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Ground state sign-changing solutions for a class of critical Kirchhoff-type equations with steep potential well

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Abstract. In this paper, we investigate the existence of ground state sign-changing solutions for Kirchhoff-type equation with critical exponent

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda V(x)u=|u|^4u+\mu f(u),\quad x\in\mathbb{R}^3,$$

where $a,b,\mu,\lambda>0$ are positive parameters. Under suitable conditions on f and V, by working on a sign-changing Nehari manifold and combining minimax arguments with deformation techniques, we establish the existence of least energy sign-changing solutions. At the same time, we also study the asymptotic behavior of sign-changing solutions as $\lambda\to\infty$.

Keywords: Kirchhoff-type equation, critical exponent, steep potential well, Nehari method, sign-changing solutions.

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1 Introduction and main results

In this work, we consider the existence of ground state sign-changing solutions for the following critical Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda V(x)u=|u|^4u+\mu f(u),\quad x\in\mathbb{R}^3,\tag{1.1}$$

where $a,b,\mu,\lambda>0$ are positive parameters, the potential V satisfy the following assumptions:

(V_1) $V\in C(\mathbb{R}^3,\mathbb{R})$ and $V(x)\geq 0$ in $\mathbb{R}^3$;

(V_2) there is $V_0>0$ such that the set $\{x\in\mathbb{R}^3: V(x)\leq V_0\}$ is nonempty and has finite measure;

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(V_3) $\Omega := \text{int}\, V^{-1}(0)$ is nonempty and has smooth boundary, $\bar{\Omega} = V^{-1}(0)$.

The above conditions of V were firstly proposed by Bartsch and Wang in [3] to study the nonlinear Schrödinger equation. They pointed out that $\lambda V(x)$ is a potential well whose depth is controlled by λ. If λ is sufficiently large, then $\lambda V(x)$ is called the steep potential well. Moreover, on the nonlinearity term f, we assume that

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$ and $f(t)t > 0$ for all $t \in \mathbb{R} \setminus \{0\}$;

(f_2) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^5} = 0$;

(f_3) $\frac{f(t)}{|t|^3}$ is an increasing function of $t \in \mathbb{R} \setminus \{0\}$;

(f_4) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^4} = +\infty$, where $F(t) = \int_0^t f(s)ds \geq 0$.

Problem (1.1) is related to the stationary analogue of the equation

$$u_{tt} - (a + b \int_\Omega |\nabla u|^2 dx)\Delta u = f(x, u),$$

which was introduced by Kirchhoff [13] as a generalization of the well-known D'Alembert wave equations for free vibration of elastic strings. For more mathematical and physical background on Kirchhoff-type equation, we refer the reader to [1,2] and the references therein.

In recent years, there has been a great deal of interests in studying the existence of positive solutions, multiple solutions and ground state solutions for Kirchhoff problem, for examples [14,16,19,20,29] and the references therein. Moreover, for more results about sign-changing solutions for Kirchhoff-type equation with subcritical growth, see [8,17,23,28,30,31]. As far as we know, there are few articles on the existence of sign-changing solutions for Kirchhoff-type equation with critical growth, such as [9,25,26,34]. More specifically, Wang [26] firstly obtained the least energy sign-changing solutions of the following critical Kirchhoff equation in a bounded domain

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = |u|^4 u + \lambda f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$, λ, a, $b > 0$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfying:

(f_5) there exist $q \in (4, 6)$ and $C > 0$ such that $|f(x, t)| \leq C(1 + |t|^{q-1})$, for almost every $x \in \Omega$ and all $t \in \mathbb{R}$;

(f_6) $\lim_{t\to 0} \frac{f(x,t)}{|t|^3} = 0$, uniformly for almost every $x \in \Omega$;

(f_7) $\frac{f(x,t)}{|t|^3}$ is a strictly increasing function of $t \in \mathbb{R}\setminus\{0\}$, for almost every $x \in \Omega$.

Using the constraint variational methods and the degree theory, he got the existence of the least energy sign-changing solutions. Gao et al. [9] proved the existence of sign-changing solutions by truncation methods for the following equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(u) + \lambda |u|^{p-2}u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

where V is coercive potential, $p \geq 6$, $\lambda \in (0, \lambda^*)$ for some $\lambda^* > 0$. When $V(x) = \lambda = 1$, $p = 6$, f satisfies (f_1)–(f_3) and the following condition:

(f_4') $\lim_{|t|\to\infty} \frac{F(t)}{|t|^5} = +\infty$.

Zhao et al. [34] also obtained the existence of the least energy sign-changing solutions of problem (1.2) by the parameter perturbation technique. Recently, Tian et al. [25] studied the following critical Kirchhoff equation with steep potential well

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda V(x)u=|u|^4u+|u|^{p-1}u,\quad x\in\mathbb{R}^3,$$

where V satisfies (V_1)–(V_3) and $p\in(4,5)$. By adopting a sign-changing Nehari manifold and perturbation technique, they obtained that the existence and asymptotic behavior of ground state sign-changing solutions for $\lambda\geq 1$ large enough and $b>0$ small.

Inspired by the above results, in this paper, we study the existence of ground state sign-changing solutions of problem (1.1) with steep potential well and more general f. Now our main results in this paper can be stated as follows.

Theorem 1.1. *Assume that (V_1)–(V_3), (f_1)–(f_4) hold, then there exists $\mu^*>0$ and $\Lambda>1$ such that problem (1.1) has a least energy sign-changing solution u_λ for all $\mu>\mu^*$ and $\lambda>\Lambda$, which has precisely two nodal domains, and its energy is strictly larger than twice that of the positive ground state solution.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, for any sequence $\lambda_n\subset[\Lambda,+\infty)$ with $\lambda_n\to+\infty$ as $n\to\infty$, the sequence of sign-changing solutions for problem (1.1) strongly converges to u_* in $H^1(\mathbb{R}^3)$ up to a subsequence, where $u_*\in H_0^1(\Omega)$ is a ground state sign-changing solution of the following problem*

$$\begin{cases}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=|u|^4u+\mu f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}\tag{1.3}$$

Remark 1.3. To our best knowledge, our results are up to date. On the one hand, we generalize [28] to the critical case. On the other hand, compared with [26] and [34], we study the steep potential well and more general f. Moreover, when $f(u)=|u|^{p-2}u$ with $p\in(4,6)$, our results also hold.

Moreover, we replace the condition (f_4) with a stronger (f_4'), and obtain the following conclusions.

Theorem 1.4. *Assume that (V_1)–(V_3), (f_1)–(f_3) and (f_4') hold, then there exists $b_0>0$ and $\Lambda>1$ such that problem (1.1) has a least energy sign-changing solution u_λ for $0<b<b_0$, $\lambda>\Lambda$ and $\mu>0$, and u_λ has the same properties as in Theorem 1.1.*

Theorem 1.5. *Under the assumption of Theorem 1.4, for any sequence $\lambda_n\subset[\Lambda,+\infty)$ with $\lambda_n\to+\infty$ as $n\to\infty$, the sequence of sign-changing solutions for problem (1.1) strongly converges to u_{**} in $H^1(\mathbb{R}^3)$ up to a subsequence, where $u_{**}\in H_0^1(\Omega)$ changing sign exactly once is a ground state sign-changing solution of problem (1.3).*

Remark 1.6. When $f(u)=|u|^{p-2}u$ with $p\in(5,6)$, our results also hold. Consequently, we extend the results of [25] to the general case of f.

2 Preliminaries

In this section, we make use of the following notations.

- Let $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ be the Sobolev space endowed with the norm
$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$
- $H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ is a Hilbert space endowed with the scalar product and norm
$$\|u\|_{H^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx\right)^{\frac{1}{2}}.$$
- Let
$$E := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty\right\},$$
be equipped with the inner product and the norm
$$(u,v) = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + V(x)uv)dx, \qquad \|u\| = (u,u)^{\frac{1}{2}}.$$
- For any $\lambda > 0$, we introduce the following work space
$$E_\lambda := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \lambda V(x)u^2 dx < \infty\right\},$$
with scalar product and norm given by
$$(u,v)_\lambda = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + \lambda V(x)uv)dx, \qquad \|u\|_\lambda = (u,u)_\lambda^{\frac{1}{2}}.$$

It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Then, for any $q \in [2,6]$, the embedding $E_\lambda \hookrightarrow L^q(\mathbb{R}^3)$ is continuous and there exists $S_q > 0$ such that
$$|u|_q \le S_q\|u\| \le S_q\|u\|_\lambda. \tag{2.1}$$
Moreover, the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous and there exists the best Sobolev constant given by
$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{\frac{1}{3}}},$$
is achieved by the Talenti function
$$u_\varepsilon = \frac{(3\varepsilon)^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}} \in D^{1,2}(\mathbb{R}^3). \tag{2.2}$$

Next, we define the energy functional of problem (1.1) by $I_\lambda^\mu : E_\lambda \to \mathbb{R}$
$$I_\lambda^\mu(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx - \mu\int_{\mathbb{R}^3} F(u)dx.$$

The functional I^μ_λ is well defined for every $u \in E_\lambda$ and $I^\mu_\lambda \in C^1(E_\lambda, \mathbb{R})$. For any $u, v \in E_\lambda$, we get that

$$\langle (I^\mu_\lambda)'(u), v\rangle = (u,v)_\lambda + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx - \int_{\mathbb{R}^3} |u|^4 uv dx - \mu \int_{\mathbb{R}^3} f(u) v dx. \quad (2.3)$$

By simple calculation, one obtains

$$I^\mu_\lambda(u) = I^\mu_\lambda(u^+) + I^\mu_\lambda(u^-) + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx, \quad (2.4)$$

$$\langle (I^\mu_\lambda)'(u), u^\pm\rangle = \langle (I^\mu_\lambda)'(u^\pm), u^\pm\rangle + b\int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx. \quad (2.5)$$

Clearly, the critical points of I^μ_λ are weak solutions of problem (1.1). Moreover, if $u \in E_\lambda$ is a solution of problem (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution, where

$$u^+ := \max\{u, 0\}, \quad u^- := \min\{u, 0\}.$$

To prove the existence of least energy sign-changing solutions of problem (1.1), we consider the following minimization problem

$$m^\mu_\lambda := \inf_{u \in \mathcal{M}^\mu_\lambda} I^\mu_\lambda(u),$$

where

$$\mathcal{M}^\mu_\lambda = \{u \in E_\lambda : u^\pm \neq 0,\ \langle (I^\mu_\lambda)'(u), u^\pm\rangle = 0\}.$$

On the other hand, we seek a ground state solution on the following Nehari manifold

$$\mathcal{N}^\mu_\lambda = \{u \in E_\lambda \setminus \{0\} : \langle (I^\mu_\lambda)'(u), u\rangle = 0\},$$

and let $c^\mu_\lambda := \inf_{u \in \mathcal{N}^\mu_\lambda} I^\mu_\lambda(u)$.

Now, we will focus on proving $\mathcal{M}^\mu_\lambda \neq \emptyset$ and giving some properties of $\mathcal{M}^\mu_\lambda$.

Lemma 2.1. *Suppose that (V_1) and (f_1)–(f_3) hold. For any $\lambda > 0$ and $u \in E_\lambda$ with $u^\pm \neq 0$, then*

(i) there exists a unique pair of positive numbers (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}^\mu_\lambda$ and $I^\mu_\lambda(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} I^\mu_\lambda(su^+ + tu^-)$;

(ii) if $\langle (I^\mu_\lambda)'(u), u^\pm\rangle \leq 0$, there exists a pair of $(s_u, t_u) \in (0,1] \times (0,1]$ such that $s_u u^+ + t_u u^- \in \mathcal{M}^\mu_\lambda$.

Proof. (i) First, we prove the existence of s_u and t_u. Let

$$\begin{aligned} g_1(s,t) &= \langle (I^\mu_\lambda)'(su^+ + tu^-), su^+\rangle \\ &= s^2\|u^+\|^2_\lambda + b\int_{\mathbb{R}^3} |\nabla(su^+ + tu^-)|^2 dx \int_{\mathbb{R}^3} |\nabla su^+|^2 dx \\ &\quad - s^6 \int_{\mathbb{R}^3} |u^+|^6 dx - \mu \int_{\mathbb{R}^3} f(su^+) su^+ dx, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} g_2(s,t) &= \langle (I^\mu_\lambda)'(su^+ + tu^-), tu^-\rangle \\ &= t^2\|u^-\|^2_\lambda + b\int_{\mathbb{R}^3} |\nabla(su^+ + tu^-)|^2 dx \int_{\mathbb{R}^3} |\nabla tu^-|^2 dx \\ &\quad - t^6 \int_{\mathbb{R}^3} |u^-|^6 dx - \mu \int_{\mathbb{R}^3} f(tu^-) tu^- dx. \end{aligned} \quad (2.7)$$

By (f_3), we have

$$f(t)t - 4F(t) \geq 0, \tag{2.8}$$

is an increasing function as $t > 0$. From (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^5, \tag{2.9}$$

for any $t \in \mathbb{R}$. Obviously, we can see that $g_1(s,s) > 0$, $g_2(s,s) > 0$ for $s > 0$ sufficiently small and $g_1(t,t) < 0$, $g_2(t,t) < 0$ for $t > 0$ sufficiently large. Then, there exist $0 < \eta_1 < \eta_2$ such that

$$g_1(\eta_1,\eta_1) > 0, \quad g_2(\eta_1,\eta_1) > 0, \quad g_1(\eta_2,\eta_2) < 0, \quad g_2(\eta_2,\eta_2) < 0. \tag{2.10}$$

Thus, we can deduce that

$$g_1(\eta_1,t) > 0, \quad g_1(\eta_2,t) < 0, \quad \forall t \in [\eta_1,\eta_2]; \tag{2.11}$$

$$g_2(s,\eta_1) > 0, \quad g_2(s,\eta_2) < 0, \quad \forall s \in [\eta_1,\eta_2]. \tag{2.12}$$

Then, using Miranda's theorem [21], there exists $(s_u,t_u) \in (0,\infty) \times (0,\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$.

Next, we prove the uniqueness of (s_u,t_u) and consider the following cases.

Case 1: $u \in \mathcal{M}_\lambda^\mu$.

For any $u \in \mathcal{M}_\lambda^\mu$, it means that

$$\|u^\pm\|_\lambda^2 + b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^\pm|^2dx = \int_{\mathbb{R}^3}|u^\pm|^6dx + \mu\int_{\mathbb{R}^3}f(u^\pm)u^\pm dx. \tag{2.13}$$

We will show that the pair $(s_u,t_u) = (1,1)$ is the unique one which such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$. Let (s_0,t_0) be a pair of numbers such that $s_0u^+ + t_0u^- \in \mathcal{M}_\lambda^\mu$ with $0 < s_0 \leq t_0$, then one has

$$\begin{aligned} &s_0^2\|u^+\|_\lambda^2 + b\int_{\mathbb{R}^3}|\nabla(s_0u^+ + t_0u^-)|^2dx\int_{\mathbb{R}^3}|\nabla s_0u^+|^2dx \\ &\quad = s_0^6\int_{\mathbb{R}^3}|u^+|^6dx + \mu\int_{\mathbb{R}^3}f(s_0u^+)s_0u^+dx \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &t_0^2\|u^-\|_\lambda^2 + b\int_{\mathbb{R}^3}|\nabla(s_0u^+ + t_0u^-)|^2dx\int_{\mathbb{R}^3}|\nabla t_0u^-|^2dx \\ &\quad = t_0^6\int_{\mathbb{R}^3}|u^-|^6dx + \mu\int_{\mathbb{R}^3}f(t_0u^-)t_0u^-dx. \end{aligned} \tag{2.15}$$

By $0 < s_0 \leq t_0$ and (2.15), we deduce

$$\frac{\|u^-\|_\lambda^2}{t_0^2} + b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx \geq t_0^2\int_{\mathbb{R}^3}|u^-|^6dx + \mu\int_{\mathbb{R}^3}\frac{f(t_0u^-)}{(t_0u^-)^3}(u^-)^4dx. \tag{2.16}$$

Combining (2.13) and (2.16), one obtains

$$\left(\frac{1}{t_0^2} - 1\right)\|u^-\|_\lambda^2 \geq (t_0^2 - 1)\int_{\mathbb{R}^3}|u^-|^6dx + \mu\int_{\mathbb{R}^3}\left[\frac{f(t_0u^-)}{(t_0u^-)^3} - \frac{f(u^-)}{(u^-)^3}\right](u^-)^4dx.$$

Under the assumption (f_3), we get $t_0 \leq 1$. Similarly, we can easily get $s_0 \geq 1$. Therefore, we have $s_0 = t_0 = 1$.

Case 2: $u \notin \mathcal{M}_\lambda^\mu$.

If $u \notin \mathcal{M}_\lambda^\mu$, we suppose that there exist (s_1, t_1) and (s_2, t_2) such that

$$u_1 = s_1 u^+ + t_1 u^- \in \mathcal{M}_\lambda^\mu, \quad u_2 = s_2 u^+ + t_2 u^- \in \mathcal{M}_\lambda^\mu.$$

Then, it is easy to see that

$$u_2 = \left(\frac{s_2}{s_1}\right) s_1 u^+ + \left(\frac{t_2}{t_1}\right) t_1 u^- = \left(\frac{s_2}{s_1}\right) u_1^+ + \left(\frac{t_2}{t_1}\right) u_1^- \in \mathcal{M}_\lambda^\mu.$$

If the first case holds and $u_1 \in \mathcal{M}_\lambda^\mu$, then $s_1 = s_2$ and $t_1 = t_2$. Thus, we complete the proof for the uniqueness of positive numbers (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$.

Next, we define $\psi_u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\psi_u(s,t) := I_\lambda^\mu(su^+ + tu^-)$. We show that the critical point (s_u, t_u) of ψ_u is its unique maximum point on $[0,\infty) \times [0,\infty)$. Since

$$\begin{aligned} \psi_u(s,t) &= I_\lambda^\mu(su^+ + tu^-) \\ &= \frac{s^2}{2}\|u^+\|_\lambda^2 + \frac{t^2}{2}\|u^-\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(su^+ + tu^-)|^2 dx\right)^2 \\ &\quad - \frac{s^6}{6}\int_{\mathbb{R}^3} |u^+|^6 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} |u^-|^6 dx - \mu \int_{\mathbb{R}^3} F(su^+ + tu^-) dx. \end{aligned} \tag{2.17}$$

It is easy to see that $I_\lambda^\mu(su^+ + tu^-) > 0$ as $|(s,t)| \to 0$ and $I_\lambda^\mu(su^+ + tu^-) \to -\infty$ as $|(s,t)| \to \infty$, which means that the maximum point cannot be achieved on the boundary of $\mathbb{R}^+ \times \mathbb{R}^+$. Without loss of generality, we only show that $I_\lambda^\mu(t_u u^-) < I_\lambda^\mu(s_u u^+ + t_u u^-)$. Clearly, $I_\lambda^\mu(su^+ + tu^-)$ is an increasing function with respect to s if $s > 0$ is small enough, which means that $I_\lambda^\mu(t_u u^-) < I_\lambda^\mu(s_u u^+ + t_u u^-)$. Thus, $I_\lambda^\mu(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} I_\lambda^\mu(su^+ + tu^-)$. We complete the proof of (i).

(ii) Finally, we prove that if $\langle (I_\lambda^\mu)'(u), u^\pm \rangle \leq 0$, then $0 < s_u, t_u \leq 1$. Since $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$, without loss of generality, we may assume that $0 < t_u \leq s_u$. Consequently, we obtain

$$s_u^2\|u^+\|_\lambda^2 + bs_u^4 \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \geq s_u^6 \int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} f(s_u u^+) s_u u^+ dx. \tag{2.18}$$

Since $\langle (I_\lambda^\mu)'(u), u^\pm \rangle \leq 0$, one has

$$\|u^+\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \leq \int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} f(u^+) u^+ dx. \tag{2.19}$$

Combining (2.18) with (2.19), we get

$$\left(1 - \frac{1}{s_u^2}\right)\|u^+\|_\lambda^2 \leq (1 - s_u^2)\int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} \left[\frac{f(u^+)}{(u^+)^3} - \frac{f(s_u u^+)}{(s_u u^+)^3}\right](u^+)^4 dx.$$

Thanks to (f_3), one gets $s_u \leq 1$. Then, $0 < s_u, t_u \leq 1$. Thus, the proof of Lemma 2.1 is completed. □

Lemma 2.2. *Assume that (V_1) and (f_1)–(f_3) hold. For any $\lambda \geq 1$ and $u \in E_\lambda$ with $u^\pm \neq 0$, (s_u, t_u) is obtained by Lemma 2.1, then*

(i) s_u, t_u are continuous functionals in E_λ with respect to u;

(ii) if $u_n^+ \to 0$ as $n \to \infty$, then $s_{u_n} \to \infty$, if $u_n^- \to 0$ as $n \to \infty$, then $t_{u_n} \to \infty$;

(iii) if $\{u_n\} \subset \mathcal{M}_\lambda^\mu$, $\lim_{n\to\infty} I_\lambda^\mu(u_n) = m_\lambda^\mu$, then $m_\lambda^\mu > 0$ and $C_1 \leq \|u_n^\pm\|_\lambda \leq C_2$ for some $C_1, C_2 > 0$.

Proof. (i) Take a sequence $\{u_n\} \subset E_\lambda$ such that $u_n \to u$ in E_λ, then $u_n^\pm \to u^\pm$ in E_λ. By Lemma 2.1, there exist (s_{u_n}, t_{u_n}) and (s_u, t_u) such that $s_{u_n}u_n^+ + t_{u_n}u_n^-$, $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$. Then, we obtain

$$\begin{aligned} &s_{u_n}^2\|u_n^+\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla(s_{u_n}u_n^+ + t_{u_n}u_n^-)|^2\,dx \int_{\mathbb{R}^3} |\nabla s_{u_n}u_n^+|^2\,dx \\ &\quad = s_{u_n}^6\int_{\mathbb{R}^3} |u_n^+|^6\,dx + \mu\int_{\mathbb{R}^3} f(s_{u_n}u_n^+)s_{u_n}u_n^+\,dx, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &t_{u_n}^2\|u_n^-\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla(s_{u_n}u_n^+ + t_{u_n}u_n^-)|^2\,dx \int_{\mathbb{R}^3} |\nabla t_{u_n}u_n^-|^2\,dx \\ &\quad = t_{u_n}^6\int_{\mathbb{R}^3} |u_n^-|^6\,dx + \mu\int_{\mathbb{R}^3} f(t_{u_n}u_n^-)t_{u_n}u_n^-\,dx. \end{aligned} \tag{2.21}$$

We claim that $\{s_{u_n}\}$ and $\{t_{u_n}\}$ are bounded in $\mathbb{R}^+$. Otherwise, if $s_{u_n} \to +\infty$ as $n \to \infty$, it follows from (2.20) that

$$\frac{t_{u_n}}{s_{u_n}} \to +\infty,$$

and we get $t_{u_n} \to +\infty$ as $n \to \infty$. Similarly, by (2.21), one also gets

$$\frac{s_{u_n}}{t_{u_n}} \to +\infty,$$

which leads to a contradiction.

Thus, passing to a subsequence if necessary, still denoted by $\{s_{u_n}\}$, $\{t_{u_n}\}$, we assume that there is a pair of non-negative numbers (s_0, t_0) such that

$$\lim_{n\to\infty} s_{u_n} = s_0, \quad \lim_{n\to\infty} t_{u_n} = t_0.$$

Passing to the limit in (2.20) and (2.21), we obtain

$$\begin{aligned} &s_0^2\|u^+\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla(s_0u^+ + t_0u^-)|^2\,dx \int_{\mathbb{R}^3} |\nabla s_0u^+|^2\,dx \\ &\quad = s_0^6\int_{\mathbb{R}^3} |u^+|^6\,dx + \mu\int_{\mathbb{R}^3} f(s_0u^+)s_0u^+\,dx, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &t_0^2\|u^-\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla(s_0u^+ + t_0u^-)|^2\,dx \int_{\mathbb{R}^3} |\nabla t_0u^-|^2\,dx \\ &\quad = t_0^6\int_{\mathbb{R}^3} |u^-|^6\,dx + \mu\int_{\mathbb{R}^3} f(t_0u^-)t_0u^-\,dx, \end{aligned} \tag{2.23}$$

which implies that $s_0u^+ + t_0u^- \in \mathcal{M}_\lambda^\mu$. By Lemma 2.1, since the uniqueness of (s_u, t_u), so we have $(s_u, t_u) = (s_0, t_0)$ and complete the proof of (i).

(ii) We only prove that $s_{u_n} \to +\infty$ if $u_n^+ \to 0$ as $n \to \infty$; similarly, we can prove the other case. It follows from (2.20) that

$$\begin{aligned} h_1(u_n) &:= \|u_n^+\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla(s_{u_n}u_n^+ + t_{u_n}u_n^-)|^2dx \int_{\mathbb{R}^3} |\nabla u_n^+|^2dx - \mu\int_{\mathbb{R}^3} f(s_{u_n}u_n^+)\frac{u_n^+}{s_{u_n}}dx \\ &= s_{u_n}^4\int_{\mathbb{R}^3} |u_n^+|^6dx =: h_2(u_n). \end{aligned}$$

By contradiction. Suppose that there exists a positive constant C such that $s_{u_n} \le C$, then by the Sobolev inequality (2.1) and (2.9), choosing $1-\mu\varepsilon S_2^2 > 0$, we have

$$\begin{aligned} h_1(u_n) - h_2(u_n) &\ge \|u_n^+\|_\lambda^2 - \frac{\mu}{s_{u_n}} \int_{\mathbb{R}^3} f(s_{u_n} u_n^+) u_n^+ dx - s_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 dx \\ &\ge \|u_n^+\|_\lambda^2 - s_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 dx - \mu\varepsilon \int_{\mathbb{R}^3} |u_n^+|^2 dx - \mu C_\varepsilon s_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 dx \\ &\ge (1-\mu\varepsilon S_2^2)\|u_n^+\|_\lambda^2 - C\|u_n^+\|_\lambda^6 > 0, \end{aligned}$$

which contradicts the fact $s_{u_n} u_n^+ + t_{u_n} u_n^- \in \mathcal{M}_\lambda^\mu$ for $\|u_n^+\|_\lambda$ small enough. Thus, $s_{u_n} \to \infty$ if $u_n^+ \to 0$ as $n \to \infty$.

(iii) Because $\{u_n\} \subset \mathcal{M}_\lambda^\mu$, one derives

$$\|u_n^\pm\|_\lambda^2 + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} |\nabla u_n^\pm|^2 dx = \mu \int_{\mathbb{R}^3} f(u_n^\pm) u_n^\pm dx + \int_{\mathbb{R}^3} |u_n^\pm|^6 dx.$$

By (2.9) and the Sobolev inequality, one has

$$\|u_n^\pm\|_\lambda^2 \le \mu \int_{\mathbb{R}^3} f(u_n^\pm) u_n^\pm dx + \int_{\mathbb{R}^3} |u_n^\pm|^6 dx \le \mu\varepsilon S_2^2 \|u_n^\pm\|_\lambda^2 + C\|u_n^\pm\|_\lambda^6. \tag{2.24}$$

Choosing $1-\mu\varepsilon S_2^2 > 0$, we get $\|u_n^\pm\|_\lambda \ge C_1$ for some $C_1 > 0$. Furthermore, it follows from $\{u_n\} \subset \mathcal{M}_\lambda^\mu \subset \mathcal{N}_\lambda^\mu$ that $\langle (I_\lambda^\mu)'(u_n), u_n\rangle \to 0$. By (2.8), we have

$$\begin{aligned} m_\lambda^\mu + o(1) &= I_\lambda^\mu(u_n) - \frac{1}{4}\langle (I_\lambda^\mu)'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_\lambda^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u_n)u_n - 4F(u_n)] dx \\ &\ge \frac{1}{4}\|u_n\|_\lambda^2 \ge \frac{1}{4}\|u_n^\pm\|_\lambda^2 > 0. \end{aligned} \tag{2.25}$$

This means that $m_\lambda^\mu > 0$ and $\{u_n\}$ is bounded in E_λ. By (2.24) and (2.25), we know that there exist positive constants C_1, C_2 such that $C_1 \le \|u_n^\pm\|_\lambda \le C_2$. This completes the proof of Lemma 2.2. □

3 Proof of Theorem 1.1

Before proving Theorem 1.1, we give some important lemmas.

Lemma 3.1. *Assume that* (V_1)–(V_3), (f_1)–(f_4) *hold. For any* $\lambda \ge 1$, *if* $m_\lambda^\mu = \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$, *then* $\lim_{\mu\to\infty} m_\lambda^\mu = 0$.

Proof. First, we shall prove that $m_\lambda^\mu = \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$ is well defined. By (iii) of Lemma 2.2, one has $\|u^\pm\|_\lambda \ge C_1 > 0$. Thanks to (2.8), we get

$$\begin{aligned} I_\lambda^\mu(u) &= I_\lambda^\mu(u) - \frac{1}{4}\langle (I_\lambda^\mu)'(u), u\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u)u - 4F(u)] dx + \frac{1}{12}\int_{\mathbb{R}^3} |u|^6 dx \\ &\ge \frac{1}{4}\|u\|_\lambda^2. \end{aligned}$$

Thus, we conclude that $I_\lambda^\mu(u) > 0$, that is, $m_\lambda^\mu = \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$ is well defined.

Next, we claim that $\lim_{\mu\to\infty} m_\lambda^\mu = 0$. According to Lemma 2.1, for any $\mu > 0$, there exists a pair of (s_μ, t_μ) such that $s_\mu u^+ + t_\mu u^- \in \mathcal{M}_\lambda^\mu$. Then, by (f_2), we have that

$$\begin{aligned}
0 \le m_\lambda^\mu &\le I_\lambda^\mu(s_\mu u^+ + t_\mu u^-)\\
&\le \frac{1}{2}\|s_\mu u^+ + t_\mu u^-\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(s_\mu u^+ + t_\mu u^-)|^2 dx\right)^2\\
&\le s_\mu^2\|u^+\|_\lambda^2 + t_\mu^2\|u^-\|_\lambda^2 + 2bs_\mu^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 + 2bt_\mu^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2.
\end{aligned}$$

To our end, we just prove that $s_\mu \to 0$ and $t_\mu \to 0$ as $\mu \to \infty$. Let

$$H_u = \{(s_\mu, t_\mu) \in [0,\infty) \times [0,\infty) : (g_1(s_\mu, t_\mu), g_2(s_\mu, t_\mu)) = (0,0), \mu > 0\},$$

where g_1, g_2 are defined by Lemma 2.1. By (f_1) and (f_4), one gets

$$\begin{aligned}
s_\mu^6 \int_{\mathbb{R}^3} |u^+|^6 dx + t_\mu^6 \int_{\mathbb{R}^3} |u^-|^6 dx &\le s_\mu^6 \int_{\mathbb{R}^3} |u^+|^6 dx + t_\mu^6 \int_{\mathbb{R}^3} |u^-|^6 dx\\
&\quad + \mu \int_{\mathbb{R}^3} f(s_\mu u^+) s_\mu u^+ dx + \mu \int_{\mathbb{R}^3} f(t_\mu u^-) t_\mu u^- dx\\
&= \|s_\mu u^+ + t_\mu u^-\|_\lambda^2 + b\left(\int_{\mathbb{R}^3} |\nabla(s_\mu u^+ + t_\mu u^-)|^2 dx\right)^2\\
&\le 2s_\mu^2\|u^+\|_\lambda^2 + 2t_\mu^2\|u^-\|_\lambda^2\\
&\quad + 4bs_\mu^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right) dx + 4bt_\mu^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right) dx.
\end{aligned}$$

Hence, H_u is bounded.

Let $\{\mu_n\} \subset (0,\infty)$ be such that $\mu_n \to \infty$ as $n \to \infty$. Then, there exist s_0 and t_0 such that

$$(s_{\mu_n}, t_{\mu_n}) \to (s_0, t_0), \tag{3.1}$$

as $n \to \infty$ (in a subsequent sense). We claim $s_0 = t_0 = 0$. By contradiction. Suppose that $s_0 > 0$ or $t_0 > 0$. Since $s_{\mu_n} u^+ + t_{\mu_n} u^- \in \mathcal{M}_\lambda^{\mu_n}$, for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\|s_{\mu_n} u^+ + t_{\mu_n} u^-\|_\lambda^2 + b\left(\int_{\mathbb{R}^3} |\nabla(s_{\mu_n} u^+ + t_{\mu_n} u^-)|^2 dx\right)^2\\
&\quad = \int_{\mathbb{R}^3} |s_{\mu_n} u^+ + t_{\mu_n} u^-|^6\, dx + \mu_n \int_{\mathbb{R}^3} f(s_{\mu_n} u^+ + t_{\mu_n} u^-)(s_{\mu_n} u^+ + t_{\mu_n} u^-)\, dx.
\end{aligned} \tag{3.2}$$

From (2.9), (3.1) and (f_1), one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(s_{\mu_n} u^+ + t_{\mu_n} u^-)(s_{\mu_n} u^+ + t_{\mu_n} u^-)\, dx\\
&\qquad \to \int_{\mathbb{R}^3} f(s_0 u^+ + t_0 u^-)(s_0 u^+ + t_0 u^-)\, dx > 0.
\end{aligned}$$

Since $\mu_n \to \infty$ as $n \to \infty$ and $\{s_{\mu_n} u^+ + t_{\mu_n} u^-\}$ is bounded in E_λ that we have a contradiction with (3.2). Therefore, we get $s_0 = t_0 = 0$, that is, $\lim_{\mu\to\infty} m_\lambda^\mu = 0$. So, we complete the proof. □

Lemma 3.2. *Assume that (V_1)–(V_3), (f_1)–(f_4) hold, there exists $\Lambda > 1$ and $\mu^* > 0$ such that m_λ^μ is achieved for all $\mu \ge \mu^*$ and $\lambda \ge \Lambda$.*

Proof. In view of definition of m_λ^μ, there exists a sequence $\{u_n\} \subset \mathcal{M}_\lambda^\mu$ such that $m_\lambda^\mu = \lim\limits_{n\to\infty} I_\lambda^\mu(u_n)$. It follows from (2.25) that $\{u_n\}$ is bounded in E_λ. Then, for the subsequence of $\{u_n\}$, there exists $u = u^+ + u^- \in E_\lambda$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & \text{in } E_\lambda, \\ u_n \to u_\lambda, & \text{in } L^q_{\text{loc}}(\mathbb{R}^3) \text{ for all } q \in [2,6), \\ u_n(x) \to u_\lambda(x), & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Next we will show $u_n \to u$ in $L^q(\mathbb{R}^3)$ with $q \in (2,6)$. Let $u_n = u_{\lambda,n}$, then there are the following two cases may happen

(1) $\liminf_{\lambda\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_{\lambda,n} - u|^2\,dx > 0$;

(2) $\liminf_{\lambda\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_{\lambda,n} - u|^2\,dx = 0$.

If (1) holds, there exists $\nu > 0$ independent of λ such that

$$\liminf_{\lambda\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_{\lambda,n} - u|^2\,dx = \nu > 0. \tag{3.3}$$

By the weak lower semicontinuity, one has

$$\|u_{\lambda,n} - u_\lambda\|_\lambda \le \|u_{\lambda,n}\|_\lambda + \|u_\lambda\|_\lambda \le \|u_{\lambda,n}\|_\lambda + \liminf_{n\to\infty} \|u_{\lambda,n}\|_\lambda.$$

From (2.25), there exists $\bar{C} > 0$ independent of λ such that $\limsup_{n\to\infty} \|u_{\lambda,n} - u\|_\lambda \le 2\bar{C}$. Define

$$D_R := \{x \in \mathbb{R}^3 \setminus B_R(0) : V(x) \ge V_0\}, \quad F_R := \{x \in \mathbb{R}^3 \setminus B_R(0) : V(x) < V_0\}.$$

Then

$$\begin{aligned} \limsup_{n\to\infty} \int_{D_R} |u_{\lambda,n} - u|^2\,dx &\le \frac{1}{\lambda V_0} \limsup_{n\to\infty} \int_{D_R} \lambda V(x) |u_{\lambda,n} - u|^2\,dx \\ &\le \frac{1}{\lambda V_0} \limsup_{n\to\infty} \|u_{\lambda,n} - u\|_\lambda^2 \le \frac{4\bar{C}^2}{\lambda V_0}. \end{aligned}$$

Furthermore, we have that $|F_R| \to 0$ as $R \to \infty$ by (V_2). Combining this with the Hölder inequality, as $R \to \infty$, one gets

$$\begin{aligned} \int_{F_R} |u_{\lambda,n} - u|^2\,dx &\le \left(\int_{F_R} |u_{\lambda,n} - u|^s dx \right)^{\frac{2}{s}} \left(\int_{F_R} 1 dx \right)^{\frac{s-2}{s}} \\ &\le C \|u_{\lambda,n} - u\|_\lambda^2 |F_R|^{\frac{s-2}{s}} \to 0, \end{aligned}$$

where $s \in (2,6)$. By $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^3)$ with $q \in [2,6)$, and taking $R > 0$ sufficiently large, we have

$$
\begin{aligned}
\nu &= \liminf_{\lambda\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_{\lambda,n}-u|^2\,dx \\
&\leq \liminf_{\lambda\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^3} |u_{\lambda,n}-u|^2\,dx \\
&\leq \liminf_{\lambda\to\infty} \limsup_{n\to\infty} \int_{B_R(0)} |u_{\lambda,n}-u|^2\,dx + \liminf_{\lambda\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R(0)} |u_{\lambda,n}-u|^2\,dx \\
&= \liminf_{\lambda\to\infty} \limsup_{n\to\infty} \left(\int_{D_R} |u_{\lambda,n}-u|^2\,dx + \int_{F_R} |u_{\lambda,n}-u|^2\,dx \right) \\
&\leq \liminf_{\lambda\to\infty} \frac{4\bar{C}^2}{\lambda V_0} = 0,
\end{aligned}
$$

which is a contradiction. Hence (1) is impossible.

If (2) holds. Then for any sequence $\{\lambda_k\}$ such that $\lambda_k\to\infty$ as $k\to\infty$. Since $n,k\in\mathbb{N}$, by a diagonalization principle, there is a subsequence $\{u_{\lambda_k,k}\}$ such that

$$
\liminf_{k\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_{\lambda_k,k}-u|^2 dx = 0.
$$

By Lions' lemma, $u_{\lambda_k,k}\to u$ in $L^s(\mathbb{R}^3)$ for $s\in(2,6)$. Then, we have $u^{\pm}_{\lambda_k,k}\to u^{\pm}$ in $L^q(\mathbb{R}^3)$ with $q\in(2,6)$.

Let $\delta := \frac{1}{3}S^{\frac{3}{2}}$, where S is the best Sobolev constant. By Lemma 3.1, there exists $\mu^*>0$ such that $m^{\mu}_{\lambda}<\delta$ for any $\mu>\mu^*$. Since $\{u_{\lambda_k,k}\}$ is a subsequence of $\{u_{\lambda,n}\}$ and $\lambda_k\to\infty$ as $k\to\infty$. Then, thanks to Lemma 2.1, one obtains $I^{\mu}_{\lambda}(su^+_{\lambda_k,k}+tu^-_{\lambda_k,k})\leq I^{\mu}_{\lambda}(u_{\lambda_k,k})$ for all $s,t>0$. Hence, according to Fatou's lemma and the Brézis–Lieb lemma, we obtain

$$
\begin{aligned}
\liminf_{k\to\infty} I^{\mu}_{\lambda}(su^+_{\lambda_k,k}+tu^-_{\lambda_k,k}) &\geq \frac{s^2}{2}\lim_{k\to\infty}\left(\|u^+_{\lambda_k,k}-u^+\|^2_{\lambda}+\|u^+\|^2_{\lambda}\right) \\
&\quad + \frac{t^2}{2}\lim_{k\to\infty}\left(\|u^-_{\lambda_k,k}-u^-\|^2_{\lambda}+\|u^-\|^2_{\lambda}\right) \\
&\quad + \frac{bs^4}{4}\left(\lim_{k\to\infty}\int_{\mathbb{R}^3}(|\nabla(u^+_{\lambda_k,k}-u^+)|^2+|\nabla u^+|^2)dx\right)^2 \\
&\quad + \frac{bt^4}{4}\left(\lim_{k\to\infty}\int_{\mathbb{R}^3}(|\nabla(u^-_{\lambda_k,k}-u^-)|^2+|\nabla u^-|^2)dx\right)^2 \\
&\quad - \frac{s^6}{6}\lim_{k\to\infty}\left(\int_{\mathbb{R}^3}|u^+_{\lambda_k,k}-u^+|^6dx+\int_{\mathbb{R}^3}|u^+|^6dx\right) \\
&\quad - \frac{t^6}{6}\lim_{k\to\infty}\left(\int_{\mathbb{R}^3}|u^-_{\lambda_k,k}-u^-|^6dx+\int_{\mathbb{R}^3}|u^-|^6dx\right) \\
&\quad + \frac{bs^2t^2}{2}\liminf_{k\to\infty}\int_{\mathbb{R}^3}|\nabla u^+_{\lambda_k,k}|^2dx\int_{\mathbb{R}^3}|\nabla u^-_{\lambda_k,k}|^2dx \\
&\quad - \mu\int_{\mathbb{R}^3}F(su^+)dx-\mu\int_{\mathbb{R}^3}F(tu^-)dx \\
&\geq I^{\mu}_{\lambda}(su^++tu^-)+\frac{s^2}{2}\lim_{k\to\infty}\|u^+_{\lambda_k,k}-u^+\|^2_{\lambda}+\frac{t^2}{2}\lim_{k\to\infty}\|u^-_{\lambda_k,k}-u^-\|^2_{\lambda} \\
&\quad + \frac{bs^4}{2}\lim_{k\to\infty}\int_{\mathbb{R}^3}|\nabla(u^+_{\lambda_k,k}-u^+)|^2dx\int_{\mathbb{R}^3}|\nabla u^+|^2dx \\
&\quad + \frac{bt^4}{2}\lim_{k\to\infty}\int_{\mathbb{R}^3}|\nabla(u^-_{\lambda_k,k}-u^-)|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{bs^4}{4}\left(\lim_{k\to\infty}\int_{\mathbb{R}^3}|\nabla(u^+_{\lambda_k,k}-u^+)|^2dx\right)^2\\
&+\frac{bt^4}{4}\left(\lim_{k\to\infty}\int_{\mathbb{R}^3}|\nabla(u^-_{\lambda_k,k}-u^-)|^2dx\right)^2\\
&-\frac{s^6}{6}\lim_{k\to\infty}\int_{\mathbb{R}^3}|u^+_{\lambda_k,k}-u^+|^6dx-\frac{t^6}{6}\lim_{k\to\infty}\int_{\mathbb{R}^3}|u^-_{\lambda_k,k}-u^-|^6dx\\
=\;& I^\mu_\lambda(su^++tu^-)+\frac{s^2}{2}A_1+\frac{bs^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx+\frac{bs^4}{4}A_1^2-\frac{s^6}{6}B_1\\
&+\frac{t^2}{2}A_2+\frac{bt^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx+\frac{bt^4}{4}A_2^2-\frac{t^6}{6}B_2,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\lim_{k\to\infty}\|u^+_{\lambda_k,k}-u^+\|^2_\lambda, && A_2=\lim_{k\to\infty}\|u^-_{\lambda_k,k}-u^-\|^2_\lambda,\\
B_1&=\lim_{k\to\infty}\int_{\mathbb{R}^3}|u^+_{\lambda_k,k}-u^+|^6dx, && B_2=\lim_{k\to\infty}\int_{\mathbb{R}^3}|u^-_{\lambda_k,k}-u^-|^6dx.
\end{aligned}
$$

Thus, one gets

$$
\begin{aligned}
&I^\mu_\lambda(su^++tu^-)+\frac{s^2}{2}A_1+\frac{bs^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx+\frac{bs^4}{4}A_1^2-\frac{s^6}{6}B_1\\
&\quad+\frac{t^2}{2}A_2+\frac{bt^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx+\frac{bt^4}{4}A_2^2-\frac{t^6}{6}B_2\le m^\mu_\lambda,
\end{aligned}
\tag{3.4}
$$

for all $s,t\ge 0$.

To handle the lack of compactness, the proof is divided in to three steps.

Step 1. We prove $u^\pm\ne 0$.

Since the argument for $u^-\ne 0$ is analogous, we just prove $u^+\ne 0$. By contradiction. Suppose $u^+\equiv 0$. Hence, let $t=0$ in (3.4), one obtains

$$
\frac{s^2}{2}A_1+\frac{bs^4}{4}A_1^2-\frac{s^6}{6}B_1\le m^\mu_\lambda, \tag{3.5}
$$

for all $s\ge 0$.

Case 1: $B_1=0$.

If $A_1=0$, we have that $u^+_{\lambda_k,k}\to u^+$ in E_λ. Then, according to Lemma 2.1, we get $\|u^+\|_\lambda>0$, which is a contradiction. If $A_1>0$, by (3.5), we get

$$
\frac{s^2}{2}A_1+\frac{bs^4}{4}A_1^2\le m^\mu_\lambda,
$$

which is impossible for all $s\ge 0$.

Case 2: $B_1>0$.

By the definition of S, we obtain that

$$
\delta=\frac{1}{3}S^{\frac{3}{2}}\le\frac{1}{3}\left(\frac{A_1}{(B_1)^{\frac{1}{3}}}\right)^{\frac{3}{2}}.
$$

It is not hard to see that

$$
\frac{1}{3}\left(\frac{A_1}{(B_1)^{\frac{1}{3}}}\right)^{\frac{3}{2}}=\max_{s\ge 0}\left\{\frac{s^2}{2}A_1-\frac{s^6}{6}B_1\right\}\le\max_{s\ge 0}\left\{\frac{s^2}{2}A_1+\frac{bs^4}{4}A_1^2-\frac{s^6}{6}B_1\right\}.
$$

Hence, according to $m_\lambda^\mu < \delta$ and (3.5), we get

$$\delta \le \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\right\} \le \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^6}{6}B_1\right\} < \delta,$$

which is a contradiction.

Therefore, we deduce that $u^\pm \neq 0$.

Step 2. We prove that $B_1 = B_2 = 0$.

Indeed, we assume that $B_1 > 0$. We discuss it with the following two cases.

Case 1: $B_2 > 0$.

According to $B_1, B_2 > 0$ and the Sobolev embedding, we obtain that $A_1, A_2 > 0$. Let

$$\varphi(s) = \frac{s^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^6}{6}B_1, \quad s \ge 0.$$

It is easy to see that $\varphi(s) > 0$ for $s > 0$ small enough and $\varphi(s) < 0$ for $s > 0$ large enough. Hence, by the continuity of $\varphi(s)$, there exists $\tilde{s} > 0$ such that

$$\frac{\tilde{s}^2}{2}A_1 + \frac{b\tilde{s}^4}{4}A_1^2 - \frac{\tilde{s}^6}{6}B_1 = \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^6}{6}B_1\right\}.$$

Similarly, there exists $\tilde{t} > 0$ such that

$$\frac{\tilde{t}^2}{2}A_2 + \frac{b\tilde{t}^4}{4}A_2^2 - \frac{\tilde{t}^6}{6}B_2 = \max_{t\ge 0}\left\{\frac{t^2}{2}A_2 + \frac{bt^4}{4}A_2^2 - \frac{t^6}{6}B_2\right\}.$$

Since $[0,\tilde{s}] \times [0,\tilde{t}]$ is compact and ψ_u(defined by (2.17)) is continuous, there exists $(s_u, t_u) \in [0,\tilde{s}] \times [0,\tilde{t}]$ such that

$$\psi_u(s_u, t_u) = \max_{(s,t)\in[0,\tilde{s}]\times[0,\tilde{t}]} \psi_u(s,t).$$

Then, we need to prove that $(s_u, t_u) \in (0,\tilde{s}) \times (0,\tilde{t})$.

Note that, if t is small enough, for any $s \in [0,\tilde{s}]$, one gets

$$\psi_u(s,0) = I_\lambda^\mu(su^+) < I_\lambda^\mu(su^+) + I_\lambda^\mu(tu^-) \le I_\lambda^\mu(su^+ + tu^-) = \psi_u(s,t).$$

Hence, for any $s \in [0,\tilde{s}]$, there exists $t_0 \in [0,\tilde{t}]$ such that $\psi_u(s,0) \le \psi_u(s,t_0)$. That is, any $(s,0)$ with $s \in [0,\tilde{s})$ is not the maximum point of ψ_u. Therefore, $(s_u, t_u) \notin [0,\tilde{s}] \times \{0\}$. Similarly, we can get $(s_u, t_u) \notin \{0\} \times [0,\tilde{t}]$.

In the following, we need to prove that $(s_u, t_u) \notin \{s\} \times [0,\tilde{t}]$ and $(s_u, t_u) \notin [0,\tilde{s}] \times \{t\}$. For any $s \in (0,\tilde{s}]$ and $t \in (0,\tilde{t}]$, we have that

$$\frac{s^2}{2}A_1 + \frac{bs^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{bs^4}{4}A_1^2 - \frac{s^6}{6}B_1 > 0, \tag{3.6}$$

$$\frac{t^2}{2}A_2 + \frac{bt^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{bt^4}{4}A_2^2 - \frac{t^6}{6}B_2 > 0. \tag{3.7}$$

Then, we get

$$\begin{aligned}\delta \le\, & \frac{\tilde{s}^2}{2}A_1 + \frac{b\tilde{s}^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{b\tilde{s}^4}{4}A_1^2 - \frac{\tilde{s}^6}{6}B_1 \\ & + \frac{t^2}{2}A_2 + \frac{bt^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{bt^4}{4}A_2^2 - \frac{t^6}{6}B_2,\end{aligned}$$

and

$$\delta \leq \frac{s^2}{2}A_1 + \frac{bs^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{bs^4}{4}A_1^2 - \frac{s^6}{6}B_1 + \frac{\tilde{t}^2}{2}A_2 + \frac{b\tilde{t}^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{b\tilde{t}^4}{4}A_2^2 - \frac{\tilde{t}^6}{6}B_2,$$

for all $s \in [0,\tilde{s}]$ and all $t \in [0,\tilde{t}]$. By (3.4) and the definition of ψ_u (that is, (2.17)), one has

$$\psi_u(s,\tilde{t}) \leq 0, \qquad \psi_u(\tilde{s},t) \leq 0,$$

for all $s \in [0,\tilde{s}]$ and all $t \in [0,\tilde{t}]$. Hence, $(s_u,t_u) \notin \{\tilde{s}\} \times [0,\tilde{t}]$ and $(s_u,t_u) \notin [0,\tilde{s}] \times \{\tilde{t}\}$. In conclusion, we have that $(s_u,t_u) \in (0,\tilde{s}) \times (0,\tilde{t})$.

By Lemma 2.1, (s_u,t_u) is a critical point of ψ_u. So, $s_uu^+ + t_uu^- \in \mathcal{M}_\lambda^\mu$. Then, by (3.4), (3.6) and (3.7), we get

$$\begin{aligned} m_\lambda^\mu &\geq I_\lambda^\mu(s_uu^+ + t_uu^-) + \frac{s_u^2}{2}A_1 + \frac{bs_u^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{bs_u^4}{4}A_1^2 - \frac{s_u^6}{6}B_1 \\ &\quad + \frac{t_u^2}{2}A_2 + \frac{bt_u^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{bt_u^4}{4}A_2^2 - \frac{t_u^6}{6}B_2 \\ &> I_\lambda^\mu(s_uu^+ + t_uu^-) \geq m_\lambda^\mu. \end{aligned}$$

This is a contradiction.

Case 2: $B_2 = 0$.

In this case, we need to prove that there exists $t_0 \in [0,\infty)$ such that $I_\lambda^\mu(s_uu^+ + t_uu^-) \leq 0$ for any $(s,t) \in [0,\tilde{s}] \times [t_0,\infty)$. That is, there exists $(s_u,t_u) \in [0,\tilde{s}] \times [0,\infty)$ such that

$$\psi_u(s_u,t_u) = \max_{(s,t)\in[0,\tilde{s}]\times[0,\infty)} \psi_u(s,t).$$

Then, we claim that $(s_u,t_u) \in (0,\tilde{s}) \times (0,\infty)$. If t is small enough, for all $s \in [0,\tilde{s}]$, we have that $\psi_u(s,0) < \psi_u(s,t)$. That is, $(s_u,t_u) \notin [0,\tilde{s}] \times \{0\}$. Similarly, if s is small enough, for all $t \in [0,\infty)$, we have that $\psi_u(0,t) < \psi_u(s,t)$. That is, $(s_u,t_u) \notin \{0\} \times [0,\infty)$.

In the following, it is easy to get that

$$\delta \leq \frac{\tilde{s}^2}{2}A_1 + \frac{b\tilde{s}^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{b\tilde{s}^4}{4}A_1^2 - \frac{\tilde{s}^6}{6}B_1 + \frac{t^2}{2}A_2 + \frac{bt^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{bt^4}{4}A_2^2,$$

for all $t \in [0,\infty)$.

Therefore, we can get that $\psi_u(\tilde{s},t) \leq 0$. That is, $(s_u,t_u) \notin \{\tilde{s}\} \times [0,\infty)$. In conclusion, $(s_u,t_u) \in (0,\tilde{s}) \times (0,\infty)$. Hence, $s_uu^+ + t_uu^- \in \mathcal{M}_\lambda^\mu$. Then, by (3.6), we have that

$$\begin{aligned} m_\lambda^\mu &\geq I_\lambda^\mu(s_uu^+ + t_uu^-) + \frac{s_u^2}{2}A_1 + \frac{bs_u^4}{2}A_1\int_{\mathbb{R}^3}|\nabla u^+|^2dx + \frac{bs_u^4}{4}A_1^2 - \frac{s_u^6}{6}B_1 \\ &\quad + \frac{t_u^2}{2}A_2 + \frac{bt_u^4}{2}A_2\int_{\mathbb{R}^3}|\nabla u^-|^2dx + \frac{bt_u^4}{4}A_2^2 \\ &> I_\lambda^\mu(s_uu^+ + t_uu^-) \geq m_\lambda^\mu, \end{aligned}$$

which implies a contradiction. Hence, $B_1 = 0$. In the same way, we can get $B_2 = 0$. Therefore, $B_1 = B_2 = 0$.

Step 3. We prove that m_λ^μ is achieved.

For any $u^\pm \neq 0$, by Lemma 2.1, there exists a pair of (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$. In addition, if $\langle (I_\lambda^\mu)'(u), u^\pm \rangle \leq 0$, there exists a pair of $(s_u, t_u) \in (0,1] \times (0,1]$.

Since $u_{\lambda_k,k} \in \mathcal{M}_\lambda^\mu$, according to Lemma 2.1, we get

$$I_\lambda^\mu(s_u u_{\lambda_k,k}^+ + t_u u_{\lambda_k,k}^-) \leq I_\lambda^\mu(u_{\lambda_k,k}^+ + u_{\lambda_k,k}^-) = I_\lambda^\mu(u_{\lambda_k,k}).$$

Thanks to (f_3), $B_1 = B_2 = 0$ and the weakly lower semicontinuity, we obtain

$$\begin{aligned}
m_\lambda^\mu &\leq I_\lambda^\mu(s_u u^+ + t_u u^-) - \frac{1}{4}\langle (I_\lambda^\mu)'(s_u u^+ + t_u u^-), s_u u^+ + t_u u^- \rangle \\
&= \frac{1}{4}(\|s_u u^+\|_\lambda^2 + \|t_u u^-\|_\lambda^2) + \frac{1}{12}\int_{\mathbb{R}^3}(|s_u u^+|^6 + |t_u u^-|^6)dx \\
&\quad + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(s_u u^+)(s_u u^+) - 4F(s_u u^+)]dx + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(s_u u^-)(s_u u^-) - 4F(s_u u^-)]dx \\
&\leq \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{12}\int_{\mathbb{R}^3}|u|^6 dx + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(u)u - 4F(u)]dx \\
&\leq \lim_{k\to\infty}\left[I_\lambda^\mu(u_{\lambda_k,k}) - \frac{1}{4}\langle (I_\lambda^\mu)'(u_{\lambda_k,k}), u_{\lambda_k,k}\rangle\right] \\
&\leq m_\lambda^\mu.
\end{aligned}$$

Hence, we have that $s_u = t_u = 1$, and m_λ^μ is achieved. We complete the proof. □

Motivated by Tian and Tang [25], with the help of the Nehari manifold, the following lemma holds. Since the proof is standard, we omit it here.

Lemma 3.3. *Assume that* (V_1)–(V_3), (f_1)–(f_4) *hold, then*

(i) *for any* $u \in E_\lambda$, *there is a unique* $\widetilde{s_u} > 0$ *such that* $\widetilde{s_u}u \in \mathcal{N}_\lambda^\mu$ *and*

$$I_\lambda^\mu(\widetilde{s_u}u) = \max_{s\geq 0} I_\lambda^\mu(su);$$

(ii) *problem* (1.1) *has a positive ground state solution* $v_0 \in \mathcal{N}_\lambda^\mu$ *and* $I_\lambda^\mu(v_0) = c_\lambda^\mu < c^* := \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4+4aS)^{\frac{3}{2}}}{24}$.

Now, we give the proof of Theorem 1.1 by the quantitative deformation lemma.

Proof of Theorem 1.1. First, we will prove that the minimizer u_λ is indeed a sign-changing solution of problem (1.1), that is, $(I_\lambda^\mu)'(u_\lambda) = 0$. Since $u_\lambda \in \mathcal{M}_\lambda^\mu$, one has $\langle (I_\lambda^\mu)'(u_\lambda), u_\lambda^+ \rangle = \langle (I_\lambda^\mu)'(u_\lambda), u_\lambda^- \rangle = 0$. By Lemma 2.1 and Step 3 in the proof of Lemma 3.2, for any $(\alpha, \beta) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ and $(\alpha, \beta) \neq (1,1)$, we get

$$I_\lambda^\mu(\alpha u_\lambda^+ + \beta u_\lambda^-) < I_\lambda^\mu(u_\lambda^+ + u_\lambda^-) = m_\lambda^\mu. \tag{3.8}$$

By contradiction. Suppose that $(I_\lambda^\mu)'(u_\lambda) \neq 0$, then there exist $\kappa > 0$ and $\tau > 0$ such that

$$\|(I_\lambda^\mu)'(v)\| \geq \kappa, \quad \text{for all } \|v - u_\lambda\| \leq 3\tau.$$

By (iii) of Lemma 2.2, there exist $L > 0$ such that $\|u_\lambda^\pm\|_\lambda > L$, and we can assume $6\tau < L$. Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(\alpha, \beta) := \alpha u_\lambda^+ + \beta u_\lambda^-$, $(\alpha, \beta) \in D$. By Lemma 2.1, we have

$$\overline{m}_\lambda^\mu := \max_{(\alpha,\beta)\in\partial D}\left(I_\lambda^\mu \circ g\right)(\alpha, \beta) < m_\lambda^\mu. \tag{3.9}$$

For $\varepsilon := \min\{\frac{m_\lambda^\mu - \overline{m}_\lambda^\mu}{4}, \frac{\kappa\tau}{8}\}$ and $S_\tau := B(u_\lambda, \tau)$, there exists a deformation η such that

(1) $\eta(t,u)=u$ if $t=0$, or $u\notin (I_\lambda^\mu)^{-1}\big([m_\lambda^\mu-2\varepsilon, m_\lambda^\mu+2\varepsilon]\big)\cap S_{2\tau}$, where $S_{2\tau}:=\{x\mid \|x-y\|\le 2\tau,\ \forall y\in S\}$;

(2) $\eta\big(1,(I_\lambda^\mu)^{m_\lambda^\mu+\varepsilon}\cap S_\tau\big)\subset (I_\lambda^\mu)^{m_\lambda^\mu-\varepsilon}$;

(3) $I_\lambda^\mu\big(\eta(1,u)\big)\le I_\lambda^\mu(u)$ for all $u\in E_\lambda$;

(4) $I_\lambda^\mu(\cdot,u)$ is non increasing for every $u\in E_\lambda$.

Now, we prove

$$\max_{(\alpha,\beta)\in D} I_\lambda^\mu\big(\eta(1,g(\alpha,\beta))\big)<m_\lambda^\mu. \tag{3.10}$$

According to Lemma 2.1, we have that $I_\lambda^\mu(g(\alpha,\beta))\le m_\lambda^\mu<m_\lambda^\mu+\varepsilon$. That is, $g(\alpha,\beta)\in (I_\lambda^\mu)^{m_\lambda^\mu+\varepsilon}$. From (1) and (4), one gets

$$I_\lambda^\mu(\eta(1,u))\le I_\lambda^\mu(\eta(0,u))=I_\lambda^\mu(u),\quad \forall u\in E_\lambda. \tag{3.11}$$

In view of (3.8) and (3.11), we obtain

$$I_\lambda^\mu(\eta(1,g(\alpha,\beta)))\le I_\lambda^\mu(g(\alpha,\beta))<m_\lambda^\mu,$$

where $(\alpha,\beta)\ne(1,1)$. On the other hand, from (2), one has

$$I_\lambda^\mu(\eta(1,g(1,1)))\le m_\lambda^\mu-\varepsilon<m_\lambda^\mu.$$

Hence (3.10) holds.

In the following, we claim that $\eta(1,g(D))\cap\mathcal{M}_\lambda^\mu\ne\emptyset$. Let $\varphi(\alpha,\beta):=\eta(1,g(\alpha,\beta))$ and

$$\Phi(\alpha,\beta):=\left(\frac{1}{\alpha}\langle (I_\lambda^\mu)'(\varphi(\alpha,\beta)),(\varphi(\alpha,\beta))^+\rangle,\frac{1}{\beta}\langle (I_\lambda^\mu)'(\varphi(\alpha,\beta)),(\varphi(\alpha,\beta))^-\rangle\right).$$

The claim holds if there exists $(\alpha_0,\beta_0)\in D$ such that $\Phi(\alpha_0,\beta_0)=(0,0)$. Since

$$\begin{aligned}\|g(\alpha,\beta)-u_\lambda\|_\lambda^2&=\|(\alpha-1)u_\lambda^++(\beta-1)u_\lambda^-\|_\lambda^2\\&\le|\alpha-1|^2\|u_\lambda\|_\lambda^2\\&\le|\alpha-1|^2(6\tau)^2,\end{aligned}$$

and $|\alpha-1|^2(6\tau)^2>4\tau^2\Leftrightarrow\alpha<\frac{2}{3}$ or $\alpha>\frac{4}{3}$, using the above item (1) and the range of α, for $\alpha=\frac12$ and for every $\beta\in[\frac12,\frac32]$, we have $g(\frac12,\beta)\notin S_{2\tau}$. So from (1), when $(\frac12,\beta)\in D$, we have $\varphi(\frac12,\beta)=g(\frac12,\beta)$, thus

$$\Phi(\frac12,\beta)=\left(2\left\langle (I_\lambda^\mu)'\left(\frac12 u_\lambda^++\beta u_\lambda^-\right),\frac12 u_\lambda^+\right\rangle,\frac{1}{\beta}\left\langle (I_\lambda^\mu)'\left(\frac12 u_\lambda^++\beta u_\lambda^-\right),\beta u_\lambda^-\right\rangle\right).$$

By (f_3), we know that

$$\begin{aligned}\left\langle (I_\lambda^\mu)'\left(\frac12 u_\lambda^++\beta u_\lambda^-\right),\frac12 u_\lambda^+\right\rangle&=\left\langle (I_\lambda^\mu)'\left(\frac12 u_\lambda^+\right),\frac12 u_\lambda^+\right\rangle+\frac{\beta^2 b}{4}\int_{\mathbb{R}^3}|\nabla u_\lambda^-|^2dx\int_{\mathbb{R}^3}|\nabla u_\lambda^+|^2dx\\&\ge\left\langle (I_\lambda^\mu)'\left(\frac12 u_\lambda\right),\frac12 u_\lambda^+\right\rangle>0,\quad\text{for all }\beta\in\left[\frac12,\frac32\right].\end{aligned}\tag{3.12}$$

Similar, when $(\frac{3}{2},\beta)\in D$, we have $\varphi(\frac{3}{2},\beta)=g(\frac{3}{2},\beta)$, so that

$$\begin{aligned}\left\langle (I_\lambda^\mu)'\left(\frac{3}{2}u_\lambda^+ + \beta u_\lambda^-\right), \frac{3}{2}u_\lambda^+\right\rangle &= \left\langle (I_\lambda^\mu)'\left(\frac{3}{2}u_\lambda^+\right), \frac{3}{2}u_\lambda^+\right\rangle + \frac{9\beta^2 b}{4}\int_{\mathbb{R}^3}|\nabla u_\lambda^-|^2dx\int_{\mathbb{R}^3}|\nabla u_\lambda^+|^2dx\\ &\le \left\langle (I_\lambda^\mu)'\left(\frac{3}{2}u_\lambda\right), \frac{3}{2}u_\lambda^+\right\rangle < 0, \quad \text{for all } \beta\in\left[\frac{1}{2},\frac{3}{2}\right]. \end{aligned} \tag{3.13}$$

Similarly, we have

$$\left\langle (I_\lambda^\mu)'\left(\alpha u_\lambda^+ + \frac{1}{2}u_\lambda^-\right), \frac{1}{2}u_\lambda^-\right\rangle > 0, \quad \text{for all } \alpha\in\left[\frac{1}{2},\frac{3}{2}\right], \tag{3.14}$$

$$\left\langle (I_\lambda^\mu)'\left(\alpha u_\lambda^+ + \frac{3}{2}u_\lambda^-\right), \frac{3}{2}u_\lambda^-\right\rangle < 0, \quad \text{for all } \alpha\in\left[\frac{1}{2},\frac{3}{2}\right]. \tag{3.15}$$

Since Φ is continuous on D, according to (3.11)–(3.15) and Miranda's theorem [21], we have $\Phi(\alpha_0,\beta_0)=0$ for some $(\alpha_0,\beta_0)\in D$. Hence $\eta(1,g(\alpha_0,\beta_0))=\varphi(\alpha_0,\beta_0)\in\mathcal{M}_\lambda^\mu$, which contradicts with (3.10). Hence $(I_\lambda^\mu)'(u_\lambda)=0$. That is, u_λ is a ground state sign-changing solution of problem (1.1).

Next, we shall prove that u_λ changes sign only once, that is, u_λ has two nodal domains. Indeed, we assume that $u_\lambda=\widetilde{u}_1+\widetilde{u}_2+\widetilde{u}_3$ with

$$\begin{gathered}\widetilde{u}_i\neq 0, \quad \widetilde{u}_1\ge 0, \quad \widetilde{u}_2\le 0, \quad \widetilde{u}_3\ge 0,\\ \operatorname{supp}(\widetilde{u}_i)\cap\operatorname{supp}(\widetilde{u}_j)=\emptyset, \quad i\neq j\ (i,j=1,2,3).\end{gathered}$$

Then, let $v=\widetilde{u}_1+\widetilde{u}_2$, $v^+=\widetilde{u}_1$, $v^-=\widetilde{u}_2$, by Lemma 2.1, there is a unique pair of $(s_v,t_v)\in(0,1]\times(0,1]$ such that $s_v v^+ + t_v v^- = s_v\widetilde{u}_1+t_v\widetilde{u}_2\in\mathcal{M}_\lambda^\mu$, $I_\lambda^\mu(s_v\widetilde{u}_1+t_v\widetilde{u}_2)\ge m_\lambda^\mu$. Thanks to $\langle (I_\lambda^\mu)'(u_\lambda),\widetilde{u}_i\rangle=0$ $(i=1,2,3)$, we get $\langle (I_\lambda^\mu)'(v),v^\pm\rangle<0$. Since

$$\begin{aligned}0&=\frac{1}{4}\langle (I_\lambda^\mu)'(u_\lambda),\widetilde{u}_3\rangle\\ &=\frac{1}{4}\|\widetilde{u}_3\|_\lambda^2+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla\widetilde{u}_1|^2dx\int_{\mathbb{R}^3}|\nabla\widetilde{u}_3|^2dx+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla\widetilde{u}_2|^2dx\int_{\mathbb{R}^3}|\nabla\widetilde{u}_3|^2dx\\ &\quad+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\widetilde{u}_3|^2dx\right)^2-\frac{\mu}{4}\int_{\mathbb{R}^3}f(\widetilde{u}_3)\widetilde{u}_3dx-\frac{1}{4}\int_{\mathbb{R}^3}|\widetilde{u}_3|^6dx\\ &< I_\lambda^\mu(\widetilde{u}_3)+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla\widetilde{u}_1|^2dx\int_{\mathbb{R}^3}|\nabla\widetilde{u}_3|^2dx+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla\widetilde{u}_2|^2dx\int_{\mathbb{R}^3}|\nabla\widetilde{u}_3|^2dx.\end{aligned}$$

By (2.8), we have

$$\begin{aligned}m_\lambda^\mu&\le I_\lambda^\mu(s_v\widetilde{u}_1+t_v\widetilde{u}_2)\\ &=I_\lambda^\mu(s_v\widetilde{u}_1+t_v\widetilde{u}_2)-\frac{1}{4}\langle (I_\lambda^\mu)'(s_v\widetilde{u}_1+t_v\widetilde{u}_2),s_v\widetilde{u}_1+t_v\widetilde{u}_2\rangle\\ &=\frac{s_v^2}{4}\|\widetilde{u}_1\|_\lambda^2+\frac{1}{12}\int_{\mathbb{R}^3}|s_v\widetilde{u}_1|^6dx+\frac{t_v^2}{4}\|\widetilde{u}_2\|_\lambda^2+\frac{1}{12}\int_{\mathbb{R}^3}|t_v\widetilde{u}_2|^6dx\\ &\quad+\frac{\mu}{4}\int_{\mathbb{R}^3}[f(s_v\widetilde{u}_1)(s_v\widetilde{u}_1)-4F(s_v\widetilde{u}_1)]dx+\frac{\mu}{4}\int_{\mathbb{R}^3}[f(t_v\widetilde{u}_2)(t_v\widetilde{u}_2)-4F(t_v\widetilde{u}_2)]dx\\ &\le\frac{1}{4}\|\widetilde{u}_1\|_\lambda^2+\frac{1}{12}\int_{\mathbb{R}^3}|\widetilde{u}_1|^6dx+\frac{1}{4}\|\widetilde{u}_2\|_\lambda^2+\frac{1}{12}\int_{\mathbb{R}^3}|\widetilde{u}_2|^6dx\\ &\quad+\frac{\mu}{4}\int_{\mathbb{R}^3}[f(\widetilde{u}_1)\widetilde{u}_1-4F(\widetilde{u}_1)]dx+\frac{\mu}{4}\int_{\mathbb{R}^3}[f(\widetilde{u}_2)(\widetilde{u}_2)-4F(\widetilde{u}_2)]dx\end{aligned}$$

$$
\begin{aligned}
&= I_\lambda^\mu(\tilde{u}_1+\tilde{u}_2) - \frac{1}{4}\langle (I_\lambda^\mu)'(\tilde{u}_1+\tilde{u}_2), (\tilde{u}_1+\tilde{u}_2)\rangle \\
&= I_\lambda^\mu(\tilde{u}_1+\tilde{u}_2) + \frac{1}{4}\langle (I_\lambda^\mu)'(u_\lambda), (\tilde{u}_3)\rangle + \frac{b}{4}\int_{\mathbb{R}^3}|\nabla\tilde{u}_1|^2dx\int_{\mathbb{R}^3}|\nabla\tilde{u}_3|^2dx \\
&\quad + \frac{b}{4}\int_{\mathbb{R}^3}|\nabla\tilde{u}_2|^2dx\int_{\mathbb{R}^3}|\nabla\tilde{u}_3|^2dx \\
&< I_\lambda^\mu(\tilde{u}_1) + I_\lambda^\mu(\tilde{u}_2) + I_\lambda^\mu(\tilde{u}_3) + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\tilde{u}_2|^2dx + \int_{\mathbb{R}^3}|\nabla\tilde{u}_3|^2dx\right)\int_{\mathbb{R}^3}|\nabla\tilde{u}_1|^2dx \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\tilde{u}_1|^2dx + \int_{\mathbb{R}^3}|\nabla\tilde{u}_3|^2dx\right)\int_{\mathbb{R}^3}|\nabla\tilde{u}_2|^2dx \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\tilde{u}_1|^2dx + \int_{\mathbb{R}^3}|\nabla\tilde{u}_2|^2dx\right)\int_{\mathbb{R}^3}|\nabla\tilde{u}_3|^2dx \\
&= I_\lambda^\mu(u_\lambda) = m_\lambda^\mu,
\end{aligned}
$$

which is impossible. Then u_λ changes sign only once, that is, u_λ has exactly two nodal domains.

Finally, we are going to prove $m_\lambda^\mu > 2c_\lambda^\mu$. Similar to the proof of Lemma 2.1, there exist $\tilde{s}, \tilde{t} > 0$ such that $\tilde{s}u_\lambda^+, \tilde{t}u_\lambda^- \in \mathcal{N}_\lambda^\mu$. Then, it follows from (3.8) that

$$
\begin{aligned}
m_\lambda^\mu = I_\lambda^\mu(u_\lambda) &= I_\lambda^\mu(u_\lambda^+ + u_\lambda^-) \\
&\geq I_\lambda^\mu(\tilde{s}u_\lambda^+ + \tilde{t}u_\lambda^-) \\
&= I_\lambda^\mu(\tilde{s}u_\lambda^+) + I_\lambda^\mu(\tilde{t}u_\lambda^-) + \frac{b\tilde{s}^2\tilde{t}^2}{2}\int_{\mathbb{R}^3}|\nabla u_\lambda^+|^2dx\int_{\mathbb{R}^3}|\nabla u_\lambda^+|^2dx \\
&> I_\lambda^\mu(\tilde{s}u_\lambda^+) + I_\lambda^\mu(\tilde{t}u_\lambda^-) \geq 2c_\lambda^\mu.
\end{aligned}
$$

Hence, the energy of ground state sign-changing solution is strictly larger than twice the energy of the positive ground state solution. Then, we complete the proof of Theorem 1.1. □

Before the proof of Theorem 1.2, we define I_∞^μ: $H_0^1(\Omega) \to \mathbb{R}$ as the energy functional of problem (1.3), where

$$
I_\infty^\mu = \frac{a}{2}\int_\Omega|\nabla u|^2dx + \frac{b}{4}\left(\int_\Omega|\nabla u|^2dx\right)^2 - \frac{1}{6}\int_\Omega|u|^6dx - \mu\int_\Omega F(u)dx.
$$

Obviously, I_∞^μ is well defined and $I_\infty^\mu \in C^1(H_0^1(\Omega), \mathbb{R})$. Define

$$
\mathcal{M}_\infty = \{u \in H_0^1(\Omega) : u^\pm \neq 0, \langle (I_\infty^\mu)'(u), u^\pm\rangle = 0\},
$$

and $m_\infty := \inf_{u\in\mathcal{M}_\infty} I_\infty^\mu(u)$. Obviously, by the definition of Ω in (V_3), one has $\mathcal{M}_\infty \subset \mathcal{M}_\lambda^\mu$ and $I_\lambda^\mu(u) = I_\infty^\mu(u)$ for any $u \in H_0^1(\Omega)$ and $\lambda > 0$. Then we get $m_\lambda^\mu \leq m_\infty$. Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1, for any sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ with $\lambda_n \to \infty$ as $n \to +\infty$, $\{u_{\lambda_n}\}$ is the sequence of sign-changing solutions to problem (1.1), which satisfy $I_\lambda^\mu(u_{\lambda_n}) = m_{\lambda_n}^\mu$, $(I_{\lambda_n}^\mu)'(u_{\lambda_n}) = 0$ and $u_{\lambda_n}^\pm \neq 0$. By Lemma 3.1, there exists $\mu^* > 0$ such that $m_{\lambda_n}^\mu \leq M$ for any $\mu > \mu_1^*$, where M is independent of λ_n. Then

$$
M \geq m_{\lambda_n}^\mu = I_{\lambda_n}^\mu(u_{\lambda_n}) - \frac{1}{4}\langle (I_{\lambda_n}^\mu)'(u_{\lambda_n}), u_{\lambda_n}\rangle \geq \frac{1}{4}\|u_{\lambda_n}\|_{\lambda_n}^2 \geq \frac{1}{4}\|u_{\lambda_n}\|^2,
$$

which implies that $\{u_{\lambda_n}\}$ is bounded in E. Up to a subsequence, there is $u_* \in E$ such that

$$\begin{cases} u_{\lambda_n} \rightharpoonup u_*, \text{ in } E; \\ u_{\lambda_n} \to u_*, \text{ in } L^q_{loc}(\mathbb{R}^3) \text{ for all } q \in (2,6); \\ u_{\lambda_n} \to u_*, \text{ a.e. in } \mathbb{R}^3. \end{cases} \tag{3.16}$$

Firstly, we claim that $u_* \in H_0^1(\Omega)$ and $(I_\infty^\mu)'(u_*) = 0$. Since $V(x) \geq 0$, it follows from Fatou's lemma that

$$0 \leq \int_{\mathbb{R}^3} V(x)u_*^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x)u_{\lambda_n}^2 dx \leq \liminf_{n\to\infty} \frac{\|u_{\lambda_n}\|_{\lambda_n}^2}{\lambda_n} = 0.$$

By the condition (V_3), we have that $u_*|_{\Omega^c} = 0$. Further, $u_* \in H_0^1(\Omega)$ because the boundary of Ω smooth.

Secondly, similar to the proof of Lemma 3.2, we can deduce that $u_{\lambda_n} \to u_*$ in $L^s(\mathbb{R}^3)$ for $s \in (2,6)$ and $u_n \to u_*$ in E. Hence, we obtain $(I_{\lambda_n}^\mu)'(u_*) \to 0$. Due to $(I_{\lambda_n}^\mu)'(u_*) = 0$, it is easy to check that

$$\begin{aligned} 0 = (I_{\lambda_n}^\mu)'(u_*), \omega) &= a\int_\Omega \nabla u_* \cdot \nabla \omega dx + b\int_\Omega |\nabla u_*|^2 dx \int_\Omega \nabla u_* \cdot \nabla \omega dx \\ &\quad - \int_\Omega |u_*|^4 u_* \omega dx - \mu \int_\Omega f(u_*)\omega dx = \langle (I_\infty^\mu)'(u_*), \omega\rangle, \end{aligned}$$

for any $\omega \in H_0^1(\Omega)$, which means that $\langle (I_\infty^\mu)'(u_*), \omega\rangle = 0$. That is, u_* is a weak solution of problem (1.3).

Finally, we only need to show that u_* is a least energy sign-changing solution of problem (1.3). Since $u_{\lambda_n} \in \mathcal{M}_{\lambda_n}^\mu$, similar to Lemma 2.2-(iii), we get $\|u_{\lambda_n}^\pm\| \geq C_1 > 0$. By $u_{\lambda_n} \to u_*$ in E, then $\|u_*^\pm\| > 0$. Hence, $u_*^\pm \neq 0$ and $u_* \in \mathcal{M}_\infty$. Combining Fatou's lemma, $(I_\infty^\mu)'(u_*) = 0$ and $\langle (I_{\lambda_n}^\mu)'(u_{\lambda_n}), u_{\lambda_n}\rangle = 0$, we have

$$\begin{aligned} m_\infty &\leq I_\infty^\mu(u_*) - \frac{1}{4}\langle (I_\infty^\mu)'(u_*), u_*\rangle \\ &= \frac{1}{4}\|u_*\|_\lambda^2 + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u_*)u_* - 4F(u_*)]dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_*|^6 dx \\ &\leq \liminf_{n\to\infty} \left(\frac{1}{4}\|u_{\lambda_n}\|_{\lambda_n}^2 + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u_{\lambda_n})u_{\lambda_n} - 4F(u_{\lambda_n})]dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_{\lambda_n}|^6 dx \right) \\ &= \liminf_{n\to\infty} \left(I_{\lambda_n}^\mu(u_{\lambda_n}) - \frac{1}{4}\langle (I_{\lambda_n}^\mu)'(u_{\lambda_n}), u_{\lambda_n}\rangle \right) \\ &= \liminf_{n\to\infty} m_{\lambda_n}^\mu \leq m_\infty. \end{aligned}$$

Hence, $I_\infty^\mu(u_*) = m_\infty$. Hence,we conclude that u_* is a ground state sign-changing solution of problem (1.3) and $u_{\lambda_n} \to u_*$ in E. Thus, we complete the proof of Theorem 1.2. □

4 Proof of Theorem 1.4

Since the problem involves critical nonlinearity, in order to get the existence of sign-changing solutions for problem (1.1), we need to construct a sign-changing $(PS)_{m_\lambda^\mu}$-sequence. Inspired by the method of [6], we give some definitions. Let the functional $\varphi(u,v)$ be defined on E_λ by

$$\varphi(u,v)=\begin{cases}\dfrac{\int_{\mathbb{R}^3}|u|^6dx+\mu\int_{\mathbb{R}^3}f(u)udx}{\|u\|_\lambda^2+b(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2+b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla v|^2dx}, & \text{if } u\neq 0;\\ 0, & \text{if } u=0.\end{cases} \tag{4.1}$$

It is obvious that $u\in\mathcal{M}_\lambda^\mu$ if and only if $\varphi(u^+,u^-)=\varphi(u^-,u^+)=1$. Set

$$U:=\left\{u\in E_\lambda:\frac{1}{2}<\varphi(u^+,u^-)<\frac{3}{2},\frac{1}{2}<\varphi(u^-,u^+)<\frac{3}{2}\right\}. \tag{4.2}$$

Let P denote the cone of non-negative functions in E_λ, $Q=[0,1]\times[0,1]$, Σ be the set of continuous maps σ with $s,t\in[0,1]$, and

(I) $\sigma(s,0)=0$, $\sigma(0,t)\in P$ and $\sigma(1,t)\in -P$;

(II) $(I_\lambda^\mu\circ\sigma)(s,1)\leq 0$, $\frac{\int_{\mathbb{R}^3}|\sigma(s,1)|^6\,dx+\mu\int_{\mathbb{R}^3}f(\sigma(s,1))\sigma(s,1)\,dx}{\|\sigma(s,1)\|_\lambda^2+\left(\int_{\mathbb{R}^3}|\nabla\sigma(s,1)|^2\,dx\right)^2}\geq 2$.

Choosing $u\in E_\lambda$ with $u^\pm\neq 0$, let $\sigma(s,t)=Ct(1-s)u^++Ctsu^-$, where $C>0$, $s,t\in[0,1]$. It is easy to see that $\sigma\in\Sigma$ for $C>0$ large enough, which implies $\Sigma\neq\emptyset$.

Lemma 4.1.

$$\inf_{\sigma\in\Sigma}\sup_{u\in\sigma(Q)}I_\lambda^\mu(u)=\inf_{u\in\mathcal{M}_\lambda^\mu}I_\lambda^\mu(u).$$

Proof. In fact, for any $u\in\mathcal{M}_\lambda^\mu$, there exists $\sigma(s,t)=Ct(1-s)u^++Ctsu^-\in\Sigma$ for $C>0$ sufficiently large. By Lemma 2.1, we can deduce that

$$I_\lambda^\mu(u)=\max_{s,t\geq 0}I_\lambda^\mu(su^++tu^-)\geq\sup_{u\in\sigma(Q)}I_\lambda^\mu(u)\geq\inf_{\sigma\in\Sigma}\sup_{u\in\sigma(Q)}I_\lambda^\mu(u).$$

Consequently,

$$\inf_{\sigma\in\Sigma}\sup_{u\in\sigma(Q)}I_\lambda^\mu(u)\leq\inf_{u\in\mathcal{M}_\lambda^\mu}I_\lambda^\mu(u). \tag{4.3}$$

Meanwhile, if for each $\sigma\in\Sigma$, there exists $u_\sigma\in\sigma(Q)\cap\mathcal{M}_\lambda^\mu$, which implies that

$$\sup_{u\in\sigma(Q)}I_\lambda^\mu(u)\geq I_\lambda^\mu(u_\sigma)\geq\inf_{u\in\mathcal{M}_\lambda^\mu}I_\lambda^\mu(u),$$

which gives

$$\inf_{\sigma\in\Sigma}\sup_{u\in\sigma(Q)}I_\lambda^\mu(u)\geq\inf_{u\in\mathcal{M}_\lambda^\mu}I_\lambda^\mu(u). \tag{4.4}$$

Hence it is sufficient to show that for each $\sigma\in\Sigma$, there holds that $\sigma(Q)\cap\mathcal{M}_\lambda^\mu\neq\emptyset$. On the one hand, for each $\sigma\in\Sigma$, $t\in[0,1]$, we have

$$\varphi(\sigma^+(0,t),\sigma^-(0,t))-\varphi(\sigma^-(0,t),\sigma^+(0,t))=\varphi(\sigma^+(0,t),\sigma^-(0,t))\geq 0; \tag{4.5}$$

$$\varphi(\sigma^+(1,t),\sigma^-(1,t))-\varphi(\sigma^-(1,t),\sigma^+(1,t))=-\varphi(\sigma^-(1,t),\sigma^+(1,t))\leq 0. \tag{4.6}$$

On the other hand, from the definition of Σ, for any $\sigma\in\Sigma$ and $s\in[0,1]$, using the elementary inequality $\frac{b}{a}+\frac{d}{c}\geq\frac{b+d}{a+c}$ for any $a,b,c,d>0$, one infers that

$$\begin{aligned}\varphi(\sigma^+(s,1),\sigma^-(s,1))+\varphi(\sigma^-(s,1),\sigma^+(s,1))&\geq\frac{\int_{\mathbb{R}^3}|\sigma(s,1)|^6\,dx+\mu\int_{\mathbb{R}^3}f(\sigma(s,1))\sigma(s,1)\,dx}{\|\sigma(s,1)\|_\lambda^2+b\left(\int_{\mathbb{R}^3}|\nabla\sigma(s,1)|^2\,dx\right)^2}\\&\geq 2.\end{aligned}$$

Therefore, by the definition of φ and Σ, one has

$$\varphi(\sigma^+(s,1),\sigma^-(s,1)) + \varphi(\sigma^-(s,1),\sigma^+(s,1)) - 2 \geq 0, \tag{4.7}$$
$$\varphi(\sigma^+(s,0),\sigma^-(s,0)) + \varphi(\sigma^-(s,0),\sigma^+(s,0)) - 2 = -2 < 0. \tag{4.8}$$

By Miranda's theorem [21] with (4.5) – (4.8), it follows that there exists $(s_\sigma, t_\sigma) \in Q$ such that

$$\varphi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) - \varphi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) = 0;$$
$$\varphi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) + \varphi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) = 2.$$

Then,

$$\varphi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) = \varphi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) = 1.$$

This means that for all $\sigma \in \Sigma$, there exists $u_\sigma = \sigma(s_\sigma, t_\sigma) \in \sigma(Q) \cap \mathcal{M}_\lambda^\mu$. Therefore, by (4.3) and (4.4), we complete the proof of Lemma 4.1. □

Lemma 4.2. *There exists a sequence $\{u_n\} \subset U$ satisfying $I_\lambda^\mu(u_n) \to m_\lambda^\mu$ and $(I_\lambda^\mu)'(u_n) \to 0$.*

Proof. First, we prove the existence of the $(PS)_{m_\lambda^\mu}$-sequence $\{u_n\} \subset E_\lambda$ for I_λ^μ. Consider a minimizing sequence $\{w_n\} \subset \mathcal{M}_\lambda^\mu$ and $\sigma_n(s,t) = Ct(1-s)w_n^+ + Ctsw_n^- \in \Sigma$, then

$$\lim_{n\to\infty} \max_{w\in\sigma_n(Q)} I_\lambda^\mu(w) = \lim_{n\to\infty} I_\lambda^\mu(w_n) = m_\lambda^\mu.$$

Using a variant form of the classical deformation lemma [22] due to Hofer [10], we claim that $\{u_n\} \subset E_\lambda$ such that, as $n \to \infty$,

$$I_\lambda^\mu(u_n) \to m_\lambda^\mu, \qquad (I_\lambda^\mu)'(u_n) \to 0, \qquad \mathrm{dist}(u_n, \sigma_n(Q)) \to 0. \tag{4.9}$$

Suppose the claim is false, then it is possible to find a $\delta_1 > 0$ such that $\sigma_n(Q) \cap \mathbb{V}_{\delta_1} = \emptyset$ for n large enough, where

$$\mathbb{V}_{\delta_1} = \{u \in E_\lambda : \exists v \in E_\lambda, \text{ s.t. } \|v-u\|_\lambda \leq \delta_1,\ \|(I_\lambda^\mu)'(v)\|_\lambda \leq \delta_1,\ |I_\lambda^\mu(v) - m_\lambda^\mu| \leq \delta_1\}.$$

By the deformation lemma [22], there exists a continuous map $\eta : [0,1] \times E_\lambda \to E_\lambda$ satisfying for some $\varepsilon \in (0, \frac{m_\lambda^\mu}{2})$ and all $t \in [0,1]$, with

(a) $\eta(0,u) = u$; $\eta(t,-u) = -\eta(t,u)$;

(b) $\eta(t,u) = u$ for $\forall u \in (I_\lambda^\mu)^{m_\lambda^\mu - \varepsilon} \cup (E_\lambda \backslash (I_\lambda^\mu)^{m_\lambda^\mu + \varepsilon})$;

(c) $\eta(1, (I_\lambda^\mu)^{m_\lambda^\mu + \frac{\varepsilon}{2}} \backslash \mathbb{V}_{\delta_1}) \subset (I_\lambda^\mu)^{m_\lambda^\mu - \frac{\varepsilon}{2}}$;

(d) $\eta(1, ((I_\lambda^\mu)^{m_\lambda^\mu + \frac{\varepsilon}{2}} \cap P)) \backslash \mathbb{V}_{\delta_1} \subset (I_\lambda^\mu)^{m_\lambda^\mu - \frac{\varepsilon}{2}} \cap P$,

where $(I_\lambda^\mu)^d = \{u \in E_\lambda : I_\lambda^\mu(u) \leq d\}$.

Since $\lim_{n\to\infty} \max_{w\in\sigma_n(Q)} I_\lambda^\mu(w) = m_\lambda^\mu$, we can find n such that

$$\sigma_n(Q) \subset (I_\lambda^\mu)^{m_\lambda^\mu + \frac{\varepsilon}{2}}, \quad \sigma_n(Q) \cap \mathbb{V}_{\delta_1} = \emptyset. \tag{4.10}$$

Let us define $\tilde{\sigma}_n : Q \to E_\lambda$ by $\tilde{\sigma}_n(s,t) = \eta(1, \sigma_n(s,t))$ for $\forall (s,t) \in Q$. Then, we know $\tilde{\sigma}_n \in \Sigma$. By the property (c) of η, we can derive that $\tilde{\sigma}_n(Q) \subset (I_\lambda^\mu)^{m_\lambda^\mu - \frac{\varepsilon}{2}}$, which leads to a contradiction because

$$m_\lambda^\mu = \inf_{\sigma\in\Sigma} \sup_{w\in\sigma(Q)} I_\lambda^\mu(w) \leq \max_{w\in\tilde{\sigma}_n(Q)} I_\lambda^\mu(w) \leq m_\lambda^\mu - \frac{\varepsilon}{2}.$$

Therefore, it is sufficient to verify that $\tilde{\sigma}_n \in \Sigma$. Indeed, due to property (b) of η and $\sigma_n \in \Sigma$, one gets

$$\tilde{\sigma}_n(s,0) = \eta(1, \sigma_n(s,0)) = \eta(1,0) = 0.$$

And it follows from $\sigma_n(0,t) \in P$, (4.10) and property (d) of η that $\tilde{\sigma}_n(0,t) \in P$. Moreover, in view of $\sigma_n(1,t) \in -P$ and (4.10), we obtain

$$-\sigma_n(1,t) \in \left((I_\lambda^\mu)^{m_\lambda^\mu + \frac{\epsilon}{2}} \cap P \right) \setminus \mathbb{V}_{\delta_1},$$

which implies that $\tilde{\sigma}_n(1,t) = -\eta(1, -\sigma_n(1,t)) \in -P$ thanks to properties (a) and (d) of η. Then, $\tilde{\sigma}_n$ satisfies property (I). Furthermore, using the fact that $(I_\lambda^\mu \circ \sigma)(s,1) \leq 0$ and property (b) of η, we have

$$\tilde{\sigma}_n(s,1) = \eta(1, \sigma_n(s,1)) = \sigma_n(s,1),$$

that is, $\tilde{\sigma}_n$ satisfies property (II). Hence, we can conclude that $\tilde{\sigma}_n \in \Sigma$ from the continuity of η and $\tilde{\sigma}_n$.

Lastly, we show that the $\{u_n\} \subset U$ for n sufficiently large. Since $(I_\lambda^\mu)'(u_n) \to 0$, then $\langle (I_\lambda^\mu)'(u_n), u_n^\pm \rangle = o(1)$. Then we need to prove that $u_n^\pm \neq 0$ because it implies that $\varphi(u_n^+, u_n^-) \to 1$, $\varphi(u_n^-, u_n^+) \to 1$ for n sufficiently large. By (4.9), there exists a sequence $\{v_n\}$ such that

$$v_n = \gamma_n w_n^+ + \beta_n w_n^- \in \sigma_n(Q), \quad \|v_n - u_n\|_\lambda \to 0. \tag{4.11}$$

Thus, in order to get $u_n^\pm \neq 0$, we can prove that $\gamma_n w_n^+ \neq 0$ and $\beta_n w_n^- \neq 0$ for n sufficiently large. By $\{w_n\} \subset \mathcal{M}_\lambda^\mu$ and (iii) of Lemma 2.2, we only need to prove that $\lim_{n\to\infty} \gamma_n \neq 0$ and $\lim_{n\to\infty} \beta_n \neq 0$. Otherwise, if $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$, it follows from the continuity of I_λ^μ and (4.11) that

$$m_\lambda^\mu = \lim_{n\to\infty} I_\lambda^\mu(v_n) = \lim_{n\to\infty} I_\lambda^\mu(\gamma_n w_n^+ + \beta_n w_n^-) = \lim_{n\to\infty} I_\lambda^\mu(\gamma_n w_n^+).$$

However, for β sufficiently small, by (2.9), we can deduce that

$$\begin{aligned}
m_\lambda^\mu &= \lim_{n\to\infty} I_\lambda^\mu(w_n) = \lim_{n\to\infty} \max_{\gamma,\beta\geq 0} I_\lambda^\mu(\gamma w_n^+ + \beta w_n^-) \geq \lim_{n\to\infty} \max_{\beta\geq 0} I_\lambda^\mu(\gamma_n w_n^+ + \beta w_n^-) \\
&= \lim_{n\to\infty} \max_{\beta\geq 0} \left\{ \frac{1}{2}\|\gamma_n w_n^+ + \beta w_n^-\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(\gamma_n w_n^+ + \beta w_n^-)|^2\, dx \right)^2 \right. \\
&\quad \left. - \frac{1}{6}\int_{\mathbb{R}^3} \left(|\gamma_n w_n^+|^6 + |\beta w_n^-|^6 \right) dx - \mu \int_{\mathbb{R}^3} F(\gamma_n w_n^+ + \beta w_n^-)\, dx \right\} \\
&\geq \lim_{n\to\infty} I_\lambda^\mu(\gamma_n w_n^+) + \lim_{n\to\infty} \max_{\beta\geq 0} \left\{ \frac{\beta^2}{2}\|w_n^-\|_\lambda^2 - \mu \int_{\mathbb{R}^3} F(\beta w_n^-)\, dx - \frac{1}{6}\int_{\mathbb{R}^3} |\beta w_n^-|^6\, dx \right\} \\
&\geq m_\lambda^\mu + \max_{\beta\geq 0} \left(C_1\beta^2 - C_2\beta^6 \right) > m_\lambda^\mu,
\end{aligned}$$

which is a contradiction. Therefore, we can assume that $\{u_n\} \subset U$ satisfies $I_\lambda^\mu(u_n) \to m_\lambda^\mu$ and $(I_\lambda^\mu)'(u_n) \to 0$ for n sufficiently large. Hence, we complete the proof of Lemma 4.2. □

Let $v_\varepsilon = \varphi u_\varepsilon$, where u_ε is defined by (2.2) and φ is a cutoff function such that $0 \leq \varphi \leq 1$, $\varphi|_{B_{r_0(0)}} \equiv 1$ and $\operatorname{supp}\varphi \subset B_{2r_0(0)}$ for some $r_0 > 0$.

Similar to [4], we have

$$\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\, dx = S^{\frac{3}{2}} + O\left(\varepsilon^{\frac{1}{2}}\right), \quad \left(\int_{\mathbb{R}^3} |v_\varepsilon|^6\, dx \right)^{\frac{1}{3}} = S^{\frac{1}{2}} + O\left(\varepsilon^{\frac{3}{2}}\right),$$

and

$$\int_{\mathbb{R}^3} |v_\varepsilon|^r\, dx = \begin{cases} O\left(\varepsilon^{\frac{r}{4}}\right) & r \in [2,3); \\ O\left(\varepsilon^{\frac{r}{4}} |\ln \varepsilon|\right) & r = 3; \\ O\left(\varepsilon^{\frac{6-r}{4}}\right) & r \in (3,6). \end{cases} \tag{4.12}$$

Then, we have the following conclusion.

Lemma 4.3. *Assume that* (V_1)*–*(V_3)*,* (f_1)*–*(f_3) *and* (f_4') *hold, then* $m_\lambda^\mu < c_\lambda^\mu + c^*$*.*

Proof. **Step 1.** We need to find an element that belongs to $\mathcal{M}_\lambda^\mu$. Actually, there exist $s_\varepsilon, t_\varepsilon > 0$ such that $s_\varepsilon v_0 - t_\varepsilon v_\varepsilon \in \mathcal{M}_\lambda^\mu$, where v_0 defined by Lemma 3.3 is a positive ground state solution of problem (1.1). By the elliptic estimates, we derive $v_0 \in L^\infty(\mathbb{R}^3)$. Denote $\varphi(a) = \frac{1}{a} v_0 - v_\varepsilon$, $a > 0$. Define

$$a_1 = \sup\left\{a \in \mathbb{R}^+ : \varphi^+(a) \neq 0\right\}, \quad a_2 = \inf\left\{a \in \mathbb{R}^+ : \varphi^-(a) \neq 0\right\},$$

we obtain $a_1 = +\infty$ and $0 < a_2 < a_1$, because of v_0 is a positive ground state solution. As $a \to a_2^+, \varphi^-(a) \to 0$, then $t(\varphi(a)) \to +\infty$ by (ii) of Lemma 2.2 and the boundedness of $s(\varphi(a))$ in $\mathbb{R}^+$. Therefore,

$$s(\varphi(a)) - t(\varphi(a)) \to -\infty. \tag{4.13}$$

Similarly, as $a \to a_1 = +\infty$, we obtain that

$$s(\varphi(a)) - t(\varphi(a)) \to +\infty. \tag{4.14}$$

Hence, by (i) of Lemma 2.2 and (4.13) and (4.14), there exists $a_\varepsilon \in (a_2, a_1)$ such that $s(\varphi(a_\varepsilon)) - t(\varphi(a_\varepsilon)) = 0$, i.e., $s(\varphi(a_\varepsilon)) = t(\varphi(a_\varepsilon))$. Let $s_\varepsilon = \frac{1}{a_\varepsilon} s(\varphi(a_\varepsilon))$ and $t_\varepsilon = t(\varphi(a_\varepsilon))$, it is easy to see that

$$s(\varphi(a_\varepsilon))\varphi(a_\varepsilon) = s(\varphi(a_\varepsilon))\varphi^+(a_\varepsilon) + t(\varphi(a_\varepsilon))\varphi^-(a_\varepsilon) = s_\varepsilon v_0 - t_\varepsilon v_\varepsilon \in \mathcal{M}_\lambda^\mu.$$

Step 2. We show that $m_\lambda^\mu < c_\lambda^\mu + c^*$. We only need to consider that s_ε, t_ε are contained in bounded interval because $I_\lambda^\mu(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon) < 0$ if s_ε or t_ε is sufficiently large. By a direct calculation, one obtains

$$\begin{aligned}
&I_\lambda^\mu(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon) - I_\lambda^\mu(s_\varepsilon v_0) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^3} (a(|\nabla t_\varepsilon v_\varepsilon|^2 - 2 s_\varepsilon t_\varepsilon \nabla v_0 \nabla v_\varepsilon) + \lambda V(x)(t_\varepsilon^2 v_\varepsilon^2 - 2 s_\varepsilon t_\varepsilon v_0 v_\varepsilon)))dx \\
&\qquad + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx\Big)^2 + b\Big(\int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx\Big)^2 + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla s_\varepsilon v_0|^2\, dx \int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx \\
&\qquad - b\int_{\mathbb{R}^3} |\nabla s_\varepsilon v_0|^2\, dx \int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx - b\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx \int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx \\
&\qquad - \frac{1}{6}\int_{\mathbb{R}^3} (s_\varepsilon v_0 - t_\varepsilon v_\varepsilon)^6\, dx - \mu\int_{\mathbb{R}^3} F(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon)\, dx + \frac{1}{6}\int_{\mathbb{R}^3} |s_\varepsilon v_0|^6\, dx + \mu\int_{\mathbb{R}^3} F(s_\varepsilon v_0)\, dx \\
&\quad = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6 + \Pi_7.
\end{aligned}$$

That is,

$$I_\lambda^\mu(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon) = I_\lambda^\mu(s_\varepsilon v_0) + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6 + \Pi_7, \tag{4.15}$$

where

$$\Pi_1 = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx\Big)^2 - \frac{1}{6}\int_{\mathbb{R}^3} |t_\varepsilon v_\varepsilon|^6\, dx,$$
$$\Pi_2 = -a\int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon dx - b\int_{\mathbb{R}^3} |\nabla s_\varepsilon v_0|^2\, dx \int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx,$$
$$\Pi_3 = -\mu\int_{\mathbb{R}^3} F(-t_\varepsilon v_\varepsilon)\, dx,$$
$$\Pi_4 = b\Big(\int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx\Big)^2 + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla s_\varepsilon v_0|^2\, dx \int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx,$$
$$\Pi_5 = \frac{1}{2}\int_{\mathbb{R}^3} \lambda V(x)|t_\varepsilon v_\varepsilon|^2 dx - b\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2\, dx \int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon\, dx - s_\varepsilon t_\varepsilon \int_{\mathbb{R}^3} \lambda V(x) v_0 v_\varepsilon dx,$$
$$\Pi_6 = \mu\int_{\mathbb{R}^3} F(s_\varepsilon v_0)\, dx + \mu\int_{\mathbb{R}^3} F(-t_\varepsilon v_\varepsilon)\, dx - \mu\int_{\mathbb{R}^3} F(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon)\, dx,$$
$$\Pi_7 = \frac{1}{6}\int_{\mathbb{R}^3} |s_\varepsilon v_0|^6\, dx + \frac{1}{6}\int_{\mathbb{R}^3} |t_\varepsilon v_\varepsilon|^6\, dx - \frac{1}{6}\int_{\mathbb{R}^3} (s_\varepsilon v_0 - t_\varepsilon v_\varepsilon)^6\, dx.$$

For Π_1, according to (4.12), as $\varepsilon \to 0$, one has

$$\Pi_1 \le \max_{t\ge 0}\Big\{\frac{a}{2}\int_{\mathbb{R}^3} |\nabla t v_\varepsilon|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla t v_\varepsilon|^2\, dx\Big)^2 - \frac{1}{6}\int_{\mathbb{R}^3} |t v_\varepsilon|^6\, dx\Big\} = c^* + O(\varepsilon^{\frac{1}{2}}). \tag{4.16}$$

For Π_2, we get

$$\Pi_2 < 0. \tag{4.17}$$

For Π_3, when ε is sufficiently small, for $|x| < \varepsilon^{\frac{1}{2}}$,

$$v_\varepsilon = u_\varepsilon = \frac{3\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}} \ge C\varepsilon^{-\frac{1}{4}}.$$

By (f_4'), we know that for $R > 0$ large enough, there exists $A_R > 0$ such that

$$F(s) \ge R|s|^5,$$

for any $|s| \in [A_R, +\infty)$.

When ε is small enough, one gets

$$\int_{|x|<\varepsilon^{\frac{1}{2}}} F(-t_\varepsilon v_\varepsilon) \ge CR\int_{|x|<\varepsilon^{\frac{1}{2}}} \varepsilon^{-\frac{5}{4}} dx = CR\varepsilon^{\frac{1}{4}}. \tag{4.18}$$

By (f_1) and (f_4'), there exist $L > 0$ and any $t > 0$ such that $F(t) + Lt^2 \ge 0$. Through (4.12), one has

$$\int_{|x|\ge\varepsilon^{\frac{1}{2}}} F(t_\varepsilon v_\varepsilon) dx \ge -C\int_{|x|\ge\varepsilon^{\frac{1}{2}}} |v_\varepsilon|^2\, \mathrm{d}x = -C\varepsilon^{\frac{1}{2}}. \tag{4.19}$$

Thus, for any $\varepsilon > 0$, through (4.18) and (4.19), we obtain

$$\begin{aligned}\Pi_3 &= -\mu\int_{\mathbb{R}^3} F(-t_\varepsilon v_\varepsilon)\, \mathrm{d}x \\ &\le -CR\int_{|x|<\varepsilon^{\frac{1}{2}}} \varepsilon^{-\frac{5}{4}}\, \mathrm{d}x + C\int_{|x|\ge\varepsilon^{\frac{1}{2}}} |v_\varepsilon|^2\, \mathrm{d}x \\ &\le C\varepsilon^{\frac{1}{2}} - CR\varepsilon^{\frac{1}{4}}.\end{aligned}$$

For Π_4, we can assume $0 < b < b_0 := \varepsilon^{\frac{1}{2}}$ with Hölder's inequality such that

$$\begin{aligned}
\Pi_4 &= b\left(\int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon \, dx\right)^2 + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla s_\varepsilon v_0|^2 \, dx \int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2 \, dx \\
&\le Cb\left[\left(\int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \, dx\right)^{\frac{1}{2}}\right]^2 + Cb\int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \, dx \\
&\le Cb\int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \, dx \\
&\le CbS^{\frac{3}{2}} \le C\varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.20}$$

For Π_5, due to $\operatorname{supp}\varphi \subset B_{2R_0}(0)$ and $B_{2R_0}(0) \subset \Omega$, we get

$$\Pi_5 = -b\int_{\mathbb{R}^3} |\nabla t_\varepsilon v_\varepsilon|^2 \, dx \int_{\mathbb{R}^3} \nabla s_\varepsilon v_0 \cdot \nabla t_\varepsilon v_\varepsilon \, dx < 0. \tag{4.21}$$

For Π_6, by [34, Lemma 3.3] and (4.12), we have

$$\begin{aligned}
\Pi_6 &\le \mu\int_{\mathbb{R}^3} \big(|f(s_\varepsilon v_0)t_\varepsilon v_\varepsilon| + |f(-s_\varepsilon v_0)t_\varepsilon v_\varepsilon| + |f(-t_\varepsilon v_\varepsilon)s_\varepsilon v_0| + |f(t_\varepsilon v_\varepsilon)s_\varepsilon v_0|\big) \, dx \\
&\le 4\mu\int_{\mathbb{R}^3} \big(\varepsilon s_\varepsilon t_\varepsilon v_0 v_\varepsilon + C_\varepsilon s_\varepsilon^5 t_\varepsilon v_0^5 v_\varepsilon + C_\varepsilon s_\varepsilon^5 v_\varepsilon v_0^5\big) \, dx \\
&\le C|v_0|_\infty \int_{\mathbb{R}^3} |v_\varepsilon| \, dx + C|v_0|_\infty^5 \int_{\mathbb{R}^3} |v_\varepsilon| \, dx + C|v_0|_\infty \int_{\mathbb{R}^3} |v_\varepsilon|^5 \, dx \\
&\le C\varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{4.22}$$

For Π_7, for the elementary inequality $x^6 + y^6 - |x-y|^6 \le C(x^5 y + x y^5)$ for any $x, y > 0$, by $v_0 \in L^\infty(\mathbb{R}^3)$, Hölder's inequality and the boundedness of $s_\varepsilon, t_\varepsilon$, one obtains

$$\begin{aligned}
\Pi_7 &\le C\int_{\mathbb{R}^3} \big(|s_\varepsilon v_0|^5 |t_\varepsilon v_\varepsilon| + |t_\varepsilon v_\varepsilon|^5 |s_\varepsilon v_0|\big) \, dx \\
&\le C|v_0|_\infty^5 \int_{\mathbb{R}^3} |v_\varepsilon| \, dx + C|v_0|_\infty \int_{\mathbb{R}^3} |v_\varepsilon|^5 \, dx \\
&\le C\varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{4.23}$$

Thus, in virtue of (4.16)–(4.23) and (ii) of Lemma 2.2, as $\varepsilon \to 0$, one obtains

$$\begin{aligned}
m_\lambda^\mu &\le I_\lambda^\mu(s_\varepsilon v_0 - t_\varepsilon v_\varepsilon) \\
&\le I_\lambda^\mu(v_0) + c^* + C\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{1}{4}} + -CR\varepsilon^{\frac{1}{4}} \\
&< c_\lambda^\mu + c^* \quad (\text{since } R > 0 \text{ large enough}),
\end{aligned}$$

we conclude that $m_\lambda^\mu < c_\lambda^\mu + c^*$. Thus, we complete the proof. □

Similar to [15, Lemma 3.6], we can obtain the following lemma by a standard argument.

Lemma 4.4. *For $\bar{s}, \bar{t} > 0$, the following system*

$$\begin{cases} l_1(\bar{s}, \bar{t}) = \bar{s} - aS(\bar{s} + \bar{t})^{\frac{1}{3}} = 0, \\ l_2(\bar{s}, \bar{t}) = \bar{t} - bS^2(\bar{s} + \bar{t})^{\frac{2}{3}} = 0, \end{cases} \tag{4.24}$$

has a unique solution $(\bar{s}_0, \bar{t}_0)$. Furthermore, if $l_1(\bar{s}, \bar{t}) \ge 0$ and $l_2(\bar{s}, \bar{t}) \ge 0$, then

$$\bar{s} \ge \bar{s}_0 \quad \text{and} \quad \bar{t} \ge \bar{t}_0,$$

where

$$\bar{s}_0 = \frac{abS^3 + a\sqrt{b^2S^6 + 4aS^3}}{2} \quad \text{and} \quad \bar{t}_0 = \frac{b^3S^6 + 2abS^3 + b^2S^3\sqrt{b^2S^6 + 4aS^3}}{2}.$$

Lemma 4.5. *Assume that (V_1)–(V_3), (f_1)–(f_3) and (f_4') hold. Then there exists $\Lambda > 1$ such that for any sequence $\{u_n\} \subset U$ satifying*

$$I_\lambda^\mu(u_n) \to m_\lambda^\mu \in (0, c_\lambda^\mu + c^*) \quad \text{and} \quad (I_\lambda^\mu)'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{4.25}$$

admits a strongly convergent subsequence for all $\lambda \geq \Lambda$.

Proof. By (4.25), we can deduce that

$$\begin{aligned} m_\lambda^\mu + o(1) &= I_\lambda^\mu(u_n) - \frac{1}{4}\langle (I_\lambda^\mu)'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_\lambda^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(u_n)u_n - 4F(u_n)]dx \\ &\geq \frac{1}{4}\|u_n\|_\lambda^2, \end{aligned} \tag{4.26}$$

which implies that $\{u_n\}$ is bounded in E_λ. Up to a subsequence $\{u_n\}$, there exists $u_\lambda \in E_\lambda$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & \text{in } E_\lambda, \\ u_n \to u_\lambda, & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } q \in [2,6), \\ u_n(x) \to u_\lambda(x), & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Set $v_n := u_n - u_\lambda$. By the Brézis–Lieb lemma, we get

$$\int_{\mathbb{R}^3} |v_n|^6\, dx = \int_{\mathbb{R}^3} |u_n|^6\, dx - \int_{\mathbb{R}^3} |u_\lambda|^6\, dx + o(1). \tag{4.27}$$

Furthermore,

$$\begin{cases} \|u_n\|_\lambda^2 = \|v_n\|_\lambda^2 + \|u_\lambda\|_\lambda^2 + o(1), \\ \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx + o(1), \\ (\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 = (\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 + (\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx)^2 + 2\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx + o(1). \end{cases} \tag{4.28}$$

Let $\rho \in \mathbb{R}$ such that $\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx \to \rho^2$. By the weak lower semicontinuity of norm, we have that $\rho^2 \geq \int_{\mathbb{R}^3} |\nabla u_\lambda|^2\, dx$. We define

$$\begin{aligned} J_\lambda^\mu(u) := \frac{a + b\rho^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} \lambda V(x) u^2\, dx \\ - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\, dx - \mu\int_{\mathbb{R}^3} F(u) dx, \end{aligned} \tag{4.29}$$

$$\begin{aligned} \langle (J_\lambda^\mu)'(u), v\rangle := (a + b\rho^2)\int_{\mathbb{R}^3} \nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^3} \lambda V(x) uv\, dx \\ - \int_{\mathbb{R}^3} |u|^4 uv\, dx - \mu\int_{\mathbb{R}^3} f(u)v\, dx. \end{aligned} \tag{4.30}$$

Since $(I_\lambda^\mu)'(u_n) \to 0$ and $u_n \rightharpoonup u_\lambda$ in E_λ, one has

$$\int_{\mathbb{R}^3} (a\nabla u_\lambda \cdot \nabla\varphi + \lambda V(x) u_\lambda \varphi)\, dx + b\rho^2\int_{\mathbb{R}^3} \nabla u_\lambda \cdot \nabla\varphi\, dx - \int_{\mathbb{R}^3} |u_\lambda|^4 u_\lambda \varphi\, dx - \mu\int_{\mathbb{R}^3} f(u_\lambda)\varphi\, dx = 0,$$

for any $\varphi \in E_\lambda$, which implies that $(J_\lambda^\mu)'(u_\lambda) = 0$. Using (4.27) and (4.28), we get

$$(a + b\rho^2) \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^3} \lambda V(x) v_n^2 \, dx - \mu \int_{\mathbb{R}^3} f(v_n) v_n \, dx - \int_{\mathbb{R}^3} |v_n|^6 \, dx = o(1). \tag{4.31}$$

By (2.1), (2.9) and (4.31), one obtains

$$\begin{aligned} \|v_n\|_\lambda^2 &\le \int_{\mathbb{R}^3} |v_n|^6 \, dx + \mu \int_{\mathbb{R}^3} f(v_n) v_n \, dx + o(1) \\ &\le \mu\varepsilon \int_{\mathbb{R}^3} |v_n|^2 \, dx + C \int_{\mathbb{R}^3} |v_n|^6 \, dx \le \mu\varepsilon S_2^2 \|v_n\|_\lambda^2 + C\|v_n\|_\lambda^6. \end{aligned}$$

Choosing $1 - \mu\varepsilon S_2^2 > 0$ and $\widetilde{C} > 0$ independent of λ, we get

$$\|v_n\|_\lambda^2 \le \tilde{C} \|v_n\|_\lambda^6. \tag{4.32}$$

Next, we show $v_n \to 0$ in E_λ. Then, one of two following cases occur:

Case (i): $\|v_n\|_\lambda^2 = 0$;

Case (ii): $\|v_n\|_\lambda^2 > l > 0$ for some $l > 0$ independent of λ.

If case (i) holds, the proof is completed.

If case (ii) holds, let $u_n = u_{\lambda,n}$ and $v_n = v_{\lambda,n}$, by the same to the prove of Lemma 3.2, one gets

$$\liminf_{\lambda\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_{\lambda,n}|^2 dx = 0,$$

then, there exists a subsequence $\{v_{\lambda_k,k}\} = \{u_{\lambda_k,k} - u_\lambda\}$ such that $v_{\lambda_k,k} \to 0$ in $L^q(\mathbb{R}^3)$ with $q \in (2,6)$.

Since

$$\lim_{n\to\infty} I_\lambda^\mu(u_{\lambda,n}) = m_\lambda^\mu, \quad \langle (I_\lambda^\mu)'(u_{\lambda,n}), u_{\lambda,n} \rangle \to 0 \quad \text{as } n \to \infty, \tag{4.33}$$

and from Lemma 3.3, we have $c_\lambda^\mu < c^*$. Then, taking the limit of (4.33), we deduce

$$\liminf_{\lambda\to\infty} \lim_{n\to\infty} I_\lambda^\mu(u_{\lambda,n}) = \liminf_{\lambda\to\infty} m_\lambda^\mu < 2c^*, \tag{4.34}$$

and

$$\liminf_{\lambda\to\infty} \lim_{n\to\infty} \langle (I_\lambda^\mu)'(u_{\lambda,n}), u_{\lambda,n} \rangle = 0. \tag{4.35}$$

By (4.34) and (4.35), as $k \to \infty$, one has

$$I_{\lambda_k}^\mu(u_{\lambda_k,k}) < 2c^*, \qquad \langle (I_{\lambda_k}^\mu)'(u_{\lambda_k,k}), u_{\lambda_k,k} \rangle \to 0.$$

Since $u_{\lambda,n} \rightharpoonup u_\lambda$, by a direct calculation, one has

$$I_\lambda^\mu(u_{\lambda,n}) \to J_\lambda^\mu(v_{\lambda,n}) + J_\lambda^\mu(u_\lambda) - \frac{bA^4}{4}. \tag{4.36}$$

From $(J_\lambda^\mu)'(u_\lambda) = 0$, one has

$$\begin{aligned} & J_\lambda^\mu(u_\lambda) - \frac{b}{4}\rho^2 \int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \, dx \\ &= \frac{1}{2}\|u_\lambda\|_\lambda^2 + \frac{b}{4}\rho^2 \int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \, dx - \mu \int_{\mathbb{R}^3} F(u_\lambda) dx - \frac{1}{6} \int_{\mathbb{R}^3} |u_\lambda|^6 \, dx \\ &\quad - \frac{1}{4} \left\langle (J_\lambda^\mu)'(u_\lambda), u_\lambda \right\rangle \\ &= \frac{1}{4}\|u_\lambda\|_\lambda^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} [f(u_\lambda)u_\lambda - 4F(u_\lambda)] dx + \frac{1}{12} \int_{\mathbb{R}^3} |u_\lambda|^6 \, dx \\ &\ge 0. \end{aligned} \tag{4.37}$$

Then, by (4.36) and (4.37), we get

$$\lim_{n\to\infty} I^{\mu}_{\lambda}(u_{\lambda,n}) \geq \lim_{n\to\infty} J^{\mu}_{\lambda}(v_{\lambda,n}).$$

Taking the limit with respect to λ in the above inequality, we obtain

$$\liminf_{\lambda\to\infty}\lim_{n\to\infty} I^{\mu}_{\lambda}(u_{\lambda,n}) \geq \liminf_{\lambda\to\infty}\lim_{n\to\infty} J^{\mu}_{\lambda}(v_{\lambda,n}),$$

and consequently,

$$\lim_{k\to\infty} I^{\mu}_{\lambda_k}(u_{\lambda_k,k}) \geq \lim_{k\to\infty} J^{\mu}_{\lambda_k}(v_{\lambda_k,k}). \tag{4.38}$$

Moreover, $\langle (J^{\mu}_{\lambda})'(v_{\lambda,n}), v_{\lambda,n}\rangle \to 0$ $(n\to\infty)$, by a similar argument as above, together with the properties of subsequences, we obtain

$$\langle (J^{\mu}_{\lambda_k})'(v_{\lambda_k,k}), v_{\lambda_k,k}\rangle, \quad \text{as } k\to\infty. \tag{4.39}$$

From Lemma 3.2, we get $u_{\lambda_k,k} \to u_\lambda$ in $L^s(\mathbb{R}^3)$ for all $s\in(2,6)$, then, by $(f_1)-(f_3)$ and (f_4'), one has

$$\int_{\mathbb{R}^3} f(u_{\lambda_k,k})u_{\lambda_k,k}dx = \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda dx + o(1); \int_{\mathbb{R}^3} F(u_{\lambda_k,k})dx = \int_{\mathbb{R}^3} F(u_\lambda)dx + o(1). \tag{4.40}$$

By (4.39)–(4.40), we get

$$(a+b\rho^2)\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx + \lambda_k \int_{\mathbb{R}^3} V(x)|v_{\lambda_k,k}|^2 dx = \int_{\mathbb{R}^3} |v_{\lambda_k,k}|^6\,dx + o(1).$$

Thus there exist $l_i \geq 0$ $(i=1,2,3)$ such that

$$a\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx + \lambda_k \int_{\mathbb{R}^3} V(x)v^2_{\lambda_k,k}dx \to l_1, \quad b\rho^2 \int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx \to l_2, \quad \int_{\mathbb{R}^3} |v_{\lambda_k,k}|^6 dx \to l_3.$$

Moreover, by the Sobolev inequality, one obtains

$$a\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx + \lambda_k \int_{\mathbb{R}^3} V(x)v^2_{\lambda_k,k}dx \geq aS\left(\int_{\mathbb{R}^3} |v_{\lambda_k,k}|^6\,dx\right)^{\frac{1}{3}},$$

and

$$b\left(\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2\,dx\right)^2 \geq bS^2\left(\int_{\mathbb{R}^3} |v_{\lambda_k,k}|^6\,dx\right)^{\frac{2}{3}},$$

then

$$l_1 \geq aS(l_1+l_2)^{\frac{1}{3}} \quad \text{and} \quad l_2 \geq bS^2(l_1+l_2)^{\frac{2}{3}}.$$

By Lemma 4.4, we get

$$\begin{aligned} \frac{1}{3}l_1 + \frac{1}{12}l_2 &\geq \frac{1}{3}\frac{abS^3 + a\sqrt{b^2S^6+4aS^3}}{2} + \frac{1}{12}\frac{b^3S^6 + 2abS^3 + b^2S^3\sqrt{b^2S^6+4aS^3}}{2} \\ &= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4+4aS)^{\frac{3}{2}}}{24} = c^*. \end{aligned} \tag{4.41}$$

Consequently, by (4.38) and (4.41), one gets

$$\begin{aligned} c^* &> \lim_{k\to\infty} I^{\mu}_{\lambda_k}(u_{\lambda_k,k}) \geq \lim_{k\to\infty}\left(J^{\mu}_{\lambda_k}(v_{\lambda_k,k}) - \frac{1}{6}\langle (J^{\mu}_{\lambda_k})'(v_{\lambda_k,k}), v_{\lambda_k,k}\rangle\right) \\ &= \lim_{k\to\infty}\left(\frac{a}{3}\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx + \frac{\lambda_k}{3}\int_{\mathbb{R}^3} V(x)v^2_{\lambda_k,k}dx + \frac{b\rho^2}{3}\int_{\mathbb{R}^3} |\nabla v_{\lambda_k,k}|^2 dx\right) \\ &\geq \frac{1}{3}l_1 + \frac{1}{12}l_2 \geq c^*, \end{aligned}$$

which is a contradiction. Hence case (ii) is impossible. Therefore only case (i) holds, i.e. $u_n \to u_\lambda$ strongly in E_λ. This completes the proof. □

Now, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. By Lemma 4.1, there exists $\{u_n\} \subset U$ such that $I_\lambda^\mu(u_n) \to m_\lambda^\mu$ and $(I_\lambda^\mu)'(u_n) \to 0$ as $n \to \infty$. Combining Lemma 4.3 with Lemma 4.5, we know that $\{u_n\}$ contains a strongly convergent subsequence. Thus, there exists $u_\lambda \in E_\lambda$ such that $u_n \to u_\lambda$ in E_λ as $n \to \infty$. We can deduce that $I_\lambda^\mu(u_\lambda) = m_\lambda^\mu$ and $(I_\lambda^\mu)'(u_\lambda) = 0$. Similar to the calculation of (2.24), we have $\|u_n^\pm\|_\lambda \geq C_1 > 0$, which means that $\|u_\lambda^\pm\|_\lambda \geq C_1 > 0$ and $u_\lambda^\pm \in \mathcal{M}_\lambda^\mu$. Then, u_λ is a least energy sign-changing solution of problem (1.1).

Next, the same to the proof of Theorem 1.1, we can prove that u_λ has exactly two nodal domains and its energy is strictly larger than twice that of the positive ground state solution. □

The proof of Theorem 1.5 is analogous to that of Theorem 1.2, so we omit the details here.

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