alpha+1} v^{\beta}}{v^{q-1}} \psi^q \right] dx \\
 &\leq \frac{\alpha+1}{p} \int_{\Omega} \frac{-\Delta_p u}{u^{p-1}} \phi^p dx + \frac{\beta+1}{q} \int_{\Omega} \frac{-\Delta_q v}{v^{q-1}} \psi^q dx.
 \end{aligned}$$

Therefore, we deduce from Lemma 2.5 that $A(\phi, \psi) = A(u, v)$ i.e

$$\lambda_1 \int_{\Omega} |\phi|^{\alpha+1} |\psi|^{\beta+1} dx = \lambda \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} dx.$$

So, by normalization, we conclude that $\lambda_1 = \lambda$.

(iii) Let $(\lambda, (u, v))$ be an eigenpair of (ES) , $u^-(x) = \min(u(x), 0)$ and $v^-(x) = \min(v(x), 0)$.

Thus, by multiplying the first equation of (ES) by u^- we have

$$\begin{aligned}
 \int_{\Omega} |\nabla u^-|^p dx &= \lambda \int_{\Omega} |u|^\alpha |v|^\beta v u^- dx \\
 &= \lambda \left[\int_{\Omega^-} |u^-|^{\alpha+1} |v^-|^{\beta+1} + \int_{\Omega} |u|^\alpha |v|^\beta u^- v^+ \right] dx \\
 &\leq |\lambda| \int_{\Omega^-} |u^-|^{\alpha+1} |v^-|^{\beta+1} dx \\
 &\leq |\lambda| |\Omega^-|^\omega \|u^-\|_{p^*}^{\alpha+1} \|v^-\|_{q^*}^{\beta+1} \\
 &\leq c |\lambda| |\Omega^-|^\omega \|u^-\|_{1,p}^{\alpha+1} \|v^-\|_{1,q}^{\beta+1} \\
 &\leq c |\lambda| |\Omega^-|^\omega \left(\frac{\alpha+1}{p} \int_{\Omega} |\nabla u^-|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla v^-|^q dx \right). \tag{3.1}
 \end{aligned}$$

Similarly, multiplying the second equation of (ES) by v^- , we obtain

$$\int_{\Omega} |\nabla v^-|^q dx \leq c |\lambda| |\Omega^-|^\omega \left(\frac{\alpha+1}{p} \int_{\Omega} |\nabla u^-|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla v^-|^q dx \right). \tag{3.2}$$

Hence by (3.1) and (3.2) we complete the proof of (iii).

(iv) The proof is a rather a simple adaptation of argument of [1] modulo suitable modification. ■

3.2. Bifurcation Result. Once we have proved in the previous subsection that λ_1 is simple and isolated, we can study the bifurcation when λ is near λ_1 .

Proposition 3.1. *If $(\bar{\lambda}; (0, 0))$ is a bifurcation point of (BS) , then $\bar{\lambda}$ is an eigenvalue of (ES) .*

Proof.

$(\bar{\lambda}, (0, 0))$ is a bifurcation point of (BS) then there is a sequence $\{\lambda_n, (u_n, v_n)\}_n \subset \mathbb{R} \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of nontrivial solutions of (ES) such that

$$\begin{aligned}
 \lambda_n &\longrightarrow \bar{\lambda} \text{ in } \mathbb{R} \\
 (u_n, v_n) &\longrightarrow (0, 0) \text{ in } W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).
 \end{aligned}$$

Thus

$$\begin{aligned} A_u(u_j, v_j) - \lambda_j B_u(u_j, v_j) &= F^{\lambda_j}(u_j, v_j) \\ A_v(u_j, v_j) - \lambda_j B_v(u_j, v_j) &= G^{\lambda_j}(u_j, v_j). \end{aligned} \tag{3.3}$$

Set $\bar{u}_j = \frac{u_j}{\|\nabla u_j\|_p^{p-1} + \|\nabla v_j\|_q^{q-1}}$ and $\bar{v}_j = \frac{v_j}{\|\nabla u_j\|_p^{p-1} + \|\nabla v_j\|_q^{q-1}}$. Then \bar{u}_j and \bar{v}_j are bounded, it follows that there exist a pair $(\bar{u}, \bar{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that $(\bar{u}_j, \bar{v}_j) \rightharpoonup (\bar{u}, \bar{v})$ weakly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and a.e in Ω . Thanks to Lemma 2.5 and Lemma 2.4 we deduce the following convergence

$$\lambda_j B_u(\bar{u}_j, \bar{v}_j) + \frac{F^{\lambda_j}(u_j, v_j)}{\|\nabla u_j\|_p^{p-1} + \|\nabla v_j\|_q^{q-1}} \longrightarrow \bar{\lambda} B_u(\bar{u}, \bar{v})$$

and

$$\lambda_j B_v(u_j, v_j) + \frac{G^{\lambda_j}(u_j, v_j)}{\|\nabla u_j\|_p^{p-1} + \|\nabla v_j\|_q^{q-1}} \longrightarrow \bar{\lambda} B_v(\bar{u}, \bar{v}).$$

According to (3.3), we have

$$A_u(\bar{u}_j, \bar{v}_j) \longrightarrow \bar{\lambda} B_u(\bar{u}, \bar{v})$$

and

$$A_v(\bar{u}_j, \bar{v}_j) \longrightarrow \bar{\lambda} B_v(\bar{u}, \bar{v}),$$

Since $-\Delta_p$ and $-\Delta_q$ are homeomorphisms, it is clear that

$$(\bar{u}_j, \bar{v}_j) \longrightarrow (A_u)^{-1}(\bar{\lambda} B_u(\bar{u}, \bar{v}))$$

and

$$(\bar{u}_j, \bar{v}_j) \longrightarrow (A_v)^{-1}(\bar{\lambda} B_v(\bar{u}, \bar{v})).$$

Consequently by the convergence a.e in Ω we conclude that

$$(\bar{u}, \bar{v}) = (A_u)^{-1}(\bar{\lambda} B_u(\bar{u}, \bar{v}))$$

and

$$(\bar{u}, \bar{v}) = (A_v)^{-1}(\bar{\lambda} B_v(\bar{u}, \bar{v})).$$

Finally, since $\|\nabla \bar{u}_j\|_p^{p-1} + \|\nabla \bar{v}_j\|_q^{q-1} = 1$ we must have $\bar{u} \neq 0$ and $\bar{v} \neq 0$. It follows that (\bar{u}, \bar{v}) solves (ES). This complete the proof in view of Remark 2.4. ■

Let $X = \mathbb{R} \times W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be equipped with the norm

$$\|(\lambda, u, v)\| = (|\lambda|^2 + \|\nabla u\|_p^2 + \|\nabla v\|_q^2)^{\frac{1}{2}}.$$

Definition 3.1. We say that

$$C = \{(\lambda, (u, v)) : (\lambda, (u, v)) \text{ solves } (BS), u \neq 0, v \neq 0\}$$

is a continuum of nontrivial solutions of (BS), if it is a connected subset in E .

Now let, $L = \frac{\alpha+1}{p}A_u + \frac{\beta+1}{q}A_v$, $G = \frac{1}{2}(B_u + B_v)$ and $F = F^\lambda + G^\lambda$. In virtue of the preceding results $Deg[L - \lambda G - F, B_\epsilon(0, 0), (0, 0)]$ is well defined. By homotopic we have

$$Deg[L - \lambda G - F, B_\epsilon(0, 0), (0, 0)] = Deg[L - \lambda G, B_\epsilon(0, 0), (0, 0)].$$

The main bifurcation result in this paper is the following

Theorem 3.2. Under the assumptions (f) and (g), the pair $(\lambda_1, (0, 0))$ is a bifurcation point of (BS). Moreover, there is a continuum of nontrivial solutions C of (BS) such that $(\lambda_1, (0, 0)) \in \bar{C}$ and C is either unbounded in E or there is $\bar{\lambda} \neq \lambda_1$, an eigenvalue of (ES), with $(\bar{\lambda}, (0, 0)) \in \bar{C}$.

Proof.

We will give only a sketch of the proof since it follows the lines of the proof of Theorem 14.18 in [9]. The key point in the proof is the fact that the value of

$$Deg[L - \lambda G; B_\epsilon(0), 0] \tag{3.4}$$

changes when λ crosses λ_1 . If this fact is proved then the result follows exactly as in the classical bifurcation result of Rabinowitz [15]. Choose $a > 0$ such that $(\lambda_1, \lambda_1 + a)$ does not contain any eigenvalue of (ES). Then the variational characterization of λ_1 and lemma 2.2 yield

$$Deg[L - \lambda G; B_\epsilon(0), 0] = 1, \tag{3.5}$$

when $\lambda \in (\lambda_1 - a, \lambda_1)$. To evaluate (3.4) for $\lambda \in (\lambda_1, \lambda_1 + a)$ we use the following trick. Fix a number $K > 0$ and define a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi(s) = \begin{cases} 0 & \text{for } s \leq K \\ \frac{2a}{\lambda_1}(s - 2K) & \text{for } s \geq 3K \end{cases}$$

and ψ is positive and strictly convex in $(K, 3K)$. Define a functional

$$\bar{\psi}_\lambda(u, v) = \langle L(u, v), (u, v) \rangle - \lambda \langle G(u, v), (u, v) \rangle + \psi(\langle L(u, v), (u, v) \rangle).$$

Then $\bar{\psi}_\lambda$ is continuously Frèchet differentiable and its critical point $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ corresponds to a solution of the following

system

$$\frac{\partial \bar{\psi}_\lambda}{\partial u} = \frac{\partial \bar{\psi}_\lambda}{\partial v} = 0,$$

where

$$\left\langle \frac{\partial \bar{\psi}_\lambda}{\partial u}(u, v), (u, v) \right\rangle = (\alpha + 1) [\langle A_u(u, v), (u, v) \rangle (1 + \psi'(\langle L(u, v), (u, v) \rangle)) - \lambda \langle B_u(u, v), (u, v) \rangle],$$

and

$$\left\langle \frac{\partial \bar{\psi}_\lambda}{\partial v}(u, v), (u, v) \right\rangle = (\beta + 1) [\langle A_v(u, v), (u, v) \rangle (1 + \psi'(\langle L(u, v), (u, v) \rangle)) - \lambda \langle B_v(u, v), (u, v) \rangle],$$

However, since $\lambda \in (\lambda_1 - a, \lambda_1)$, only nontrivial critical points of the derivative of $\bar{\psi}$ noted $D\bar{\psi}_\lambda$ occur if

$$\psi'(\langle L(u_0, v_0), (u_0, v_0) \rangle) = \frac{\lambda}{\lambda_1} - 1. \quad (3.6)$$

Due to the definition of ψ we have $\langle L(u_0, v_0), (u_0, v_0) \rangle \in (K, 3K)$; by (3.6) and the simplicity of λ_1 , we conclude that $(u_0, v_0) \in \{(u_1, v_1), (-u_1, -v_1)\}$, where (u_1, v_1) is an eigenvector associated to λ_1 . Therefore, for $\lambda \in (\lambda_1, \lambda_1 + a)$, the derivative $D\bar{\psi}_\lambda$ has precisely three isolated critical point .

The existence of such isolated critical points is ensured by the fact $\bar{\psi}_\lambda$ is weakly lower semicontinuous and

$$\lim_{\|(u,v)\| \rightarrow +\infty} \|\bar{\psi}_\lambda\|_{1,p} = \infty,$$

due to the definition of ψ . So, $\bar{\psi}_\lambda$ attains local minima at (u_1, v_1) and $(-u_1, -v_1)$. Note that $(0, 0)$ is an obvious isolated critical point. It follows from Lemma 2.1 that

$$\text{ind}(D\bar{\psi}_\lambda, (u_1, v_1)) = \text{ind}(D\bar{\psi}_\lambda, (-u_1, -v_1)) = 1. \quad (3.7)$$

Since also

$$\langle D\bar{\psi}_\lambda((u, v)), (u, v) \rangle > 0$$

for $\|u\|_p + \|v\|_q = R$, with $R > 0$, sufficiently large, we have according to Lemma 2.2 that

$$\text{Deg}[D\bar{\psi}_\lambda; B_R(0), 0] = 1. \quad (3.8)$$

Additivity property of the degree, (3.7) and (3.8) yield

$$\text{Deg}[L - \lambda G; B_\epsilon(0), 0] = -1. \quad (3.9)$$

for any $\lambda \in (\lambda_1, \lambda_1 + a)$ and $\epsilon > 0$ sufficiently small. Since (3.5) and (3.9) establish the "jump" of the degree the proof is completed. ■

- Remarks 3.1.** (i) For the case of one equation and when $\partial\Omega$ of class $C^{2,\nu}$, a similar bifurcation theorem has proved by Del Pino and Manasevich [11] and Binding and Huang [2]. The first authors used the continuity of $\lambda_1(p)$ with respect to p ; on the other hand, the second authors considered the following particular case $f \equiv f(x, s)$ which satisfies $f(x, s) \in C(\bar{\Omega} \times \mathbb{R})$ an odd function in s and $|f(x, s)| \leq c|s|^{q-1}$ uniformly in Ω ; where $p < q < \bar{q} = p + \frac{p^2}{N}$. They use $C^{1,\nu}(\bar{\Omega})$ regularity of solutions.
- (ii) Fleckinger, Manásevich and De Thelin show a similar bifurcation theorem for the following system

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + b|v|^{p-2}v + f(\lambda, |u|^{p-2}u) & \text{in } \Omega \\ -\Delta_p v = c|u|^{p-2}u + d|v|^{p-2}v & \text{in } \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases}$$

where Ω is sufficiently regular bounded open subset of \mathbb{R}^N . The function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and λ is a parameter; b, c and d are constants satisfying $bc > 0$, $d \leq 0$.

Note that the methods used in all papers are not applicable in our case (any bounded domain). So the method used to evaluate the degree is different.

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