



Osgood-type barriers for fractional differential equations with regularly varying coefficients

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Abstract. We study finite-time blow-up phenomena for nonlinear fractional differential equations with superlinear source terms and time-dependent coefficients. For $0 < \alpha < 1$ and $p > 1$, we consider both the Riemann–Liouville and Caputo models

$$\mathcal{D}^\alpha u(t) = \lambda(t) u(t)^p + f(t),$$

where $\lambda \geq 0$ is measurable and locally integrable on $(0, \infty)$, while $f \geq 0$ is locally integrable.

We extend the barrier method recently introduced by Webb [J. R. L. Webb, *Electron. J. Differential Equations* **2024**, No. 40, 1–16] and derive a sharp Osgood-type integral criterion for blow-up. In both fractional settings, this criterion provides an explicit upper bound for the maximal interval of existence and, in particular, rules out global nontrivial solutions whenever the condition is satisfied.

We then apply the general framework to coefficients of regularly varying type $\lambda(t) = t^\beta \ell(t)$, where ℓ is slowly varying, and more broadly to the logarithmic-order class $\mathcal{M}(\beta)$. In the subcritical regime, the relevant barrier integral diverges with the same critical exponent $p_0 = \frac{1+\beta}{1-\alpha}$ as in the pure power case. At the critical exponent, however, the qualitative behavior depends on the borderline integrability of $\frac{\ell(s)}{s}$, or on finer oscillatory effects within $\mathcal{M}(\beta)$.

Several examples are included to illustrate the sharpness of the results and the subtle character of the critical threshold.

Keywords: fractional differential equations, blow-up, variable coefficients, Caputo derivative, Riemann–Liouville derivative, Volterra equations, barrier method, slowly varying functions, Karamata theory.

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1 Introduction

Fractional differential equations have been widely studied as models for memory-driven phenomena arising in physics, biology, and viscoelasticity. Among the fundamental nonlinear mechanisms is the interplay between the smoothing effect of the fractional integral and the growth generated by a superlinear source term. A prototypical example is the initial value problem

$$\mathcal{D}^\alpha u(t) = \lambda_0 t^\beta u(t)^p + f(t), \quad t > 0, \quad (1.1)$$

where $0 < \alpha < 1$, $p > 1$, $\beta \in \mathbb{R}$, $\lambda_0 > 0$, and $f \geq 0$. Here $\mathcal{D}^\alpha$ denotes either the Riemann–Liouville derivative or the Caputo derivative. We supplement (1.1) with the initial condition appropriate both to the choice of fractional derivative *and* to the functional setting in which the problem is formulated. Thus, in the Caputo case, we impose

$$u(0) = u_0.$$

For the Riemann–Liouville derivative, when the problem is posed in the weighted space $C_{\alpha-1}[0, T]$ introduced in (2.4), the appropriate initial condition is

$$\lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u_0,$$

whereas, when the problem is considered in $L^1(0, T)$, we prescribe

$$(\mathcal{I}^{1-\alpha} u)(0^+) = \Gamma(\alpha) u_0.$$

The central qualitative issue is to determine whether (1.1) admits global-in-time solutions on $(0, \infty)$ or whether solutions blow up in finite time.

The study of nonexistence and blow-up for (1.1) goes back at least to Laskri and Tatar [9], who identified a critical exponent for a Riemann–Liouville inequality and proved the nonexistence of global solutions in a subcritical range. Subsequently, Zhang–Liu–Wu–Cui [21] claimed a correction with a counterexample. A recent paper by Webb [18] revisited the problem and clarified that the correction and counterexample of [21] are not accurate. Webb introduced a simple and robust barrier method that yields sharp nonexistence results for both Riemann–Liouville and Caputo inequalities and provides explicit upper bounds on any possible interval of existence in the power coefficient case.

The key idea in [18] is to transform the fractional differential problem into an equivalent Volterra integral equation and then derive a one-sided differential inequality for an auxiliary function. For the Riemann–Liouville problem, posed in $C_{\alpha-1}[0, T]$, the initial condition reads

$$\lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u_0,$$

which encodes the characteristic singular behavior $u(t) \sim u_0 t^{\alpha-1}$ as $t \downarrow 0$. This singularity plays a decisive role in the Riemann–Liouville theory and leads to an instantaneous nonexistence phenomenon at the origin in the supercritical range. In contrast, for Caputo problems, one prescribes $u(0) = u_0$ and solutions are continuous at $t = 0$; this shifts the criticality mechanism away from $t = 0$. Remarkably, both derivative types share the same critical exponent

$$p_0 = \frac{1 + \beta}{1 - \alpha},$$

as already observed by Webb [18] in the pure power case; what differs between the two settings is not the value of p_0 but the mechanism of nonexistence.

In the Caputo setting, blow-up criteria and upper bounds have also been studied in recent work, including Shan and Lv [14] for the case $\beta = 0$ and Villa-Morales [15] for systems. As emphasized in [18], however, for fractional inequalities one must distinguish between the nonexistence of global solutions and the genuine blow-up of solutions of an equation. In the Caputo equation case, blow-up at a finite endpoint of the maximal interval of existence can be justified using continuation results, for example those of Elloe–Masthay [4] and Wu–Liu [20].

The present paper extends Webb’s barrier method from the power coefficient $\lambda_0 t^\beta$ to general variable coefficients $\lambda(t) \geq 0$. The correct replacement of the explicit power-law integration appearing in [18] is an integral barrier involving the quantity

$$\Phi(t) := \int_0^t \lambda(s) s^{-p(1-\alpha)} ds,$$

with the lower limit depending on the derivative type. More precisely, the integral defining Φ starts at 0 in the Riemann–Liouville case, where the weighted function $t^{1-\alpha}u(t)$ has the positive limit u_0 at the origin, whereas in the Caputo case $t^{1-\alpha}u(t) \rightarrow 0$ as $t \rightarrow 0^+$, so that no positive lower bound is available near the origin and the integration must start from a positive time; see Subsection 4.1 for a detailed discussion. This leads to a sharp and flexible criterion for the nonexistence of global solutions, and yields explicit maximal-time upper bounds when Φ can be estimated. In particular, our results cover the important case of coefficients of regularly varying type $\lambda(t) = t^\beta \ell(t)$, where ℓ is slowly varying at infinity, as well as the larger class $\mathcal{M}(\beta)$ in the sense of logarithmic order.

We recall that a measurable function ℓ is slowly varying at infinity if $\frac{\ell(cx)}{\ell(x)} \rightarrow 1$ as $x \rightarrow \infty$ for every $c > 0$, and we write $\ell \in \mathcal{RV}_0$. More generally, the class $\mathcal{M}(\rho)$ consists of functions U such that

$$\lim_{x \rightarrow \infty} \frac{\log U(x)}{\log x} = \rho,$$

which captures coefficients that behave like x^ρ up to logarithmic fluctuations. This class strictly contains the regularly varying functions of index ρ and allows oscillatory coefficients. The theory of regularly varying functions originates in the foundational work of Karamata [5, 6]; for systematic treatments, we refer to the monographs of Bingham–Goldie–Teugels [1] and Seneta [13]. The class $\mathcal{M}(\rho)$ was introduced by Cadena, Kratz, and Omey [2]. Recently, P. Řehák [11] developed a comprehensive fractional extension of several fundamental results in the theory of regularly varying functions, thereby yielding a characterization of regular variation via fractional operators; see also [10].

Our main contributions can be summarized as follows.

Riemann–Liouville barrier for variable coefficients

For the Riemann–Liouville problem

$$\mathcal{D}^\alpha u(t) = \lambda(t)u(t)^p + f(t), \quad \lim_{t \rightarrow 0^+} t^{1-\alpha}u(t) = u_0 > 0,$$

we prove a necessary barrier inequality of the form

$$\int_0^t \lambda(s) s^{-p(1-\alpha)} ds < \frac{\Gamma(\alpha)}{(p-1)u_0^{p-1}},$$

valid on any interval of existence. This yields an instantaneous nonexistence criterion at $t = 0$ and, when the barrier integral diverges as $t \rightarrow \infty$, it implies nonexistence of global solutions.

Caputo barrier for variable coefficients

For the Caputo problem

$$\mathbf{D}_*^\alpha u(t) = \lambda(t)u(t)^p + f(t), \quad u(0) = u_0 > 0,$$

we obtain an analogous barrier inequality. In particular, if

$$\int_1^\infty \lambda(s) s^{-p(1-\alpha)} ds = \infty,$$

then no global solution exists. This recovers Webb's result in the special case $\lambda(t) = \lambda_0 t^\beta$ and provides a precise formulation for borderline cases when $\lambda(t) = t^\beta \ell(t)$.

Consequences for slowly varying modulation

Using Karamata's theorem (recalled in Appendix A) and elementary comparisons, we show that for $\lambda(t) = t^\beta \ell(t)$ with ℓ slowly varying, the barrier integral diverges in the same subcritical range as in the pure power case. In the critical borderline case, the result depends on whether $\int_1^\infty \frac{\ell(s)}{s} ds$ diverges or converges. For coefficients in $\mathcal{M}(\beta)$ we obtain sharp subcritical global nonexistence, while the critical case remains sensitive to finer oscillations.

We emphasize that, even if the barrier integral remains bounded, global existence is not guaranteed by our analysis. In the Caputo equation case, blow-up at a finite endpoint can often be deduced from a continuation theorem under additional regularity assumptions on the nonlinearity; see [4, 20]. For the Riemann–Liouville equation, we are not aware of continuation results comparable to those available in the Caputo case. In particular, to the best of our knowledge, the corresponding finite-endpoint blow-up alternative has not been established in the literature, and we therefore do not claim such a result here.

The paper is organized as follows. In Section 2, we recall the precise definitions of fractional integrals, the Riemann–Liouville and Caputo derivatives, the equivalence between fractional initial value problems and their Volterra integral counterparts, and we fix the notation for the rest of the paper. In Section 3, we prove the variable-coefficient barrier inequality for the Riemann–Liouville case together with its consequences. In Section 4, we treat the Caputo case. Section 5 is devoted to several illustrative examples highlighting the sharpness of the results and the subtlety of the critical borderline. Finally, Appendix A collects, for the reader's convenience, the facts from the theory of regularly varying functions and from the class $\mathcal{M}(\rho)$ – in particular Karamata's theorem – that are used throughout the paper.

2 Caputo and Riemann–Liouville derivatives

Throughout this section, let $\alpha \in (0, 1)$ and $T > 0$. We work on the bounded interval $[0, T]$, with left endpoint fixed at 0; extending the results to a general left endpoint requires only minor notational changes. Our notation follows [8, 17, 18]; see also the classical references [3, 7, 12].

The Gamma and Beta functions arise naturally in many problems of fractional calculus. For $z > 0$, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau. \quad (2.1)$$

A basic identity satisfied by the Gamma function is

$$\Gamma(z+1) = z\Gamma(z), \quad z > 0.$$

For $\xi > 0$ and $\eta > 0$, the Beta function is defined by

$$B(\xi, \eta) = \int_0^1 (1-\tau)^{\xi-1} \tau^{\eta-1} d\tau. \quad (2.2)$$

Both integrals are well defined in the Lebesgue sense, and it is well known that

$$B(\xi, \eta) = \frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi+\eta)}.$$

We shall also use the standard scaling identity

$$\int_0^t (t-\tau)^{\xi-1} \tau^{\eta-1} d\tau = t^{\xi+\eta-1} B(\xi, \eta), \quad t, \xi, \eta > 0, \quad (2.3)$$

which follows by the simple change of variable $\tau \mapsto \sigma$, $\tau = \sigma t$.

We recall the following standard spaces of real-valued functions on $[0, T]$. The space $C[0, T]$ denotes the set of continuous functions on $[0, T]$. The space $L^1(0, T)$ denotes the space of (equivalence classes of) Lebesgue integrable functions on $(0, T)$. The space $AC[0, T]$ denotes the set of absolutely continuous functions on $[0, T]$. Recall that $u \in AC[0, T]$ if and only if u' exists almost everywhere on $[0, T]$, $u' \in L^1(0, T)$, and the fundamental theorem of calculus holds in the Lebesgue sense,

$$u(t) = u(0) + \int_0^t u'(s) ds, \quad t \in [0, T].$$

Following [18], for $\alpha \in (0, 1)$ we also introduce the weighted space of continuous functions

$$C_{\alpha-1}[0, T] := \left\{ u \in C(0, T) : \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) \text{ exists} \right\}, \quad (2.4)$$

equipped with the norm

$$\|u\|_{\alpha-1} := \sup_{t \in [0, T]} t^{1-\alpha} |u(t)|.$$

Note that

$$C_{\alpha-1}[0, T] \subset L^1(0, T).$$

This is the ambient space adopted for the Riemann–Liouville problems considered in Section 3. This choice is not unique, however: the same equation can also be studied in $L^1(0, T)$, with the prescribed initial condition taking a corresponding different form; see Subsection 2.4 below.

2.1 The Riemann–Liouville fractional integral

Definition 2.1. For $\alpha > 0$ and $u \in L^1(0, T)$, the *Riemann–Liouville fractional integral* of order α is the function

$$\mathcal{I}^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad (2.5)$$

defined for almost every $t \in (0, T)$.

For $u \in L^1(0, T)$, the integral on the right-hand side of (2.5) is finite for almost every $t \in (0, T)$, and the resulting function $\mathcal{I}^\alpha u$ belongs to $L^1(0, T)$.

A detailed treatment of these operators can be found in [3, 7, 12]; see also [16] for a survey of several important results. The following proposition summarizes some useful properties.

Proposition 2.2. *Let $\alpha, \nu > 0$.*

- (i) *The map $\mathcal{I}^\alpha : L^1(0, T) \rightarrow L^1(0, T)$ is linear, bounded, and preserves positivity.*
- (ii) *$\mathcal{I}^\alpha$ maps $C[0, T]$ into $C[0, T]$ and $AC[0, T]$ into $AC[0, T]$.*
- (iii) *For every $u \in L^1(0, T)$,*

$$\mathcal{I}^\alpha \mathcal{I}^\nu u(t) = \mathcal{I}^{\alpha+\nu} u(t) \quad \text{for a.e. } t \in [0, T]. \quad (2.6)$$

If u is continuous or $\alpha + \nu \geq 1$, the above equality holds for all $t \in [0, T]$.

- (iv) *For every $\mu > -1$,*

$$\mathcal{I}^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}. \quad (2.7)$$

For the convenience of the reader we indicate the precise sources: parts (i) and (ii) are [8, Proposition 1(1) and (5)], part (iii) is [8, Lemma 2], and part (iv) follows at once from the scaling identity recalled at the beginning of this section together with the Beta integral. We shall also use the Hardy–Littlewood mapping property $\mathcal{I}^\alpha : L^r(0, T) \rightarrow C^{0, \alpha-1/r}[0, T]$ for $r > 1/\alpha$, recalled in [8, Proposition 1(3)], which in particular gives $\mathcal{I}^\alpha f \in C[0, T]$ and $(\mathcal{I}^\alpha f)(0) = 0$ for such f .

2.2 The Riemann–Liouville derivative

Definition 2.3. Let $\alpha \in (0, 1)$ and let $u : (0, T] \rightarrow \mathbb{R}$ be such that $u \in L^1(0, T)$ and $\mathcal{I}^{1-\alpha} u \in AC[0, T]$. The *Riemann–Liouville fractional derivative* of order α of u is the function

$$\mathcal{D}^\alpha u(t) := \frac{d}{dt} (\mathcal{I}^{1-\alpha} u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds, \quad (2.8)$$

defined for almost every $t \in (0, T)$.

The hypothesis $\mathcal{I}^{1-\alpha} u \in AC[0, T]$ is the regularity condition guaranteeing that the right-hand side of (2.8) exists as the a.e. derivative of a Lebesgue integral.

Proposition 2.4. *Let $\alpha \in (0, 1)$.*

- (i) *If $u \in L^1(0, T)$, then $\mathcal{D}^\alpha \mathcal{I}^\alpha u(t) = u(t)$ for almost every $t \in (0, T)$ (see [7, Lemma 2.4]).*

(ii) If $u \in L^1(0, T)$ and $\mathcal{I}^{1-\alpha}u \in AC[0, T]$, then

$$(\mathcal{I}^\alpha \mathcal{D}^\alpha u)(t) = u(t) - \frac{(\mathcal{I}^{1-\alpha}u)(0^+)}{\Gamma(\alpha)} t^{\alpha-1} \quad (2.9)$$

for almost every $t \in (0, T)$ (see [7, Lemma 2.5(b)]).

The constant $b := (\mathcal{I}^{1-\alpha}u)(0^+) \in \mathbb{R}$ appearing in (2.9) is the initial condition that is prescribed when an initial value problem driven by $\mathcal{D}^\alpha$ is posed in $L^1(0, T)$, as in (2.16) below.

2.3 The Caputo derivative

Definition 2.5. Let $\alpha \in (0, 1)$ and let $u \in C[0, T]$ be such that $\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T]$. The *Caputo fractional derivative* of order α of u is defined as an L^1 function by

$$\mathbf{D}_*^\alpha u(t) := \mathcal{D}^\alpha(u - u(0))(t). \quad (2.10)$$

When $u \in AC[0, T]$, formula (2.10) reduces to

$$\mathbf{D}_*^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds. \quad (2.11)$$

Expression (2.11) is often adopted as the definition of the Caputo derivative. We stress, however, that the more general formulation in (2.10) requires only the weaker regularity condition

$$\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T],$$

rather than $u \in AC[0, T]$. For a detailed discussion and illustrative examples, see [8, 19].

2.4 Relations between the two derivatives

The Caputo and Riemann–Liouville derivatives are connected through the identity

$$\mathcal{D}^\alpha x(t) = \mathbf{D}_*^\alpha x(t) + \frac{x(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \quad t \in (0, T), \quad (2.12)$$

which holds for almost every t whenever both sides are well defined [3, Lemma 3.4].

In particular, for any constant c one has

$$\mathbf{D}_*^\alpha c = 0, \quad \mathcal{D}^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}.$$

This distinction is one of the main reasons why initial conditions for Caputo problems are usually prescribed in the classical form $u(0) = u_0$, whereas in the L^1 formulation Riemann–Liouville problems involve the initial condition

$$(\mathcal{I}^{1-\alpha}u)(0^+) = b.$$

In the weighted setting $C_{\alpha-1}[0, T]$, adopted throughout Section 3, the initial condition is instead prescribed as

$$\lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u_0.$$

For functions in this space, the two formulations are equivalent, with

$$b = \Gamma(\alpha) u_0.$$

Indeed, by [18, Lemma 3.3], if $u \in L^1(0, T)$ and

$$\lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = c,$$

then

$$(\mathcal{I}^{1-\alpha} u)(0^+) = \Gamma(\alpha) c.$$

Conversely, if $u \in C_{\alpha-1}[0, T]$, then the limit

$$c := \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t)$$

exists by the definition of the space, and the same lemma yields

$$(\mathcal{I}^{1-\alpha} u)(0^+) = \Gamma(\alpha) c.$$

Hence, prescribing

$$(\mathcal{I}^{1-\alpha} u)(0^+) = \Gamma(\alpha) u_0$$

necessarily gives $c = u_0$. We emphasize, however, that only the first implication holds for a general $u \in L^1(0, T)$; the converse relies essentially on the additional assumption $u \in C_{\alpha-1}[0, T]$. We also record the Caputo counterpart of (2.9).

Proposition 2.6. *Let $\alpha \in (0, 1)$. Then:*

(i) *If $u \in C[0, T]$, then*

$$\mathbf{D}_*^\alpha \mathcal{I}^\alpha u = u \quad \text{on } [0, T].$$

(ii) *If $u \in C[0, T]$ and $\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T]$, then $u - u(0) \in \mathcal{I}^\alpha(L^1(0, T))$ and*

$$(\mathcal{I}^\alpha \mathbf{D}_*^\alpha u)(t) = u(t) - u(0) \tag{2.13}$$

for almost every $t \in [0, T]$. If, in addition, $\mathbf{D}_*^\alpha u \in L^r(0, T)$ for some $r > 1/\alpha$, then $\mathcal{I}^\alpha \mathbf{D}_*^\alpha u \in C[0, T]$ and (2.13) holds for every $t \in [0, T]$.

Both assertions of part (ii) are consequences of results of Lan–Webb [8]; we indicate how, since the precise form of the hypotheses matters. The first rests on the characterisation of the range of $\mathcal{I}^\alpha$ on L^1 : for $v \in L^1(0, T)$ one has $v \in \mathcal{I}^\alpha(L^1(0, T))$ if and only if $\mathcal{I}^{1-\alpha} v \in AC[0, T]$ and $(\mathcal{I}^{1-\alpha} v)(0^+) = 0$; see [8, Proposition 2]. Applied to $v := u - u(0)$, the second condition is automatic, since v is continuous with $v(0) = 0$ and

$$|(\mathcal{I}^{1-\alpha} v)(t)| \leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \left(\sup_{0 \leq s \leq t} |v(s)| \right) \longrightarrow 0 \quad (t \rightarrow 0^+).$$

Hence $v = \mathcal{I}^\alpha g$ for some $g \in L^1(0, T)$, and, by the semigroup property of Proposition 2.2(iii),

$$\mathbf{D}_*^\alpha u = \mathcal{D}^\alpha v = \frac{d}{dt} \mathcal{I}^{1-\alpha} \mathcal{I}^\alpha g = \frac{d}{dt} \mathcal{I}^1 g = g \quad \text{a.e. on } (0, T),$$

which identifies g with $\mathbf{D}_*^\alpha u$ and gives (2.13). Some additional hypothesis is indeed needed for the identity to hold at *every* point: under the sole assumptions of part (ii) one knows only that $\mathbf{D}_*^\alpha u \in L^1(0, T)$, and for a general $g \in L^1(0, T)$ the integral defining $(\mathcal{I}^\alpha g)(t)$ may be infinite at individual points of a Lebesgue null set; thus (2.13) holds at every t at which its left-hand side is defined, but that exceptional set cannot be removed. The Hardy–Littlewood

mapping property recalled after Proposition 2.2 removes this obstruction when $\mathbf{D}_*^\alpha u \in L^r(0, T)$ with $r > 1/\alpha$, and yields in addition $(\mathcal{I}^\alpha \mathbf{D}_*^\alpha u)(0) = 0$; the resulting equivalence between the Caputo initial value problem and its Volterra form is [8, Lemma 4]. Finally, the regularity condition used here may be written in either of the two forms occurring in the literature: since the constant function $u(0)$ satisfies $\mathcal{I}^{1-\alpha}u(0) = \frac{u(0)}{\Gamma(2-\alpha)} t^{1-\alpha} \in AC[0, T]$ by Proposition 2.2(iv), the requirement $\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T]$ is the same as $\mathcal{I}^{1-\alpha}u \in AC[0, T]$.

The regularity assumption $\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T]$ in (2.13) is essential and, in general, cannot be replaced merely by the existence of $\mathbf{D}_*^\alpha u$. If the Caputo derivative is instead defined in the frequently used form $\mathcal{I}^{1-\alpha}u'$ for $u \in AC[0, T]$, the equivalence between the initial value problem and the integral equation fails: as shown in [8, Section 9], the corresponding initial value problem with datum f can have a solution only if $f \in \mathcal{I}^{1-\alpha}(L^1(0, T))$, and this necessary condition, which is violated by some continuous f , has often been overlooked; see [8, Theorem 9 and Proposition 3]. This is why we work throughout with the operator $\mathbf{D}_*^\alpha$ of Definition 2.5.

2.5 Equivalence between fractional IVPs and Volterra integral equations

We now state the precise functional framework under which a fractional differential equation is equivalent to its associated Volterra integral equation. Although this equivalence is frequently used, the necessary regularity assumptions are often only implicit in the literature. Our presentation follows [7, 8, 17].

CAPUTO CASE. Let $\alpha \in (0, 1)$ and $f \in L^1(0, T)$. Suppose that a function $u \in C[0, T]$ with

$$\mathcal{I}^{1-\alpha}(u - u(0)) \in AC[0, T]$$

satisfies the initial value problem

$$\mathbf{D}_*^\alpha u(t) = f(t) \quad \text{for a.e. } t \in (0, T), \quad u(0) = u_0, \quad (2.14)$$

then it also satisfies the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \text{a.e. } t \in [0, T]. \quad (2.15)$$

When the data are continuous, the implication (2.14) $\Rightarrow$ (2.15) is classical; see [7, Theorem 3.24]. Under the present L^1 assumptions, a proof is given in [18, Lemma 5.1]. It is also exactly Proposition 2.6 (ii): applying $\mathcal{I}^\alpha$ to (2.14) and using (2.13) gives (2.15).

Suppose now that $u \in C[0, T]$ satisfies (2.15) with $f \in L^1(0, T)$, so that

$$u - u_0 = \mathcal{I}^\alpha f$$

a.e. on $(0, T)$. By the semigroup property in Proposition 2.2(iii),

$$\mathcal{I}^{1-\alpha}(u - u_0) = \mathcal{I}^{1-\alpha} \mathcal{I}^\alpha f = \mathcal{I}^1 f = \int_0^t f(s) ds \in AC[0, T].$$

Thus, the required regularity condition follows directly from the Volterra formulation. Differentiating the above identity yields $\mathbf{D}_*^\alpha u = f$ a.e. on $(0, T)$, while $u(0) = u_0$ follows from the continuity of u . Hence, under the stated L^1 assumptions, the two formulations are equivalent, with (2.15) understood a.e. Moreover, if $f \in L^r(0, T)$ for some $r > 1/\alpha$, then $\mathcal{I}^\alpha f$ is continuous, and (2.15) holds for every $t \in [0, T]$. See also [8, Lemma 4].

RIEMANN–LIOUVILLE CASE. Let $\alpha \in (0, 1)$ and $f \in L^1(0, T)$. A function $u \in L^1(0, T)$ with

$$\mathcal{I}^{1-\alpha}u \in AC[0, T]$$

satisfies the initial value problem

$$\mathcal{D}^\alpha u(t) = f(t) \text{ a.e. } t \in [0, T], \quad (\mathcal{I}^{1-\alpha}u)(0^+) = b \in \mathbb{R}, \quad (2.16)$$

if and only if it satisfies, for a.e. $t \in (0, T)$, the Volterra integral equation

$$u(t) = \frac{b}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \quad (2.17)$$

The implication (2.16) $\Rightarrow$ (2.17) is proved in [7, Theorem 3.1 and Lemma 2.5(b)]; the equivalence in the L^1 framework, with initial condition prescribed as $(\mathcal{I}^{1-\alpha}u)(0^+) = b$, is [16, Proposition 6.1]. For the converse: if u is given by (2.17) with $f \in L^1(0, T)$, then by Proposition 2.2(iii)–(iv),

$$(\mathcal{I}^{1-\alpha}u)(t) = b + (\mathcal{I}^1 f)(t) = b + \int_0^t f(s) ds,$$

which belongs to $AC[0, T]$; hence $\mathcal{D}^\alpha u = (\mathcal{I}^{1-\alpha}u)' = f$ a.e. and $(\mathcal{I}^{1-\alpha}u)(0^+) = b$, that is, (2.16).

Whether the solution of (2.17) belongs to the weighted space $C_{\alpha-1}[0, T]$, and not merely to $L^1(0, T)$, is a separate question: it depends on f and is not settled by (2.17) alone. Since

$$t^{1-\alpha}u(t) = \frac{b}{\Gamma(\alpha)} + t^{1-\alpha}(\mathcal{I}^\alpha f)(t),$$

one has $u \in C_{\alpha-1}[0, T]$ precisely when $\mathcal{I}^\alpha f$ is continuous on $(0, T]$ and $t^{1-\alpha}(\mathcal{I}^\alpha f)(t)$ admits a finite limit as $t \rightarrow 0^+$. This is the case, for instance, when $f \in L^r(0, T)$ for some $r > 1/\alpha$, or when $f \in C[0, T]$; the limit is then 0, so that $b = \Gamma(\alpha) \lim_{t \rightarrow 0^+} t^{1-\alpha}u(t)$. It may fail for a general $f \in L^1(0, T)$. The weighted space is used systematically in [18], whose notation we follow.

3 Blow-up for Riemann–Liouville differential equations

Throughout this section, we fix $\alpha \in (0, 1)$ and $p > 1$. We use the notation of Section 2: $\mathcal{I}^\alpha$ denotes the Riemann–Liouville fractional integral, $\mathcal{D}^\alpha$ the Riemann–Liouville fractional derivative, and $C_{\alpha-1}[0, T]$ the weighted space defined in (2.4).

We consider the Riemann–Liouville initial value problem

$$\mathcal{D}^\alpha u(t) = \lambda(t) u(t)^p + f(t) \text{ a.e. } t \in (0, T), \quad \lim_{t \rightarrow 0^+} t^{1-\alpha}u(t) = u_0. \quad (3.1)$$

The initial condition is prescribed in weighted form because the problem is posed in $C_{\alpha-1}[0, T]$. For functions in this space, this condition is equivalent to

$$(\mathcal{I}^{1-\alpha}u)(0^+) = \Gamma(\alpha)u_0,$$

as discussed in Subsection 2.4. We stress, however, that the weighted limit implies the above identity for every $u \in L^1(0, T)$ by [18, Lemma 3.3], whereas the converse relies essentially on the assumption $u \in C_{\alpha-1}[0, T]$ and does not follow from the Volterra representation (2.17) alone. We assume:

- $u_0 > 0$ is given;

- $\lambda : (0, \infty) \rightarrow [0, \infty)$ is measurable and belongs to $L^1_{\text{loc}}(0, \infty)$, that is, is integrable on every compact subinterval of $(0, \infty)$;
- $f \in L^1_{\text{loc}}([0, \infty))$ with $f \geq 0$ a.e.;
- $p > 1$.

By *solution* of (3.1) on an interval $[0, T]$ we mean a function

$$u \in C_{\alpha-1}[0, T] \quad \text{with} \quad \mathcal{I}^{1-\alpha}u \in AC[0, T], \quad \lambda(\cdot)u(\cdot)^p \in L^1(0, T),$$

satisfying (3.1) almost everywhere on $(0, T)$. By the Riemann–Liouville equivalence (2.16) $\Leftrightarrow$ (2.17), applied with $b = \Gamma(\alpha)u_0$, u is a solution of (3.1) on $[0, T]$ if and only if

$$u(t) = u_0 t^{\alpha-1} + \mathcal{I}^\alpha(\lambda(\cdot)u(\cdot)^p)(t) + (\mathcal{I}^\alpha f)(t) \quad (3.2)$$

for almost every $t \in (0, T)$. Two points are worth clarifying here. First, (3.2) is an identity in $L^1(0, T)$, valid for a.e. $t \in (0, T)$; no pointwise identity is asserted. The continuity properties used below follow from the assumption $u \in C_{\alpha-1}[0, T]$, rather than from (3.2). Second, this membership is part of the definition of a solution. Although it is consistent with (3.2), it is not implied by that representation, as noted in Subsection 2.5. Within this class, the initial condition in (3.1) simply means that the function $w(t) := t^{1-\alpha}u(t)$ extends continuously to $[0, T]$ and satisfies $w(0) = u_0$. Throughout the section, we work directly with (3.2).

Theorem 3.1 (Riemann–Liouville barrier; variable coefficient). *Under the assumptions above, suppose $u \geq 0$ is a solution of (3.1) on $[0, T]$ in the sense of (3.2), with $T > 0$. Define*

$$\Phi(t) := \int_0^t \lambda(s) s^{-p(1-\alpha)} ds, \quad t > 0. \quad (3.3)$$

Then:

- (A) (Instantaneous nonexistence criterion) *If $\Phi(t) = \infty$ for all $t > 0$, then (3.1) has no solution in $C_{\alpha-1}[0, T]$ for any $T > 0$.*
- (B) (Barrier inequality on the interval of existence) *If a solution exists on $[0, T]$, then*

$$\Phi(t) < \frac{\Gamma(\alpha)}{(p-1)u_0^{p-1}} \quad \text{for every } t \in (0, T]. \quad (3.4)$$

- (C) (No global solution if the barrier integral diverges at ∞) *If $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$, then (3.1) admits no global solution.*
- (D) (Maximal-time upper bound) *Let*

$$T_* := \inf \left\{ t > 0 : \Phi(t) \geq \frac{\Gamma(\alpha)}{(p-1)u_0^{p-1}} \right\} \in [0, \infty]. \quad (3.5)$$

Then any solution of (3.1) can exist only on $[0, T]$ with $T < T_*$.

Proof. Assume a solution $u \geq 0$ exists on $[0, T]$. Since $f \geq 0$, dropping the $\mathcal{I}^\alpha f$ term in (3.2) yields, for a.e. $t \in (0, T)$,

$$u(t) \geq u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(s) u(s)^p ds. \quad (3.6)$$

Set $w(t) := t^{1-\alpha}u(t)$ for $t \in (0, T]$. Since $u \in C_{\alpha-1}[0, T]$, w extends to a continuous function on $[0, T]$, and $w(0) = u_0$ by the initial condition in (3.1). Multiplying (3.6) by $t^{1-\alpha}$ and using $(t-s)^{\alpha-1} = t^{\alpha-1}(1-s/t)^{\alpha-1} \geq t^{\alpha-1}$ (because $\alpha-1 < 0$ and $0 < s < t$), we obtain, for a.e. $t \in (0, T)$,

$$\begin{aligned} w(t) &\geq u_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(s) u(s)^p ds \\ &\geq u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(s) s^{-p(1-\alpha)} w(s)^p ds, \end{aligned} \quad (3.7)$$

using $u(s)^p = s^{-p(1-\alpha)}w(s)^p$. Since both sides of (3.7) are continuous on $[0, T]$, the inequality (3.7) extends to every $t \in [0, T]$.

Case 1: instantaneous nonexistence (A). Suppose that $\Phi(t) = \infty$ for all $t > 0$, and assume, by contradiction, that a solution u exists on $[0, T]$ for some $T > 0$. From (3.7) and $w(s) \geq u_0$ we get, for every $t \in (0, T]$,

$$w(t) \geq u_0 + \frac{u_0^p}{\Gamma(\alpha)} \Phi(t),$$

and the right-hand side is infinite, which contradicts the finiteness of $w(t)$. Hence no solution exists on $[0, T]$ for any $T > 0$. We point out that, since $\lambda \in L_{\text{loc}}^1(0, \infty)$ and $s \mapsto s^{-p(1-\alpha)}$ is continuous, hence bounded, on every compact subinterval of $(0, \infty)$, the integrand $\lambda(s)s^{-p(1-\alpha)}$ is integrable on every compact subinterval of $(0, \infty)$; the divergence of Φ can therefore only originate at $s = 0$, so that $\Phi(t) = \infty$ for some $t > 0$ if and only if $\Phi(t) = \infty$ for all $t > 0$.

Case 2: barrier inequality (B). Assume a solution exists on $[0, T]$, so that $\Phi(t) < \infty$ for every $t \in (0, T]$. Define

$$g(t) := u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(s) s^{-p(1-\alpha)} w(s)^p ds, \quad t \in [0, T]. \quad (3.8)$$

Then $g \in AC[0, T]$, $g(0) = u_0 > 0$, g is non-decreasing, and (3.7) reads $w(t) \geq g(t)$. Differentiating (3.8) for a.e. $t \in (0, T)$ yields

$$g'(t) = \frac{1}{\Gamma(\alpha)} \lambda(t) t^{-p(1-\alpha)} w(t)^p \geq \frac{1}{\Gamma(\alpha)} \lambda(t) t^{-p(1-\alpha)} g(t)^p.$$

Since $g(t) \geq u_0 > 0$, we may divide by $g(t)^p$. Using $\frac{d}{dt}(g^{1-p}) = (1-p)g^{-p}g'$,

$$\frac{d}{dt}(g(t)^{1-p}) \leq -\frac{p-1}{\Gamma(\alpha)} \lambda(t) t^{-p(1-\alpha)} \quad \text{a.e. on } (0, T). \quad (3.9)$$

Integrating (3.9) over $(0, t)$ gives, for every $t \in (0, T]$,

$$g(t)^{1-p} \leq u_0^{1-p} - \frac{p-1}{\Gamma(\alpha)} \Phi(t). \quad (3.10)$$

The left-hand side is in $(0, u_0^{1-p}]$, hence the right-hand side must be strictly positive, which is (3.4).

Case 3: consequences (C)–(D). If $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$, then (3.4) fails for t large, contradicting global existence. Conclusion (D) is the contrapositive of (B): if $T \geq T_*$, there exists $t \leq T$ with $\Phi(t) \geq \Gamma(\alpha)/((p-1)u_0^{p-1})$, contradicting (3.4). $\square$

In the remainder of this section, we present some straightforward consequences for power and regularly varying coefficients.

Corollary 3.2 (Webb's critical exponent for $\lambda(t) = \lambda_0 t^\beta$). *Let $\lambda(t) = \lambda_0 t^\beta$ with $\lambda_0 > 0$ and $\beta \in \mathbb{R}$, and let $u_0 > 0$. The integral $\Phi(t) = \int_0^t \lambda_0 s^{\beta-p(1-\alpha)} ds$ is finite for some (equivalently, every) $t > 0$ if and only if*

$$\beta - p(1 - \alpha) > -1 \iff p < \frac{1 + \beta}{1 - \alpha}. \quad (3.11)$$

Consequently:

- (i) If $p \geq \frac{1 + \beta}{1 - \alpha}$, then (3.1) has no solution in $C_{\alpha-1}[0, T]$ for any $T > 0$.
- (ii) If $1 < p < \frac{1 + \beta}{1 - \alpha}$, then any solution exists only on $[0, T]$ with $T < T_*$, where setting $\gamma := p(1 - \alpha) - \beta < 1$,

$$T_* = \left(\frac{(1 - \gamma) \Gamma(\alpha)}{(p - 1) \lambda_0 u_0^{p-1}} \right)^{1/(1-\gamma)}.$$

Proof. This follows by direct evaluation of Φ and Theorem 3.1. □

Corollary 3.3 (Coefficient $t^\beta \ell(t)$). *Let $\lambda(t) = t^\beta \ell(t)$ with $\ell(t) > 0$ for all $t > 0$.*

- (i) (Nonexistence near 0) *If there exist $c > 0$ and $t_0 > 0$ such that $\ell(t) \geq c$ on $(0, t_0]$, then $\Phi(t) = \infty$ for every $t > 0$ whenever $p \geq (1 + \beta)/(1 - \alpha)$; if in addition ℓ is bounded above on $(0, t_0]$, the converse implication also holds. Under this lower bound on ℓ near 0, the same critical exponent as in Corollary 3.2 yields instantaneous nonexistence.*
- (ii) (No global solution from growth at ∞) *If $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$ (equivalently $\int_0^\infty t^{\beta-p(1-\alpha)} \ell(t) dt = \infty$), then no global solution exists.*
- (iii) (Critical borderline at ∞) *If $p = (1 + \beta)/(1 - \alpha)$, then the large- t part of Φ is $\int_t^\infty \frac{\ell(s)}{s} ds$, so global existence is ruled out whenever $\int_0^\infty \frac{\ell(s)}{s} ds = \infty$.*

Proof. All three assertions follow by direct comparison in $\Phi(t) = \int_0^t s^{\beta-p(1-\alpha)} \ell(s) ds$ and Theorem 3.1. For (i), the lower bound $\ell \geq c$ on $(0, t_0]$ gives $\Phi(t) \geq c \int_0^{\min(t, t_0)} s^{\beta-p(1-\alpha)} ds$, which is infinite exactly when $\beta - p(1 - \alpha) \leq -1$, that is, when $p \geq (1 + \beta)/(1 - \alpha)$; if moreover ℓ is bounded above on $(0, t_0]$, the same integral bounds Φ from above near the origin, while $\lambda \in L_{\text{loc}}^1((0, \infty))$ makes the remaining part of Φ finite, whence the converse. Assertions (ii) and (iii) are immediate from the definition of Φ , using $\beta - p(1 - \alpha) = -1$ at $p = (1 + \beta)/(1 - \alpha)$. □

Corollary 3.4 (Subcritical global nonexistence under $\mathcal{M}(\beta)$). *Assume $\lambda \in \mathcal{M}(\beta)$ and $\lambda(t) > 0$ for all sufficiently large t . If*

$$1 < p < \frac{1 + \beta}{1 - \alpha}, \quad (3.12)$$

then $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$, hence (3.1) admits no global solution.

Proof. Fix $\varepsilon \in (0, 1 + \beta - p(1 - \alpha))$, possible by (3.12). By the characterization of $\mathcal{M}(\beta)$ recalled in Theorem A.5 of Appendix A (see (A.3)), there exists $t_\varepsilon > 0$ such that $\lambda(t) \geq t^{\beta-\varepsilon}$ for all $t \geq t_\varepsilon$. Hence

$$\Phi(t) \geq \int_{t_\varepsilon}^t s^{\beta-\varepsilon-p(1-\alpha)} ds \rightarrow \infty \quad (t \rightarrow \infty),$$

and Theorem 3.1(C) applies. $\square$

Remark 3.5 (Critical case under $\mathcal{M}(\beta)$). If $p = (1 + \beta)/(1 - \alpha)$, then $\mathcal{M}(\beta)$ alone does not decide whether $\Phi(t)$ diverges at ∞ ; this is analogous to the slowly varying borderline of Corollary 3.3(iii).

4 Blow-up for Caputo differential equations

We now turn to the Caputo problem. With the notation of Section 2, $\mathbf{D}_*^\alpha$ denotes the Caputo derivative of order $\alpha \in (0, 1)$. Throughout this section we fix $p > 1$ and consider

$$\mathbf{D}_*^\alpha u(t) = \lambda(t) u(t)^p + f(t) \quad \text{a.e. } t \in (0, T), \quad u(0) = u_0 > 0, \quad (4.1)$$

where $\lambda : (0, \infty) \rightarrow [0, \infty)$ is measurable and belongs to $L_{\text{loc}}^1(0, \infty)$, and $f \in L_{\text{loc}}^1([0, \infty))$ with $f \geq 0$ a.e.

By a *solution* of (4.1) on $[0, T]$ we mean a function

$$u \in C[0, T] \quad \text{with} \quad \mathcal{I}^{1-\alpha}(u - u_0) \in AC[0, T], \quad \lambda(\cdot)u(\cdot)^p \in L^1(0, T),$$

satisfying (4.1) a.e. on $(0, T)$. Since $\lambda(\cdot)u(\cdot)^p + f \in L^1(0, T)$, the Caputo implication (2.14) $\implies$ (2.15) of Subsection 2.5 shows that u solves the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\lambda(s)u(s)^p + f(s)) ds \quad \text{for a.e. } t \in [0, T]. \quad (4.2)$$

Theorem 4.1 (Caputo barrier; variable coefficient). *Under the assumptions above, suppose $u \geq 0$ is a solution of (4.1) on $[0, T]$ with $T > 1$. Define, for $t \geq 1$,*

$$\Phi(t) := \int_1^t \lambda(s) s^{-p(1-\alpha)} ds. \quad (4.3)$$

Then the following necessary estimate holds:

$$\Phi(t) < \frac{\Gamma(\alpha)}{(p-1)u_0^{p-1}} \quad \text{for every } t \in [1, T]. \quad (4.4)$$

In particular, if $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$, then (4.1) admits no global solution. More precisely, setting

$$T_1 := \inf \left\{ t > 1 : \Phi(t) \geq \frac{\Gamma(\alpha)}{(p-1)u_0^{p-1}} \right\} \in (1, \infty], \quad (4.5)$$

any solution of (4.1) can exist only on $[0, T]$ with $T < T_1$.

Proof. Fix $T > 1$ and suppose $u \geq 0$ is a solution on $[0, T]$. Since $f \geq 0$, (4.2) yields

$$u(t) \geq u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(s) u(s)^p ds \quad \text{for a.e. } t \in [0, T]. \quad (4.6)$$

Using $(t-s)^{\alpha-1} \geq t^{\alpha-1}$ for $0 < s < t$ (valid since $\alpha - 1 < 0$), we get, for a.e. $t \in (0, T]$,

$$u(t) \geq u_0 + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \lambda(s) u(s)^p ds. \quad (4.7)$$

Since both sides of (4.7) are continuous on $(0, T]$, the inequality (4.7) holds for every $t \in (0, T]$. Set $w(t) := t^{1-\alpha} u(t)$ for $t \in (0, T]$. Then $w \in C(0, T]$ and (4.7) becomes

$$w(t) \geq t^{1-\alpha} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(s) s^{-p(1-\alpha)} w(s)^p ds, \quad (4.8)$$

since $u(s)^p = s^{-p(1-\alpha)} w(s)^p$. For $t \geq 1$, $t^{1-\alpha} u_0 \geq u_0$, hence

$$w(t) \geq u_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \lambda(s) s^{-p(1-\alpha)} w(s)^p ds \quad \text{for } t \in [1, T]. \quad (4.9)$$

Define, for $t \in [1, T]$,

$$g(t) := u_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \lambda(s) s^{-p(1-\alpha)} w(s)^p ds. \quad (4.10)$$

Then $g \in AC[1, T]$, $g(1) = u_0 > 0$, g is non-decreasing, and $w(t) \geq g(t) \geq u_0$. Differentiating (4.10) a.e. yields

$$g'(t) = \frac{1}{\Gamma(\alpha)} \lambda(t) t^{-p(1-\alpha)} w(t)^p \geq \frac{1}{\Gamma(\alpha)} \lambda(t) t^{-p(1-\alpha)} g(t)^p.$$

Dividing by $g(t)^p$ and integrating from 1 to t gives

$$g(t)^{1-p} \leq u_0^{1-p} - \frac{p-1}{\Gamma(\alpha)} \Phi(t). \quad (4.11)$$

The left-hand side lies in $(0, u_0^{1-p}]$, so the right-hand side must be strictly positive, giving (4.4). The remaining conclusions follow as in the Riemann–Liouville case. $\square$

Corollary 4.2 (Subcritical nonexistence for $\lambda \in \mathcal{M}(\beta)$). *Under the assumptions of Theorem 4.1, suppose $\lambda(t) > 0$ for all sufficiently large t and $\lambda \in \mathcal{M}(\beta)$. If $1 < p < (1 + \beta)/(1 - \alpha)$, then (4.1) admits no global solution.*

Proof. Identical to the proof of Corollary 3.4, with Φ replaced by (4.3). $\square$

Corollary 4.3 (Regular variation with slow modulation, Caputo). *Under the assumptions of Theorem 4.1, suppose $\lambda(t) = t^\beta \ell(t)$ for all large t , with ℓ slowly varying at infinity and $\ell(t) > 0$ for all large t . Set $\gamma := p(1 - \alpha) - \beta$.*

- (i) *If $\gamma < 1$ (equivalently $p < (1 + \beta)/(1 - \alpha)$), then $\Phi(t) \rightarrow \infty$ and no global solution exists.*
- (ii) *If $\gamma = 1$ (equivalently $p = (1 + \beta)/(1 - \alpha)$), then $\Phi(t) = \int_1^t \frac{\ell(s)}{s} ds$. If $\int_1^\infty \frac{\ell(s)}{s} ds = \infty$ then no global solution exists; if $\int_1^\infty \frac{\ell(s)}{s} ds < \infty$, the barrier argument is silent.*

Proof. For (i), Karamata's theorem (Theorem A.2 in Appendix A; see (A.2)) gives

$$\int_1^t s^{-\gamma} \ell(s) ds \sim \frac{t^{1-\gamma} \ell(t)}{1-\gamma} \rightarrow \infty,$$

so $\Phi(t) \rightarrow \infty$. The statement (ii) is direct from the definition of Φ . $\square$

4.1 Comparison between the Riemann–Liouville and Caputo cases

1. In the Riemann–Liouville case, the auxiliary function $w(t) = t^{1-\alpha}u(t)$ extends continuously to $[0, T]$ with $w(0) = u_0 > 0$. The barrier integral can therefore be initiated at $t = 0$, and instantaneous nonexistence at 0 is captured. In the Caputo case, $u(0) = u_0$ but $w(t) = t^{1-\alpha}u(t) \rightarrow 0$ as $t \rightarrow 0^+$, so w does not provide a positive lower bound on $(0, 1)$. As in [18], restricting the integration to $t \geq 1$ is a technically convenient device that yields a clean inequality $t^{1-\alpha}u_0 \geq u_0$; it is not the only choice. Indeed, the same argument, applied on $[t_0, T]$ with $t_0 > 0$, yields

$$\int_{t_0}^t \lambda(s) s^{-p(1-\alpha)} ds < \frac{\Gamma(\alpha)}{(p-1)u(t_0)^{p-1}} \quad (t \in [t_0, T]),$$

provided one has an a priori lower bound on $u(t_0)$.

2. For $\lambda(t) = \lambda_0 t^\beta$, both the Riemann–Liouville and Caputo barriers yield the same critical exponent

$$p_0 = \frac{1+\beta}{1-\alpha},$$

since in both cases the threshold arises from the condition $\beta - p(1-\alpha) = -1$ in the barrier integral. This agrees with [18, Theorems 4.2 and 5.3].

3. What differs between the two settings is not p_0 but the mechanism by which nonexistence occurs:

- **Riemann–Liouville:** for $p \geq p_0$ (with ℓ bounded below near 0, in the slowly varying setting), $\Phi(t) = \infty$ already near $t = 0$, forcing instantaneous nonexistence; no solution exists in $C_{\alpha-1}[0, T]$ for any $T > 0$ (Theorem 3.1(A)). This reflects the singular profile $u(t) \sim u_0 t^{\alpha-1}$ encoded by the Riemann–Liouville initial condition.
- **Caputo:** for $p \geq p_0$, nonexistence is of global type. A local solution can exist (since the Caputo Volterra integral (4.2) has a regular initial datum $u(0) = u_0$), but $\Phi(t) \rightarrow \infty$ at infinity forces the maximal interval of existence to be finite (Theorem 4.1).

4. Theorem 4.1 is best viewed as an ODE-type barrier inequality for Φ . In the pure power case $\lambda(s) = \lambda_0 s^\beta$, Φ reduces to an explicit power integral, but for general λ there is no canonical closed form. The correct threshold depends on the precise divergence rate of Φ , not just on a single exponent.
5. Since the Caputo barrier integral (4.3) starts at $t = 1$, it does not detect singular behavior of λ near 0. For example, $\lambda(t) = t^{-\eta}$ with $\eta \in (0, 1)$ on $(0, 1)$ does not enter the inequality (4.4).
6. At the critical Caputo exponent $p = (1+\beta)/(1-\alpha)$ with $\lambda \in \mathcal{M}(\beta)$, the definition of $\mathcal{M}(\beta)$ controls λ only up to sub-power modulation and does not decide whether $\Phi(t) \rightarrow \infty$. Hence at the critical exponent, global nonexistence may hold or fail depending on the fine asymptotics of λ .
7. In Theorems 3.1 and 4.1, the forcing term $f \geq 0$ is discarded to obtain a lower bound. The barriers are therefore independent of f : adding any nonnegative forcing can only make nonexistence more likely. If f is allowed to change sign, the argument breaks down.

8. The barriers are necessary conditions for existence and yield nonexistence when violated. They do not, however, provide a sufficient condition for existence: even if $\Phi(\infty) < \infty$, the theorems make no claim of global existence, since they only address one specific obstruction.

5 Illustrative examples

Except for Example 5.1, which concerns the Riemann–Liouville problem (3.1), all the examples below are formulated within the Caputo framework (4.1). The conclusions, however, often extend to both settings, as they ultimately depend on the divergence or convergence of the barrier integral Φ .

Example 5.1 (Pure power coefficient: explicit T_*). Let $\lambda(t) = \lambda_0 t^\beta$ with $\lambda_0 > 0$ and $1 < p < (1 + \beta)/(1 - \alpha)$. Then Corollary 3.2 gives the explicit upper bound

$$T < T_* = \left(\frac{(1 - \gamma) \Gamma(\alpha)}{(p - 1) \lambda_0 u_0^{p-1}} \right)^{1/(1-\gamma)}, \quad \gamma = p(1 - \alpha) - \beta < 1,$$

on the existence interval of any Riemann–Liouville solution. The same coefficient yields the analogous Caputo bound T_1 from Theorem 4.1, with integration starting at 1.

Example 5.2 (An $\mathcal{M}(\beta)$ coefficient that oscillates). Let $\lambda(t) = t^\beta \exp(\sin \log \log t)$ for $t > e^e$. Then $\lambda \in \mathcal{M}(\beta)$ since $\log \lambda(t) = \beta \log t + O(1)$. The factor $\exp(\sin \log \log t)$ is bounded in $[e^{-1}, e]$, so a direct comparison shows that at the critical Caputo exponent $p = (1 + \beta)/(1 - \alpha)$, $\Phi(t) \gtrsim \log t \rightarrow \infty$, and Theorem 4.1 yields nonexistence of global Caputo solutions.

Example 5.3 (Subcritical $\mathcal{M}(\beta)$). Fix $\beta > 0$ and define

$$\lambda(t) = t^\beta \exp\left(\sqrt{\log \log(e^e + t)}\right).$$

Then $\lambda \in \mathcal{M}(\beta)$ since $\log \lambda(t) = \beta \log t + o(\log t)$. If $1 < p < (1 + \beta)/(1 - \alpha)$, Corollary 4.2 gives $\Phi(t) \rightarrow \infty$ and hence no global Caputo solution exists. This illustrates that $\mathcal{M}(\beta)$ is sufficient for the subcritical range.

Example 5.4 (Critical exponent: slowly varying modulation giving divergence). Let $\lambda(t) = t^\beta (\log(e + t))^\theta$ with $\theta \in \mathbb{R}$, so $\lambda(t) = t^\beta \ell(t)$ with ℓ slowly varying. At the critical Caputo exponent $p = (1 + \beta)/(1 - \alpha)$,

$$\Phi(t) = \int_1^t \frac{(\log(e + s))^\theta}{s} ds.$$

Hence:

- if $\theta \geq -1$, then $\Phi(\infty) = \infty$ and Theorem 4.1 yields nonexistence of global Caputo solutions;
- if $\theta < -1$, then $\Phi(\infty) < \infty$ and the barrier is silent.

Remark 5.5. Several borderline coefficients that occur frequently are particular cases of Example 5.4, all of them lying in $\mathcal{M}(\beta)$. Taking $\theta = 0$ gives the pure power coefficient $\lambda(t) = t^\beta$, for which $\Phi(\infty) = \infty$ and Theorem 4.1 yields global nonexistence at the critical exponent. Taking

$\theta = -2$ and $\theta = -3$ gives $\lambda(t) = t^\beta(\log(e+t))^{-2}$ and $\lambda(t) = t^\beta(\log(e+t))^{-3}$, and equally $\lambda(t) = t^\beta(\log t)^{-2}$ for $t \geq e$; in each of these cases

$$\Phi(\infty) < \infty,$$

so that at the critical Caputo exponent Theorem 4.1 is silent and does not preclude global solutions, whose existence has to be decided by other techniques.

Example 5.6 (A coefficient in $\mathcal{M}(\beta)$ but not regularly varying). For $t \geq e^e$, let $\lambda(t) = t^\beta \exp(\sin(\log \log t))$ be the coefficient of Example 5.2, examined here from a different angle. Then $\lambda \in \mathcal{M}(\beta)$, but λ is not regularly varying because of its persistent oscillation. In the subcritical Caputo range $p < (1 + \beta)/(1 - \alpha)$, Corollary 4.2 forces global nonexistence. At the critical exponent, the bounded oscillatory factor still gives $\Phi(t) \gtrsim \log t \rightarrow \infty$; for generic $\lambda \in \mathcal{M}(\beta)$, however, the critical case is sensitive to cancellation effects.

Example 5.7 (A spiky coefficient). Let $\beta \in \mathbb{R}$ and define

$$\lambda(t) = \begin{cases} n^{\beta+p(1-\alpha)} & t \in [n, n + n^{-2}], \\ 0 & \text{otherwise,} \end{cases} \quad n \in \mathbb{N}.$$

Then λ vanishes outside a set of finite Lebesgue measure $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, but

$$\Phi(\infty) \gtrsim \sum_{n=1}^{\infty} n^{\beta-2}.$$

If $\beta \geq 1$, then $\Phi(\infty) = \infty$ and no global Caputo solution exists, even though λ is zero except on a set of measure $\pi^2/6$. The barrier depends on the integrated mass of $\lambda(s)s^{-p(1-\alpha)}$, not on pointwise size.

A Regularly varying functions and the class $\mathcal{M}(\rho)$

For the reader's convenience and to make the paper as self-contained as possible, we briefly summarize the main facts on regularly varying functions and the logarithmic-order class $\mathcal{M}(\rho)$ that are needed in Sections 3 and 4. In particular, we include the version of Karamata's theorem used in the proof of Corollary 4.3. The theory of regular variation was initiated by Karamata in his pioneering works [5, 6]; comprehensive accounts can be found in the classical monographs of Bingham–Goldie–Teugels [1] and Seneta [13].

Definition A.1. Let $a \geq 0$. A measurable function $U : (a, \infty) \rightarrow (0, \infty)$ is said to be *regularly varying at infinity with index* $\rho \in \mathbb{R}$ if

$$\lim_{x \rightarrow \infty} \frac{U(cx)}{U(x)} = c^\rho \quad \text{for every } c > 0.$$

We then write $U \in \mathcal{RV}_\rho$. In the special case $\rho = 0$, U is called *slowly varying at infinity* and is denoted by ℓ .

Every $U \in \mathcal{RV}_\rho$ can be written as $U(x) = x^\rho \ell(x)$ with ℓ slowly varying at infinity, so slow variation captures the “sub-power” part of regular variation. Typical examples of slowly

varying functions are the constants, $(\log x)^a$ with $a \in \mathbb{R}$, and iterated logarithms. In contrast, bounded oscillatory factors such as $2 + \sin x$ are *not* slowly varying; this is precisely the point at which the larger class $\mathcal{M}(\rho)$ discussed below becomes useful.

The following result is the version of Karamata's theorem employed throughout the paper to characterize the asymptotic behavior of the barrier integral Φ as $t \rightarrow \infty$; see, for example, [1].

Theorem A.2. *Let $a \geq 0$ and let $U : (a, \infty) \rightarrow (0, \infty)$ be measurable, Lebesgue integrable on finite subintervals of (a, ∞) , and $U \in \mathcal{RV}_\rho$.*

(i) *If $\rho > -1$, then $x \mapsto \int_a^x U(t) dt$ is regularly varying of index $\rho + 1$, and*

$$\lim_{x \rightarrow \infty} \frac{x U(x)}{\int_a^x U(t) dt} = \rho + 1.$$

(ii) *If $\rho < -1$, then $\int_x^\infty U(t) dt < \infty$, $x \mapsto \int_x^\infty U(t) dt$ is regularly varying of index $\rho + 1$, and*

$$\lim_{x \rightarrow \infty} \frac{x U(x)}{\int_x^\infty U(t) dt} = -\rho - 1. \quad (\text{A.1})$$

(iii) *If $\rho = -1$ and $\int_a^\infty U(t) dt = \infty$, then $x \mapsto \int_a^x U(t) dt$ is slowly varying at infinity and*

$$\lim_{x \rightarrow \infty} \frac{x U(x)}{\int_a^x U(t) dt} = 0.$$

If instead $\int_a^\infty U(t) dt < \infty$, the conclusion (A.1) remains valid.

Remark A.3. Written in the form most directly used in the proof of Corollary 4.3, Theorem A.2 gives: for ℓ slowly varying at infinity and $\gamma < 1$,

$$\int_1^x \frac{\ell(t)}{t^\gamma} dt \sim \frac{x^{1-\gamma} \ell(x)}{1-\gamma} \quad (x \rightarrow \infty), \quad (\text{A.2})$$

whereas for $\gamma = 1$ the function $x \mapsto \int_1^x \frac{\ell(t)}{t} dt$ is slowly varying at infinity, and for $\gamma > 1$ one has $\int_1^\infty \ell(t) t^{-\gamma} dt < \infty$.

A.1 The logarithmic-order class $\mathcal{M}(\rho)$

The class $\mathcal{M}(\rho)$ was introduced by Cadena–Kratz–Omey [2] to extend the notion of regular variation by encompassing functions with a well-defined power-law asymptotic order, even when the accompanying multiplicative factor oscillates too strongly to be slowly varying in the sense of Definition A.1.

Definition A.4. Let $a \geq 0$ and $U : (a, \infty) \rightarrow (0, \infty)$ be measurable and bounded on finite intervals. We say that $U \in \mathcal{M}(\rho)$, where $\rho \in \mathbb{R}$, if

$$\lim_{x \rightarrow \infty} \frac{\log U(x)}{\log x} = \rho.$$

The following characterization, which is central to the proofs of Corollaries 3.4 and 4.2, asserts that for every $\varepsilon > 0$, the function U is eventually trapped between two power-law envelopes with exponents arbitrarily close to ρ . We refer to [2] for a detailed discussion.

Theorem A.5 (Characterization of $\mathcal{M}(\rho)$ via power envelopes). *Let $a \geq 0$ and let $U : (a, \infty) \rightarrow (0, \infty)$ be measurable and bounded on finite intervals, and let $\rho \in \mathbb{R}$. The following are equivalent.*

(i) $U \in \mathcal{M}(\rho)$.

(ii) For every $\varepsilon > 0$,

$$\frac{U(x)}{x^{\rho+\varepsilon}} \longrightarrow 0, \quad \frac{U(x)}{x^{\rho-\varepsilon}} \longrightarrow +\infty, \quad (x \rightarrow \infty).$$

(iii) There exist slowly varying functions ℓ_1, ℓ_2 at infinity such that, as $x \rightarrow \infty$,

$$\frac{U(x)}{x^\rho \ell_1(x)} \longrightarrow 0, \quad \frac{U(x)}{x^\rho \ell_2(x)} \longrightarrow +\infty.$$

In particular, if $U \in \mathcal{M}(\rho)$, then for every $\varepsilon > 0$ there exists $x_\varepsilon > 0$ such that

$$x^{\rho-\varepsilon} \leq U(x) \leq x^{\rho+\varepsilon} \quad \text{for all } x \geq x_\varepsilon. \quad (\text{A.3})$$

Remark A.6. One has $\mathcal{RV}_\rho \subsetneq \mathcal{M}(\rho)$: every regularly varying function of index ρ belongs to $\mathcal{M}(\rho)$, but the converse fails. Typical members of $\mathcal{M}(\rho)$ include

$$x^\rho, \quad x^\rho (\log x)^a \quad (a \in \mathbb{R}), \quad x^\rho \exp(\sin \log \log x),$$

the last of which is not regularly varying because of its persistent oscillation; it is the coefficient used in Examples 5.2 and 5.6.

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References

- [1] N. H. BINGHAM, C. M. GOLDIE, J. L. TEUGELS, *Regular variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, Cambridge, 1987. <https://doi.org/10.1017/CB09780511721434>; MR0898871; Zbl 0617.26001
- [2] M. CADENA, M. KRATZ, E. OMEY, On the order of functions at infinity, *J. Math. Anal. Appl.* **452**(2017), No. 1, 109–125. <https://doi.org/10.1016/j.jmaa.2017.02.042>; MR3628010; Zbl 1379.26004
- [3] K. DIETHELM, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, Vol. 2004, Springer, Berlin, 2010. <https://doi.org/10.1007/978-3-642-14574-2>; MR2680847; Zbl 1215.34001
- [4] P. W. ELOE, T. MASTHAY, Initial value problems for Caputo fractional differential equations, *J. Fract. Calc. Appl.* **9**(2018), 178–195. MR3772633; Zbl 1499.34044
- [5] J. KARAMATA, Sur un mode de croissance régulière des fonctions, *Math. (Cluj)* **4**(1930), 38–53. Zbl 56.0907.01

- [6] J. KARAMATA, Sur un mode de croissance régulière. Théorèmes fondamentaux, *Bull. Soc. Math. France* **61**(1933), 55–62. <https://doi.org/10.24033/bsmf.1196> MR1504998; Zbl 0008.00807
- [7] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006. MR2218073; Zbl 1092.45003
- [8] K. Q. LAN, J. R. L. WEBB, A new Bihari inequality and initial value problems of first order fractional differential equations, *Fract. Calc. Appl. Anal.* **26**(2023), No. 3, 962–988. <https://doi.org/10.1007/s13540-023-00152-5>; MR4599223; Zbl 1522.34025
- [9] Y. LASKRI, N.-E. TATAR, The critical exponent for an ordinary fractional differential problem, *Comput. Math. Appl.* **59**(2010), No. 3, 1266–1270. <https://doi.org/10.1016/j.camwa.2009.06.031>; MR2579488; Zbl 1189.34013
- [10] P. ŘEHÁK, Superlinear solutions of sublinear fractional differential equations and regular variation, *Fract. Calc. Appl. Anal.* **26**(2023), No. 3, 989–1015. <https://doi.org/10.1007/s13540-023-00156-1>; MR4599224; Zbl 1522.34027
- [11] P. ŘEHÁK, Regular variation and fractional operators, *Fract. Calc. Appl. Anal.* **29**(2026), No. 2, 662–682. <https://doi.org/10.1007/s13540-026-00501-0>; MR5057405; Zbl 08212605
- [12] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, Yverdon, 1993. MR1347689; Zbl 0818.26003
- [13] E. SENETA, *Regularly varying functions*, Lecture Notes in Mathematics, Vol. 508, Springer, Berlin, 1976. <https://doi.org/10.1007/BFb0079658>; MR0453936; Zbl 0324.26002
- [14] Y. SHAN, G. LV, New criteria for blow-up of fractional differential equations, *Filomat* **38**(2024), No. 4, 1305–1315. <https://doi.org/10.2298/fil2404305s>; MR4678454
- [15] J. VILLA-MORALES, Upper bounds for the blow-up time of a system of fractional differential equations with Caputo derivatives, *Results Appl. Math.* **20**(2023), Paper No. 100408, 7 pp. <https://doi.org/10.1016/j.rinam.2023.100408>; MR4665017; Zbl 1532.34020
- [16] J. R. L. WEBB, Initial value problems for Caputo fractional equations with singular nonlinearities, *Electron. J. Differential Equations* **2019**, No. 117, 1–32. MR4028821; Zbl 1425.34029
- [17] J. R. L. WEBB, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.* **471**(2019), No. 1–2, 692–711. <https://doi.org/10.1016/j.jmaa.2018.11.004>; MR3906348; Zbl 1404.26022
- [18] J. R. L. WEBB, Nonexistence results for fractional differential inequalities, *Electron. J. Differential Equations* **2024**, No. 40, 1–16. <https://doi.org/10.58997/ejde.2024.40>; MR4781353; Zbl 1544.34017
- [19] J. R. L. WEBB, Inequalities for fractional derivatives via the Marchaud derivative, *Electron. J. Differential Equations* **2025**, No. 20, 1–26. <https://doi.org/10.58997/ejde.2025.20>; MR4873454; Zbl 1556.34014 .

- [20] C. WU, X. LIU, The continuation of solutions to systems of Caputo fractional order differential equations, *Fract. Calc. Appl. Anal.* **23**(2020), No. 2, 591–599. <https://doi.org/10.1515/fca-2020-0029>; MR4098664; Zbl 1451.34016
- [21] X. ZHANG, L. LIU, Y. WU, Y. CUI, New result on the critical exponent for solution of an ordinary fractional differential problem, *J. Funct. Spaces* **2017**, Art. ID 3976469, 4 pp. <https://doi.org/10.1155/2017/3976469>; MR3713074; Zbl 1377.34018