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Homoclinic bifurcation in a discontinuous delay differential equation

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Abstract. We consider the delay differential equation (E_g) $x'(t) = -ax(t) + bg(x(t-1))$ where $b > a > 0$ and $g : [0,\infty) \to [0,1]$ is given such that it is smooth on $[0,1]$, $\frac{g(\xi)}{\xi}$ increases on $(0,1]$, $g(0) = g'(0) = 0$, and $g(\xi) = 0$ for $\xi > 1$. For a certain range of parameters $a > 0$, we prove the existence of $b^* = b^*(a) > a$ such that equation (E_g) with $b = b^*$ has a homoclinic solution to its unique positive stationary point $\xi_{a,b^*} \in (0,1)$. The homoclinic solution crosses the discontinuity of the feedback twice, both times transversally. We then establish a homoclinic bifurcation at $b = b^*$: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $b \in (b^*, b^* + \delta)$, equation (E_g) has a periodic solution $p^{a,b} : \mathbb{R} \to (0,\infty)$ whose minimal period tends to infinity as $b \to b^*+$, and whose orbit has Hausdorff distance less than ε from the homoclinic orbit. This bifurcation result does not require the usual saddle-quantity condition involving the leading stable and unstable characteristic values. Moreover, the periodic orbit has a neighborhood from which every solution reaches the orbit in finite time. Finally, (E_g) arises as a limiting equation for a class of smooth Mackey–Glass type equations, and the above results yield, as a corollary, orbitally asymptotically stable periodic orbits with arbitrarily large minimal periods for equations in this class.

Keywords: delay differential equation, homoclinic bifurcation, Mackey–Glass type equation, stable periodic orbit.

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1 Introduction

Homoclinic bifurcations constitute one of the classical mechanisms for the creation of periodic and more complicated dynamics. Starting with the seminal work of Shilnikov [28, 29], an extensive theory has been developed for finite-dimensional smooth dynamical systems, and related results have subsequently been established for infinite-dimensional systems and delay differential equations; see, among others, the work of Chow, Hale and Mallet-Paret [5], Hale and Lin [9], Lin [19], Walther [32], Chow and Deng [4], and Ignatenko [13]. In particular,

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classical results on the bifurcation of a stable periodic orbit from a homoclinic orbit typically involve a spectral condition of saddle-quantity type, requiring that the sum of the leading unstable characteristic value and the real part of the leading stable characteristic value be negative.

The situation considered here is different in two respects. First, the feedback function is discontinuous: although the stationary point to which the homoclinic orbit is asymptotic lies in the region where the feedback is smooth, the homoclinic solution crosses the discontinuity twice, both times transversally. Under our conditions, the characteristic values of the linearization at the stationary point, to which the homoclinic orbit is asymptotic, are $\lambda_0 \in \mathbb{R}$ and a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_{j=1}^{\infty}$ with $\lambda_0 > 0 > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \dots$. In contrast to the above-mentioned results, we do not require the spectral condition $\lambda_0 + \operatorname{Re}\lambda_1 < 0$. Instead, the special form of the feedback permits a direct analysis of the global return and yields periodic solutions whose minimal periods tend to infinity as the parameter approaches the homoclinic value. Moreover, these periodic orbits possess a stronger stability property than ordinary orbital asymptotic stability: each has a neighborhood such that every solution starting in this neighborhood reaches the periodic orbit after a finite time and thereafter coincides with it up to a phase shift. The particular nonlinearity, see condition (C_g) below, provides a homoclinic bifurcation mechanism that differs substantially from the standard smooth theory and, through approximation, yields orbitally asymptotically stable periodic orbits with arbitrarily large minimal periods for a class of smooth Mackey–Glass type equations.

Consider the delay differential equation

$$x'(t) = -ax(t) + bg\left(x(t-1)\right) \tag{E_g}$$

with $b > a > 0$, and a nonlinear discontinuous feedback function g satisfying the condition:

(C_g) The function $g : [0,\infty) \to [0,1]$ is given such that the restriction $g|_{[0,1]}$ is C^1-smooth, $g(0) = g'(0) = 0$, $g(1) = 1$, $g(\xi) = 0$ for all $\xi > 1$, and $g'(\xi) > \frac{g(\xi)}{\xi}$ holds for all $\xi \in (0,1]$, where $g'(1)$ denotes the left hand derivative of g at 1.

Throughout this paper, $b > a > 0$ and (C_g) are assumed.

Equation (E_g) is the limiting version of the sequence of equations

$$y'(t) = -ay(t) + bf_n\left(y(t-1)\right) \tag{E_{f_n}}$$

as $n \to \infty$ provided f_n approaches g in a sense described in condition (C_{f_n}), see also [2,3].

Equations (E_g) and (E_{f_n}) are particular delay differential equations with nonmonotone delayed feedback. In equation (E_{f_n}), replacing f_n with the function $f(\xi) = \xi^k e^{-\xi}$ or $f(\xi) = \frac{\xi^k}{1+\xi^n}$ with $n > k > 0$, we get the well-known equations introduced by Lasota [18] and by Mackey and Glass [23] in 1977 to model the feedback control of blood cells. The papers [18,23] motivated a lot of research on nonlinear dynamics, see [33], because numerical results demonstrated that the above nonlinearities could create complex dynamical behavior. Despite the large number of theoretical results, see, e.g. [10,12,20–22,27,30,34], so far, rigorous mathematical results have explained very little about the high complexity of the dynamics obtained numerically; see [7,25].

The papers [1–3,14] studied equations (E_{f_n}) together with their limiting version (E_g). Periodic orbits, connecting orbits, and information on the global dynamics were obtained first for equation (E_g), and then it was proved that these objects persist for equations (E_{f_n}) for all large n.

Let $C = C([-1,0],\mathbb{R})$ denote the continuous real-valued functions equipped with the norm $\|\phi\| = \max_{\theta\in[-1,0]} |\phi(\theta)|$. For $\xi \in \mathbb{R}$, let $\hat{\xi} \in C$ be the constant function with value ξ. For $a \in \mathbb{R}$, set $\eta^a \in C$ by $\eta^a(\theta) = e^{-a(1+\theta)}$, $\theta \in [-1,0]$. If $u : I \to \mathbb{R}$ is a function on $I \subset \mathbb{R}$, then set $\|u\|_I = \sup_{t\in I} |u(t)|$. In the case $u : I \to \mathbb{R}$ is differentiable in $I \subset \mathbb{R}$, then $\|u\|_I^1 = \sup_{t\in I}(|u(t)| + |u'(t)|)$. If $u : I \to \mathbb{R}$ is continuous, $t \in I$ and $t-1 \in I$, then $u_t \in C$ is given by $u_t(\theta) = u(t+\theta)$, $\theta \in [-1,0]$.

Solutions of equations (E_g) and (E_{f_n}) are defined as solutions of the corresponding integral equations; see Section 2. As we are interested in nonnegative solutions, the natural phase space is $C^+ = \{\psi \in C : \psi(s) \geq 0,\ -1 \leq s \leq 0\}$. For all $\phi, \psi \in C^+$, there exist unique solutions $x^\phi : [-1,\infty) \to \mathbb{R}$ and $y^\psi : [-1,\infty) \to \mathbb{R}$ of equations (E_g) and (E_{f_n}) with $x_0^\phi = \phi$ and $y_0^\psi = \psi$, respectively. The solutions of equations (E_g) and (E_{f_n}) define the semiflows

$$\Psi_{a,b} : [0,\infty) \times C^+ \ni (t,\phi) \mapsto x_t^\phi \in C^+ \quad \text{and} \quad \Phi_{a,b}^n : [0,\infty) \times C^+ \ni (t,\psi) \mapsto y_t^\psi \in C^+.$$

We remark that $\Phi_{a,b}^n$ is continuous; however, $\Psi_{a,b}$ is not continuous.

Note that, by (C_g), $\frac{g(\xi)}{\xi}$ is strictly increasing in $(0,1]$, and $\frac{g(\xi)}{\xi} \to g'(0) = 0$ as $\xi \to 0+$. Then, by $b > a > 0$, the function $[0,1] \ni \xi \mapsto -a\xi + bg(\xi) \in \mathbb{R}$ has exactly two zeros, 0 and $\xi_{a,b} \in (0,1)$. If $b > a > 0$ then $\hat{\xi}_{a,b}$ is an unstable stationary point of $\Psi_{a,b}$.

In [2], considering the 1-dimensional leading unstable set of $\hat{\xi}_{a,b}$, we proved that, for all $b > a > 0$, equation (E_g) has a unique solution $x^{+,a,b} : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} &\text{the restriction } x^{+,a,b}|_{(-\infty,1]} \text{ is } C^1\text{-smooth with positive derivatives,} \\ &x^{+,a,b}(0) = 1, \quad \text{and} \quad x^{+,a,b}(t) \to \xi_{a,b} \quad \text{as } t \to -\infty. \end{aligned} \tag{1.1}$$

From condition (C_g) and (1.1), it easily follows that there exists a unique $\tau_{0,a,b} > 1$ such that

$$\begin{aligned} &x^{+,a,b}(t) > 1 \quad \text{for all } t \in (0,\tau_{0,a,b}), \quad x^{+,a,b}(\tau_{0,a,b}) = 1, \\ &x^{+,a,b}(t) = x^{+,a,b}(1)e^{-a(t-1)} \quad \text{for all } t \in [1,\tau_{0,a,b}+1], \quad \text{and } x^{+,a,b}_{\tau_{0,a,b}+1} = \eta^a. \end{aligned} \tag{1.2}$$

Note that $x^{+,a,b}_{\tau_{0,a,b}+1} = \eta^a$ does not depend on b. The papers [2, 3] considered the solutions $z^{a,b} : [-1,\infty) \to \mathbb{R}$ of equation (E_g) with the initial condition $z_0^{a,b} = \eta^a$. By uniqueness of solutions of equation (E_g),

$$x^{+,a,b}(t + \tau_0(a,b) + 1) = z^{a,b}(t) \quad \text{for all } t \geq 0. \tag{1.3}$$

In the paper [2], it is proved that for each $a > 0$ there exists a unique $b^*(a) > a$ such that

$$b \in (a, b^*(a)) \Longrightarrow z^{a,b}(t) \in (0,1) \quad \text{for all } t \geq 0, \quad \text{and} \quad z^{a,b}(t) \to 0 \quad \text{as } t \to \infty; \tag{1.4}$$

and

$$b > b^*(a) \Longrightarrow \text{ the set } \{t \geq 0 : z^{a,b}(t) = 1\} \text{ is unbounded.} \tag{1.5}$$

The threshold value $b^*(a)$ plays a key role in the global dynamics of equations (E_g) and (E_{f_n}) for large n, see [2].

In this paper, we consider the critical case $b = b^*(a)$. A crucial hypothesis is as follows.

(H_a) For the parameter $a > 0$, $\lim_{t\to\infty} z^{a,b^*(a)}(t) = \xi_{a,b^*(a)}$ holds.

From (1.1) and (1.3) it is clear that, under hypothesis (H_a), the solution $x^{a,b^*(a)}$ of equation (E_g) satisfies

$$\lim_{|t|\to\infty} x^{a,b^*(a)}(t) = \xi_{a,b^*(a)},$$

that is, $x^{a,b^*(a)}$ is a homoclinic solution of (E_g). Let

$$\mathcal{H} = \left\{ x_t^{a,b^*(a)} : t \in \mathbb{R} \right\}$$

be the homoclinic orbit, provided (H_a) is satisfied.

In Section 3, see Proposition 3.1, we prove that (H_a) holds provided a is less than a number given as a function of $\max_{\xi\in[0,1]} |g'(\xi)|$. In the particular case $g(\xi) = \xi^2$, $\xi \in [0,1]$, the condition is $a < \log \frac{3}{2}$. In the forthcoming paper [15], we verify that (H_a) is satisfied for all $a \in (0, a^*)$ with some $a^* \in (2, \frac{5\pi}{3\sqrt{3}})$, provided $g(\xi) = \xi^2$, $\xi \in [0,1]$.

For $S_1, S_2 \subset C$, let $\mathrm{dist}_H(S_1, S_2)$ denote the Hausdorff distance of S_1, S_2.

The papers [1–3, 14] assumed a hypothesis on periodic solutions of (E_g). We recall it in an equivalent form.

(P) Let $b > a > 0$ and suppose that equation (E_g) admits a periodic solution $p = p^{a,b} : \mathbb{R} \to (0,\infty)$, with minimal period $\omega_{a,b} > 0$, such that $p_0^{a,b} = \eta^a$ and $g(p^{a,b}(t-1)) \neq \frac{a}{b}$ for all $t \in (0, \omega_{a,b}]$ with $p^{a,b}(t) = 1$.

Notice that condition $g(p(t-1)) \neq \frac{a}{b}$ for all $t \in (0, \omega_p]$ with $p(t) = 1$ means that p intersects the discontinuity point 1 transversally. Observe that, by condition (C_g), if p is a periodic solution of (E_g) with $p(t) > 1 = p(-1)$ for all $t \in [-2, -1)$, then $p_0 = \eta^a$.

For a periodic solution $p : \mathbb{R} \to \mathbb{R}$ of (E_g) with minimal period $\omega > 0$, let $\mathcal{O} = \{p_t : t \in [0,\omega]\}$ denote the corresponding periodic orbit. We say that $\mathcal{O}$ has finite-time orbital attraction if there is an open neighborhood of $\mathcal{O}$ such that for any ϕ from this neighborhood there exists a finite $T = T(\phi)$ such that $\Psi_{a,b}(T + t, \phi) = p_t$ for all $t \geq 0$.

The main result of this paper shows for equation (E_g) that, under hypothesis (H_a), stable periodic orbits bifurcate from the homoclinic orbit $\mathcal{H}$ for $b > b^*(a)$.

Theorem 1.1. *Let $a > 0$ be given so that (H_a) holds. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(a,\varepsilon) > 0$ such that for all $b \in (b^*(a), b^*(a) + \delta)$ equation (E_g) has a unique periodic solution $p^{a,b} : \mathbb{R} \to \mathbb{R}$ with property (P) and, for $\mathcal{O}^{a,b} = \{p_t^{a,b} : t \in [0, \omega_{a,b}]\}$,*

$$\mathrm{dist}_H\{\mathcal{O}^{a,b}, \mathcal{H}\} < \varepsilon.$$

Each periodic orbit $\mathcal{O}^{a,b}$ has finite-time orbital attraction. Moreover, $\omega_{a,b} \to \infty$ as $b \to b^*(a)+$ for the minimal period $\omega_{a,b}$ of $p^{a,b}$.

No spectral condition at the stationary point $\hat{\xi}_{a,b^*(a)}$ appears explicitly in Theorem 1.1. Condition (C_g) implies $b^* g'(\xi_{a,b^*(a)}) > a$, and hence the spectrum of the generator of the linearized semigroup at $\hat{\xi}_{a,b^*(a)}$ has a leading real eigenvalue $\lambda_0 > 0$; see Section 2. This fact is important in the construction of the unique solution $x^{+,a,b} : \mathbb{R} \to \mathbb{R}$ satisfying (1.1); see [2]. We emphasize that the proof of Theorem 1.1 does not require information on the rest of the spectrum. That is, we do not use the spectral condition $\lambda_0 + \mathrm{Re}\,\lambda_1 < 0$, which is a key assumption in previous papers for homoclinic bifurcation; see [4, 5, 9, 13, 19, 28, 29, 32]. In this paper, the fact that the nonlinearity g is zero on $(1,\infty)$ guarantees, by direct estimations, that stable periodic orbits approach the homoclinic orbit.

In the papers [2, 3], condition (P) was the key hypothesis to guarantee the existence of stable periodic orbits of equation (E_{f_n}) for large n. Since Theorem 1.1 shows that (P) holds for the obtained periodic solution $p^{a,b}$, applying Proposition 2.3 of [2] yields the following result.

Corollary 1.2. *Assume $a > 0$ is given so that (H_a) holds. Let $b \in (b^*(a), b^*(a) + \delta(a,1))$, where $\delta(a,1)$ is given in Theorem 1.1. Suppose that the sequence of functions $(f_n)_{n=1}^{\infty}$ satisfies:*

(C_{f_n}) *Each $f_n : [0,\infty) \to [0,1)$ is C^1-smooth with $f_n(0) = 0$ and $f_n'(\xi) > 0$ for $\xi \in (0, 1/2]$. For any $\kappa \in (0,1)$ and $m \in \mathbb{N}$,*

$$\|f_n - g\|^1_{[0,1-\kappa]\cup[1+\kappa,\infty)} \to 0, \quad \text{and} \quad \|f_n'\|_{[1+\kappa,\infty)} \cdot \left(\|f_n'\|_{[0,\infty)}\right)^m \to 0 \quad \text{as } n \to \infty.$$

Then, for all sufficiently large n, equation (E_{f_n}) admits a hyperbolic, orbitally asymptotically stable, exponentially attractive periodic orbit $\mathcal{O}^{a,b_n}$ with $\operatorname{dist}_H\left(\mathcal{O}^{a,b_n}, \mathcal{O}^{a,b}\right) \to 0$ as $n \to \infty$

The paper is organized as follows. Section 2 contains some preliminary results for equation (E_g), and recalls some facts for equations with monotone feedback functions. Proposition 3.1 in Section 3 gives an interval where (H_a) holds. The rest of Section 3 proves Theorem 1.1. Section 4 contains examples showing that our sufficient condition (3.1) for hypothesis (H_a) does not imply any of the spectral conditions $\lambda_0 + \operatorname{Re}\lambda_1 < 0$ or $\lambda_0 + \operatorname{Re}\lambda_1 > 0$.

2 Preliminary results

Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Recall $C = C([-1,0], \mathbb{R})$ and $C^+ = \{\varphi \in C : \varphi(\theta) \geq 0,\ \theta \in [-1,0]\}$, $\|\varphi\| = \max_{\theta\in[-1,0]}|\varphi(\theta)|$. For $\xi \in [0,\infty)$, set $\hat{\xi} \in C$ by $\hat{\xi}(s) = \xi$, $s \in [-1,0]$.

Let t_1, $t_2 \in \mathbb{R}$ with $t_1 < t_2 \leq \infty$. By using (C_g), it is easy to see that for a continuous $x : [t_1 - 1, t_2) \to [0,\infty)$ the map $[t_1, t_2) \ni s \mapsto g(x(s-1)) \in [0,1]$ is measurable. Then a solution $x : [t_1 - 1, t_2) \to [0,\infty)$ of (E_g) on the interval $[t_1 - 1, t_2)$ is defined so that x is continuous on the interval $[t_1 - 1, t_2)$ and the integral equation

$$x(t) = e^{-a(t-\tau)}x(\tau) + b\int_\tau^t e^{-a(t-s)}g(x(s-1))ds \tag{2.1}$$

holds for all $t_1 \leq \tau < t < t_2$. For given $\varphi \in C^+$, the method of steps allows to define a unique solution $x^\varphi : [-1,\infty) \to [0,\infty)$ with $x_0^\varphi = \varphi$. Moreover, if $t > 0$ and $x^\varphi(t-1) \neq 1$, then x^φ is differentiable at t, and equation (E_g) holds at t. The map

$$\Psi_{a,b} : [0,\infty) \times C^+ \ni (t,\varphi) \mapsto x_t^\varphi \in C^+$$

defines a semiflow. $\Psi_{a,b}$ is not continuous; see [3], but its restriction to $[0,\infty) \times \{\varphi \in C^+ : \varphi(s) < 1,\ s \in [-1,0]\}$ is continuous.

By the smoothness of f_n, classical results from [6, 8] apply to equation (E_{f_n}) for existence, uniqueness, continuous dependence. For $t_1 < t_2 \leq \infty$, a solution of (E_{f_n}) on $[t_1 - 1, t_2)$ is a continuous $y : [t_1 - 1, t_2) \to [0,\infty)$ so that y is continuously differentiable on (t_1, t_2) and equation (E_{f_n}) holds for all $t \in (t_1, t_2)$. Equivalently, a continuous $y : [t_1 - 1, t_2) \to [0,\infty)$ satisfying (2.1), with f_n instead of g, is a solution of (E_{f_n}) on $[t_1 - 1, t_2)$. The method of steps

is applied as well to obtain that for any $\psi \in C^+$, there is a unique solution $y^\psi : [-1, \infty) \to \mathbb{R}$ of equation (E_{f_n}) with $y_0^\psi = \psi$, and

$$\Phi_{a,b}^n : [0, \infty) \times C^+ \ni (t, \psi) \mapsto y_t^\psi \in C^+$$

is a continuous semiflow.

The feedback function g is monotone on the interval $[0, 1]$. This fact allows the application of well-known results from the theory of monotone semiflows; see e.g. [11, 17, 30]. To apply results from the theory of monotone semiflows for equation (E_g), define $G : [0, \infty) \to \mathbb{R}$ so that

$$G(\xi) = \begin{cases} g(\xi) & \text{if } 0 \le \xi \le 1, \\ 2 - \exp\left[g'(1)(1-\xi)\right] & \text{if } \xi > 1, \end{cases}$$

and consider the modified equation

$$x'(t) = -ax(t) + bG\left(x(t-1)\right). \tag{E_G}$$

It is easy to check that G is C^1-smooth, $G'(\xi) > 0$ for all $\xi > 0$, and $G''(\xi)$ exists for $\xi > 1$ with $G''(\xi) < 0$. For $-\infty < t_1 < t_2 \le \infty$, the function $x : [t_1 - 1, t_2) \to [0, \infty)$ of (E_G) is a solution on the interval $[t_1 - 1, t_2)$ if it is continuous and the integral equation (2.1) with g replaced by G holds for all $t_1 \le \tau < t < t_2$. The method of steps can be applied to find a unique solution $x^\varphi : [-1, \infty) \to [0, \infty)$ with $x_0^\varphi = \varphi$. Moreover, x^φ is differentiable on $(0, \infty)$ and right differentiable at 0. Denote by $\Gamma_{a,b} : [0, \infty) \times C^+ \to C^+$ the continuous semiflow generated by the solutions of equation (E_G). For $b > a > 0$, by the properties of G, the map $[0, \infty) \ni \xi \mapsto -a\xi + bG(\xi) \in \mathbb{R}$ has three zeros: 0, $\xi_{a,b}$ and $\eta_{a,b} \in (1, 2\frac{b}{a})$. The semiflows $\Psi_{a,b}$ and $\Gamma_{a,b}$, restricted to $[0, \infty) \times \{\varphi \in C^+ : \varphi(\theta) \le 1,\ \theta \in [-1, 0]\}$, coincide.

The following continuous dependence argument can be verified by using the method of steps and the integral equation version of (E_G):

$$\begin{aligned} &\text{For any } \varepsilon > 0,\ T > 0 \text{ and } b > a > 0 \text{ there exists a } \delta > 0 \text{ such that} \\ &\phi, \tilde{\phi} \in C^+,\ \|\phi - \tilde{\phi}\| < \delta,\ |b - \tilde{b}| < \delta,\ t \in [0, T] \Longrightarrow \|\Gamma_{a,b}(t, \phi) - \Gamma_{a,\tilde{b}}(t, \tilde{\phi})\| < \varepsilon. \end{aligned} \tag{2.2}$$

The linear variational equations of (E_g) and (E_G) at the stationary point $\hat{\xi}_{a,b}$ coincide. The spectrum of the generator associated with $\left(D_2\Gamma_{a,b}(t, \hat{\xi}_{a,b})\right)_{t \ge 0}$ is given by the zeros of the characteristic function $\mathbb{C} \ni \lambda \mapsto \lambda + a - bg'(\xi_{a,b})e^{-\lambda} \in \mathbb{C}$. From ($C_g$) it follows that

$$bg'(\xi_{a,b}) > b\frac{g(\xi_{a,b})}{\xi_{a,b}} = a,$$

and hence, the characteristic function has a unique positive $\lambda_0 > 0$, and the remaining zeros occur in complex conjugate pairs $\lambda_j, \overline{\lambda_j}$, $j \in \mathbb{N}$, with $(2j-1)\pi < \operatorname{Im}\lambda_j < 2j\pi$, $\lambda_0 > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \ldots$, and $\operatorname{Re}\lambda_j \to -\infty$ as $j \to \infty$. All zeros are simple; see e.g. [6].

We remark that hypothesis (H_a) implies $\operatorname{Re}\lambda_1 \le 0$. Suppose $\operatorname{Re}\lambda_1 > 0$. Proposition 2.2 below shows that $z^{a,b^*(a)}(t) \in (0, 1)$ for all $t > -1$. Hence, $u(t) = z^{a,b^*(a)}(t) - \xi_{a,b^*(a)}$, $t \ge -1$, satisfies the equation

$$u'(t) = -au(t) + b^*(a)G(\xi_{a,b^*(a)} + u(t-1)) - G(\xi_{a,b^*(a)})$$

for all $t > 0$, which is an equation with monotone positive feedback. The initial function $u_0 = \eta^a - \hat{\xi}_{a,b^*(a)}$ has one sign change in $[-1, 0]$. Applying a discrete Lyapunov functional

$V(u_t)$, which basically counts the number of sign changes of u on $[t-1,t]$, we obtain $V(u_t) \le 2$ for all $t \ge 0$; see e.g. [17, 24, 26]. On the other hand, by Proposition 4.6 in [16], we have $V(u_t) \ge 4$ for all $t \ge 0$, whenever $\operatorname{Re}\lambda_1 > 0$ and $\lim_{t\to\infty} u(t) = 0$; a contradiction.

For $\phi, \psi \in C$, introduce $\phi \le \psi$ and $\phi \ll \psi$ if $\phi(s) \le \psi(s)$, $s \in [-1,0]$, and $\phi(s) < \psi(s)$, $s \in [-1,0]$, respectively. We write $\phi < \psi$ if $\phi \le \psi$ and $\phi \ne \psi$. The relations $\ge$, $\gg$, $>$ are defined analogously.

The semiflow $\Gamma_{a,b}$ is monotone because $G'(\xi) > 0$ for $\xi > 0$.

Proposition 2.1.

(i) *$\Gamma_{a,b}$ is monotone in the sense that $\phi, \psi \in C^+$, $\phi \le \psi$ ($\phi \ll \psi$) imply $\Gamma_{a,b}(t,\psi) \le \Gamma(t,\psi)$ ($\Gamma_{a,b}(t,\psi) \ll \Gamma(t,\psi)$) for all $t \ge 0$; and $\phi, \psi \in C^+$, $\phi < \psi$ imply $\Gamma_{a,b}(t,\psi) \ll \Gamma(t,\psi)$ for all $t \ge 2$.*

(ii) *If $\phi, \psi \in C^+$ and $\phi \ll \psi$, then either $\Gamma_{a,b}(t,\phi) \to \hat{0}$ or $\Gamma_{a,b}(t,\psi) \to \hat{\eta}_{a,b}$ as $t \to \infty$.*

Proof. Statement *(i)* is a standard result in the theory of monotone semiflows for delay differential equations; see [11, 17, 30]. We prove *(ii)*.

The set of stationary points of $\Gamma_{a,b}$ is $E = \{\hat{0}, \hat{\xi}_{a,b}, \hat{\eta}_{a,b}\}$, and $\hat{0}, \hat{\eta}_{a,b}$ are stable and attract all points from their small neighborhoods in the set $\{\phi \in C : \phi(s) \in [0, \eta_{a,b}],\ s \in [-1,0]\}$. Using Theorem 4.3 in [30], see also [31], there exist $\phi^*, \psi^* \in C^+$ such that $\phi \ll \phi^* \ll \psi^* \le \psi$, and $\omega(\phi^*) \subset E$, $\omega(\psi^*) \subset E$.

If $\hat{0}$ is in $\omega(\phi^*)$ (or in $\omega(\psi^*)$), then monotonicity of the semiflow and the attracting property of $\hat{0}$ imply $\omega(\phi) = \hat{0}$. In the same way, if $\hat{\eta}_{a,b}$ is in $\omega(\phi^*)$ (or in $\omega(\psi^*)$) then $\omega(\psi) = \hat{\eta}_{a,b}$ follows. Thus, it remains to consider the case $\omega(\phi^*) = \omega(\psi^*) = \hat{\xi}_{a,b}$.

Define $u(t) = x^{\phi^*}(t) - x^{\psi^*}(t)$, $t \ge -1$, where x^{ϕ^*}, x^{ψ^*} are the solutions of (E_G) with $x_0^{\phi^*} = \phi^*$, $x_0^{\psi^*} = \psi^*$, respectively. Then $u_t \gg \hat{0}$ for all $t \ge 0$, and

$$u'(t) = -au(t) + b(t)u(t-1) \quad (t > 0)$$

with $b(t) = b\int_0^1 G'((1-s)x^{\psi^*}(s-1) + sx^{\phi^*}(s-1))\,ds > 0$, $t \ge 0$. From $x^{\phi^*}(t) \to \xi_{a,b}$, $x^{\psi^*} \to \xi_{a,b}$ as $t \to \infty$, it follows that $b(t) \to bG'(\xi_{a,b}) > a$ as $t \to \infty$. For any $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}[bG'(\xi_{a,b}) - a]$ there is a $T > 2$ so that $b(t) > a + \varepsilon$ for $t \ge T$. Then, for all $t \ge T$,

$$u'(t) \ge -au(t) + [a+\varepsilon]u(t-1).$$

There is a $\mu > 0$ with $\mu = -a + (a+\varepsilon)e^{-\mu}$ and, for any $c \in \mathbb{R}$, $v(t) = ce^{\mu t}$ satisfies $v'(t) = -av(t) + (a+\varepsilon)v(t-1)$. Choose $c > 0$ such that $u(t) > ce^{\mu t}$ for $t \in [T, T+1]$. Then, for the function $w(t) = u(t) - ce^{\mu t}$, $t \ge T$, the inequality

$$w'(t) \ge -aw(t) + [a+\varepsilon]w(t-1) \quad \text{for all } t \ge T+1$$

holds with $w_{T+1} \gg \hat{0}$. It is straightforward to check that $w(t) > 0$ follows for all $t \ge T$. Hence $u(t) > ce^{\mu t}$ for all $t \ge T$. This contradicts $u(t) \to 0$, $t \to \infty$.

Therefore, the case $\omega(\phi^*) = \omega(\psi^*) = \hat{\xi}_{a,b}$ cannot happen. This completes the proof. □

As in the introduction, let $z^{a,b} : [-1,\infty) \to \mathbb{R}$ denote the unique solution of equation (E_g) with the initial conditions $z_0^{a,b} = \eta^a$. By Section 5 in [2], there is a unique $b^*(a) > a$ so that properties (1.4) and (1.5) hold.

Next, we characterize $b^*(a)$ by using equation (E_G), and give a bound for $z^{a,b^*(a)}$.

Proposition 2.2. *For $b^*(a) > a$, the following properties hold.*

(i) $a < b < b^*(a)$ *implies* $\Gamma_{a,b}(t, \eta^a) \to \hat{0}$ *as* $t \to \infty$.

(ii) $b > b^*(a)$ *implies* $\Gamma_{a,b}(t, \eta^a) \to \hat{\eta}_{a,b}$ *as* $t \to \infty$.

(iii) $b_2 > b_1 > b^*(a)$ *implies* $\Gamma_{a,b_2}(t, \eta^a)(0) > \Gamma_{a,b_1}(t, \eta^a)(0) > \Gamma_{a,b^*(a)}(t, \eta^a)(0)$ *for all* $t > 0$.

(iv) $z^{a,b^*(a)}(t) = \Gamma_{a,b^*(a)}(t, \eta^a)(0) \in (e^{-a}, 1)$ *for all* $t > 0$.

Proof. Let $a > 0$ be fixed. For $b > a > 0$, let $Z^{a,b} : [-1, \infty) \to \mathbb{R}$ denote the solution of (E_G) with $Z^{a,b}_0 = \eta^a$. Then $\Gamma_{a,b}(t, \eta^a)(0) = Z^{a,b}(t)$ for all $t \geq 0$. Define

$$B_- = \{b > a : Z^{a,b}(t) \to 0 \text{ as } t \to \infty\}, \quad B_+ = \{b > a : Z^{a,b}(t) \to \eta_{a,b} \text{ as } t \to \infty\}.$$

Step 1. Let $a < b_1 < b_2$. We claim $Z^{a,b_1}(t) < Z^{a,b_2}(t)$ for all $t > 0$. Indeed, from (E_G), for the right-hand derivatives we have

$$\left(Z^{a,b_1}\right)'(0+) < \left(Z^{a,b_2}\right)'(0+).$$

So, the claim holds for small $t > 0$. Suppose that $t_0 > 0$ is minimal at which $Z^{a,b_1}(t_0) = Z^{a,b_2}(t_0)$. Then $\left(Z^{a,b_1}\right)'(t_0) \geq \left(Z^{a,b_2}\right)'(t_0)$ holds. On the other hand, from (E_G), the opposite inequality $\left(Z^{a,b_1}\right)'(t_0) < \left(Z^{a,b_2}\right)'(t_0)$ can be obtained, a contradiction. This proves *(iii)*.

Step 2. For all sufficiently large b, $\eta^a \gg \hat{\xi}_{a,b}$. Hence $\Gamma_{a,b}(t, \eta^a) \to \hat{\eta}_{a,b}$ follows as $t \to \infty$, by Proposition 2.1. Thus, $b \in B_+$ for all large b. If $b \in B_+$ then there is a $T > 0$ so that $\Gamma_{a,b}(T, \eta^a) \gg \hat{\xi}_{a,b}$. The continuity of the map $b \mapsto \hat{\xi}_{a,b}$ and the continuous dependency (2.2) imply $\Gamma_{a,\tilde{b}}(T, \eta^a) \gg \hat{\xi}_{a,\tilde{b}}$ for $\tilde{b}$ in a small neighborhood of b. Using Proposition 2.1, we see that $\tilde{b} \in B_+$. Thus, B_+ is nonempty and open.

Step 3. Suppose $b > a$. For $t \in [0, 1]$, we have

$$\begin{aligned} Z^{a,b}(t) &= e^{-at}e^{-a} + b\int_0^t e^{-a(t-s)}G(Z^{a,b}(s-1))\,ds \\ &\leq e^{-at}e^{-a} + b\int_0^t e^{-a(t-s)}\,ds = e^{-at}e^{-a} + \frac{b}{a}(1 - e^{-at}) \\ &= \frac{b}{a} - e^{-at}\left(\frac{b}{a} - e^{-a}\right) \leq \frac{b}{a} - e^{-a}\left(\frac{b}{a} - e^{-a}\right) = \frac{b}{a}(1 - e^{-a}) + e^{-2a}. \end{aligned}$$

Observe $\frac{b}{a}(1 - e^{-a}) + e^{-2a} \to 1 - e^{-a} + e^{-2a} < 1$ and $\hat{\xi}_{a,b} \to \hat{1}$ as $b \to a$. Therefore, $\Gamma(1, \eta^a) \ll \hat{\xi}_{a,b}$, and Proposition 2.1 implies $Z^{a,b}(t) \to 0$, $t \to \infty$, provided $b - a > 0$ is sufficiently small. Thus, B_- is nonempty.

Assume $b \in B_-$. Then $Z^{a,b}_T \ll \hat{\xi}_{a,b}$ for some $T > 0$. By the continuous dependence argument in (2.2), and the continuity of $b \mapsto \hat{\xi}_{a,b}$, there is $\delta > 0$ so that $Z^{a,\tilde{b}}_T \ll \hat{\xi}_{a,\tilde{b}}$ for all $\tilde{b} \in (b - \delta, b + \delta)$. Hence, by Proposition 2.1, $\tilde{b} \in B_-$ follows. Therefore, B_- is nonempty and open.

Step 4. Steps 1–3 imply the existence of a unique $B^* = B^*(a) > a$ such that

$$B_- = (a, B^*) \text{ and } B_+ = (B^*, \infty).$$

Clearly, $Z^{a,B^*}(t) \not\to 0$ and $Z^{a,B^*}(t) \not\to \eta_{a,B^*}$ as $t \to \infty$.

Step 5. The proof of $Z^{a,B^*}(t) \in (e^{-a}, 1)$, $t > 0$.

If $Z^{a,B^*}(t) < 1$ does not hold for all $t > 0$, then there is a minimal $t_0 > 0$ with $Z^{a,B^*}(t_0) = 1$. Clearly, $\left(Z^{a,B^*}\right)'(t_0) \geq 0$. Then, by $(\eta^a)'(s) = -a\eta^a(s) < 0$, $s \in [-1, 0]$, there exists a $\delta > 0$ such that $Z^{a,B^*}(t_0 + s) > \eta^a(-1 + s)$ for $0 < s < \delta$. Suppose $Z^{a,B^*}(t_0 + s) > \eta^a(-1 + s)$ for all $s \in (0, s_0)$ and $Z^{a,B^*}(t_0 + s_0) = \eta^a(-1 + s_0)$ for some $s_0 \in (0, 1)$. Then $\left(Z^{a,B^*}\right)'(t_0 + s_0) \leq (\eta^a)'(-1 + s_0) = -a\eta^a(-1 + s_0)$. On the other hand, by (E_G),

$$\begin{aligned}\left(Z^{a,B^*}\right)'(t_0 + s_0) &= -aZ^{a,B^*}(t_0 + s_0) + B^*G(Z^{a,B^*}(t_0 + s_0 - 1)) \\ &> -aZ^{a,B^*}(t_0 + s_0) = -a\eta^a(-1 + s_0),\end{aligned}$$

a contradiction. Consequently,

$$Z^{a,B^*}_{t_0+1} > \eta^a.$$

By Proposition 2.1, $Z^{a,B^*}_{t_0+3} \gg \Gamma_{a,B^*}(2, \eta^a) = Z^{a,B^*}_2$, and necessarily either $Z^{a,B^*}_t \to \hat{0}$ or $Z^{a,B^*}_t \to \hat{\eta}_{a,B^*}$ as $t \to \infty$, a contradiction. Thus, $Z^{a,B^*}(t) < 1$ for all $t > 0$.

Suppose $Z^{a,B^*}(t) > e^{-a}$ does not hold for all $t > 0$. From $(Z^{a,B^*})'(0+) = -ae^{-a} + B^* > 0$ it follows that $Z^{a,B^*}(t) > e^{-a}$ for all $t \in (0, \delta_0)$ for some $\delta_0 > 0$. Then we find a $t_0 > 0$ with $Z^{a,B^*}(t_0) = e^{-a}$ and $(Z^{a,B^*})'(t_0) > -ae^{-a} = (\eta^a)'(0-)$. By continuity, there exists a $\delta_1 > 0$ such that $Z^{a,B^*}(t_0 + s) < \eta^a(s)$ for $-\delta_1 < s < 0$. If $Z^{a,B^*}(t_0 + s_0) = \eta^a(s_0)$ for some $s_0 \in [-1, 0)$, then $(Z^{a,B^*})'(t_0 + s_0) \leq (\eta^a)'(s_0) = -a\eta^a(s_0)$. On the other hand, from (E_G),

$$\begin{aligned}\left(Z^{a,B^*}\right)'(t_0 + s_0) &= -aZ^{a,B^*}(t_0 + s_0) + B^*G(Z^{a,B^*}(t_0 + s_0 - 1)) \\ &> -aZ^{a,B^*}(t_0 + s_0) = -a\eta^a(s_0),\end{aligned}$$

a contradiction. Therefore,

$$Z^{a,B^*}_{t_0+1} < \eta^a,$$

and, by applying Proposition 2.1, we arrive at a contradiction as above. Thus, $Z^{a,B^*}(t) > e^{-a}$ for all $t > 0$.

Step 6. Notice that Step 4 with Step 1 implies $\Gamma_{a,b}(t, \eta^a) = \Psi_{a,b}(t, \eta^a)$ for all $t \geq 0$, provided $b \leq B^*(a)$. Then, by the definitions of B_- and B_+, we conclude $b^*(a) = B^*(a)$, and $z^{a,b^*(a)}(t) = Z^{a,b^*(a)}(t)$, $t \geq -1$.

The proof is complete. □

The following result is a simplified version of Proposition 3.5 in [2].

Proposition 2.3. *Let $a > 0$ be fixed and suppose $B > a > 0$. For each $b > B$, let $\nu_b > 0$, $m_b \in (0, 1)$ and a continuous function $x^b : (-\infty, 0] \to [0, 1]$ be given such that $\nu_b \to \infty$, $m_b \to \xi_{a,B}$ as $b \to B_+$, $x^b(t) \in [m_b, 1]$ for all $t \leq 0$, $x^b(0) = 1$, and x^b is a solution of the equation*

$$x'(t) = -ax(t) + bG\left(x(t-1)\right), \tag{$E_{G,n}$}$$

on the interval $[-\nu_b - 1, 0]$. Then, for every $\varepsilon > 0$ and $T > 0$ there exists a $\delta = \delta(\varepsilon, T) > 0$ such that

$$b \in (B, B + \delta) \Longrightarrow \sup_{t \in [-T, 0]} |x^b(t) - x^{+,a,B}(t)| < \varepsilon.$$

There is a slight modification in the conditions of Proposition 2.3 compared to Proposition 3.5 in [2]. Here we require x^b to be a solution on the interval $[-\nu_b - 1, 0]$, while Proposition 3.5 of [2] assumes solutions on $(-\infty, 0]$. However, this change does not cause any extra difficulty in the proof. For the proof, we refer to Proposition 3.5 of [2], which is a standard application of the Arzelà–Ascoli theorem and the diagonalization process. For any sequence $(b_n)_{n=1}^{\infty}$ with $b_n \to B_+$, there is a subsequence $(b_{n_k})_{k=1}^{\infty}$ such that $(x^{b_{n_k}})_{k=1}^{\infty}$ converges to a continuous function $u : (-\infty, 0] \to \mathbb{R}$, uniformly on compact subintervals of $(-\infty, 0]$ satisfying the integral equation

$$u(t) = e^{-a(t-\tau)}u(\tau) + B\int_{\tau}^{t} e^{-a(t-s)}G\left(u(s-1)\right)ds$$

for all τ, t with $-\infty < \tau < t \le 0$, and $u((-\infty, 0]) \subset [\xi_{a,B}, 1]$, $u(0) = 1$. Then Proposition 3.4 of [2] implies $u(t) = x^{+,a,B}(t)$ for all $t \le 0$. Convergence for the full sequence $(x^{b_n})_{n=1}^{\infty}$ is a consequence of the fact that for arbitrary subsequence we obtain the same limit function $x^{+,a,B}$ due to Proposition 3.4 of [2].

3 Proofs

In this section, first we prove that condition (H_a) holds for some $a > 0$. The forthcoming paper [15] verifies (H_a) for a larger interval of the parameter a in the particular case $g(\xi) = \xi^2$, $0 \le \xi \le 1$. Define

$$A : [0, \infty) \ni a \mapsto \frac{e^{-a}(1 - e^{-a})}{g(e^{-a})} \in \mathbb{R}.$$

By using the condition $g'(\xi) > \frac{g(\xi)}{\xi}$, $0 < \xi \le 1$, in (C_g), it is easy to obtain that $A'(a) > 0$ for all $a > 0$. Clearly, $A(0) = 0$. From $\frac{g(e^{-a})}{e^{-a}} \to g'(0) = 0$, $a \to \infty$, we find $A(a) \to \infty$ as $a \to \infty$. Let $A^{-1} : [0, \infty) \to [0, \infty)$ denote the inverse of A.

Proposition 3.1. *Assume $a > 0$. Condition (H_a) holds if*

$$a < A^{-1}\left(\frac{1}{\|g'\|_{[0,1]}}\right). \tag{3.1}$$

In the particular case $g(\xi) = \xi^2$ for $0 \le \xi \le 1$, condition (3.1) reduces to $a < \log\frac{3}{2}$.

Proof. Set $z^* = z^{a,b^*}$, $\xi_* = \xi_{a,b^*}$. By Proposition 2.2, $z^*(t) \in (e^{-a}, 1)$ for all $t > 0$.

1. We claim that the function $u : [-1, \infty) \ni t \mapsto z^*(t) - \xi_* \in \mathbb{R}$ is oscillatory. By Proposition 2.1*(i)*, an equivalent statement is that, for any $t \ge 0$, neither $z_t^* > \hat{\xi}_*$ nor $z_t^* < \hat{\xi}_*$ can appear. By Proposition 2.1*(ii)*, $z_t^* = \Gamma_{a,b^*}(t, \eta^a)$ for $t \ge 0$, and $z^*(t) \in (e^{-a}, 1)$ for $t > 0$ combined imply that, for every $t \ge 0$, neither $z_t^* > \hat{\xi}_*$ nor $z_t^* < \hat{\xi}_*$ is satisfied.

2. The inequality $\xi_* > e^{-a}$ follows as well, because $\xi_* \le e^{-a}$ and Proposition 2.1*(i)* yield $\hat{\xi}_* \ll z_2^*$, which is impossible. Thus $e^{-a} < \xi_*$. Then (C_g) and $\frac{a}{b^*(a)} = \frac{g(\xi_*)}{\xi_*}$ imply

$$b^*(a) = a\frac{\xi_*}{g(\xi_*)} < \frac{ae^{-a}}{g(e^{-a})}.$$

3. The function u satisfies

$$u'(t) = -au(t) + b(t)u(t-1) \quad (t > 0) \tag{3.2}$$

with

$$b(t) = b^*(a) \int_0^1 g'(\xi_* + su(t-1))\, ds \in \left(0, b^*(a)\|g'\|_{[0,1]}\right] \quad \text{for all } t \geq 0,$$

and $u_0 = \eta^a - \hat{\xi}_*$ The function u has one sign change in $[-1,0]$; it is oscillatory and satisfies (3.2) with a continuous and positive $b(t)$. For this situation, a discrete Lyapunov functional $V : C \setminus \{\hat{0}\} \to \{0,2,4,\ldots\} \cup \{\infty\}$, which essentially counts the number of sign changes per unit interval, can be applied; see e.g. [24] and Appendix VI from [17]. For $u : [-1,\infty) \to \mathbb{R}$, using $V(u_0) = 1$, the monotone decreasing property of V, and the fact that u is oscillatory, we obtain $V(u_t) = 2$ for all $t \geq 0$. A consequence is that there is a $T > 0$ such that all zeros of u are simple on $[T,\infty)$, the distance between consecutive zeros is less than 1, and there are at most 2 zeros on each subinterval $[t-1,t] \subset [T,\infty)$. Therefore, we find an increasing sequence $(t_j)_{j=1}^{\infty}$ in $[T,\infty)$ such that

$$\begin{aligned} &\text{for every } j \in \mathbb{N}, \quad t_{j+1} - t_j < 1,\ t_{j+2} - t_j > 1,\ u(t_j) = 0, \\ &u(t) > 0 \quad \text{for } t \in (t_{2j-1}, t_{2j}), \quad u(t) < 0 \quad \text{for } t \in (t_{2j}, t_{2j+1}). \end{aligned} \tag{3.3}$$

Hence we can choose $\sigma_j \in (t_j, t_{j+1})$ with

$$m_j = |u(\sigma_j)| = \max_{t \in [t_j, t_{j+1}]} |u(t)|.$$

Clearly, $u'(\sigma_j) = 0$, and $u(\sigma_j)u(\sigma_j - 1) > 0$ for all $j \in \mathbb{N}$. Then, from (3.3), $\sigma_{j+2} - 1 \in (t_j, t_{j+1})$ follows for all $j \in \mathbb{N}$.

4. Applying the integral equation version of (3.2), obtained from the variation-of-constants formula, to u on $[t_{j+2}, \sigma_{j+2}]$, using $t_{j+2} - 1 \in (t_j, t_{j+1})$ and $\sigma_{j+2} - 1 \in (t_j, t_{j+1})$, we obtain

$$\begin{aligned} m_{j+2} = |u(\sigma_{j+2})| &= \left| \int_{t_{j+2}}^{\sigma_{j+2}} e^{-a(\sigma_{j+2}-s)} b(s) u(s-1)\, ds \right| \\ &\leq b^*(a)\|g'\|_{[0,1]} \int_{t_{j+2}-1}^{\sigma_{j+2}-1} e^{-a(\sigma_{j+2}-1-s)} |u(s)|\, ds \\ &\leq b^*(a)\|g'\|_{[0,1]} \frac{1 - e^{-a(\sigma_{j+2} - t_{j+2})}}{a} m_j \leq b^*(a)\|g'\|_{[0,1]} \frac{1 - e^{-a}}{a} m_j \\ &< \frac{ae^{-a}}{g(e^{-a})} \frac{1 - e^{-a}}{a} \|g'\|_{[0,1]} m_j = A(a)\|g'\|_{[0,1]} m_j. \end{aligned} \tag{3.4}$$

Setting $\kappa = A(a)\|g'\|_{[0,1]}$, condition (3.1) on a implies $\kappa \in (0,1)$. Then, by induction, from (3.4) it follows that

$$m_{2j+1} \leq \kappa^j m_1 \quad \text{and} \quad m_{2j+2} \leq \kappa^j m_2 \quad \text{for all } j \in \mathbb{N}.$$

By using $\kappa < 1$, we conclude $u(t) \to 0$ as $t \to \infty$. The proof is complete. □

Remark 3.2. We saw in Section 2 that conditions (C_g) and (3.1) imply $\lambda_0 > 0$ and $\operatorname{Re}\lambda_1 \leq 0$ for the linearization of (E_g) at $\hat{\xi}_{a,b^*(a)}$. In Section 4, we give a set of examples where (C_g) and (3.1) hold, and either $\lambda_0 + \operatorname{Re}\lambda_1 < 0$ or $\lambda_0 + \operatorname{Re}\lambda_1 > 0$ can happen.

Now we prove Theorem 1.1.

The proof of Theorem 1.1. Let $a > 0$ be fixed such that (H_a) holds. As a is fixed, we omit the dependence on a. Set $\xi_* = \xi_{a,b^*}$, $z^* = z^{a,b^*}$, $x^{+,*} = x^{+,a,b^*}$.

Step 1. By $b^*g'(\xi_*) > a$, $B_0 = \frac{1}{2}[a + b^*g'(\xi_*)] \in (a, b^*g'(\xi_*))$. Condition (C_g) implies that $(0,1] \ni \xi \mapsto \frac{g(\xi)}{\xi} \in \mathbb{R}$ is increasing. Therefore, by using (C_g) again,

$$b^*g'(\xi) > b^*\frac{g(\xi)}{\xi} \geq b^*\frac{g(\xi_*)}{\xi_*} = a \quad \text{for all } \xi \in [\xi_*, 1].$$

Using the continuity of g', there is an $\alpha_0 \in (0, 1-\xi_*)$ such that

$$b^*g'(\xi) \geq B_0 \quad \text{for all } \xi \in [\xi_* - \alpha_0, 1]. \tag{3.5}$$

Let $\varepsilon > 0$ be given. For $z^* : [-1, \infty) \to \mathbb{R}$ we have $z_0^* = \eta^a$, $\lim_{t\to\infty} z^*(t) = \xi_*$, and $z^*(t) \in (e^{-a}, 1)$ for all $t > 0$. Recall $x^{+,*}$ from Section 1 with property (1.1). Choose $\varepsilon_0 \in (0, \varepsilon)$ such that

$$\varepsilon_0 < 1 - \sup_{t\geq 0} z^*(t), \quad \varepsilon_0 < \left(1 - e^{-a}\right) x^{+,*}(1) \quad \text{and} \quad \varepsilon_0 < \min_{t\in[0,1]} (x^{+,*})'(t), \tag{3.6}$$

where $(x^{+,*})'(1)$ denotes the left-hand derivative. Define $\varepsilon_1 > 0$ and $\delta_0 > 0$ such that

$$\varepsilon_1 = \frac{1}{2(1+a)b^*\|g'\|_{[0,1]}}\varepsilon_0 \quad \text{and} \quad \delta_0 = \frac{1}{2(1+a)}\varepsilon_0. \tag{3.7}$$

By (H_a) and (1.1), we can choose $T_0 > 0$ so that

$$|z^*(t) - \xi_*| < \min\{\alpha_0, \tfrac{1}{2}\varepsilon\} \quad \text{for all } t \geq T_0, \quad \text{and} \quad x^{+,*}(t) - \xi_* < \tfrac{1}{2}\varepsilon \quad \text{for all } t \leq -T_0. \tag{3.8}$$

Step 2. For $b > b^*$, recall the definition of $z^b = z^{a,b}$ as a solution of (E_g) with $z_0^b = \eta^a$. Notice that $z^b(t) = \Gamma_{a,b}(t, \eta^a)(0)$ for all $t \geq 0$ whenever $z^b(s) \leq 1$ for all $s \in [0, t]$, and hence, by Proposition 2.2,

$$z^*(t) < z^{b_1}(t) < z^{b_2}(t) \quad \text{for all } b_2 > b_1 > b^*,\ t > 0, \text{ provided } z^{b_2}(s) \leq 1 \text{ for all } s \in [0, t].$$

Moreover, by Proposition 2.2 and $z_0^b = \eta^a = z_0^*$, for every $b > b^*$ there is a $t_1 > 0$ such that $z^b(t_1) = 1$ and $z^*(t) < z^b(t) < 1$ for all $t \in (0, t_1)$. There is a minimal time $\sigma_{0,b} > 0$ such that

$$z^b(\sigma_{0,b}) - z^*(\sigma_{0,b}) = \varepsilon_0, \quad \text{and} \quad 0 < z^b(t) - z^*(t) < \varepsilon_0 \quad \text{for all } t \in (0, \sigma_{0,b}). \tag{3.9}$$

Using the continuous dependence argument in (2.2) with $z_0^b = \eta^a = z_0^*$, there exists a $\delta_1 \in (0, \delta_0)$ such that $\sigma_{0,b} > T_0 + 1$ for every $b \in (b^*, b^* + \delta_1)$. For $b \in (b^*, b^* + \delta_1)$ define $\sigma_{1,b} > \sigma_{0,b}$ such that

$$z^b(t) < 1 \quad \text{for all } t \in [0, \sigma_{1,b}), \quad \text{and} \quad z^b(\sigma_{1,b}) = 1.$$

Combining the continuous dependence argument in (2.2), $z_0^b = \eta^a = z_0^*$, $z^*(t) \in (0,1)$ for $t > -1$, and $\lim_{t\to\infty} z^*(t) = \xi_*$, it follows that $\sigma_{1,b} \to \infty$ as $b \searrow b^*$.

For $b \in (b^*, b^* + \delta_1)$, define

$$u : [\sigma_{0,b}, \infty) \ni t \mapsto z^b(t) - z^*(t) \in \mathbb{R}.$$

By the definition of T_0 in (3.8) and by the choices of $\sigma_{1,b} > \sigma_{0,b} > T_0 + 1$, we have $z^b(t) > z^*(t) \geq \xi_* - \alpha_0$ for all $t \in [\sigma_{0,b} - 1, \sigma_{1,b}]$. Then, with B_0 defined in Step 1 satisfying (3.5), for $t \in [\sigma_{0,b}, \sigma_{1,b} + 1)$, we have

$$\begin{aligned} u'(t) &= -au(t) + b^* \int_0^1 g'\left(sz^b(t-1) + (1-s)z^*(t-1)\right) ds\, u(t-1) + (b - b^*)g(z^b(t-1)) \\ &\geq -au(t) + B_0 u(t-1), \end{aligned}$$

and $u(\sigma_{0,b}) = \varepsilon_0$. Clearly, $u(t) \geq e^{-a(t-\sigma_{0,b})}\varepsilon_0$, $t \in [\sigma_{0,b}, \sigma_{1,b}+1)$, and thus $u(t) \geq e^{-a}\varepsilon_0$ for $t \in [\sigma_{0,b}, \sigma_{0,b}+1]$.

From $B_0 > a$ it follows that there is a $\mu > 0$ so that $\mu = -a + B_0 e^{-\mu}$, and the function $v : \mathbb{R} \ni t \mapsto \frac{1}{2}e^{-a}e^{\mu(t-\sigma_{0,b}-1)}\varepsilon_0 \in \mathbb{R}$ satisfies the equation $v'(t) = -av(t) + B_0 v(t-1)$. Hence it is easy to see that $w : [\sigma_{0,b}, \sigma_{1,b}+1) \ni t \mapsto u(t) - v(t) \in \mathbb{R}$ satisfies

$$w'(t) \geq -aw(t) + B_0 w(t-1) \quad \text{for all } t \in (\sigma_{0,b}+1, \sigma_{1,b}+1),$$

and $w(t) > 0$ for $t \in [\sigma_{0,b}, \sigma_{0,b}+1]$. Then, it is elementary to check that $w(t) > 0$ follows for all $t \in [\sigma_{0,b}, \sigma_{1,b}+1)$. This means that

$$u(t) > \tfrac{1}{2}e^{-a}e^{\mu(t-\sigma_{0,b}-1)}\varepsilon_0 \quad \text{for all } t \in [\sigma_{0,b}, \sigma_{1,b}+1).$$

Taking into account

$$v\left(\sigma_{0,b} + 1 + \tfrac{1}{\mu}\log\tfrac{2e^a}{\varepsilon_0}\right) = 1,$$

it follows that

$$\sigma_{1,b} < \sigma_{0,b} + 1 + \tfrac{1}{\mu}\log\tfrac{2e^a}{\varepsilon_0}.$$

Then, for the constant

$$T(\varepsilon_0) = \max\left\{T_0 + 1, 2 + \tfrac{1}{\mu}\log\tfrac{2e^a}{\varepsilon_0}\right\},$$

the inequality

$$\sigma_{1,b} - T(\varepsilon_0) < \sigma_{0,b} - 1 \tag{3.10}$$

holds.

Step 3. For $b \in (b^*, b^*+\delta_1)$, define the function $y^b : \mathbb{R} \to \mathbb{R}$ by

$$y^b(t) = \begin{cases} z^b(\sigma_{1,b}+t) & \text{if } t \in \left[-\frac{1}{2}\sigma_{1,b}, \infty\right), \\ z^b\left(-\frac{1}{2}\sigma_{1,b}\right) & \text{if } t < -\frac{1}{2}\sigma_{1,b}. \end{cases}$$

Setting $B = b^*$, $\nu_b = \frac{1}{2}\sigma_{0,b}$, $x^b = y^b|_{(-\infty,0]}$, $m_b = \min_{t\in[\nu_b,\sigma_{1,b}]} z^b(t)$ in Proposition 2.3, the conditions $\nu_b \to \infty$, $m_b \to \xi_*$ as $b \searrow b^*$, and $x^b(t) \in [m_b, 1]$ for $t \leq 0$ hold because of (H_a), and $\sigma_{1,b} \to \infty$ as $b \to b^*+$, and $z^b(t) > z^*(t)$ for $t \in (0, \sigma_{1,b}]$. Therefore, for ε_1, defined in (3.7), and for $T(\varepsilon_0)$ there exists a $\delta_2 \in (0, \delta_1)$ such that $b \in (b^*, b^*+\delta_2)$ implies

$$|y^b(t) - x^{+,*}(t)| < \varepsilon_1 \quad \text{for all } t \in [-T(\varepsilon_0), 0]. \tag{3.11}$$

Step 4. Suppose $b \in (b^*, b^*+\delta_2)$. By using $y^b(0) = x^{+,*}(0) = 1$, $y^b(t) \in (0,1]$ and $x^{+,*}(t) \in (0,1]$ for all $t \in [-1, 0]$, and the consequence $\max_{t\in[0,1]} |y^b(t) - x^{+,*}(t)| < \varepsilon_1$ of (3.11), we can apply the integral equations for y^b and $x^{+,*}$ to obtain that, for all $t \in [0,1]$,

$$\begin{aligned} |y^b(t) - x^{+,*}(t)| &\leq b^* \int_0^t e^{-a(t-s)} \left| g(y^b(s-1)) - g(x^{+,*}(s-1)) \right| ds \\ &\quad + (b - b^*) \int_0^t e^{-a(t-s)} g(y^b(s-1))\, ds \\ &\leq b^* \|g'\|_{[0,1]} \varepsilon_1 + \delta_2 < \varepsilon_0. \end{aligned}$$

For the derivatives of $y^b(t)$ and $x^{+,*}(t)$ for $t \in [0,1]$, with the left-hand derivatives at 1, we have

$$\begin{aligned}|(y^b)'(t) - (x^{+,*})'(t)| &\le ab^*\|g'\|_{[0,1]}\varepsilon_1 + a\delta_2 + b^*\|g'\|_{[0,1]}\varepsilon_1 + \delta_2 \\ &= \varepsilon_1(1+a)b^*\|g'\|_{[0,1]} + (1+a)\delta_2 < \varepsilon_0.\end{aligned}$$

By the choice of ε_0, it follows that $(y^b)'(t) > 0$ for all $t \in [0,1)$, and hence $y^b(t) > 1$ for all $t \in (0,1]$. In particular, $(y^b)'(0) > 0$ implies $(z^b)'(\sigma_{1,b}) > 0$ and

$$g\left(z^b(\sigma_{1,b} - 1)\right) > \frac{a}{b}. \tag{3.12}$$

It is clear that $\tau_{0,a,b^*} = 1 + \frac{1}{a}\log x^{+,*}(1)$ for the constant in (1.2). Set $\tau_{0,*} = \tau_{0,a,b^*}$. Defining $\tau_{0,b} = 1 + \frac{1}{a}\log y^b(1)$, from (E_g) it is easy to see that

$$\begin{aligned}x^{+,*}(t) &= e^{-a(t-1)}x^{+,*}(1) \quad \text{for } 1 \le t \le \tau_{0,*} + 1, \\ y^b(t) &= e^{-a(t-1)}y^b(1) \qquad \text{for } 1 \le t \le \tau_{0,b} + 1.\end{aligned} \tag{3.13}$$

Then

$$|y^b(t) - x^{+,*}(t)| < \varepsilon_0 \quad \text{for all } t \in [0, \min\{\tau_{0,*} + 1, \tau_{0,b} + 1\}]. \tag{3.14}$$

Therefore, from (3.11) and (3.14), we obtain

$$|y^b(t) - x^{+,*}(t)| < \varepsilon_0 \quad \text{for all } t \in [-T(\varepsilon_0), \min\{\tau_{0,*} + 1, \tau_{0,b} + 1\}]. \tag{3.15}$$

By the choice of ε_0,

$$|\tau_{0,b} - \tau_{0,*}| = \frac{1}{a}\left|\log\frac{y^b(1)}{x^{+,*}(1)}\right| \le \frac{1}{a}\log\frac{x^{+,*}(1)}{x^{+,*}(1) - \varepsilon_0} < 1.$$

Therefore, from (3.13), for any t from the closed interval with endpoints $\tau_{0,*} + 1$ and $\tau_{0,b} + 1$, there is an s from the same interval such that

$$y^b_t = x^{+,*}_s. \tag{3.16}$$

Step 5. Let $b \in (b^*, b^* + \delta_2)$. Recalling the definition of y^b from Step 3, and the results in Step 4, in particular (3.16), it follows that

$$\begin{aligned}&z^b(t) < 1 \quad \text{for } t \in (0, \sigma_{1,b}), \quad z^b(t) > 1 \quad \text{for } t \in (\sigma_{1,b}, \sigma_{1,b} + \tau_{0,b}), \\ &z^b(\sigma_{1,b}) = z^b(\sigma_{1,b} + \tau_{0,b}) = 1, \quad \text{and} \quad z^b_{\sigma_{1,b}+\tau_{0,b}+1} = \eta^a = z^b_0.\end{aligned}$$

Setting $\omega_{a,b} = \sigma_{1,b} + \tau_{0,b} + 1$, the $\omega_{a,b}$-periodic extension of $z^b|_{[-1,\sigma_{1,b}+\tau_{0,b}]}$ gives a periodic solution $p^{a,b} : \mathbb{R} \to \mathbb{R}$. From $(z^b)'(\sigma_{1,b}) > 0$ and $(z^b)'(\sigma_{1,b} + \sigma_{0,b}) < 0$ it follows that

$$g(p^{a,b}(t-1)) \ne \frac{a}{b} \quad \text{provided } p^{a,b}(t) = 1.$$

Therefore, condition (P) is satisfied.

Step 6. We claim that $b \in (b^*, b^* + \delta_2)$ implies

$$\operatorname{dist}_H(\mathcal{O}^{a,b}, \mathcal{H}) < \varepsilon. \tag{3.17}$$

First, we prove that $\operatorname{dist}\{p_t^{a,b}, \mathcal{H}\} < \varepsilon$ for all $t \in [0, \omega_{a,b}]$.

If $t \in [0, \sigma_{0,b}]$ then (3.9) and $p_0^{a,b} = z_0^b = \eta^a = x^{+,*}_{\tau_{0,*}+1}$ give $\operatorname{dist}\{p_t^{a,b}, \mathcal{H}\} < \varepsilon_0 < \varepsilon$.

Let $t \in [\sigma_{0,b}, \sigma_{1,b} + \min\{\tau_{0,b}, \tau_{0,*}\}]$. By (3.10), $t > \sigma_{1,b} - T(\varepsilon_0) + 1$. Then, using $z_t^b = y^b_{t-\sigma_{1,b}}$, from (3.15) one concludes $\operatorname{dist}\{p_t^{a,b}, \mathcal{H}\} < \varepsilon_0 < \varepsilon$.

In the remaining case, that is when $\tau_{0,b} > \tau_{0,*}$, and t is in the closed interval with endpoints $\tau_{0,*} + 1$ and $\tau_{0,b} + 1$, equality (3.16) yields $p_t^{a,b} \in \mathcal{H}$. Therefore, (3.17) holds.

Note that if s is given in the closed interval with endpoints $\tau_{0,a,b^*} + 1$ and $\tau_{0,b} + 1$ then we find t in the same interval with (3.16). This fact is used in the proof of $\operatorname{dist}_H(x_t^{+,*}, \mathcal{O}^{a,b}) < \varepsilon$, as well.

If $t \in [-T(\varepsilon_0) + 1, \tau_{0,*} + 1 + \tau_{1,b}]$ then

$$\operatorname{dist}_H(x_t^{+,*}, \mathcal{O}^{a,b}) < \varepsilon \tag{3.18}$$

is obtained analogously to the proof of (3.17). For $t \le T(\varepsilon_0) + 1$ or $t \ge \tau_{0,*} + 1 + \sigma_{0,b}$ the choice of T_0 satisfying (3.8) guarantees (3.18).

Step 7. The finite-time orbital attraction.

We have $p^{a,b}(t) > 1$ for $t \in (\sigma_{1,b}, \sigma_{1,b} + \tau_{0,b})$, and $\tau_{0,b} > 1$. Thus, there are a $t_1 \in (\sigma_{1,b} + 1, \sigma_{1,b} + \tau_{0,b})$ and a $\gamma > 0$ such that

$$\min_{t \in [t_1 - 1, t_1]} p^{a,b}(t) > 1 + 2\gamma.$$

The periodic solution $p^{a,b}$ intersects the discontinuity $\xi = 1$ transversally, and there are two intersections in every period. This fact is the key to showing the existence of a $\beta > 0$ so that for $s \in [0, \omega_{a,b}]$ and $\phi \in C^+$ with $\|\phi - p_s^{a,b}\| < \beta$ it follows that

$$\|\Psi_{a,b}(t, \phi) - p^{a,b}_{s+t}\| < \gamma \quad \text{for all } t \in [0, \omega_{a,b}].$$

For the proof, we refer to a simplified version of Proposition 2.6 in [3]. This result shows that there is a $t_2 \in [0, \omega_{a,b}]$ such that either $s + t_2 = t_1$ or $s + t_2 = t_1 + \omega_{a,b}$. Hence

$$\Psi_{a,b}(t_2, \phi) \gg \widehat{1 + \gamma}$$

follows. Then, using $g(\xi) = 0$ for $\xi > 1$, there exists a $t_3 > t_2 + 1$ such that $\Phi_{a,b}(t_3, \phi) = \eta^a$. Consequently, by uniqueness, $\Phi(t_3 + t, \phi) = p_t^{a,b}$ for all $t \ge 0$.

The proof of Theorem 1.1 is complete. □

4 Examples

In this section, we consider equation (E_g) with $\kappa > 0$ and $g(\xi) = \xi^{1+\kappa}$, $\xi \in [0, 1]$. Clearly, condition (C_g) holds, $\|g'\|_{[0,1]} = 1 + \kappa$, and $\xi_{a,b} = \left(\frac{a}{b}\right)^{1/\kappa}$ for $b > a > 0$. Then the sufficient condition (3.1) for (H_a) in Proposition 3.1 reads as

$$e^{a\kappa}\left(1 - e^{-a}\right) < \frac{1}{1 + \kappa}. \tag{4.1}$$

The linear variational equation of (E_g) at $\hat{\xi}_{a,b}$ is $u'(t) = -au(t) + a(1 + \kappa)u(t - 1)$ with characteristic equation

$$H : \mathbb{C} \ni \lambda \mapsto \lambda + a - a(1 + \kappa)e^{-\lambda} \in \mathbb{C}. \tag{4.2}$$

By Section 2, H has a unique real zero $\lambda_0 = \lambda_0(a,\kappa) > 0$, there is a zero $\lambda_1 = \lambda_1(a,\kappa) \in \mathbb{C}$ of H with $\operatorname{Im}\lambda_1(a,\kappa) \in (\pi, 2\pi)$, and λ_1 has the maximal real part among all complex zeros of H. We know from Section 2 that (H_a) implies $\operatorname{Re}\lambda_1(a,\kappa) \le 0$.

The sign of the quantity

$$\sigma(a,\kappa) = \lambda_0(a,\kappa) + \operatorname{Re}\lambda_1(a,\kappa)$$

is a spectral condition at the unstable stationary point $\hat{\xi}_{a,b}$. Such a quantity plays an important role in the study of the dynamics of smooth semiflows in a neighborhood of a homoclinic orbit. For this paper, this is not the case. However, it is expected that, for smooth nonlinearities f_n close to g in the sense of condition (C_{f_n}), homoclinic orbits exist for equation (E_{f_n}), and the sign of the quantity $\lambda_0 + \operatorname{Re}\lambda_1$ will be crucial, analogously to the results in [4,5,9,13,19,28,29,32].

For the particular case of g considered above, we ask whether condition (4.1) contains information about the sign of $\sigma(a,\kappa)$. The answer is no by the following result.

Proposition 4.1.

(i) For every $\kappa > 0$ there exists an $a_0(\kappa) > 0$ such that, for all $a \in (0, a_0(\kappa))$, (4.1) and $\sigma(a,\kappa) < 0$ hold simultaneously.

(ii) If $0 < a \le 1/2$ and $\kappa \ge 4/a$, then (4.1) and $\sigma(a,\kappa) > 0$ hold simultaneously.

Proof. 1. The proof of (i). Let $\kappa > 0$ be fixed. Clearly, for small $a > 0$, (4.1) is satisfied. A graphical analysis shows that $\lambda_0(a,\kappa) \to 0+$ as $a \to 0+$. From $\operatorname{Im}\lambda_1(a,\kappa) \in (\pi, 2\pi)$ and (4.2) it follows that

$$\pi < |\lambda_1(a,\kappa)| = a|(1+\kappa)e^{-\lambda_1(a,\kappa)} - 1|.$$

Hence, $a \to 0+$ implies $\operatorname{Re}\lambda_1(a,\kappa) \to -\infty$, and we obtain *(i)*.

2. The proof of (ii).

Put $p = a\kappa$. Then $p \ge 4$ and $\alpha = a(1+\kappa) = a + p \ge a + 4$. The characteristic equation $H(\lambda) = 0$ can be written as

$$(\lambda + a)e^{\lambda} = \alpha. \tag{4.3}$$

Since $1 - e^{-a} < a$, we have

$$(1+\kappa)a^{a\kappa}(1 - e^{-a}) < (1+\kappa)a^{p+1} = (a+p)a^p.$$

For fixed $a \in (0, 1/2]$, set

$$h_a(p) = (a+p)a^p, \qquad p \ge 4.$$

The logarithmic derivative of $p \mapsto h_a(p)$ is

$$\frac{h_a'(p)}{h_a(p)} = \frac{1}{a+p} + \log a \le \frac{1}{4} - \log 2 < 0.$$

Hence h_a is decreasing on $[4, \infty)$, and therefore

$$(a+p)a^p \le (a+4)a^4.$$

The function $(a+4)a^4$ is increasing for $a > 0$, so, using $a \le 1/2$,

$$(a+4)a^4 \le \left(4 + \frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{9}{32} < 1.$$

Consequently

$$(1+\kappa)a^{a\kappa}(1-e^{-a}) < 1,$$

which gives (4.1).

For real u, equation (4.3) becomes

$$(u+a)e^{u} = \alpha. \tag{4.4}$$

The function $u \mapsto (u+a)e^{u}$ is strictly increasing on $[-a,\infty)$. Moreover, at $u=0$ its value is $a<\alpha$, while it tends to infinity as $u\to\infty$. Hence (4.4) has a unique real solution $\lambda_0>0$.

We shall need the stronger estimate $\lambda_0>1$. Indeed, the function $a\mapsto\frac{1+a}{4+a}$ is strictly increasing on $(0,\infty)$ and its value at $a=1/2$ is $1/3$, and hence, for all $a\in(0,1/2]$,

$$(1+a)e \le \frac{e}{3}(4+a) < a+4 \le \alpha.$$

Since the left-hand side of (4.4) is strictly increasing,

$$\lambda_0 > 1. \tag{4.5}$$

For λ_1 write

$$\lambda = \mu + i\nu, \qquad \nu\in(\pi,2\pi)\,.$$

Separating real and imaginary parts in (4.3) gives

$$e^{\mu}\big((\mu+a)\cos\nu - \nu\sin\nu\big) = \alpha, \tag{4.6}$$

$$e^{\mu}\big((\mu+a)\sin\nu + \nu\cos\nu\big) = 0. \tag{4.7}$$

From (4.7)

$$\mu + a = -\nu\cot\nu. \tag{4.8}$$

Substitution into (4.6) yields

$$\alpha = \beta_a(\nu) := -\frac{\nu}{\sin\nu}\exp\big(-a-\nu\cot\nu\big). \tag{4.9}$$

For $\nu\in(\pi,2\pi)$, the right-hand side is positive. Moreover, β_a is strictly increasing on this interval. In fact,

$$\frac{d}{d\nu}\log\beta_a(\nu) = \frac{1}{\nu} - 2\cot\nu + \nu\csc^2\nu,$$

and, after multiplication by the positive quantity $\nu\sin^2\nu$, the numerator becomes

$$\sin^2\nu - 2\nu\sin\nu\cos\nu + \nu^2 = (\nu-\sin\nu\cos\nu)^2 + \sin^4\nu > 0.$$

Also,

$$\lim_{\nu\downarrow\pi}\beta_a(\nu) = 0, \qquad \lim_{\nu\uparrow 2\pi}\beta_a(\nu) = +\infty.$$

Thus there is a unique $\nu_*\in(\pi,2\pi)$ satisfying $\beta_a(\nu_*)=\alpha$, and (4.8) then defines a nonreal zero $\lambda_* = \mu_* + i\nu_*$ of H. We have $\lambda_1=\lambda_*$.

We claim that

$$\mu_* > -1. \tag{4.10}$$

To see this, let $\widehat{\nu}\in(\pi,3\pi/2)$ be the unique solution of

$$\widehat{\nu}\cot\widehat{\nu} = 1-a. \tag{4.11}$$

Such a solution exists because $\nu \cot \nu$ decreases continuously from $+\infty$ to 0 on $(\pi, 3\pi/2)$. At $\nu = \widehat{\nu}$, formula (4.8) gives $\mu = -1$. Furthermore, using (4.11),

$$H_a(\widehat{\nu}) = -\frac{\widehat{\nu}}{\sin \widehat{\nu}} e^{-1} = e^{-1}\sqrt{\widehat{\nu}^2 + (1-a)^2}.$$

Here we used $\sin \widehat{\nu} < 0$ and $\widehat{\nu} \cot \widehat{\nu} = 1 - a$. Since $\widehat{\nu} < 3\pi/2$ and $0 < 1 - a < 1$,

$$\beta_a(\widehat{\nu}) < e^{-1}\sqrt{\left(\frac{3\pi}{2}\right)^2 + 1} < 2 < 4 \leq \alpha.$$

As β_a is strictly increasing and $\beta_a(\nu_*) = \alpha$, it follows that

$$\nu_* > \widehat{\nu}.$$

On $(\pi, 2\pi)$ the function $\nu \cot \nu$ is not monotone across $3\pi/2$, but the desired conclusion follows directly in two possible cases. If $\nu_* \geq 3\pi/2$, then $\cot \nu_* \leq 0$, and hence

$$\mu_* = -a - \nu_* \cot \nu_* \geq -a > -1.$$

If $\widehat{\nu} < \nu_* < 3\pi/2$, then $\nu \cot \nu$ is strictly decreasing on $(\pi, 3\pi/2)$, so

$$\nu_* \cot \nu_* < \widehat{\nu} \cot \widehat{\nu} = 1 - a,$$

and therefore again

$$\mu_* = -a - \nu_* \cot \nu_* > -1.$$

This proves (4.10).

Combining $\operatorname{Re} \lambda_1 = \mu_* > -1$ with (4.5), we obtain

$$\sigma(a, \kappa) = \lambda_0 + \operatorname{Re} \lambda_1 > 0.$$

This completes the proof. □

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