

Existence for semi-infinite cline with neutral belt

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Abstract. In this paper we study the semi-infinite Neumann problem, which models the allele frequency variation in a semi-infinite habitat with a neutral belt. We prove that there exists an unbounded solution branch emanating from $(k_1, 0)$, where k_1 is the smallest root of $k = \arctan[\alpha/(1 + \alpha\beta k)]$. The system admits only the zero solution when $k \leq k_1$, while it possesses a positive solution when $k > k_1$; this solution does not exceed 1 in $C[0, +\infty)$ and decays exponentially to 0. These results confirm the necessity part of Nagylaki's conjecture.

Keywords: bifurcation, cline, Nagylaki's conjecture, neutral belt.

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1 Introduction

A cline is a spatial gradient in allele or genotype frequency across a geographic region, shaped by the interplay of natural selection, gene flow, and environmental heterogeneity. The study of clines advances the theory of spatially varying selection and has implications for conservation biology, range dynamics, and habitat fragmentation.

A fundamental problem in cline theory is the existence of nontrivial equilibria in spatially heterogeneous environments. Consider a single locus with alleles A and a in a monoecious population, in the absence of mutation and random drift. Let p denote the frequency of allele A . We consider the following problem

$$\begin{cases} \frac{d^2 p}{d\zeta^2} + k^2 g(\zeta) p(1-p) = 0, & \zeta > 0, \\ \frac{dp}{d\zeta} = 0 & \text{at } 0, +\infty, \end{cases} \quad (1.1)$$

where $k^2 = 2sa^2/\sigma^2$ combines the selection intensity $s > 0$, the spatial scale $a > 0$, and the migration variance $\sigma^2/2$. The function $g(\zeta)$ is a piecewise constant selection coefficient, which will be specified below.

For the semi-infinite habitat with a piecewise selection coefficient $g(\zeta) = 1$ for $0 \leq \zeta \leq 1$ and $g(\zeta) = -\alpha^2$ for $\zeta > 1$, Nagylaki [16] proved that a unique nontrivial cline exists if and only if $k > \arctan \alpha$, but his analysis relied on the boundary value $p(0)$ (the allele frequency at

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$\xi = 0$) and left the stability as an open conjecture. Using global bifurcation theory, the authors [3] recently established the uniqueness of the full solution $p(\xi)$, identified $(\arctan \alpha, 0)$ as the bifurcation point, and rigorously proved the stability of both trivial and nontrivial solutions.

In biological terms, the neutral belt models the “transition zone” in an environmental gradient. The present paper extends this investigation to a habitat containing a neutral belt, where the selection coefficient $g(\xi)$ takes the form

$$g(\xi) = \begin{cases} 1, & 0 \leq \xi \leq 1, \\ 0, & 1 < \xi \leq 1 + \beta, \\ -\alpha^2, & \xi > 1 + \beta, \end{cases} \quad (1.2)$$

with $\alpha > 0$, $\beta \geq 0$. The neutral belt $(1, 1 + \beta]$ introduces two novel features absent in the case $\beta = 0$ studied in [3]: (i) the linearized eigenvalue problem yields an implicit equation

$$k = \arctan \left(\frac{\alpha}{1 + \alpha \beta k} \right), \quad (1.3)$$

in contrast to the explicit relation $k = \arctan \alpha$ for $\beta = 0$; this indicates that even if the selection intensities in the favorable region ($0 \leq \xi \leq 1$) and the unfavorable region ($\xi > 1 + \beta$) remain unchanged, the width of the neutral belt itself significantly affects the threshold for cline formation; (ii) the presence of a selection-free interval complicates the analysis of solution behavior across this region.

Let k_1 be the smallest positive root of (1.3) and $p_0 = p(0)$. By analyzing how p_0 depends on the parameter k , Nagylaki [16] obtained that $k > k_1$ is a sufficient condition for the existence of at least one cline, and such a cline corresponds to a solution of (1.1). In fact, he only derived an equation for p_0 but did not prove the existence of a full solution $p(\xi)$. Meanwhile, he also conjectured that the condition $k > k_1$ is necessary and the corresponding cline is unique: necessity and uniqueness have not been proved, but are highly plausible. To the best of our knowledge, this conjecture remains open. The main aim of this paper is to affirm the necessity part of Nagylaki’s conjecture. Furthermore, we address two additional questions: does the cline p (rather than p_0) form a continuous solution branch with respect to the parameter k ? And what is the decay rate of $p(\xi)$ as $\xi \rightarrow +\infty$? We provide complete and satisfactory answers to these questions.

The mathematical analysis of clines has a rich history. Fisher’s 1937 contribution marked the inception of mathematical treatments combining dispersal and natural selection [7]. This line of inquiry was advanced by Haldane [9], who, in 1948, formulated a one-dimensional reaction-diffusion equation describing equilibrium allele frequencies at a diallelic locus under spatially heterogeneous selection, an achievement commonly acknowledged as the earliest mathematical theory of cline. Subsequent decades saw extensive developments (e.g., [16–26, 31]); foundational results concerning existence, uniqueness, and stability were established [1, 5, 6, 8, 10]. More recently, the framework has been extended to general elliptic operators [11–15, 27, 28], multi-locus dynamics [32], and bifurcation analysis [29]. The present work differs from these studies in its focus on an unbounded domain with a discontinuous piecewise-constant weight (including a neutral region) and logistic nonlinearity $p(1 - p)$.

Since the habitat is unbounded and the selection coefficient is negative for large ξ (the unfavorable region), it is natural to impose that the cline decays to zero. Consequently, we henceforth require that the solution p to (1.1) satisfies $\lim_{\xi \rightarrow +\infty} p(\xi) = 0$. We introduce the

function space X as

$$X = \left\{ p \in C[0, +\infty) : p'(0) = 0, \lim_{\xi \rightarrow +\infty} \frac{p(\xi) e^{\frac{\alpha k_1}{\sqrt{3}} \xi}}{\xi} = 0 \right\}$$

equipped with the supremum norm $\|p\| = \sup_{\xi \in [0, +\infty)} |p(\xi)|$. It is straightforward to verify that X is a Banach space. The nonnegative cone of X is denoted by $X^+ = \{p \in X : p(\xi) \geq 0 \text{ for all } \xi\}$, equipped with the induced norm, and P^+ represents the subset of functions in X^+ that are strictly positive on $[0, +\infty)$. Finally, we define the parameter-function set $K^+ = (0, +\infty) \times P^+$ endowed with the product topology.

The main result of this paper is the following theorem.

Theorem 1.1. *The pair $(k_1, 0)$ is a bifurcation point of problem (1.1) and the associated bifurcation branch $\mathcal{C} \subseteq (K^+ \cup \{(k_1, 0)\})$ is unbounded and satisfies $0 < p(\xi) \leq 1$ for all $\xi \geq 0$ for every $(k, p) \in \mathcal{C}$. Moreover, for $k \leq k_1$, problem (1.1) has only the trivial solution $p(\xi) \equiv 0$; while for $k > k_1$, there exists a solution $p(\xi)$ with $0 < p(\xi) \leq 1$. In addition, for each $(k, p) \in \mathcal{C}$, $p(\xi)$ decays exponentially to 0 as $\xi \rightarrow +\infty$.*

Theorem 1.1 establishes that $k > k_1$ is both necessary and sufficient for the existence of a nontrivial cline in a habitat with a neutral belt, thereby resolving the necessity part of Nagylaki's conjecture in this more general setting. Furthermore, we obtain the global bifurcation structure (see Figure 1.1) and the decay rate of the cline at infinity. The special case $\beta = 0$ was studied in [3].

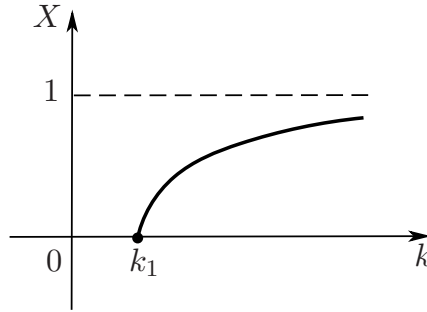


Figure 1.1: Bifurcation diagram of Theorem 1.1.

The bifurcation diagram in Figure 1.1 illustrates that a branch of positive solutions emanates from the trivial solution at $k = k_1$ and remains bounded above by 1 as $k \rightarrow +\infty$.

The rest of this paper is arranged as follows. Section 2 begins with a study of the spectrum of the linearized problem. Through an auxiliary problem, we then show that the cline satisfies $0 < p(\xi) \leq 1$ and further obtain its decay behavior at infinity. Section 3 establishes the global bifurcation structure and completes the proof of Theorem 1.1.

2 Preliminaries

We start by analyzing the spectral properties of the linearized problem. Recall that $g(\xi)$ is defined in (1.2). Linearizing system (1.1) about the trivial equilibrium 0 yields

$$\begin{cases} \frac{d^2 p}{d\tilde{\xi}^2} + k^2 g(\tilde{\xi}) p = 0, & \tilde{\xi} > 0, \\ \frac{dp}{d\tilde{\xi}} = 0 & \text{at } 0, +\infty. \end{cases} \quad (2.1)$$

Regarding the spectrum of problem (2.1), we have the following conclusion.

Proposition 2.1. *Problem (2.1) possesses a sequence of positive eigenvalues k_i such that $\tan(k_i) = \alpha / (1 + \alpha \beta k_i)$ and*

$$(i-1)\pi < k_i < \frac{\pi}{2} + (i-1)\pi, \quad i \geq 1.$$

The eigenfunction φ_i corresponding to k_i has exactly $i-1$ simple zeros $\pi / (2k_i), (3\pi) / (2k_i), \dots, ((2i-3)\pi) / (2k_i)$ in $(0, 1)$. In particular, φ_1 is positive.

Proof. Solving the constant-coefficient ODEs on each interval and using the boundary conditions in (2.1), we obtain

$$p(\tilde{\xi}) = \begin{cases} A \cos(k\tilde{\xi}), & 0 \leq \tilde{\xi} \leq 1, \\ a\tilde{\xi} + b, & 1 < \tilde{\xi} \leq 1 + \beta, \\ B e^{-\alpha k \tilde{\xi}}, & \tilde{\xi} > 1 + \beta. \end{cases}$$

Since $p(\tilde{\xi})$ and $p'(\tilde{\xi})$ are continuous at $\tilde{\xi} = 1$ and $\tilde{\xi} = 1 + \beta$, we find that

$$\begin{cases} A \cos(k) = a + b, \\ a(1 + \beta) + b = B e^{-\alpha k(1 + \beta)}, \\ -A k \sin(k) = a, \\ a = -B \alpha k e^{-\alpha k(1 + \beta)}. \end{cases}$$

In order for $p(\tilde{\xi})$ to be nontrivial, the above linear system in the unknowns A, B, a, b must have a nonzero solution. Hence its coefficient matrix must be singular, leading to

$$0 = \begin{vmatrix} \cos(k) & 0 & -1 & -1 \\ 0 & -e^{-\alpha k(1 + \beta)} & 1 + \beta & 1 \\ -k \sin(k) & 0 & -1 & 0 \\ 0 & \alpha k e^{-\alpha k(1 + \beta)} & 1 & 0 \end{vmatrix}.$$

Expanding the determinant gives

$$\begin{aligned} & \cos(k) \cdot (\alpha k e^{-\alpha k(1 + \beta)}) - k \sin(k) \cdot (e^{-\alpha k(1 + \beta)}(1 + \alpha k \beta)) = 0, \\ & \Rightarrow \alpha \cos(k) - (1 + \alpha k \beta) \sin(k) = 0, \\ & \Rightarrow \tan(k) = \frac{\alpha}{1 + \alpha \beta k}. \end{aligned} \quad (2.2)$$

Thus, problem (2.1) possesses a sequence of positive eigenvalues k_i such that k_i satisfies (2.2) and

$$(i-1)\pi < k_i < \frac{\pi}{2} + (i-1)\pi, \quad i = 1, 2, \dots$$

Furthermore, the corresponding eigenfunction is given by

$$\varphi_i(\tilde{\xi}) = \begin{cases} \cos(k_i \tilde{\xi}), & 0 \leq \tilde{\xi} \leq 1, \\ -k_i \sin(k_i) \tilde{\xi} + \cos(k_i) + k_i \sin(k_i), & 1 < \tilde{\xi} \leq 1 + \beta, \\ \frac{\sin(k_i)}{\alpha} e^{\alpha k_i(1 + \beta)} e^{-\alpha k_i \tilde{\xi}}, & \tilde{\xi} > 1 + \beta. \end{cases}$$

From the range of k_i , one sees that $\cos(k_i) \neq 0$. Further, since $\tan(k_i) = \alpha / (1 + \alpha\beta k_i) > 0$, we can derive that $\cos(k_i)$ and $\sin(k_i)$ have the same sign. Hence $\varphi_i(\xi)$ does not change sign for $\xi > 1$. Using the range of k_i again, we see that $\cos(k_i\xi)$ has exactly $i - 1$ simple zeros in $(0, 1)$. In particular, φ_1 is positive. For $i \geq 2$, the zeros of φ_i are $\pi / (2k_i), (3\pi) / (2k_i), \dots, ((2i - 3)\pi) / (2k_i)$. $\square$

The locations of the eigenvalues k_i from Proposition 2.1 are shown in Figure 2.1.

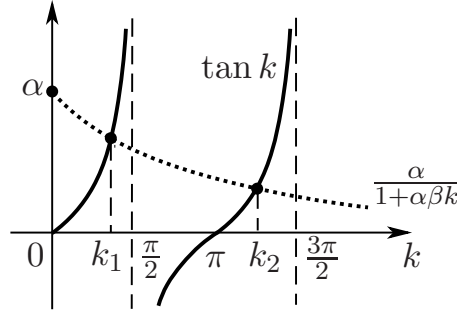


Figure 2.1: The location of the eigenvalues k_i .

To study the decay rate of solutions at infinity, we first study the range of solutions to problem (1.1). Define

$$f(s) = \begin{cases} 0 & \text{if } s < 0, \\ s(1 - s) & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s > 1 \end{cases}$$

and consider the following auxiliary problem

$$\begin{cases} \frac{d^2 p}{d\xi^2} + k^2 g(\xi) f(p) = 0, & \xi > 0, \\ \frac{dp}{d\xi} = 0 & \text{at } 0, +\infty. \end{cases} \quad (2.3)$$

Following the same argument as in [3, Lemma 2.2], with obvious modifications, we can show that $0 \leq p(\xi) \leq 1$ for all $\xi \geq 0$ for any nonconstant solution $p(\xi)$ of problem (2.3); hence it is also a solution of the original problem (1.1). So we only need to study the existence of nontrivial solutions for the auxiliary problem (2.3). Further, as in [3, Lemma 2.3 and 2.4], we can show that $0 < p(\xi) \leq 1$ for every nonconstant solution $p(\xi)$ of problem (2.3).

Now we return to the decay rate of a nonconstant solution $p(\xi)$ of problem (2.3) at $+\infty$. Define $p_* := p(1 + \beta)$.

Proposition 2.2. *Let $p(\xi)$ be a nonconstant solution of (2.3). Then for $\xi > 1 + \beta$, one has that*

$$p_* e^{-\alpha k(\xi - 1 - \beta)} < p(\xi) \leq p_* e^{-\frac{\alpha k}{\sqrt{3}}(\xi - 1 - \beta)}.$$

Proof. For $\xi > 1 + \beta$, from (2.3) we find that

$$\frac{d^2 p}{d\xi^2} = k^2 \alpha^2 p(1 - p).$$

Set $u(\xi) = \frac{dp}{d\xi}$. Then

$$\frac{d^2p}{d\xi^2} = \frac{du}{d\xi} = \frac{du}{dp} \cdot \frac{dp}{d\xi} = u \frac{du}{dp}.$$

Thus

$$u \frac{du}{dp} = k^2 \alpha^2 p(1-p).$$

Integrating with respect to ξ from ξ to $+\infty$ (which is equivalent to integrating with respect to $p(\xi)$ from $p(\xi)$ to 0, since $p(\xi) \rightarrow 0$ and $u(\xi) \rightarrow 0$ as $\xi \rightarrow +\infty$), we obtain

$$\frac{1}{2}u^2 = k^2 \alpha^2 \int_0^p s(1-s) ds = k^2 \alpha^2 p^2 \left(\frac{1}{2} - \frac{1}{3}p \right).$$

As noted by Nagylaki [16], for this neutral belt model the allele frequency $p(\xi)$ is decreasing for $\xi \geq 0$, and hence $u(\xi) = p'(\xi) \leq 0$ for $\xi \geq 0$, so one sees that

$$u = -k\alpha p \left(1 - \frac{2}{3}p \right)^{1/2} \neq 0$$

for $\xi > 1 + \beta$. Thus

$$d\xi = \frac{dp}{u} = -\frac{dp}{k\alpha p \left(1 - \frac{2}{3}p \right)^{1/2}}.$$

Integrating from $1 + \beta$ to ξ gives

$$\alpha k(\xi - 1 - \beta) = \int_{p(\xi)}^{p_*} \frac{1}{s \left(1 - \frac{2}{3}s \right)^{1/2}} ds.$$

Since $0 < p(\xi) \leq 1$, we have

$$\ln \frac{p_*}{p(\xi)} = \int_{p(\xi)}^{p_*} \frac{1}{s} ds < \int_{p(\xi)}^{p_*} \frac{1}{s \left(1 - \frac{2}{3}s \right)^{1/2}} ds = \alpha k(\xi - 1 - \beta)$$

and

$$\alpha k(\xi - 1 - \beta) = \int_{p(\xi)}^{p_*} \frac{1}{s \left(1 - \frac{2}{3}s \right)^{1/2}} ds \leq \sqrt{3} \int_{p(\xi)}^{p_*} \frac{1}{s} ds = \sqrt{3} \ln \frac{p_*}{p(\xi)},$$

which implies the desired inequalities. $\square$

Proposition 2.2 indicates that any nontrivial solution of problem (2.3) that satisfies $\lim_{\xi \rightarrow +\infty} p(\xi) = 0$ decays exponentially to 0 at infinity. Further we have the following result.

Lemma 2.3. *If $p(\xi)$ is any nonconstant solution of problem (2.3), one has $k > k_1$ and*

$$\lim_{\xi \rightarrow +\infty} \frac{p(\xi) e^{\frac{\alpha k_1}{\sqrt{3}} \xi}}{\xi} = 0,$$

where k_1 is the first eigenvalue of problem (2.1).

Proof. Multiplying the equation for p by φ_1 and integrating by parts, we obtain

$$\int_0^{+\infty} p'(\xi) \varphi_1'(\xi) d\xi = \int_0^{+\infty} k^2 g(\xi) p(\xi) \varphi_1(\xi) (1 - p(\xi)) d\xi. \quad (2.4)$$

Note that $p(\xi)$ is nonconstant, $p'(\xi) \leq 0$ and $p'(\xi) \not\equiv 0$. Since φ_1 is strictly decreasing, we conclude that

$$\int_0^{+\infty} p'(\xi) \varphi_1'(\xi) d\xi > 0. \quad (2.5)$$

It follows from (2.4) and (2.5) that

$$\int_0^{+\infty} g(\xi) p(\xi) \varphi_1(\xi) (1 - p(\xi)) d\xi > 0. \quad (2.6)$$

Then, multiplying the equation for φ_1 by $p(1 - p)$ and integrating by parts, then subtracting from (2.4), we obtain

$$2 \int_0^{+\infty} p(\xi) p'(\xi) \varphi_1'(\xi) d\xi = (k^2 - k_1^2) \int_0^{+\infty} g(\xi) p(\xi) \varphi_1(\xi) (1 - p(\xi)) d\xi. \quad (2.7)$$

By (2.6) and (2.7) we see that

$$k > k_1.$$

On the other hand, from Proposition 2.2 we have

$$p(\xi) \leq p_* e^{-\frac{ak}{\sqrt{3}}(\xi-1-\beta)}.$$

Since $k > k_1$, for $\xi > 1 + \beta$ we have

$$p(\xi) < p_* e^{-\frac{ak_1}{\sqrt{3}}(\xi-1-\beta)}.$$

It follows that

$$\frac{p(\xi) e^{\frac{ak_1}{\sqrt{3}}\xi}}{\xi} < \frac{p_* e^{\frac{ak_1(1+\beta)}{\sqrt{3}}}}{\xi} \rightarrow 0$$

as $\xi \rightarrow +\infty$, which gives the desired conclusion. $\square$

We conclude this section by stating a Rabinowitz-type global bifurcation theorem. Let E be a real Banach space with norm $\|\cdot\|$, and let $\mathcal{O}$ be an open subset of $\mathbb{R} \times E$. Denote by $\text{pr}_E(\mathcal{O})$ and $\text{pr}_{\mathbb{R}}(\mathcal{O})$ the projections of $\mathcal{O}$ onto E and $\mathbb{R}$, respectively. Consider an operator equation

$$u = L(\lambda)u + H(\lambda, u) =: G(\lambda, u), \quad (2.8)$$

where $\lambda \in \text{pr}_{\mathbb{R}}(\overline{\mathcal{O}})$. The map $\lambda \mapsto L(\lambda)$ is continuous, with $L(\cdot) : \text{pr}_{\mathbb{R}}(\overline{\mathcal{O}}) \rightarrow \text{pr}_E(\overline{\mathcal{O}})$ a homogeneous, completely continuous operator. Moreover, $H : \overline{\mathcal{O}} \rightarrow E$ is compact and satisfies $H = o(\|u\|)$ as $u \rightarrow 0$, uniformly over bounded λ -intervals in $\overline{\mathcal{O}}$. Define the solution set

$$\mathcal{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ satisfies (2.8) and } u \neq 0\}}^{\overline{\mathcal{O}}}.$$

We say μ is an eigenvalue of

$$u = L(\mu)u, \quad u \in E, \quad (2.9)$$

if there exists a nontrivial $\varphi \in E \setminus \{0\}$ such that $\varphi = L(\mu)\varphi$. The set of all real eigenvalues of (2.9) is denoted by Σ . For any $\lambda \notin \Sigma$ and any open ball $B_R(0) \subset \mathcal{O}$, the Leray–Schauder degree $\deg(I - L(\lambda), B_R(0), 0)$ is well-defined.

We recall the following global bifurcation result, stated in [2, Lemma 2.5], which complements and generalizes the famous theorem of Rabinowitz [30, Theorem 1.3].

Proposition 2.4. *If $\mu \in \text{pr}_{\mathbb{R}}(\mathcal{O}) \cap \Sigma$ such that the Leray–Schauder degree $\deg(I - L(\lambda), B_R(0))$ changes when λ passes μ , then $\mathcal{S}$ possesses a maximal sub-continuum $C_\mu \subset \overline{\mathcal{O}}$ with $(\mu, 0) \in C_\mu$ and one of the following holds:*

- (i) C_μ is unbounded in $\overline{\mathcal{O}}$,
- (ii) C_μ meets $\partial\mathcal{O} \setminus \{(\mu, 0)\}$,
- (iii) C_μ meets $(\bar{\mu}, 0)$ where $\bar{\mu} \in \text{pr}_{\mathbb{R}}(\overline{\mathcal{O}}) \cap \Sigma$ and $\bar{\mu} \neq \mu$.

3 Proof of Theorem 1.1

Problem (2.3) is equivalent to

$$p(\xi) = k^2 F(p)(\xi),$$

where

$$F(p)(\xi) = \int_{\xi}^{+\infty} \int_{+\infty}^s g(\tau) f(p(\tau)) d\tau ds, \quad \xi \geq 0.$$

Define also

$$T(p)(\xi) = \int_{\xi}^{+\infty} \int_{+\infty}^s g(\tau) p(\tau) d\tau ds, \quad \xi \geq 0.$$

Then, in view of Lemma 2.3, as in [3], $F, T : X \rightarrow X$ are completely continuous. Hence, for any ball $B_R(0) \subset X$ and any $k \in (0, k_2) \setminus \{k_1\}$ where k_2 is the second eigenvalue given in Proposition 2.1, the Leray–Schauder degree $\deg(I - \lambda T, B_R(0), 0)$ is well-defined for $\lambda = k^2$.

Now we prove the main result.

Proof of Theorem 1.1. Problem (2.3) can be written as

$$p(\xi) = k^2 T(p)(\xi) + H(k, p)(\xi),$$

with

$$H(k, p)(\xi) = -k^2 \int_{\xi}^{+\infty} \int_{+\infty}^s g(\tau) p(\tau)^2 d\tau ds.$$

For any $p \in X \setminus \{0\}$,

$$\frac{p(\xi)^2}{\|p\|} \leq \frac{|p(\xi)|}{\|p\|} |p(\xi)| \leq \|p\| \rightarrow 0 \quad \text{as } \|p\| \rightarrow 0.$$

Thus $H(k, p) = o(\|p\|)$ uniformly for k in any bounded interval of $(0, +\infty)$ as $p \rightarrow 0$ in X . Since T is compact and linear, it follows from [4, Theorem 8.10] that

$$\deg(I - k^2 T, B_R(0), 0) = \begin{cases} 1 & \text{if } k \in (0, k_1), \\ -1 & \text{if } k \in (k_1, k_2). \end{cases}$$

With $\mathcal{O} = (0, +\infty) \times X$, Proposition 2.4 implies that $(k_1, 0)$ is a bifurcation point of (2.3). Moreover, the associated bifurcation branch $\mathcal{C}$ in $(0, +\infty) \times X$ (whose closure contains $(k_1, 0)$) satisfies at least one of: (i) $\mathcal{C}$ is unbounded in $(0, +\infty) \times X$; (ii) $\mathcal{C}$ meets X at $k = 0$; (iii) $\mathcal{C}$ contains a point $(\bar{k}, 0)$ where $\bar{k} \neq k_1$ is an eigenvalue of (2.1).

For any $(k, p) \in \mathcal{C}$ with $p \neq 0$, we have $0 < p(\xi) \leq 1$ on $[0, +\infty)$, so

$$\mathcal{C} \subseteq ((0, +\infty) \times X^+) \cup \{(k_1, 0)\}.$$

Lemma 2.3 gives $k > k_1$ for all nontrivial $(k, p) \in \mathcal{C}$. Hence case (ii) is impossible; moreover, $k > k_1$ prevents $\mathcal{C}$ from extending to the left of k_1 . Therefore, for $k \leq k_1$, problem (1.1) admits only the trivial solution $p \equiv 0$. Following the argument in [3], case (iii) can also be excluded. Consequently, $\mathcal{C}$ is unbounded. It neither returns to the trivial axis nor extends into the region $k < k_1$. Since $0 < p(\xi) \leq 1$ for every nontrivial $(k, p) \in \mathcal{C}$, the branch is bounded in the norm of X by 1 (i.e., $\|p\| \leq 1$ for every $(k, p) \in \mathcal{C}$), and therefore must be unbounded in the k -direction. Thus a cline exists if and only if $k > k_1$. This completes the proof of Theorem 1.1. $\square$

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