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Refined oscillation criteria for second-order mixed neutral equations: A finite-sum iterative approach

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Abstract. This paper establishes refined oscillation criteria for second-order mixed neutral differential equations. For equations with variable delays, we construct a bridge function to prove a comparison theorem, which reduces the study of the second-order equation to a first-order delay differential equation. For equations with constant delays, we develop a finite-sum iterative method. By applying an integral operator prior to the iteration steps and iterating the neutral feedback inside this integral bound, we generate increasing sequences that yield sharper oscillation conditions. Finally, we provide two illustrative examples. The first validates the criteria for variable delays, while the second shows that the iterative process detects oscillation even when classical results and the initial iteration step fail.

Keywords: second-order neutral differential equations, oscillation criteria, mixed coefficients, finite-sum iterative method.

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1 Introduction

The oscillation theory of differential equations is motivated by applications in physics [26], population dynamics [14, 19], and control systems [27, 33]. Among the most notable classical monographs in this field are those by Győri and Ladas [15], Erbe et al. [11], Ladde et al. [20], and Koplatadze [17].

Oscillation criteria for first-order equations have a long history. We refer to the aforementioned classical monographs, the early work of Koplatadze and Chanturija [18], and subsequent papers. This theory has been further studied for various complex models. For delay equations, there has been recent progress, such as improved criteria for variable delays [10,12], explicit oscillation bounds [29], and relating the speed of oscillation and stability [32]. For neutral equations, studies address mixed coefficients [13], alongside the asymptotic behavior

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and periodic solutions of nonhomogeneous models [36]. Furthermore, higher-order equations have also been investigated systematically (see [8,22] and references therein).

Second-order equations model diverse dynamic phenomena, such as mechanical inertia, wave propagation, and complex oscillatory processes. Manojlović et al. [23] and Yıldız et al. [37] investigated neutral equations with mixed coefficients of the following form

$$\left[x(t) \pm \sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right]'' + \sum_{i=1}^{m} p_i(t)x(t-\delta_i) - \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0.$$

To handle negative terms, they utilized double integral bounds and reduced the problem to first-order inequalities by integrating on $[t,\infty)$. This approach requires point-wise conditions, such as $p_i(t) - q_i(t-\delta_i) \geq 0$, alongside $\sum \int_0^\infty \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi ds \leq 1$. Karpuz et al. [16] and Shoukaku [31] extended this method to equations with variable delays by introducing a mapping function to relate the deviating arguments. Defining a combined coefficient relaxes this point-wise condition. However, these single-step criteria often yield loose bounds, failing when the positive coefficients are not sufficiently large.

An alternative approach to handling negative coefficients is to incorporate them into the highest derivative, leading to neutral equations of the form:

$$\big(a(t)[x(t) - p(t)x(t-\tau)]'\big)' + q(t)x(t-\sigma) = 0.$$

For such equations, the Riccati transformation is widely used to reduce the problem to first-order inequalities [2,21,25]. This technique has also been extended to noncanonical forms [3,34,35], multiple delays [1], and delayed damping models [24]:

$$\big(a(t)z'(t)\big)' + p(t)z'(\beta(t)) + q(t)x(\delta(t)) = 0, \quad \text{where } z(t) = x(t) + b(t)x(\tau(t)).$$

However, applying the Riccati substitution to neutral equations with mixed coefficients is difficult. The main obstacle lies in estimating the ratio $x(\sigma(t))/z(t)$. Consequently, existing studies often restrict the delays to be constant or impose strong auxiliary conditions [4,5].

Similarly, comparison principles for mixed coefficients exist for first-order equations [13], but their extensions to second-order cases seem to be incomplete. Moreover, iterative methods tighten oscillation bounds without relying on the choice of auxiliary functions [6,7,9]. These iterations are usually applied point-wise to the differential inequalities. However, when negative coefficients are present, this point-wise approach often leads to a loss of sharpness in the resulting estimates.

Handling negative terms thus remains an active area of study [28,38]. Prabaharan et al. [30] investigated hybrid equations with a non-delay negative term

$$\big(a(t)x'(t)\big)' - q_1(t)x(t) + q_2(t)x^\alpha(\sigma(t)) = 0.$$

They absorbed this term using the transformation $x(t) = u(t)z(t)$, where $u(t)$ solves the equation $(a(t)u'(t))' = q_1(t)u(t)$. However, obtaining positive solutions for the auxiliary equation is not easy for complex coefficients. Furthermore, extending this transformation to delayed negative terms has not been considered.

Motivated by these observations, and acknowledging the significant contributions of Garab and Tóth [13], we study the oscillation of the second-order neutral differential equation with mixed coefficients and multiple variable delays:

$$z''(t) + \sum_{i=1}^{n_2} \Phi_i(t)x(t-\tau_i(t)) - \sum_{i=1}^{n_3} \Psi_i(t)x(t-\sigma_i(t)) = 0, \tag{1.1}$$

for $t \geq t_0$, where

$$z(t) = x(t) - \sum_{i=1}^{n_1} \Theta_i(t) x(t - \kappa_i(t)), \tag{1.2}$$

The coefficients $\Theta_i(t)$, $\Phi_i(t)$, and $\Psi_i(t)$ are continuous non-negative functions. The delays $\kappa_i(t)$, $\tau_i(t)$, and $\sigma_i(t)$ are continuous, non-negative, and satisfy $\lim_{t\to\infty}(t - \kappa_i(t)) = \lim_{t\to\infty}(t - \tau_i(t)) = \lim_{t\to\infty}(t - \sigma_i(t)) = \infty$.

To clarify the mathematical novelty of our results, we outline the analytical challenges resolved. Methods utilizing double integral bounds (e.g., [31,37]) rely on restrictive point-wise conditions (such as $p(t) - q(t - \alpha + \beta) \geq 0$) that lead to loose estimates. Conversely, Riccati techniques (e.g., [24]) encounter difficulties in estimating the delayed ratio $x(\sigma(t))/z(t)$ and are limited to constant delays. Furthermore, negative coefficients lead to a loss of monotonicity in the solution's derivatives. We resolve this by constructing a bridge function (Lemma 2.2) that absorbs negative terms via an implicit delay relation to restore monotonicity and provide a dynamic lower bound.

Using this bound, Theorem 3.1 establishes a comparison principle reducing the second-order equation to a first-order inequality. Theorem 4.1 then introduces our principal novelty: a finite-sum iterative framework. It differs from existing techniques, as standard second-order iterations (e.g., [6, 7]) operate point-wise on differential inequalities and lose sharpness for mixed equations.

Moreover, our method cannot be derived from first-order schemes (e.g., [13]). First-order methods construct iterations by extracting the tail of an infinite series applied to the first derivative. Extending this approach to second-order equations fails, as the second derivative must be mapped to an integral operator via the bound $z(t) \geq (t - T)w(t)$. We address this by substituting the neutral feedback as a finite sum inside the mapped integral before iterating. This integration maintains the tightness of the estimates, generating sharp bounds that capture oscillatory behavior which standard single-step or point-wise conditions fail to detect.

The paper is organized as follows. Section 2 is devoted to introducing the fundamental hypotheses, as well as the statement and proof of the principal lemma (Lemma 2.2). We construct an auxiliary function to absorb the negative coefficients through an implicit relation between the delays. This lemma establishes monotonicity properties and provides lower-bound estimates for the behavior of solutions.

In Section 3, we prove a comparison theorem for variable delays (Theorem 3.1). This result reduces the oscillation of the second-order equation (1.1) to that of a first-order differential equation. Consequently, we apply classical first-order criteria to obtain oscillation conditions for the studied equation (Corollary 3.2).

Section 4 is devoted to equations with constant delays. Here, we develop a novel finite-sum iterative technique (Theorem 4.1). Instead of iterating point-wise, we apply an integral operator prior to the iteration steps. Substituting the neutral feedback directly into this integral ensures the tightness of the inequalities. This yields an increasing sequence of oscillation bounds, detecting oscillatory solutions that classical conditions cannot capture.

Finally, Section 5 provides two examples to illustrate the theoretical results and demonstrate how our iterative criteria successfully detect oscillation in cases where classical conditions fail.

2 Hypotheses and principal lemma

This section introduces the fundamental hypotheses, alongside the statement and proof of the main auxiliary lemma.

Naturally, we assume that $n_2 \geq n_3$. However, without loss of generality, we may set

$$n_2 = n_3 = N$$

by defining $\Psi_i(t) \equiv 0$ for $i > n_3$ if necessary.

Next, we assume the following fundamental condition on the delays:

(H1) For $1 \leq i \leq N$, $\tau_i(t) \geq \sigma_i(t)$ and there exists a constant $\sigma^* < 1$ such that $\sigma_i'(t) \leq \sigma^*$.

Based on the properties established in [13, Proposition 2.1], condition (H1) ensures the existence of non-negative, differentiable mapping c_i for each index i. This mapping is implicitly governed by the identity

$$c_i = \tau_i(t) - \sigma_i(t - c_i(t)). \tag{2.1}$$

Here, we assume that the functions $c_i(t)$ are uniformly bounded for all $t \geq t_0$, and

$$C_i := \sup_{t \geq t_0} c_i(t) < \infty.$$

Incorporating the derivative $c_i'(t)$, we construct the modified positive coefficient as follows:

$$\tilde{\Phi}_i(t) := \Phi_i(t) - \Psi_i(t - c_i(t))(1 - c_i'(t)).$$

With the bounds established, we impose the following core hypotheses for our main oscillation criteria.

(H2) $\tilde{\Phi}_i(t) \geq 0$ for sufficiently large t. Furthermore, the system satisfies the boundedness condition:

$$\sum_{i=1}^{n_1} \Theta_i(t) + \sum_{i=1}^{N} C_i \int_{t_0}^{\infty} \Psi_i(s)ds \leq \theta_0 < 1, \quad \forall\, t \geq t_0.$$

(H3) For the case of constant delays (i.e., $\kappa_i(t) \equiv \kappa_i$, $\tau_i(t) \equiv \tau_i$), there exists a constant $v > 0$ such that

$$\Theta_j(t - \tau_i)\tilde{\Phi}_i(t) \geq v\tilde{\Phi}_i(t - \kappa_j),$$

for all i, j and for sufficiently large t.

Remark 2.1. The validity of (H3) and the choice of $v > 0$ depend on the asymptotic behaviors of $\Theta_j(t)$ and $\tilde{\Phi}_i(t)$:

(i) *Non-vanishing neutral coefficients* ($\liminf_{t\to\infty} \Theta_j(t) = \theta_* > 0$)*:* The asymptotic behavior of $\tilde{\Phi}_i(t)$ is governed by $\Phi_i(t)$ when $\Psi_i(t) = o(\Phi_i(t))$, or by a positive remainder term such as

$$\Psi_i(t) \sim \Phi_i(t) - ct^{-\alpha} \quad \text{or} \quad \Psi_i(t) \sim \Phi_i(t) - ce^{-\alpha t} \quad (c > 0).$$

This yields three cases:

(a) *Dominant positive coefficients* ($\Psi_i(t) = o(\Phi_i(t))$)*:* This condition yields $\tilde{\Phi}_i(t) \sim \Phi_i(t)$. Provided that $\Phi_i(t) \sim \Phi_i(t - \kappa_j)$, this implies

$$\lim_{t\to\infty} \frac{\tilde{\Phi}_i(t)}{\tilde{\Phi}_i(t - \kappa_j)} = 1.$$

Thus, any constant $v \in (0, \theta_*)$ is an admissible choice.

(b) *Decay with positive remainder:* If $\tilde{\Phi}_i(t)$ has a power-law ($\sim ct^{-\alpha}$) or exponential ($\sim ce^{-\alpha t}$) decay, we define $c_0 = \min_{1\le j\le n_1}\{1, e^{-\alpha\kappa_j}\} > 0$. It follows that

$$\liminf_{t\to\infty} \Theta_j(t-\tau_i)\frac{\tilde{\Phi}_i(t)}{\tilde{\Phi}_i(t-\kappa_j)} \ge \theta_* c_0 > 0.$$

Therefore, any constant $v \in (0, \theta_* c_0)$ is an admissible choice.

(c) *Oscillatory decay:* If $\Phi_i(t)$ contains a periodic factor, choosing the delay as an integer multiple of the period eliminates the effect of this periodicity on the ratio of the modified coefficients. For instance, in Example 5.2 (see Section 5), taking $\kappa_m = 2$ and $\Phi_1(t) \sim ct^{-1}(1 + a\sin(\pi t))$ yields

$$\lim_{t\to\infty} \frac{\tilde{\Phi}_1(t)}{\tilde{\Phi}_1(t-\kappa_m)} = 1.$$

Thus, any constant $v \in (0, \theta_*)$ is an admissible choice.

(ii) *Vanishing neutral coefficients ($\liminf_{t\to\infty} \Theta_j(t) = 0$):* Condition (H3) requires the delayed ratio to diverge:

$$\frac{\tilde{\Phi}_i(t)}{\tilde{\Phi}_i(t-\kappa_j)} \ge \frac{v}{\Theta_j(t-\tau_i)} \to \infty \quad \text{as } t \to \infty.$$

Functions with power-law or exponential behavior yield bounded delayed ratios. Such divergence arises from faster growth rates, such as super-exponential growth (e.g., $\sim ce^{t^2}$). Thus, (H3) fails when the modified coefficients $\tilde{\Phi}_i(t)$ have power-law or exponential bounds. The restriction $\liminf_{t\to\infty} \Theta_j(t) > 0$ is necessary.

Next, to state and prove the auxiliary lemma, we define

$$w(t) = z'(t) - \sum_{i=1}^{N} \int_{t-c_i(t)}^{t} \Psi_i(s)x(s-\sigma_i(s))ds. \tag{2.2}$$

Lemma 2.2. *Let $x(t)$ be an eventually positive solution of Eq. (1.1). If hypothesis (H1) and (H2) hold, then for all sufficiently large t, the function $w(t)$ defined in (2.2) satisfies*

$$w(t) > 0 \quad \textit{and} \quad w'(t) \le 0.$$

Furthermore, there exists a constant $T \ge t_0$ such that the following inequality holds

$$z(t) \ge (t-T)w(t), \quad \textit{for all } t > T.$$

Proof. Since $x(t)$ is an eventually positive solution of Eq. (1.1) and the delayed arguments satisfy

$$\lim_{t\to\infty}(t-\kappa_i(t)) = \lim_{t\to\infty}(t-\tau_i(t)) = \lim_{t\to\infty}(t-\sigma_i(t)) = \infty, \quad \text{for all } i,$$

there exists a constant $t_1 \ge t_0$ such that $x(t) > 0$ and

$$x(t-\kappa_i(t)) > 0, \quad x(t-\tau_i(t)) > 0, \quad \text{and} \quad x(t-\sigma_i(t)) > 0,$$

for all $t \ge t_1$.

Claim 1. $w'(t) \leq 0$.
From the definition of $w(t)$, it follows that

$$w'(t) = z''(t) - \sum_{i=1}^{N} \Psi_i(t)x(t-\sigma_i(t)) + \sum_{i=1}^{N} \Psi_i(t-c_i(t))x(t-c_i(t)-\sigma_i(t-c_i(t)))(1-c_i'(t)). \tag{2.3}$$

On the other hand, equation (1.1) can be rewritten as

$$z''(t) = -\sum_{i=1}^{N} \Phi_i(t)x(t-\tau_i(t)) + \sum_{i=1}^{N} \Psi_i(t)x(t-\sigma_i(t)).$$

By substituting $z''(t)$ into (2.3) and noting the identity $t - c_i(t) - \sigma_i(t - c_i(t)) = t - \tau_i(t)$, we obtain

$$\begin{aligned} w'(t) &= -\sum_{i=1}^{N} \Phi_i(t)x(t-\tau_i(t)) + \sum_{i=1}^{N} \Psi_i(t-c_i(t))(1-c_i'(t))x(t-\tau_i(t)) \\ &= -\sum_{i=1}^{N} \left[\Phi_i(t) - \Psi_i(t-c_i(t))(1-c_i'(t))\right] x(t-\tau_i(t)) \\ &= -\sum_{i=1}^{N} \tilde{\Phi}_i(t)x(t-\tau_i(t)). \end{aligned}$$

Since $x(t-\tau_i(t)) > 0$ and $\tilde{\Phi}_i(t) \geq 0$ for $t \geq t_1$, it follows from the expression of $w'(t)$ that $w'(t) \leq 0$. Consequently, $w(t)$ is monotonically non-increasing for $t \geq t_1$.

Claim 2. $w(t) > 0$.
Suppose, for the sake of contradiction, that $\lim_{t\to\infty} w(t) = 2L < 0$. Since $w'(t) \leq 0$, the function $w(t)$ decreases. Thus, there exists a sufficiently large t_2 satisfying $t_2 - \max_i\{C_i\} \geq t_1 \geq t_0$, and $w(t) \leq L < 0$ for all $t \geq t_2$. An integration of (2.2) over the interval $[t_2, t]$ yields

$$\begin{aligned} z(t) - z(t_2) &= \int_{t_2}^{t} w(u)du + \sum_{i=1}^{N} \int_{t_2}^{t} \left(\int_{u-c_i(u)}^{u} \Psi_i(s)x(s-\sigma_i(s))ds \right) du \\ &\leq L(t-t_2) + \sum_{i=1}^{N} \int_{t_2}^{t} \left(\int_{u-c_i(u)}^{u} \Psi_i(s)x(s-\sigma_i(s))ds \right) du. \end{aligned} \tag{2.4}$$

By Fubini's theorem and the domain bounds $s \leq u \leq s + c_i(u) \leq s + C_i$ and $t_2 - C_i \leq t_2 - c_i(t_2) \leq s \leq t$, we obtain

$$\begin{aligned} \int_{t_2}^{t} \int_{u-c_i(u)}^{u} \Psi_i(s)x(s-\sigma_i(s))ds\,du &\leq \int_{t_2-C_i}^{t} \Psi_i(s)x(s-\sigma_i(s)) \left(\int_{s}^{s+C_i} 1\,du \right) ds \\ &= C_i \int_{t_2-C_i}^{t} \Psi_i(s)x(s-\sigma_i(s))ds. \end{aligned}$$

Substituting this bound into (2.4), we arrive at

$$z(t) \leq z(t_2) + L(t-t_2) + \sum_{i=1}^{N} C_i \int_{t_2-C_i}^{t} \Psi_i(s)x(s-\sigma_i(s))ds. \tag{2.5}$$

We consider two cases:

Case 1: $x(t)$ is bounded. There exists a constant $M > 0$ such that $x(t) \leq M$ for all sufficiently large t. From the definition of $z(t)$ and hypothesis (H2), we have

$$z(t) = x(t) - \sum_{i=1}^{n_1} \Theta_i(t)x(t - \kappa_i(t)) \geq -\sum_{i=1}^{n_1} \Theta_i(t)M \geq -\theta_0 M.$$

On the other hand, evaluating the upper bound for $z(t)$ yields

$$z(t) \leq z(t_2) + L(t - t_2) + M\sum_{i=1}^{N} C_i \int_{t_2 - C_i}^{t} \Psi_i(s)ds \leq z(t_2) + L(t - t_2) + M\theta_0.$$

Since $L < 0$, the right-hand side tends to $-\infty$ as $t \to \infty$. This contradicts the fact that $z(t) \geq -\theta_0 M$.

Case 2: $x(t)$ is unbounded. We can choose an increasing sequence $\{\hat{t}_k\}_{k=1}^{\infty}$ such that $\hat{t}_k \to \infty$ as $k \to \infty$ and $x(\hat{t}_k) = \max_{\xi \leq \hat{t}_k} x(\xi)$. Since $\hat{t}_k - \kappa_i(\hat{t}_k) \leq \hat{t}_k$, we have $x(\hat{t}_k - \kappa_i(\hat{t}_k)) \leq x(\hat{t}_k)$. From the definition (1.2), evaluating $z(t)$ at $t = \hat{t}_k$ yields

$$z(\hat{t}_k) = x(\hat{t}_k) - \sum_{i=1}^{n_1} \Theta_i(\hat{t}_k)x(\hat{t}_k - \kappa_i(\hat{t}_k)) \geq x(\hat{t}_k)\left[1 - \sum_{i=1}^{n_1} \Theta_i(\hat{t}_k)\right]. \tag{2.6}$$

Evaluating (2.5) at $t = \hat{t}_k$ and noting that $x(s - \sigma_i(s)) \leq \max_{\xi \leq \hat{t}_k} x(\xi) = x(\hat{t}_k)$ for all $s \leq \hat{t}_k$, we obtain

$$x(\hat{t}_k)\left[1 - \sum_{i=1}^{n_1} \Theta_i(\hat{t}_k)\right] \leq z(t_2) + L(\hat{t}_k - t_2) + x(\hat{t}_k)\sum_{i=1}^{N} C_i \int_{t_2 - C_i}^{\hat{t}_k} \Psi_i(s)ds.$$

Since $x(\hat{t}_k) > 0$, dividing by $x(\hat{t}_k)$ yields

$$1 - \left(\sum_{i=1}^{n_1} \Theta_i(\hat{t}_k) + \sum_{i=1}^{N} C_i \int_{t_2 - C_i}^{\hat{t}_k} \Psi_i(s)ds\right) \leq \frac{z(t_2) + L(\hat{t}_k - t_2)}{x(\hat{t}_k)}.$$

Since $L < 0$, we have $z(t_2) + L(\hat{t}_k - t_2) < 0$ for sufficiently large k. By hypothesis (H2), we obtain

$$0 < 1 - \theta_0 \leq 1 - \left(\sum_{i=1}^{n_1} \Theta_i(\hat{t}_k) + \sum_{i=1}^{N} C_i \int_{t_2 - C_i}^{\hat{t}_k} \Psi_i(s)ds\right) \leq \frac{z(t_2) + L(\hat{t}_k - t_2)}{x(\hat{t}_k)} < 0.$$

This yields a contradiction. Hence, $w(t) > 0$ must hold.

Claim 3. $z(t) \geq (t - T)w(t)$.
From (2.6) and hypothesis (H2), we have

$$z(\hat{t}_k) \geq x(\hat{t}_k)(1 - \theta_0) \geq 0.$$

We choose k^* sufficiently large such that $T_0 = \hat{t}_{k^*} > t_2$ and define

$$y(t) = \int_{T_0}^{t} w(s)ds. \tag{2.7}$$

Then, we have

$$y'(t) = w(t) > 0 \quad \text{and} \quad y''(t) = w'(t) \leq 0.$$

This implies that $y(t)$ is a concave function. Hence, for any $T_1 > T_0$, we have

$$y(T_1) \leq y(t) + y'(t)(T_1 - t) \text{ implies } y(t) \geq y(T_1) + y'(t)(t - T_1).$$

Since $y(t)$ is increasing and $T_1 > T_0$, we obtain $y(T_1) > y(T_0) = 0$. Thus,

$$y(t) \geq y(T_1) + y'(t)(t - T_1) \geq y'(t)(t - T_1) = (t - T_1)w(t). \tag{2.8}$$

Moreover, pushing (2.2) into (2.7), we have

$$\begin{aligned} y(t) &= z(t) - z(T_0) - \sum_{i=1}^{N} \int_{T_0}^{t} \left(\int_{s-c_i(s)}^{s} \Psi_i(u)x(u - \sigma_i(u))du \right) ds \\ &\leq z(t) - z(T_0). \end{aligned}$$

Combining this with (2.8), we obtain

$$z(t) - z(T_0) \geq y(t) \geq (t - T_1)w(t)$$

It follows from $z(T_0) \geq 0$ that

$$z(t) \geq y(t) \geq (t - T_1)w(t).$$

By setting $T = T_1$, we arrive at the desired inequality

$$z(t) \geq (t - T)w(t).$$

This completes the proof. □

3 Comparison theorem for variable delays

We now present the main comparison theorem, establishing a connection between the second-order neutral equation with multiple variable delays and a first-order variable-delay equation. Comparison principles for positive equations rely on bounds of its derivatives. Mixed neutral equations are different. Negative terms cause a loss of monotonicity, requiring a new bridge function. Theorem 3.1 resolves this by combining the modified derivative $w(t)$ and the dynamic bound from Lemma 2.2.

Theorem 3.1. *Assume that hypotheses (H1) and (H2) hold. If there exists a parameter $\lambda \in (0, 1)$ such that the first-order differential equation with multiple variable delays*

$$v'(t) + \lambda \sum_{i=1}^{N} \tilde{\Phi}_i(t)(t - \tau_i(t))v(t - \tau_i(t)) = 0 \tag{3.1}$$

has no eventually positive solutions, then every solution of Eq. (1.1) is oscillatory.

Proof. We argue by contradiction. Suppose that Eq. (1.1) admits a nonoscillatory solution $x(t)$. By symmetry, it suffices to consider the case where $x(t) > 0$ for all t large enough. Then, there exists a constant $t_1 \geq t_0$ such that $x(t) > 0$ and

$$x(t - \kappa_i(t)) > 0, \quad x(t - \tau_i(t)) > 0, \quad \text{and} \quad x(t - \sigma_i(t)) > 0,$$

for all $t \geq t_1$. We divide the remainder of the proof into three steps.

Step 1. Construction of an auxiliary operator and estimation of its lower bound.
Consider the function $w(t)$ defined in (2.2). By Lemma 2.2, there exists $t_1 > t_0$ such that $w(t) > 0$ and

$$w'(t) = -\sum_{i=1}^{N} \tilde{\Phi}_i(t)x(t-\tau_i(t)) \leq 0$$

for $t \geq t_1$. Integrating this expression from t to u (with $u > t$) and as u tends to infinity, it follows that

$$w(t) = \lim_{u\to\infty} w(u) + \int_t^{\infty} \sum_{i=1}^{N} \tilde{\Phi}_i(s)x(s-\tau_i(s))ds.$$

Since $w(t)$ is positive and monotonically decreasing, we have $\lim_{u\to\infty} w(u) \geq 0$. Thus, the latter equality yields

$$w(t) \geq \int_t^{\infty} \sum_{i=1}^{N} \tilde{\Phi}_i(s)x(s-\tau_i(s))ds := W(t) > 0, \quad \text{for } t \geq t_1. \tag{3.2}$$

By direct computation, we have

$$W'(t) = -\sum_{i=1}^{N} \tilde{\Phi}_i(t)x(t-\tau_i(t)) \leq 0.$$

Hence, $W(t)$ is non-increasing. Furthermore, from (1.2) and the fact that $\Theta_i(t) \geq 0$, we obtain $z(t) \leq x(t)$. Substituting this lower bound for x into (3.2), we have

$$W(t) \geq \int_t^{\infty} \sum_{i=1}^{N} \tilde{\Phi}_i(s)z(s-\tau_i(s))ds, \quad \text{for } t \geq t_1. \tag{3.3}$$

Step 2. Construction of the bridge function.
In view of Lemma 2.2, there exists a constant $T \geq t_0$ such that

$$z(\xi) \geq (\xi - T)w(\xi)$$

for all $\xi > T$. Without loss of generality, we may assume $T > 0$. For any given $\lambda \in (0,1)$, we define

$$T_\lambda = \frac{T}{1-\lambda}.$$

For any $\xi \geq T_\lambda > 0$, it is easy to verify that $\xi - T \geq \lambda\xi$. Consequently, we obtain

$$z(\xi) \geq \lambda\xi w(\xi), \quad \text{for } \xi \geq T_\lambda. \tag{3.4}$$

Since $\lim_{t\to\infty}(t-\tau_i(t)) = \infty$, there exists a sufficiently large $T_1 \geq t_1$ such that

$$t - \tau_i(t) \geq \max\{T_\lambda, t_1\}, \quad \forall t \geq T_1.$$

Combining (3.2) with (3.4), we obtain

$$z(s-\tau_i(s)) \geq \lambda(s-\tau_i(s))w(s-\tau_i(s)) \geq \lambda(s-\tau_i(s))W(s-\tau_i(s))$$

for all $s \geq t \geq T_1$. Substituting this lower bound into (3.3), we obtain

$$W(t) \geq \lambda \int_t^{\infty} \sum_{i=1}^{N} \tilde{\Phi}_i(s)(s-\tau_i(s))W(s-\tau_i(s))ds > 0, \quad \text{for } t \geq T_1. \tag{3.5}$$

We define the integral operator on the right-hand side as $U(t)$:

$$U(t) := \lambda \int_t^\infty \sum_{i=1}^N \tilde{\Phi}_i(s)(s-\tau_i(s))W(s-\tau_i(s))ds.$$

It follows from (3.5) that

$$W(t) \geq U(t) > 0, \quad \text{for } t \geq T_1. \tag{3.6}$$

Differentiating $U(t)$ with respect to t yields

$$U'(t) = -\lambda \sum_{i=1}^N \tilde{\Phi}_i(t)(t-\tau_i(t))W(t-\tau_i(t)) \leq 0, \quad \text{for } t \geq T_1. \tag{3.7}$$

Step 3. Construction of the differential inequality with multiple delays and conclusion. We choose $T_2 \geq T_1$ such that

$$t - \tau_i(t) \geq T_1, \quad \text{for all } t \geq T_2.$$

Then, it follows from (3.6) that

$$W(t-\tau_i(t)) \geq U(t-\tau_i(t)) > 0, \quad \text{for all } t \geq T_2.$$

Applying this bound to (3.7) yields

$$U'(t) + \lambda \sum_{i=1}^N \tilde{\Phi}_i(t)(t-\tau_i(t))U(t-\tau_i(t)) \leq 0, \quad \text{for all } t \geq T_2. \tag{3.8}$$

Consequently, $U(t)$ is an eventually positive solution of the differential inequality (3.8). Invoking the well-known criterion from [15, Corollary 3.2.2], the aforementioned inequality forces the associated differential equation to possess a solution that remains eventually positive. However, this deduction yields a direct contradiction regarding the established non-existence of such positive solutions. Consequently, we conclude that all solutions to Eq. (1.1) must oscillate. □

Combining Theorem 3.1 with classical results for first-order equations (see, e.g., [15, Theorem 3.4.1 and 3.4.3] or [11, Theorems 2.6.5]), we obtain the following oscillation criteria for Eq. (1.1).

Corollary 3.2. *Assume that hypotheses (H1) and (H2) hold. Denote* $\tau_{\max}(t) = \max_{1\leq i\leq N}\{\tau_i(t)\}$ *and* $\tau_{\min}(t) = \min_{1\leq i\leq N}\{\tau_i(t)\}$. *Provided that at least one of the subsequent strict inequalities is satisfied:*

(i) $\liminf_{t\to\infty} \int_{t-\tau_{\max}(t)}^t \sum_{i=1}^N \tilde{\Phi}_i(s)(s-\tau_i(s))\,ds > \frac{1}{e}$;

(ii) $\liminf_{t\to\infty} \sum_{i=1}^N \tilde{\Phi}_i(t)(t-\tau_i(t))\tau_i(t) > \frac{1}{e}$ *and furthermore, there exists* $I \subset \{1,2,\ldots,N\}$ *such that* $\liminf_{t\to\infty} \tau_*(t) > 0$ *and* $\liminf_{t\to\infty} \int_{t-\tau_*(t)}^t \sum_{i\in I} \tilde{\Phi}_i(s)(s-\tau_i(s))\,ds > 0$, *where* $\tau_*(t) = \min_{i\in I}\{\tau_i(t)\}$;

(iii) $\limsup_{t\to\infty} \int_{t-\tau_{\min}(t)}^t \sum_{i=1}^N \tilde{\Phi}_i(s)(s-\tau_i(s))\,ds > 1$,

we conclude that Eq. (1.1) is oscillatory.

Proof. Condition (i), (ii), or (iii) guarantees the existence of a parameter $\lambda \in (0,1)$ sufficiently close to 1 such that

$$\lambda \liminf_{t\to\infty} \int_{t-\tau_{\max}(t)}^{t} \sum_{i=1}^{N} \tilde{\Phi}_i(s)(s-\tau_i(s))\,ds > \frac{1}{e},$$

$$\lambda \liminf_{t\to\infty} \sum_{i=1}^{N} \tilde{\Phi}_i(t)(t-\tau_i(t))\tau_i(t) > \frac{1}{e},$$

or

$$\lambda \limsup_{t\to\infty} \int_{t-\tau_{\min}(t)}^{t} \sum_{i=1}^{N} \tilde{\Phi}_i(s)(s-\tau_i(s))\,ds > 1,$$

respectively. According to [15, Theorems 3.4.1 and 3.4.3] or [11, Theorem 2.6.5], this implies that Eq. (3.1) has no eventually positive solutions. Therefore, by Theorem 3.1, we conclude that every solution of Eq. (1.1) is oscillatory. □

4 Finite-sum oscillation iteration for constant delays

In this section, we focus on the case where the delays are constants (i.e., $\tau_i(t) \equiv \tau_i$ and $\kappa_i(t) \equiv \kappa_i$). By utilizing Lemma 2.2 and a finite-sum iterative technique on the integral operator, we derive new oscillation criteria. For an integer $n \geq 1$, we denote

$$\kappa_m = \min_{1\leq j\leq n_1}\{\kappa_j\}, \quad \tau_{\max} = \max_{1\leq i\leq N}\{\tau_i\}, \quad \text{and} \quad \tau_{\min} = \min_{1\leq i\leq N}\{\tau_i\},$$

and define the maximum and minimum delays of the iterative process as

$$\Delta_{\max} = (n-1)\kappa_m + \tau_{\max} \quad \text{and} \quad \Delta_{\min} = \tau_{\min},$$

respectively. We introduce the following iterative sequences:

$$\alpha_n = \liminf_{t\to\infty} \int_{t-\Delta_{\max}}^{t} \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(s-j\kappa_m)(s-j\kappa_m-\tau_i)\,ds, \tag{4.1}$$

$$\beta_n = \limsup_{t\to\infty} \int_{t-\Delta_{\min}}^{t} \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(s-j\kappa_m)(s-j\kappa_m-\tau_i)\,ds, \tag{4.2}$$

$$\gamma_n = \liminf_{t\to\infty} \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(t-j\kappa_m)(t-j\kappa_m-\tau_i)(j\kappa_m+\tau_i). \tag{4.3}$$

Theorem 4.1. *Assume that hypotheses (H1), (H2), and (H3) hold. If there exists an integer $n \geq 1$ such that one of the following conditions is valid:*

(i) $\alpha_n > \frac{1}{e}$;

(ii) $\beta_n > 1$;

(iii) $\gamma_n > \frac{1}{e}$,

then every solution of Eq. (1.1) is oscillatory.

Remark 4.2. From definitions (4.1)–(4.3), it is clear that each iteration step adds a positive delayed term to the summations. Consequently, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are strictly increasing sequences with respect to n. This monotonic property is the core advantage of our finite-sum technique: the criteria in Theorem 4.1 become sharper as n increases, finding oscillations that classical conditions ($n = 1$) fail to detect.

Proof. We argue by contradiction. To obtain a contradiction, let us assume that Eq. (1.1) possesses an eventually positive solution $x(t)$. As shown in the proof of Theorem 3.1, there exist $t_1 \geq t_0$ and a positive, non-increasing function $W(t)$ defined by

$$W(t) = \int_t^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)x(s-\tau_i)\,ds,$$

which satisfies (3.2) and (3.3) for $t \geq t_1$.

Step 1. Establishment of the finite-sum iterative lower bound.
Substituting the relation

$$x(s-\tau_i) = z(s-\tau_i) + \sum_{j=1}^{n_1} \Theta_j(s-\tau_i)x(s-\tau_i-\kappa_j)$$

into the definition of $W(t)$ and applying condition (H3), we obtain

$$W(t) \geq \int_t^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)z(s-\tau_i)ds + v\sum_{j=1}^{n_1} W(t-\kappa_j).$$

Since $W(t)$ is non-increasing and $\kappa_j \geq \kappa_m$ for all j, it follows that

$$\sum_{j=1}^{n_1} W(t-\kappa_j) \geq n_1 W(t-\kappa_m), \quad \text{for } t \geq t_1 + \max_{1\leq j\leq n_1}\{\kappa_j\},$$

which implies

$$W(t) \geq \int_t^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)z(s-\tau_i)ds + (vn_1)W(t-\kappa_m), \quad \text{for } t \geq t_1 + \max_{1\leq j\leq n_1}\{\kappa_j\}.$$

Continuing the above process n times, and noting that $(n_1 v)^n W(t-n\kappa_m) > 0$, we obtain

$$\begin{aligned} W(t) &\geq \sum_{j=0}^{n-1}(n_1 v)^j \int_{t-j\kappa_m}^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)z(s-\tau_i)ds + (n_1 v)^n W(t-n\kappa_m) \\ &\geq \sum_{j=0}^{n-1}(n_1 v)^j \int_{t-j\kappa_m}^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)z(s-\tau_i)ds, \quad \text{for } t \geq t_1 + n\max_{1\leq j\leq n_1}\{\kappa_j\}. \end{aligned} \tag{4.4}$$

Step 2. Construction of a delay differential equation and contradiction.
As in Step 2 of Theorem 3.1, for any $\lambda \in (0,1)$, there exists $T_1 \geq t_1$ such that

$$z(s-\tau_i) \geq \lambda(s-\tau_i)W(s-\tau_i).$$

Substituting this into (4.4), we obtain

$$W(t) \geq \lambda \sum_{j=0}^{n-1}(n_1 v)^j \int_{t-j\kappa_m}^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)(s-\tau_i)W(s-\tau_i)ds.$$

Define

$$U(t) = \lambda \sum_{j=0}^{n-1}(n_1 v)^j \int_{t-j\kappa_m}^\infty \sum_{i=1}^{N} \tilde{\Phi}_i(s)(s-\tau_i)W(s-\tau_i)ds \leq W(t), \quad t \geq T_1.$$

Differentiating $U(t)$ yields

$$U'(t) = -\lambda \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(t - j\kappa_m)(t - j\kappa_m - \tau_i) W(t - j\kappa_m - \tau_i).$$

Since $W(\xi) \geq U(\xi)$ for $\xi \geq T_1$, we choose T_2 such that $t - j\kappa_m - \tau_i \geq T_1$ for all $t \geq T_2$. Replacing W with U gives

$$U'(t) + \lambda \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(t - j\kappa_m)(t - j\kappa_m - \tau_i) U(t - j\kappa_m - \tau_i) \leq 0, \quad t \geq T_2.$$

Hence, $U(t)$ is an eventually positive solution of the above inequality. According to [15, Corollary 3.2.2], this implies that the corresponding differential equation

$$v'(t) + \lambda \sum_{i=1}^{N} \sum_{j=0}^{n-1} (n_1 v)^j \tilde{\Phi}_i(t - j\kappa_m)(t - j\kappa_m - \tau_i) v(t - j\kappa_m - \tau_i) = 0 \tag{4.5}$$

also has an eventually positive solution.

Since conditions (i), (ii), and (iii) are strict inequalities, one can choose $\lambda \in (0, 1)$ sufficiently close to 1 such that $\lambda\alpha_n > 1/e$, $\lambda\beta_n > 1$, or $\lambda\gamma_n > 1/e$, respectively. Invoking the well-known criteria from [15, Theorem 3.4.1, Theorem 3.4.3, Corollary 3.4.1], any of these conditions guarantees that every solution of Eq. (4.5) oscillates. This establishes a contradiction. Thus, Eq. (1.1) cannot possess nonoscillatory solutions. □

As an application of Theorem 4.1, we consider a class of equations where the positive coefficients decay as $1/t^\alpha$. For $\alpha \in (0, 1)$, the oscillation is unconditional. In the critical case $\alpha = 1$, however, the oscillation requires a specific threshold condition that involves the neutral term.

Corollary 4.3. *Assume that hypotheses (H1), (H2), and (H3) hold with constants delays. Consider the mixed neutral differential equation with generalized decaying coefficients for $t \geq t_0 > 0$:*

$$\left[x(t) - \sum_{i=1}^{n_1} \Theta_i x(t - \kappa_i) \right]'' + \sum_{i=1}^{N} \frac{p_i}{t^\alpha} x(t - \tau_i) - \sum_{i=1}^{N} \Psi_i(t) x(t - \sigma_i) = 0, \tag{4.6}$$

where $\alpha \in (0, 1]$, $\Theta_i \geq 0$, $p_i > 0$, and the parameters satisfy $n_1 v < 1$. The negative coefficients $\Psi_i(t)$ satisfy $\int_{t_0}^{\infty} \Psi_i(s) ds < \infty$ and $\lim_{t\to\infty} t^\alpha \Psi_i(t) = 0$.

(i) If $\alpha \in (0, 1)$, then every solution of Eq. (4.6) is oscillatory.

(ii) If $\alpha = 1$, and the following inequality holds:

$$\sum_{i=1}^{N} p_i \left(\kappa_m \frac{v_0}{(1 - v_0)^2} + \tau_i \frac{1}{1 - v_0} \right) > \frac{1}{e}, \quad v_0 := n_1 v < 1,$$

then every solution of Eq. (4.6) is oscillatory.

Proof. For Eq. (4.6), we have

$$\tilde{\Phi}_i(t) = \frac{p_i}{t^\alpha} - \Psi_i(t - c_i).$$

Since $\lim_{t\to\infty} t^{\alpha}\Psi_i(t-c_i) = 0$, we have asymptotically $\tilde{\Phi}_i(t) \sim \frac{p_i}{t^{\alpha}}$ for sufficiently large t. Observe that

$$L_i := \liminf_{t\to\infty} \tilde{\Phi}_i(t-j\kappa_m)(t-j\kappa_m-\tau_i) = \liminf_{t\to\infty} t\tilde{\Phi}_i(t) = \liminf_{t\to\infty} p_i t^{1-\alpha}.$$

Case (i): $\alpha \in (0,1)$. In this case, $1-\alpha > 0$, which implies

$$L_i = \lim_{t\to\infty} p_i t^{1-\alpha} = \infty.$$

Consequently, $\gamma_1 = \infty$, which strictly exceeds the threshold $1/e$. By Theorem 4.1(iii) for $n = 1$, every solution oscillates.

Case (ii): $\alpha = 1$. In this case, $1-\alpha = 0$, which yields $L_i = p_i$. Taking the lim inf of the sum in γ_n gives

$$\gamma_n \geq \sum_{i=1}^{N} p_i \sum_{j=0}^{n-1} v_0^j (j\kappa_m + \tau_i) = \sum_{i=1}^{N} p_i \left(\kappa_m \sum_{j=0}^{n-1} j v_0^j + \tau_i \sum_{j=0}^{n-1} v_0^j \right).$$

Since $v_0 < 1$, by direct computation, we obtain

$$\lim_{n\to\infty} \gamma_n \geq \sum_{i=1}^{N} p_i \left(\kappa_m \frac{v_0}{(1-v_0)^2} + \tau_i \frac{1}{1-v_0} \right) > \frac{1}{e}.$$

Thus, there exists a finite integer n^* such that $\gamma_{n^*} > 1/e$. By Theorem 4.1(iii), every solution is oscillatory. □

5 Illustrative examples

In this section, we present examples to illustrate the applicability of our results. In particular, we emphasize the effectiveness of the iterative method compared to standard techniques in establishing the oscillation of such equations.

Example 5.1. We consider the following mixed neutral differential equation with time-variable delays:

$$[x(t) - \Theta(t)x(t-\kappa(t))]'' + \Phi(t)x(t-\tau(t)) - \Psi(t)x(t-\sigma(t)) = 0, \quad t \geq 10, \tag{5.1}$$

where

$$\begin{aligned} \Theta(t) &= 0.5 + 0.1\sin t, & \kappa(t) &= 1.5 + 0.2\cos t, \\ \Phi(t) &= \frac{0.2}{t}, & \tau(t) &= 2.0 + 0.1\sin t, \\ \Psi(t) &= \frac{0.05}{t^2}, & \sigma(t) &= 1. \end{aligned}$$

First, we verify the hypotheses. Since $\sigma(t) = 1$ is a constant, condition (H1) is clearly satisfied with $\sigma'(t) = 0 < 1$, and $\tau(t) \geq 1.9 > 1 = \sigma(t)$. The delay function $c(t)$ defined in (2.1) takes the form:

$$c(t) = \tau(t) - \sigma(t - c(t)) = 2.0 + 0.1\sin t - 1 = 1.0 + 0.1\sin t.$$

We have $C_1 = \sup_{t\geq 10} c(t) = 1.1$ and $c'(t) = 0.1\cos t \leq 0.1 < 1$.

Next, we check the contraction bound (H2). Note that

$$\int_{10}^{\infty} \Psi(s)ds = \int_{10}^{\infty} \frac{0.05}{s^2}ds = 0.005,$$

hence, we have

$$\sup_{t\geq 10} \Theta(t) + C_1 \int_{10}^{\infty} \Psi(s)ds = 0.6 + 1.1(0.005) = 0.6055 \leq \theta_0 < 1.$$

Thus, hypothesis (H2) is satisfied.

By direct computation, we have

$$\lim_{t\to\infty} t\tilde{\Phi}(t) = \lim_{t\to\infty} t\left(\frac{0.2}{t} - \frac{0.05}{(t-c(t))^2}(1-0.1\cos t)\right) = 0.2,$$

and $\lim_{t\to\infty} \frac{t-\tau(t)}{t} = 1$. Thus,

$$\begin{aligned} L &= \liminf_{t\to\infty} \tilde{\Phi}(t)(t-\tau(t))\tau(t) \\ &= \liminf_{t\to\infty} \left[t\tilde{\Phi}(t)\right]\left[\frac{t-\tau(t)}{t}\right]\tau(t) \\ &= 0.2 \times 1 \times \liminf_{t\to\infty}(2.0 + 0.1\sin t) \\ &= 0.2 \times 1.9 = 0.38 > 1/e \approx 0.3678. \end{aligned}$$

Furthermore, we choose $I = \{1\}$. Then $\liminf_{t\to\infty} \tau(t) = 1.9 > 0$, and

$$\begin{aligned} \liminf_{t\to\infty} \int_{t-\tau(t)}^{t} \tilde{\Phi}(s)(s-\tau(s))\,ds &= \liminf_{t\to\infty} \int_{t-\tau(t)}^{t} \left[s\tilde{\Phi}(s)\right]\left[\frac{s-\tau(s)}{s}\right]ds \\ &= \liminf_{t\to\infty} \int_{t-\tau(t)}^{t} 0.2\times 1\,ds \\ &= 0.2\liminf_{t\to\infty} \tau(t) = 0.38 > 0. \end{aligned}$$

Therefore, all conditions of Corollary 3.2(ii) hold, and we conclude that every solution of Eq. (5.1) is oscillatory.

Example 5.2. Consider the following mixed neutral differential equation:

$$[x(t) - 0.5x(t-2)]'' + \frac{0.39(1.1+\sin \pi t)}{t}x(t-1) - \frac{0.02}{t^2}x(t-0.5) = 0, \quad t \geq 10. \tag{5.2}$$

We identify the parameters: $n_1 = 1$, $\Theta_1 = 0.5$, $\kappa_m = 2$, $\tau_1 = 1$, $\Phi_1(t) = 0.39(1.1+\sin \pi t)/t$, $\Psi_1(t) = 0.02/t^2$, and $\sigma_1 = 0.5$.

First, we have $c_1 = \tau_1 - \sigma_1 = 0.5$ and

$$\int_{10}^{\infty} \frac{0.02}{s^2}ds = 0.002.$$

It follows that

$$\Theta_1 + c_1 \int_{10}^{\infty} \Psi_1(s)ds = 0.5 + 0.5(0.002) = 0.501 \leq \theta_0 < 1.$$

Furthermore, since $(t-0.5)^2 = t(t-2) + t + 0.25 > t$ for all $t \geq 10$, we have

$$\tilde{\Phi}_1(t) = \Phi_1(t) - \Psi_1(t-c_1) \geq \frac{0.039}{t} - \frac{0.02}{(t-0.5)^2} > \frac{0.019}{t} > 0.$$

Thus, (H2) is satisfied.

Next, we verify condition (H3). We show that

$$\Theta_1\tilde{\Phi}_1(t) \geq v\tilde{\Phi}_1(t-2),$$

where $v = 0.4$. In fact, we will prove

$$\Delta(t) = \Theta_1\tilde{\Phi}_1(t) - 0.4\tilde{\Phi}_1(t-2) > 0, \quad \text{for } t \geq 10.$$

Since $\sin(\pi(t-2)) = \sin(\pi t)$, by direct computation, we have

$$\Delta(t) = \frac{0.39(1.1+\sin\pi t)(0.1t-1)}{t(t-2)} + \frac{0.008}{(t-2.5)^2} - \frac{0.01}{(t-0.5)^2}.$$

To prove $\Delta(t) \geq 0$ for all $t \geq 10$, we estimate the terms by considering two sub-intervals. For $t \in [10, 19]$, we have $0.1t - 1 \geq 0$. This implies

$$\Delta(t) \geq \frac{0.008}{(t-2.5)^2} - \frac{0.01}{(t-0.5)^2} = \frac{0.002(t-10)(19-t) + (0.3195 - 0.016t)}{(t-2.5)^2(t-0.5)^2} > 0.$$

For $t > 19$, we have $1.1 + \sin\pi t \geq 0.1$, $0.1t - 1 > 0.9$, and $t(t-2) < (t-0.5)^2$. This implies, by dropping the positive term $\frac{0.008}{(t-2.5)^2}$, that

$$\Delta(t) \geq \frac{0.39(0.1)(0.9)}{(t-0.5)^2} - \frac{0.01}{(t-0.5)^2} = \frac{0.0251}{(t-0.5)^2} > 0.$$

Thus, the inequality $\Theta_1\tilde{\Phi}_1(t) \geq v\tilde{\Phi}_1(t-2)$, where $v = 0.4$, holds for all $t \geq 10$. Consequently,

$$v_0 = n_1 v = 1 \times 0.4 = 0.4 < 1.$$

We now investigate the oscillation using the sequence β_n defined in Eq. (4.2). We have $N = 1$ and $\Delta_{\min} = \tau_1 = 1$. To calculate the sequence β_n in Eq. (4.2), let us define

$$J_{t,j} = \int_{t-1}^{t} \tilde{\Phi}_1(s-2j)(s-2j-1)\,ds.$$

We have

$$J_{t,j} = \int_{t-1}^{t} \tilde{\Phi}_1(s-2j)(s-2j)\,ds - \int_{t-1}^{t} \tilde{\Phi}_1(s-2j)\,ds. \tag{5.3}$$

To estimate these integrals, we note that if $f(t) \geq 0$ for all $t \geq t_0$ and $\lim_{t\to\infty} f(t) = 0$, then

$$\lim_{t\to\infty} \int_{t-1}^{t} f(s)\,ds = 0. \tag{5.4}$$

First, we show that the second integral tends to zero as $t \to \infty$. Since

$$\lim_{t\to\infty} \tilde{\Phi}_1(t-2j) = \lim_{t\to\infty} \left(\frac{0.39(1.1+\sin\pi(t-2j))}{t-2j} - \frac{0.02}{(t-2j-0.5)^2} \right) = 0,$$

we deduce from (5.4) that

$$\lim_{t\to\infty} \int_{t-1}^{t} \tilde{\Phi}_1(s-2j)\,ds = 0 \tag{5.5}$$

for any fixed j.

Next, we evaluate the first integral. We have

$$\begin{aligned}
&\int_{t-1}^{t} \tilde{\Phi}_1(s-2j)(s-2j)\,ds \\
&\quad= \int_{t-1}^{t} \left[\frac{0.39(1.1+\sin\pi(s-2j))}{s-2j} - \frac{0.02}{(s-2j-0.5)^2}\right](s-2j)\,ds \\
&\quad= \int_{t-1}^{t} 0.39(1.1+\sin\pi(s-2j))\,ds - \int_{t-1}^{t} \frac{0.02(s-2j)}{(s-2j-0.5)^2}\,ds.
\end{aligned}$$

Using (5.4), we have

$$\lim_{t\to\infty} \int_{t-1}^{t} \frac{0.02(s-2j)}{(s-2j-0.5)^2}\,ds = 0.$$

It follows that

$$\limsup_{t\to\infty} \int_{t-1}^{t} \tilde{\Phi}_1(s-2j)(s-2j)\,ds = \limsup_{t\to\infty} \int_{t-1}^{t} 0.39(1.1+\sin\pi(s-2j))\,ds. \tag{5.6}$$

Since $\sin(\pi(s-2j)) = \sin(\pi s)$, combining (5.3), (5.5), and (5.6), we obtain

$$\begin{aligned}
\limsup_{t\to\infty} J_{t,j} &= \limsup_{t\to\infty} \int_{t-1}^{t} 0.39(1.1+\sin\pi s)\,ds \\
&= \limsup_{t\to\infty} 0.39\left[1.1s - \frac{1}{\pi}\cos\pi s\right]_{t-1}^{t} \\
&= \limsup_{t\to\infty} 0.39\left(1.1 - \frac{2}{\pi}\cos\pi t\right) \\
&= 0.39(1.1+2/\pi) \approx 0.677 =: I_0.
\end{aligned}$$

Since the principal periodic parts of $J_{t,j}$ are the same for all j, we deduce from Eq. (4.2) that

$$\beta_n = \limsup_{t\to\infty} \sum_{j=0}^{n-1} v_0^j J_{t,j} = \left(\sum_{j=0}^{n-1} v_0^j\right) \limsup_{t\to\infty} J_{t,0} = I_0 \sum_{j=0}^{n-1} v_0^j.$$

We now evaluate the sequence β_n against the required threshold of 1:

For $n = 1$ (the classical criterion without iteration):

$$\beta_1 = I_0 \approx 0.677 < 1.$$

Thus, the classical criterion fails to detect oscillation.

For $n = 2$ (one iteration step):

$$\beta_2 = I_0(1+v_0) \approx 0.677 \times 1.4 = 0.948 < 1.$$

While the value increases, it still falls short of the threshold.

For $n = 3$ (two iteration steps):

$$\beta_3 = I_0(1+v_0+v_0^2) = 0.677 \times 1.56 \approx 1.056 > 1.$$

At the third step, the sequence completely exceeds the threshold. By Theorem 4.1(ii), we conclude that every solution of Eq. (5.2) is oscillatory.

6 Conclusion

This paper establishes refined oscillation criteria for second-order mixed neutral differential equations. Constructing an appropriate bridge function resolves the analytical challenges of negative coefficients, reducing the problem to a first-order delay equation. Furthermore, we introduce a finite-sum iterative technique. This operator generates an increasing sequence of oscillation bounds. As illustrated by Example 5.2, this approach successfully captures oscillatory behavior that single-step classical methods cannot detect.

Future studies could extend this iterative framework to mixed neutral equations incorporating delayed damping terms. Inspired by recent developments [24], combining an appropriate integrating factor to absorb the damping derivative with our iterative technique offers a promising tool to study the complex interaction between damping forces and mixed neutral feedbacks.

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