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Series and asymptotic analysis of the *k*-confluent hypergeometric differential equation

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Abstract. This study investigates both the homogeneous and non-homogeneous forms of the k-confluent hypergeometric differential equation. In the neighborhood of the regular singular point, series solutions are constructed using the Frobenius method. For the irregular singular point at infinity, an asymptotic analysis is carried out to describe the behavior of the solutions in that region. These analyses lead to the definition and characterization of k-Kummer and k-Tricomi-type hypergeometric functions, which are shown to constitute fundamental solutions of the differential equation. Moreover, an explicit series representation is obtained for the non-homogeneous k-confluent hypergeometric differential equation with a polynomial right-hand side. The results provide analytical tools for obtaining explicit and asymptotic solutions of differential equations of confluent hypergeometric type, including models arising in wave propagation, heat conduction, and quantum mechanics.

Keywords: k-confluent hypergeometric differential equation, k-confluent hypergeometric function, Frobenius method, asymptotic analysis, k-Kummer function, k-Tricomi function.

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1 Introduction and preliminaries

Many differential equations arising in physical models can be expressed in closed form by means of special functions. Classical examples include the Bessel, Legendre, and Hermite functions, which provide critical analytical solutions to problems in wave propagation, heat conduction, and quantum mechanics, respectively [7, 12, 13]. A large class of these functions appears as analytic solutions associated with linear differential equations of second order and is characterized by well-established structural properties. In this broader context, confluent hypergeometric functions naturally arise in the solution of differential equations encountered in the mathematical modeling of physical processes such as wave propagation, heat conduction, and quantum mechanics. In particular, models describing wave propagation in inhomogeneous media, heat conduction with variable coefficients, and the Schrödinger equation under certain classes of potentials can, through suitable scaling and variable transformations, be

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reduced to confluent hypergeometric-type differential equations. Related studies addressing the solution of differential equations via generalized integral transform techniques, including Laplace-type transforms and kernels involving special functions, can be found in [1,2]. The corresponding confluent hypergeometric functions make it possible to express the solutions of these systems in closed form, thereby allowing a detailed analysis of their singular structure, asymptotic behavior, and dependence on the relevant parameters. From this perspective, confluent hypergeometric functions not only enable the analytical treatment of a wide range of physical problems but also provide a robust mathematical framework for investigating the structural and asymptotic properties of the associated solutions.

Within this general setting, hypergeometric-type functions serve as a unifying analytical framework for a wide class of second-order linear differential equations. Among the classical representatives, the Gauss hypergeometric function ${}_2F_1(\alpha,\beta;\zeta;w)$ plays a central role and is frequently encountered in both theoretical studies and practical applications. It can be expressed through the power-series expansion

$$ {}_2F_1(\alpha,\beta;\zeta;w) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\zeta)_n}\frac{w^n}{n!}, \quad |w|<1, $$

where the notation $(\alpha)_n$ denotes the rising factorial that appears in the series coefficients and is understood in the standard sense,

$$ (\alpha)_n = \begin{cases} \prod_{j=0}^{n-1}(\alpha+j), & n\in\mathbb{N} \\ 1, & n=0 \end{cases}, \quad \alpha\in\mathbb{C}. $$

This series solves the Gauss hypergeometric differential equation,

$$ w(1-w)\gamma'' + [\zeta-(\alpha+\beta+1)w]\gamma' - \alpha\beta\gamma = 0, \tag{1.1} $$

for $\gamma(w)$, with $w\in\mathbb{C}$ and parameters $\alpha,\beta,\zeta\in\mathbb{R}$ or $\mathbb{C}$ [6]. The equation is characterized by its three regular singular points at $w=0,1,\infty$, a structure that is central to understanding the nature of its solutions.

The existence of three singularities can complicate the analysis of the solution in a number of applications. Therefore, two singularities are combined in order to simplify the analysis, particularly in limiting cases and physical problems. In particular, the singularities at $w=1$ and $w=\infty$ in equation (1.1) are combined by replacing w with w/β and taking the limit as $\beta\to\infty$. With one regular and one irregular singular point, this method simplifies the differential equation and lessens the complexity of the singular structure. The resulting equation is known as the Kummer (or confluent hypergeometric) differential equation [4,10]:

$$ w\gamma'' + (\beta-w)\gamma' - \alpha\gamma = 0. \tag{1.2} $$

The Kummer differential equation has a regular singularity at $w=0$ and has an irregular singularity at $w=\infty$. It is widely used in mathematical analysis and physical problems because of these characteristics. This equation's solutions show particular asymptotic behaviors at infinity as well as close to the origin. There are two basic solutions to equation (1.2). The first is the Kummer function, also known as the confluent hypergeometric function of the first kind. It is represented in closed form as $M(\alpha,\beta;w)$, ${}_1F_1(\alpha,\beta;w)$, or $\Phi(\alpha,\beta;w)$ and expressed by the series

$$ M(\alpha,\beta;w) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n}\frac{w^n}{n!}. $$

The Kummer function $M(\alpha,\beta;w)$ serves as the reference solution in this region, characterized by its initial value $M(\alpha,\beta;0) = 1$. A second solution, linearly independent of $M(\alpha,\beta;w)$, is given by

$$w^{1-\beta}M(\alpha+1-\beta,2-\beta;w).$$

Provided $\beta \notin \mathbb{Z}$, this solution admits a well-defined asymptotic expansion as $w \to \infty$. The confluent hypergeometric function of the second kind, denoted by $U(\alpha,\beta;w)$ and known as the Tricomi function, is defined through a specific linear combination of two linearly independent solutions of the corresponding differential equation.

$$U(\alpha,\beta;w) = \frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}M(\alpha,\beta;w) + w^{1-\beta}\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}M(\alpha-\beta+1,2-\beta;w).$$

This function unifies the asymptotic behaviors of the Kummer differential equation at the origin and at infinity, and provides a detailed, parameter-dependent analysis of the solutions' behavior.

The confluent hypergeometric differential equation is one of the cornerstones of the classical theory of special functions, thereby successfully characterizing a broad class of problems. However, due to its structure with fixed parameters, it remains limited in certain analytical and physical modeling contexts. In particular, it does not allow for a flexible representation of parameter-dependent behaviors. In such cases, a more comprehensive mathematical framework becomes necessary.

As a result, several generalization approaches have been developed. One of these approaches is the k-generalization. This topic has attracted considerable attention from many researchers, enriching both the analytical theory and the modeling aspects in applied mathematics.

The idea of k-generalization has been widely developed, especially for different special functions [8,9], and is derived from the work of Diaz and Pariguan [5] on the Pochhammer k-symbol and k-gamma function. The subsequent development of k-hypergeometric functions and their differential equations was a significant result. Mubeen, Naz, Rehman, and Rahman [11] built directly on this foundation by focusing their research on the solutions of the k-hypergeometric differential equation, which is defined as

$$kw\,(1-kw)\,\gamma'' + [\mu - (k+\alpha+\beta)\,kw]\,\gamma' - \alpha\beta\gamma = 0. \tag{1.3}$$

Applying the variable transformation $w = \frac{w}{k\beta}$ and taking the limit as $\beta \to \infty$ with w held fixed, we find the equation (1.3) simplifies to

$$kw\gamma'' + (\beta - kw)\gamma' - \alpha\gamma = 0, \tag{1.4}$$

which is the k-confluent hypergeometric differential equation.

The aim of this study is to provide solutions to the second-order linear differential equation known as the k-confluent hypergeometric differential equation at both regular and irregular singular points.

Our analysis makes use of the Frobenius method for a regular singular point, which enables us to create convergent series solutions nearby to the singularity. Examining the indicial equation is the foundation of this approach; the number and form of linearly independent solutions we can find are directly determined by the relationship between its roots. An asymptotic expansion of the solution is derived for $w \to \infty$, revealing the structural properties of the underlying special functions in this limit.

The paper is organized as follows. The first section introduces the terminology and fundamental definitions used throughout the study. The second section analyzes the singularities of the associated linear differential equation, and the final section addresses the non-homogeneous case.

Definition 1.1. The Pochhammer k-symbol [5] is defined by

$$(\alpha)_{n,k} := \begin{cases} \prod_{j=0}^{n-1} (\alpha + jk), & n \in \mathbb{N};\ \alpha \in \mathbb{C} \\ 1, & n = 0;\ \alpha \in \mathbb{C} \setminus \{0\} \end{cases} \tag{1.5}$$

where $k > 0$. It is connected to the classical Pochhammer symbol through the identity [5]

$$(\alpha)_{n,k} = k^n \left(\frac{\alpha}{k}\right)_n . \tag{1.6}$$

In particular, for $k = 1$, identity (1.6) coincides with the definition of the classical Pochhammer symbol.

Definition 1.2. For $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus k\mathbb{Z}_0^-$, the k-Kummer hypergeometric function is defined as

$${}_1\mathrm{F}_{1,\mathrm{k}}\left(\alpha, \beta; w\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n,k}(1)_n} w^n, \tag{1.7}$$

where $w \in \mathbb{C}$, also denoted by $M_k(\alpha, \beta; w)$ and $\Phi_k(\alpha, \beta; w)$.

F. G. Frobenius defined a new solution method for a second-order differential equation [3] of the form

$$R_2(w)\gamma'' + R_1(w)\gamma' + R_0(w)\gamma = 0,$$

where $w = w_0$ is a regular singular point or equivalently,

$$\gamma'' + \frac{R_1}{R_2}\gamma' + \frac{R_0}{R_2}\gamma = 0.$$

The Frobenius method extends the power series method to cases where the standard approach fails due to the non-analyticity of R_1/R_2 or R_0/R_2 at w_0. This extension is permissible provided that $R_1(w)$ and $R_0(w)$ are analytic at w_0, or at least have well-defined limits there.

This equation serves as a cornerstone in the theory of special functions, particularly in the exploration of hypergeometric functions and their manifold applications.

2 Analysis of the behavior at singular points of the homogeneous k-confluent hypergeometric equation

The linear k-confluent hypergeometric equation (1.4) is investigated in both the local and asymptotic frameworks. Frobenius expansions are developed about the regular singular point $w = 0$, while the behavior as $w \to \infty$ is characterized through asymptotic techniques.

2.1 Solution analysis in the case of "$w = 0$"

The equation admits a Frobenius series expansion about the regular singular point $w = 0$. Accordingly, we consider

$$\gamma(w,s) = \sum_{n=0}^{\infty} \gamma_n w^{n+s}$$

where $\gamma_0 \neq 0$. Then

$$\gamma'(w,s) = \sum_{n=0}^{\infty} \gamma_n (n+s) w^{n+s-1}$$

and

$$\gamma''(w,s) = \sum_{n=0}^{\infty} \gamma_n (n+s)(n+s-1) w^{n+s-2}.$$

After substituting the derivatives into the equation (1.4), we obtain

$$\sum_{n=0}^{\infty} \gamma_n (n+s)\left[k(n+s-1)+\beta\right] w^{n+s-1} - k\sum_{n=0}^{\infty} \gamma_n (n+s) w^{n+s} - \alpha \sum_{n=0}^{\infty} \gamma_n w^{n+s} = 0.$$

After writing all the terms with the same power of w, corresponding to the smallest exponent $n+s-1$, and separating the initial terms corresponding to $n = 0$, the following expression is obtained:

$$\gamma_0(ks(s-1)+\beta s)w^{s-1} + \sum_{n=1}^{\infty} (\gamma_n (n+s)[k(n+s-1)+\beta] - \gamma_{n-1}[k(n+s-1)-\alpha]) w^{n+s-1} = 0.$$

Since the powers of w are linearly independent, the coefficient corresponding to each power must vanish. Therefore, based on the first term and under the assumption $\gamma_0 \neq 0$, the indicial equation

$$s(k(s-1)+\beta) = 0$$

is obtained, giving the roots

$$s_1 = 0, \quad s_2 = 1 - \frac{\beta}{k}.$$

Equating the coefficients of the remaining powers of w yields the recurrence relation

$$\gamma_n = \frac{k(n+s-1)+\alpha}{(n+s)\left[k(n+s-1)+\beta\right]} \gamma_{n-1}, \quad n \geq 1. \tag{2.1}$$

Iterating this relation expresses the coefficients in terms of the initial constant γ_0, leading to the general formula

$$\gamma_n = \frac{(\alpha+sk)_{n,k}}{(1+s)_n (\beta+sk)_{n,k}} \gamma_0, \quad n \geq 1.$$

Thus, the assumed series solution takes the form

$$\gamma(w,s) = \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}} \frac{w^{n+s}}{(1+s)_n}. \tag{2.2}$$

After deriving the general form of the series solution, we proceed to examine the cases determined by the roots of the indicial equation.

2.1.1 Analysis of the solutions for “$s_1 - s_2 = \beta/k - 1$”

Since $s_1 - s_2 = \beta/k - 1$, the nature of the solution space depends on whether $\beta/k \in \mathbb{Z}$.

Case (i). Assume that $\beta/k \notin \mathbb{Z}$. In this case, the Frobenius construction yields two linearly independent solutions, namely,

$$\gamma_1(w) = \gamma(w,s)|_{s=0} \quad \text{and} \quad \gamma_2(w) = \gamma(w,s)|_{s=1-\beta/k}.$$

Taking $s = 0$ in the assumed series solution (2.2), we obtain the first solution as

$$\begin{aligned} \gamma_1(w) &= \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n,k}} \frac{w^n}{(1)_n} \\ &= \gamma_0 \, {}_1\mathrm{F}_{1,\mathrm{k}}(\alpha, \beta; w). \end{aligned} \tag{2.3}$$

Similarly, by taking $s = 1 - \beta/k$ in (2.2), the second solution is obtained as follows:

$$\begin{aligned} \gamma_2(w) &= \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha + (1-\beta/k)k)_{n,k}}{(\beta + (1-\beta/k)k)_{n,k}} \frac{w^{n+1-\beta/k}}{(1-\beta/k+1)_n} \\ &= \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha + k - \beta)_{n,k}}{(k)_{n,k}} \frac{w^{n+1-\beta/k}}{\left(\frac{2k-\beta}{k}\right)_n} \end{aligned}$$

and using the equality (1.6), we obtain

$$\begin{aligned} \gamma_2(w) &= \gamma_0 w^{1-\beta/k} \sum_{n=0}^{\infty} \frac{(\alpha + k - \beta)_{n,k}}{(2k-\beta)_{n,k}} \frac{w^n}{n!} \\ &= \gamma_0 w^{1-\beta/k} \, {}_1\mathrm{F}_{1,\mathrm{k}}(\alpha + k - \beta, 2k - \beta; w). \end{aligned}$$

The general solution of the equation (1.4) is of the form

$$\begin{aligned} \gamma(w) &= A\gamma_1(w) + B\gamma_2(w) \\ &= \gamma_0 \left[A \, {}_1\mathrm{F}_{1,\mathrm{k}}(\alpha, \beta; w) + B w^{1-\beta/k} {}_1\mathrm{F}_{1,\mathrm{k}}(\alpha + k - \beta, 2k - \beta; w) \right]. \end{aligned}$$

Case (ii). Let $\beta/k = 1$. Then

$$\gamma_1(w) = \gamma(w,s)|_{s=0} \quad \text{and} \quad \gamma_2(w) = \left.\frac{\partial \gamma}{\partial s}\right|_{s=0}.$$

Using $\beta = k$ and $s = 0$ in the series solution (2.2), the first solution is derived below, as in (2.3):

$$\gamma_1(w) = \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(k)_{n,k}} \frac{w^n}{(1)_n} = \gamma_0 {}_1\mathrm{F}_{1,k}(\alpha, k; w).$$

To calculate $\gamma_2(w)$, we require $\dfrac{\partial \gamma}{\partial s}$. For convenience, after setting $\beta = k$ in the series solution (2.2), let us denote

$$\delta_n = \frac{(\alpha + sk)_{n,k}}{((s+1)k)_{n,k}(s+1)_n} = \frac{(\alpha + sk)_{n,k}}{k^n \left((s+1)_n\right)^2}. \tag{2.4}$$

Then

$$\frac{\partial \gamma}{\partial s} = \gamma_0 w^s \left(\ln w \sum_{n=0}^{\infty} \delta_n w^n + \sum_{n=0}^{\infty} \frac{\partial \delta_n}{\partial s} w^n \right). \tag{2.5}$$

To determine the derivative of δ_n with respect to s, we first take the logarithm of both sides of (2.4),

$$\begin{aligned}
\ln(\delta_n) &= \ln \frac{(\alpha+sk)_{n,k}}{k^n\left((s+1)_n\right)^2} \\
&= \ln(\alpha+sk)_{n,k} - \ln\left(k^n((s+1)_n)^2\right) \\
&= \sum_{m=0}^{n-1} \left[\ln\left(\alpha+(s+m)k\right) - \ln\left(k^n(s+1+m)^2\right)\right].
\end{aligned}$$

Differentiating with respect to s, we obtain

$$\frac{\partial \delta_n}{\partial s} = \frac{(\alpha+sk)_{n,k}}{k^n\left((s+1)_n\right)^2} \sum_{m=0}^{n-1} \left[\frac{k}{\alpha+(s+m)k} - \frac{2}{s+1+m}\right].$$

Substituting the equality given above into (2.5) and setting $s=0$, and then applying the equality (1.6) yield:

$$\gamma_2(w) = \gamma_0 \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(k)_{n,k}} \left[\ln w + \sum_{m=0}^{n-1} \left(\frac{k}{\alpha+mk} - \frac{2}{1+m}\right)\right] \frac{w^n}{n!}.$$

Accordingly, the general solution takes the form

$$\begin{aligned}
\gamma(w) &= C\gamma_1(w) + D\gamma_2(w) \\
&= \gamma_0 \left(C_1 F_{1,k}(\alpha, k; w) + D \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(k)_{n,k}} \left[\ln w + \sum_{m=0}^{n-1} \left(\frac{k}{\alpha+mk} - \frac{2}{1+m}\right)\right] \frac{w^n}{n!} \right).
\end{aligned}$$

Case (iii). Let β/k be an integer but $\beta/k \neq 1$. Then there are two cases:

(1) Assume that $\beta/k < 1$. Substituting $s = 0$, the smaller indicial root, into the recurrence relation (2.1) yields

$$\gamma_n = \frac{\alpha + k(n-1)}{n\left(k(n-1)+\beta\right)} \gamma_{n-1}.$$

As $n \to 1-\beta/k$, the denominator vanishes and consequently $\gamma_{1-\beta/k} \to \infty$, indicating the emergence of singular behavior in the series expansion. To regularize the expansion at the corresponding root s_i, the initial coefficient is redefined according to

$$\gamma_0 = h_0(s-s_i)k,$$

where s_i denotes the root at which the series becomes unbounded.

Accordingly, we set

$$\gamma_0 = h_0 sk$$

and our presumed solution in equation (2.2) takes the new form

$$\gamma_h(w,s) = h_0 w^s \sum_{n=0}^{\infty} \frac{(sk)(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}} \frac{w^n}{(s+1)_n}. \tag{2.6}$$

From the obtained series representation, a pair of linearly independent solutions can be extracted as

$$\gamma_1(w) = \gamma_h(w,s)|_{s=0} \quad \text{and} \quad \gamma_2(w) = \frac{\partial \gamma_h}{\partial s}\Big|_{s=1-\beta/k}. \tag{2.7}$$

For the first solution, all terms preceding

$$\frac{(sk)(\alpha+sk)_{1-\beta/k,k}}{(\beta+sk)_{1-\beta/k,k}(s+1)_{1-\beta/k}}$$

vanish due to the presence of the factor "sk" in the numerator. Indeed, observe that

$$(\beta+sk)_{1-\beta/k,k} = (\beta+sk)(\beta+(s+1)k)\cdots(sk).$$

Hence, the new form becomes

$$\begin{aligned}\gamma_h(w,s) &= \frac{h_0 w^s}{(\beta+sk)_{-\beta/k,k}} \sum_{n=1-\beta/k}^{\infty} \frac{(\alpha+sk)_{n,k}}{k^{n-1+\beta/k}(s+1)_{n-1+\beta/k}(s+1)_n} w^n \\ &= \frac{h_0 w^s}{(\beta+sk)_{-\beta/k,k}} \sum_{n=1-\beta/k}^{\infty} \frac{(\alpha+sk)_{n,k}}{((s+1)k)_{n-1+\beta/k,k}(s+1)_n} w^n\end{aligned}$$

and for $s=0$, the solution takes the following form:

$$\gamma_1(w) = \frac{h_0}{(\beta)_{-\beta/k,k}} \sum_{n=1-\beta/k}^{\infty} \frac{(\alpha)_{n,k}}{(k)_{n-1+\beta/k,k}(1)_n} w^n.$$

Now, we will examine the second solution (2.7). To simplify the notation, we denote the term inside the series (2.6) by

$$\delta_n = \frac{(sk)(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n}.$$

Following the procedure outlined in the previous case $\beta/k=1$, we obtain

$$\frac{\partial \delta_n}{\partial s} = \frac{(sk)(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n}\left[\frac{1}{s} + \sum_{m=0}^{n-1}\left(\frac{k}{\alpha+(s+m)k} - \frac{k}{\beta+(s+m)k} - \frac{1}{1+s+m}\right)\right].$$

Since

$$\gamma_h(w,s) = h_0 w^s \sum_{n=0}^{\infty} \delta_n w^n,$$

we have

$$\begin{aligned}\frac{\partial \gamma_h}{\partial s} &= h_0 w^s \sum_{n=0}^{\infty} \frac{(sk)(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n} \\ &\quad \times \left[\ln w + \frac{1}{s} + \sum_{m=0}^{n-1}\left(\frac{k}{\alpha+(s+m)k} - \frac{k}{\beta+(s+m)k} - \frac{1}{1+s+m}\right)\right] w^n.\end{aligned}$$

By taking $s = 1-\beta/k$ in the above expression and using the equality (1.6), we get

$$\begin{aligned}\gamma_2(w) &= h_0(k-\beta) w^{1-\beta/k} \sum_{n=0}^{\infty} \frac{(\alpha+k-\beta)_{n,k}}{(2k-\beta)_{n,k}} \\ &\quad \times \left[\ln w + \frac{k}{k-\beta} + k\sum_{m=0}^{n-1}\left(\frac{1}{(\alpha+k-\beta)+mk} - \frac{1}{(1+m)k} - \frac{1}{2k-\beta+mk}\right)\right] \frac{w^n}{n!}.\end{aligned}$$

Then the general solution is obtained as

$$\begin{aligned}\gamma(w) &= E\gamma_1(w) + F\gamma_2(w)\\ &= h_0 \frac{E}{(\beta)_{-\beta/k,k}} \sum_{m=1-\beta/k}^{\infty} \frac{(\alpha)_{m,k}}{(k)_{m-1+\beta/k,k}} \frac{w^m}{(1)_m} + h_0(k-\beta)w^{1-\beta/k} F \sum_{n=0}^{\infty} \frac{(\alpha+k-\beta)_{n,k}}{(2k-\beta)_{n,k}}\\ &\quad \times \left(\ln w + \frac{k}{k-\beta} + k\sum_{m=0}^{n-1} \left[\frac{1}{(\alpha+k-\beta)+mk} - \frac{1}{(1+m)k} - \frac{1}{2k-\beta+mk} \right] \right) \frac{w^n}{n!}.\end{aligned}$$

(2) Let $\beta/k > 1$. Evaluating the recurrence relation (2.1) at the smaller root $s = 1-\beta/k$ gives

$$\gamma_n = \frac{\alpha + k(n-\beta/k)}{(n+1-\beta/k)\,(k(n-\beta/k)+\beta)} \gamma_{n-1}.$$

It follows that as $n \to \beta/k - 1$, the denominator vanishes, and consequently $\gamma_{\beta/k-1} \to \infty$, indicating the presence of a singular term in the series. As in the preceding case, the singular term associated with the corresponding root is removed by redefining the initial coefficient. Accordingly, the substitution

$$\gamma_0 = h_0(s-s_i)k$$

is introduced. Hence

$$\gamma_0 = h_0\,(s+\beta/k-1)\,k.$$

The series representation takes the form

$$\gamma_h(w,s) = h_0 \sum_{n=0}^{\infty} \frac{((s+\beta/k-1)\,k)\,(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n} w^n. \tag{2.8}$$

Then

$$\gamma_1(w) = \gamma_h(w,s)|_{s=1-\beta/k} \quad \text{and} \quad \gamma_2(w) = \left.\frac{\partial \gamma_h}{\partial s}\right|_{s=0}.$$

For $\gamma_1(w)$, all the terms before

$$\frac{((s+\beta/k-1)\,k)\,(\alpha+sk)_{\beta/k-1,k}}{(\beta+sk)_{\beta/k-1,k}(s+1)_{\beta/k-1}}$$

vanish due to the factor $s+\beta/k-1$ in the numerator. Indeed,

$$(s+1)_{\beta/k-1} = (s+1)(s+2)\cdots(s+\beta/k-1)$$

which shows that the zero of the numerator cancels the corresponding singular contribution.

Consequently, the solution assumes the form

$$\begin{aligned}\gamma_1(w) &= \frac{h_0 w^{1-\beta/k}}{(2-\beta/k)_{\beta/k-2}} \sum_{n=\beta/k-1}^{\infty} \frac{k\,(\alpha+(1-\beta/k)k)_{n,k}}{(\beta+(1-\beta/k)\,k)_{n,k}\,(1-\beta/k+\beta/k)_{n+1-\beta/k}}\\ &= \frac{h_0 w^{1-\beta/k}}{(2k-\beta)_{\beta/k-2,k}} \sum_{n=\beta/k-1}^{\infty} \frac{(\alpha+k-\beta)_{n,k}}{(k)_{n+1-\beta/k}} \frac{w^n}{(1)_n}.\end{aligned}$$

For the second solution, we need to find the derivative of γ_h with respect to s. So, we denote the term inside the series (2.8) by

$$\delta_n = \frac{((s+\beta/k-1)\,k)\,(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n}.$$

Then following the method in the previous case "$\beta/k<1$", we get

$$\begin{aligned}\frac{\partial\delta_n}{\partial s} &= \frac{((s+\beta/k-1)\,k)\,(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n}\\ &\quad\times\left[\frac{1}{s+\beta/k-1}+\sum_{m=0}^{n-1}\left(\frac{k}{\alpha+(s+m)k}-\frac{k}{\beta+(s+m)k}-\frac{1}{1+s+m}\right)\right],\end{aligned}$$

and therefore we have

$$\begin{aligned}\frac{\partial\gamma_h}{\partial s} &= h_0w^s\sum_{n=0}^{\infty}\frac{((s+\beta/k-1)\,k)\,(\alpha+sk)_{n,k}}{(\beta+sk)_{n,k}(s+1)_n}\\ &\quad\times\left[\ln w+\frac{1}{s+\beta/k-1}+\sum_{m=0}^{n-1}\left(\frac{k}{\alpha+(s+m)k}-\frac{k}{\beta+(s+m)k}-\frac{1}{1+s+m}\right)\right]w^n.\end{aligned}$$

Setting $s=0$, we obtain

$$\gamma_2(w) = h_0(\beta-k)\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}}{(\beta)_{n,k}}\left[\ln w+\frac{k}{\beta-k}+\sum_{m=0}^{n-1}\left(\frac{k}{\alpha+mk}-\frac{k}{\beta+mk}-\frac{1}{1+m}\right)\right]\frac{w^n}{n!}.$$

Therefore, the general solution can be written as:

$$\begin{aligned}\gamma(w) &= G\gamma_1(w)+H\gamma_2(w)\\ &= \frac{h_0w^{1-\beta/k}G}{(2k-\beta)_{\beta/k-2,k}}\sum_{m=\beta/k-1}^{\infty}\frac{(\alpha+k-\beta)_{m,k}}{(k)_{m+1-\beta/k}}\frac{w^m}{(1)_m}\\ &\quad+h_0(\beta-k)H\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}}{(\beta)_{n,k}}\left[\ln w+\frac{k}{\beta-k}+\sum_{m=0}^{n-1}\left(\frac{k}{\alpha+mk}-\frac{k}{\beta+mk}-\frac{1}{1+m}\right)\right]\frac{w^n}{n!}.\end{aligned}$$

2.2 Solution analysis in the case of "$w\to\infty$"

The behavior of the k-confluent hypergeometric equation as $w\to\infty$ is analyzed. Since infinity is an irregular singularity of the equation, a Frobenius-type power series expansion is not applicable in this setting. To overcome this difficulty, a suitable change of variable is introduced in order to carry out an asymptotic analysis. This approach determines the large w behavior of the solutions, particularly in situations where a convergent power series representation fails. The resulting asymptotic form therefore characterizes the solution in the limit $w\to\infty$. We now apply this asymptotic framework to the equation

$$y''+\left(\frac{\beta}{kw}-1\right)y'-\frac{\alpha}{kw}y=0. \tag{2.9}$$

We assume that the solutions have asymptotic forms

$$y(w)\sim e^{\phi(w)}w^{\rho}T(w)$$

and

$$T(w) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{w^n}$$

with coefficients c_n obtained by a natural recursion. Taking the first and second derivatives of assumed solution, we get

$$y'(w) = e^{\phi(w)} w^{\rho} \left(\phi'(w)T(w) + \frac{\rho}{w}T(w) + T'(w)\right)$$
$$y''(w) = e^{\phi(w)} w^{\rho} \left[T(w)\left(\left(\phi'(w) + \frac{\rho}{w}\right)^2 + \phi''(w) - \frac{\rho}{w^2}\right) + 2T'(w)\left(\phi'(w) + \frac{\rho}{w}\right) + T''(w)\right].$$

Substitution into (2.9) and simplification with respect to $T(w)$ yield

$$T(w)\left[(\phi'(w))^2 + \phi''(w) - \phi'(w) + \frac{2\phi'(w)\rho + \frac{\beta}{k}\phi'(w) - \rho - \frac{\alpha}{k}}{w} + \frac{\rho^2 - \rho + \frac{\rho\beta}{k}}{w^2}\right]$$
$$+ T'(w)\left[2\phi'(w) - 1 + \frac{2\rho + \frac{\beta}{k}}{w}\right] + T''(w) = 0. \tag{2.10}$$

Since $T(w) \to 1$, $T'(w) \to 0$ and $T''(w) \to 0$ as $w \to \infty$, it follows that

$$\phi''(w) + (\phi'(w))^2 - \phi'(w) = 0.$$

Let $\phi'(w) = u(w)$. Then $u^2 + u' - u = 0$. After calculation, we find

$$u(w) = 1 - \frac{1}{1 + ce^{w}}.$$

When $w \to \infty$, we have $u(w) = \phi'(w) = 1$ and consequently $\phi(w) = w$. On the other hand, $\phi(w) = 0$ corresponds to a trivial solution of the equation. Now we can apply these results to (2.10).

In the case $\phi(w) = 0$, the equation takes the form

$$T''(w) + \left(\frac{2\rho + \frac{\beta}{k}}{w} - 1\right)T'(w) + \left(\frac{-\rho - \frac{\alpha}{k}}{w} + \frac{\rho^2 + \rho\left(\frac{\beta}{k} - 1\right)}{w^2}\right)T(w) = 0. \tag{2.11}$$

We know that

$$T(w) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{w^n},$$

so

$$T'(w) = -\sum_{n=1}^{\infty} n c_n w^{-n-1} \quad \text{and} \quad T''(w) = \sum_{n=1}^{\infty} n(n+1) c_n w^{-n-2}.$$

By inserting these into the equation (2.11), we get

$$\sum_{n=1}^{\infty} n(n+1)c_n w^{-n-2} - \left(\frac{2\rho + \frac{\beta}{k}}{w} - 1\right)\sum_{n=1}^{\infty} n c_n w^{-n-1}$$
$$+ \left(\frac{-\rho - \frac{\alpha}{k}}{w} + \frac{\rho^2 + \rho\left(\frac{\beta}{k} - 1\right)}{w^2}\right)\left(1 + \sum_{n=1}^{\infty} c_n w^{-n}\right) = 0$$

and

$$\sum_{n=1}^{\infty} w^{-n-2}\Big[n(n+1)c_n - \left(2\rho+\frac{\beta}{k}\right) nc_n + (n+1)c_{n+1} - \left(\rho+\frac{\alpha}{k}\right) c_{n+1} + \left(\rho^2+\rho\left(\frac{\beta}{k}-1\right)\right) c_n\Big]$$
$$+\frac{c_1}{w^2} - \frac{\rho+\frac{\alpha}{k}}{w} - \left(\rho+\frac{\alpha}{k}\right)\frac{c_1}{w^2} + \frac{\rho^2+\rho\left(\frac{\beta}{k}-1\right)}{w^2} = 0.$$

The coefficient of w^{-r} $(r = 1, 2, \ldots)$ must vanish individually, and from the remaining terms, we obtain

$$\rho = -\frac{\alpha}{k}, \quad c_1 = -\frac{\alpha}{k^2}(\alpha-\beta+k)$$

and

$$c_{n+1} = -\frac{(\alpha+nk)((\alpha-\beta)+(n+1)k)}{(n+1)k^2} c_n.$$

The recurrence relation becomes

$$c_{n+1} = \frac{(-1)^n}{k^{2n}} \frac{(\alpha)_{n,k}(\alpha-\beta+k)_{n,k}}{n!}.$$

As a result, for $\phi(w) = 0$, we find the asymptotic solution as

$$y(w) \sim w^{-\alpha/k} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\alpha-\beta+k)_{n,k}}{n!} \left(-\frac{1}{k^2 w}\right)^n. \tag{2.12}$$

For $\phi(w) = w$, the equation reduces to

$$T''(w) + \left(1+\frac{2\rho+\frac{\beta}{k}}{w}\right) T'(w) + \left(\frac{\rho+\frac{\beta-\alpha}{k}}{w} + \frac{\rho^2+\rho\left(\frac{\beta}{k}-1\right)}{w^2}\right) T(w) = 0.$$

As in the case $\phi(w) = 0$, substituting the series $T(w)$ and its derivatives into the differential equation yields

$$\sum_{n=1}^{\infty} n(n+1)c_n w^{-n-2} - \left(1+\frac{2\rho+\frac{\beta}{k}}{w}\right) \sum_{n=1}^{\infty} nc_n w^{-n-1}$$
$$+\left(\frac{\rho+\frac{\beta-\alpha}{k}}{w} + \frac{\rho^2+\rho\left(\frac{\beta}{k}-1\right)}{w^2}\right)\left(1+\sum_{n=1}^{\infty} c_n w^{-n}\right) = 0$$

and

$$\sum_{n=1}^{\infty} w^{-n-2}\left[n(n+1)c_n - (n+1)c_{n+1} - \left(2\rho+\frac{\beta}{k}\right) nc_n + \left(\rho+\frac{\beta-\alpha}{k}\right) c_{n+1} + \left(\rho^2+\rho\left(\frac{\beta}{k}-1\right) c_n\right]$$
$$-\frac{c_1}{w^2} + \frac{\rho+\frac{\beta-\alpha}{k}}{w} + \left(\rho+\frac{\beta-\alpha}{k}\right)\frac{c_1}{w^2} + \frac{\rho^2+\rho\left(\frac{\beta}{k}-1\right)}{w^2} = 0.$$

The coefficients of w^{-r} must each vanish, and from the remaining terms we obtain the corresponding relation among the series coefficients as

$$\rho = \frac{\alpha-\beta}{k}, \quad c_1 = \frac{(\alpha-\beta)(\alpha-k)}{k^2}$$

and

$$c_{n+1} = \frac{(kn+\beta-\alpha)(kn+k-\alpha)}{k^2(n+1)} c_n.$$

After calculation, the recurrence relation becomes

$$c_n = \frac{(\beta-\alpha)_{n,k}(k-\alpha)_{n,k}}{n!k^{2n}}.$$

Then the other asymptotic solution of the differential equation is

$$y(w) \sim e^w w^{\frac{\alpha-\beta}{k}} \sum_{n=0}^{\infty} \frac{(\beta-\alpha)_{n,k}(k-\alpha)_{n,k}}{n!} \left(\frac{1}{k^2 w}\right)^n.$$

Definition 2.1. The solution given in (2.12) is called the k-Tricomi confluent hypergeometric function, and the representation of this function is given as

$$U_k(\alpha,\beta;w) = w^{-\frac{\alpha}{k}} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\alpha-\beta+k)_{n,k}}{k^{2n}n!}(-w)^{-n}, \qquad \Re(w) > 0.$$

3 Explicit solutions of the k-confluent hypergeometric model with polynomial right-hand side

The k-confluent hypergeometric equation is investigated when the non-homogeneity is polynomial. By seeking solutions in the form of finite polynomials, we obtain closed-form formulas for the coefficients of the resulting series. This approach leads to closed-form representations of particular solutions corresponding to polynomial terms, which complement the homogeneous solutions obtained in the previous section.

Accordingly, we consider the non-homogeneous k-confluent hypergeometric differential equation

$$kw\tilde{\gamma}'' + (\beta - kw)\tilde{\gamma}' - \alpha\tilde{\gamma} = \sum_{n=0}^{m} a_n w^n = f(w) \tag{3.1}$$

where $f(w)$ is a given non-zero polynomial of degree m.

A polynomial representation is considered in the form

$$\tilde{\gamma}(w) = \sum_{n=0}^{m} \tilde{\gamma}_n w^n. \tag{3.2}$$

Differentiation of this expression leads to

$$\tilde{\gamma}'(w) = \sum_{n=0}^{m-1} \tilde{\gamma}_{n+1}(n+1)w^n \tag{3.3}$$

and

$$\tilde{\gamma}''(w) = \sum_{n=0}^{m-2} \tilde{\gamma}_{n+2}(n+2)(n+1)w^n. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1) yields

$$k\sum_{n=0}^{m-2} \tilde{\gamma}_{n+2}(n+2)(n+1)w^{n+1} + \beta \sum_{n=0}^{m-1} \tilde{\gamma}_{n+1}(n+1)w^n - k\sum_{n=0}^{m-1} \tilde{\gamma}_{n+1}(n+1)w^{n+1} - \alpha \sum_{n=0}^{m} \tilde{\gamma}_n w^n = \sum_{n=0}^{m} a_n w^n.$$

Expressing this relation in powers of w gives

$$\sum_{n=0}^{m-1}\left[(n+1)(\beta+kn)\tilde{\gamma}_{n+1}-(\alpha+kn)\tilde{\gamma}_n\right]w^n-(\alpha+km)\tilde{\gamma}_m w^m=\sum_{n=0}^{m-1}a_n w^n+a_m w^m. \tag{3.5}$$

After rearranging the above equation and comparing the coefficients of the corresponding terms for $n=0,1,\ldots,m-1$, we obtain

$$a_n=(n+1)(\beta+kn)\tilde{\gamma}_{n+1}-(\alpha+kn)\tilde{\gamma}_n,$$

and

$$a_m=-(\alpha+km)\tilde{\gamma}_m.$$

Consequently, we get

$$\tilde{\gamma}_m=-\frac{a_m}{km+\alpha}$$

and

$$\begin{aligned}\tilde{\gamma}_{m-1}&=-\frac{(\alpha)_{m-1,k}}{(m-1)!(\beta)_{m-1,k}}\left[\frac{m!(\beta)_{m,k}}{(\alpha)_{m+1,k}}a_m+\frac{(m-1)!(\beta)_{m-1,k}}{(\alpha)_{m,k}}a_{m-1}\right]\\&=-\frac{(\alpha)_{m-1,k}}{(m-1)!(\beta)_{m-1,k}}\sum_{i=m-1}^{m}\frac{i!(\beta)_{i,k}}{(\alpha)_{i+1,k}}a_i.\end{aligned}$$

That being so, the coefficients of the solution given in (3.2) are found as

$$\tilde{\gamma}_i=-\frac{(\alpha)_{i,k}}{i!(\beta)_{i,k}}\sum_{j=i}^{m}\frac{j!(\beta)_{j,k}}{(\alpha)_{j+1,k}}a_j \tag{3.6}$$

for $i=0,1,2,\ldots,m$. Substituting equation (3.6) into equation (3.2), we obtain the solution

$$\tilde{\gamma}(w)=-\sum_{i=0}^{m}\left[\frac{(\alpha)_{i,k}}{i!(\beta)_{i,k}}\sum_{j=i}^{m}\frac{j!(\beta)_{j,k}}{(\alpha)_{j+1,k}}a_j\right]w^i. \tag{3.7}$$

To summarize briefly, equation (3.7) provides the explicit polynomial solution determined by the non-homogeneous term in (3.1).

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