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Existence of solutions for mixed Dirichlet–Neumann problems involving the $p(x)$-Laplacian

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Abstract. In this paper, we investigate the existence of solutions for a class of nonlinear elliptic boundary value problems involving the variable exponent $p(x)$-Laplace operator, subject to mixed Dirichlet–Neumann boundary conditions. Under suitable assumptions on the nonlinearity f and by applying an abstract critical point theorem due to Bonanno and D'Aguì, we obtain the existence of at least two nontrivial weak solutions with opposite energy sign.

Keywords: critical points theory, $p(x)$-Laplacian, variable exponents, mixed boundary conditions.

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1 Introduction

This paper is devoted to the study of a class of elliptic problems governed by the $p(x)$-Laplace operator. Specifically, we investigate the existence of solutions to the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a nonempty, connected, and bounded open set with C^1-boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, Γ_1 and Γ_2 are two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that

$$\Gamma_1 \cap \Gamma_2 = \emptyset, \qquad \overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega, \qquad \overline{\Gamma_1} \cap \overline{\Gamma_2} = \Sigma,$$

with Σ smooth $(N-2)$-dimensional submanifold of $\partial\Omega$, $p \in C(\overline{\Omega})$ and

$$\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$$

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denotes the $p(x)$-Laplace operator. The nonlinear term $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a real positive parameter.

A key aspect of our study is the generalization to variable exponent growth, which allows for a more precise modeling of nonlinear phenomena. In fact, such problems generalize the classical $p-$Laplacian framework by allowing the diffusion exponent to vary pointwise in the domain, thus introducing a spatially dependent anisotropic structure. The introduction of a variable exponent in the elliptic equation extends its applicability to problems in material mechanics, fluid dynamics, wave propagation, and heat conduction in heterogeneous media. In addition to these physical applications, the study of function spaces with variable exponent growth has led to significant advancements in the mathematical theory of partial differential equations and variational problems. These developments have found applications in electrorheological fluid modeling, elasticity, image processing, etc.

The presence of mixed boundary conditions introduces additional analytical complexity. The mixed conditions describe systems where part of the boundary is fixed while the remainder is free or insulated, a configuration common in elasticity, heat transfer, and heterogeneous materials. In such applications, the physical properties of the boundary of a domain are modeled by imposing Dirichlet conditions on one portion of the surface and Neumann conditions on the remaining part. For further details on applied problems leading to mixed Dirichlet–Neumann boundary conditions, we refer to [5,11,12] and the references therein. Owing to this wide range of applications, elliptic problems with mixed boundary conditions have been extensively studied. Several authors have established regularity results [14], as well as existence and multiplicity of solutions (see, for example, [3] and [4]).

This paper aims to establish existence and multiplicity results for problem (1.1) using variational methods. One of the main features of the variational approach adopted in this paper is that it provides a natural framework for proving the existence of multiple solutions. More precisely, using a critical point theorem due to Bonanno and D'Aguì [2], we prove the existence of at least two weak solutions to the problem. Here, a weak solution is understood as a function belonging to the suitable Sobolev space considered in this paper and satisfying the corresponding variational formulation of the problem (we refer the reader to Section 2 for the precise definition).

The paper is structured as follows: Section 2 introduces fundamental definitions, notations, and preliminary results that are used throughout the paper. Section 3 investigates the existence of weak solutions for (1.1) employing critical point theorems. As a special case, we obtain Theorem 3.5 where the algebraic condition is replaced by a superlinear condition at zero. We also provide an example illustrating the applicability of this result.

2 Basic properties and preliminaries

In order to introduce the suitable variational framework for studying the existence of weak solutions for problem (1.1), we provide some basic definitions and preliminary results concerning Lebesgue and Sobolev spaces with variable exponent. For more details we refer to the book of Diening–Harjulehto–Hästö–Růžička [6].

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a nonempty bounded open set with C^1 boundary $\partial\Omega$. Throughout the paper, for any function $r \in C(\overline{\Omega})$, we use the notation

$$r^- = \min_{x\in\overline{\Omega}} r(x), \qquad r^+ = \max_{x\in\overline{\Omega}} r(x),$$

and assume $p \in C(\overline{\Omega})$ such that

$$1 < p^- \leq p(x) \leq p^+ < \infty.$$

Denote by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$ and consider the variable exponent Lebesgue space defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \in M(\Omega) : \rho_{p(\cdot)}(u) < +\infty \right\},$$

where the modular is given by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x.$$

We endow this space with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} \, \mathrm{d}x \leq 1 \right\}.$$

Note that, if r is a positive constant, $L^r(\Omega)$ and $W^{1,r}(\Omega)$ are the usual Lebesgue and Sobolev spaces endowed with their standard norms, denoted respectively by $\|\cdot\|_r$ and $\|\cdot\|_{1,r}$. It is well known that $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a separable and reflexive Banach space. Denoting by p' the conjugate exponent of p, that is

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega},$$

then the following Hölder's inequality holds (see Diening–Harjulehto–Hästö–Růžička [6, Lemma 3.2.20])

$$\|uv\|_1 \leq 2 \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \quad \text{for all } u \in L^{p(\cdot)}(\Omega),\ v \in L^{p'(\cdot)}(\Omega). \tag{2.1}$$

Now, we summarize the properties concerning the relationship between the norm $\|\cdot\|_{p(\cdot)}$ and the modular $\rho_{p(\cdot)}$, see Fan–Zhao [10, Theorem 1.3].

Proposition 2.1. *Let $u \in L^{p(\cdot)}(\Omega)$. Then:*

(i) $\|u\|_{p(\cdot)} < 1 \,(\textit{resp. } = 1, > 1) \iff \rho_{p(\cdot)}(u) < 1 \,(\textit{resp. } = 1, > 1)$;

(ii) $\|u\|_{p(\cdot)} < 1 \implies \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$;

(iii) $\|u\|_{p(\cdot)} > 1 \implies \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$.

We now extend this framework defining the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \ : \ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

where clearly

$$\|\nabla u\|_{p(\cdot)} = \||\nabla u|\|_{p(\cdot)}.$$

We denote by

$$p^*(x) = \begin{cases} \frac{N\,p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$

for all $x \in \overline{\Omega}$ the critical Sobolev exponent to $p(\cdot)$. Given the space $C_c^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω, we denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. Furthermore, let σ be the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$ and denote by $L^{p(\cdot)}(\partial\Omega)$ the boundary Lebesgue space, endowed with the usual norm $\|\cdot\|_{p(\cdot),\partial\Omega}$ and corresponding modular $\rho_{p(\cdot),\partial\Omega}(\cdot)$. We also consider the trace operator

$$\gamma : W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\partial\Omega)$$

which is a continuous linear operator such that

$$\gamma(u) = u|_{\partial\Omega}, \quad \text{for all } u \in C^\infty(\Omega).$$

Here, we avoid explicitly using the trace operator notation, and we consider all restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces. The following theorem recalls the basic properties and the well-known Sobolev embedding results of the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, see Diening–Harjulehto–Hästö–Růžička [6, Corollary 8.3.2], Fan [7] and Fan–Shen–Zhao [8, Theorem 1.3].

Theorem 2.2 (see [9, Proposition 2.5]). *Assume that $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. Then*

(i) *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces;*

(ii) *If $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;*

(iii) *There exists a constant $C > 0$ such that*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Statement (iii) of Theorem 2.2 provides the variable exponent version of the classical Poincaré inequality, ensuring that $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Thus, for our problem we consider the Sobolev space

$$X = W_{0,\Gamma_1}^{1,p(\cdot)}(\Omega) = \{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \Gamma_1\},$$

endowed with the standard norm $\|u\|_{1,p(\cdot)}$.

The following proposition shows that, under additional geometric assumptions on the set Ω and on Γ_1, the Poincaré inequality still holds (see for instance Sidi Ammi–Dahi [15]).

Proposition 2.3 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^N$ be a connected open bounded set and $\Gamma_1 \subset \partial\Omega$ with positive $(N-1)$-dimensional Hausdorff measure. Then there exists a constant C, depending on Ω and $p(\cdot)$, such that*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in X.$$

Proof. Arguing by contradiction, for each $C = j \in \mathbb{N}$ there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq X \subset W^{1,p(\cdot)}(\Omega)$ such that

$$\|u_j\|_{p(\cdot)} > j\|\nabla u_j\|_{p(\cdot)} \quad \text{for all } j \in \mathbb{N}. \tag{2.2}$$

Without loss of generality, we can suppose $\|u_j\|_{p(\cdot)} = 1$. Indeed, (2.2) still holds true with the sequence $v_j = \frac{u_j}{\|u_j\|_{p(\cdot)}}$. Therefore, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in the reflexive Sobolev space $W^{1,p(\cdot)}(\Omega)$ and from Theorem 2.2, there exists a subsequence, renamed again with u_j, such that

$$u_j \rightharpoonup u \quad \text{in } W^{1,p(\cdot)}(\Omega), \quad u_j \to u \quad \text{in } L^{p(\cdot)}(\Omega), \quad \nabla u_j \rightharpoonup \nabla u \quad \text{in } L^{p(\cdot)}(\Omega).$$

Moreover, from (2.2) one has

$$\|\nabla u_j\|_{p(\cdot)} < \frac{1}{j}\|u_j\|_{p(\cdot)} = \frac{1}{j} \longrightarrow 0 \quad \text{in } L^{p(\cdot)}(\Omega).$$

Hence, $\nabla u = 0$ almost everywhere in Ω. Since Ω is a connected set, u is constant in Ω. Because $u \in X$, we must have $u = 0$ and so $u_j \longrightarrow 0$ in $L^{p(\cdot)}(\Omega)$ which contradicts the fact that $\|u_j\|_{p(\cdot)} = 1$. □

As a consequence of the previous inequality, we have that

$$\|u\| := \|\nabla u\|_{p(\cdot)}$$

is an equivalent norm on X.

Remark 2.4. Since $X \subset W^{1,p(\cdot)}(\Omega)$, the statements of Theorem 2.2 also apply to the space X. Then, for any $q \in C(\overline{\Omega})$ we will denote by k_q the best constant for which one has

$$\|u\|_{q(\cdot)} \leq k_q \|u\|, \quad \text{for all } u \in X.$$

In the sequel, for any $a, b > 0$ and $q \in C(\overline{\Omega})$ with $q^- > 1$, we set

$$[a]^q = \max\{a^{q^-}, a^{q^+}\}, \quad [a]_q = \min\{a^{q^-}, a^{q^+}\},$$

and it is easy to verify that

(i) $[a]^{\frac{1}{q}} = \max\{a^{\frac{1}{q^-}}, a^{\frac{1}{q^+}}\}$,

(ii) $[a]_{\frac{1}{q}} = \min\{a^{\frac{1}{q^-}}, a^{\frac{1}{q^+}}\}$,

(iii) $[a]_{\frac{1}{q}} = m \iff a = [m]^q$, $[a]^{\frac{1}{q}} = m \iff a = [m]_q$, where $m > 0$.

(iv) $[a]_q[b]_q \leq [ab]_q \leq [ab]^q \leq [a]^q[b]^q$.

We also denote

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}) : 1 < r(x) \text{ for all } x \in \overline{\Omega}\}.$$

From now on, we assume the following hypotheses on the data of problem (1.1):

(H) $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a nonempty, connected, and bounded open set with C^1 boundary $\partial\Omega$, Γ_1 and Γ_2 are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset, \overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega, \overline{\Gamma_1} \cap \overline{\Gamma_2} = \Sigma$, with Σ smooth $(N-2)$-dimensional submanifold of $\partial\Omega$ and $p \in C(\overline{\Omega})$ is such that $1 < p(x) < \infty$ for all $x \in \overline{\Omega}$.

(H_f) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that:

(f_1) f is Carathéodory;

(f_2) there exist $a_1, a_2 > 0$ and $\alpha \in C_+(\overline{\Omega})$ with $\alpha(x) < (p^-)^*$ for each $x \in \overline{\Omega}$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{\alpha(x)-1} \quad \text{for all } t \in \mathbb{R};$$

(f_3) there exist $a_3 > 0$ and $\beta, \theta \in C_+(\overline{\Omega})$ with

$$\min\{\beta^-, \theta^-\} \in \left((\alpha^+ - p^-)\frac{N}{p^-}, \alpha^+\right)$$

such that

$$0 < a_3 \leq \lim_{t\to\infty} \frac{f(x,t)t - p^+F(x,t)}{|t|^{\beta(x)}},$$
$$0 < a_3 \leq \lim_{t\to-\infty} \frac{f(x,t)t - p^+F(x,t)}{|t|^{\theta(x)}},$$

uniformly for a.a. $x \in \Omega$;

(f_4) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$.

Now, we introduce the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x, \qquad \Psi(u) = \int_\Omega F(x,u(x))\,\mathrm{d}x,$$

where

$$F(x,t) = \int_0^t f(x,\tau)\,\mathrm{d}\tau \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

As a consequence of Proposition 2.1, one can obtain the following conditions (see for instance Bonanno–Chinnì [1]).

Proposition 2.5. *Let $u \in X$, then the following hold:*

(i) $\|u\| < 1\ (=1;>1) \iff \rho_{p(\cdot)}(|\nabla u|) < 1\ (=1;>1)$;

(ii) $\|u\| > 1 \implies \frac{1}{p^+}\|u\|^{p^-} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^+}$;

(iii) $\|u\| < 1 \implies \frac{1}{p^+}\|u\|^{p^+} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^-}$;

(iv) $\|u\|_{p(\cdot)} \to \infty \iff \rho_{p(\cdot)}(u) \to \infty$.

Also, Φ is Gateaux differentiable with derivative

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\,\mathrm{d}x \quad \text{for all } u, v \in X.$$

Moreover, the functional Φ is of class C^1 and Proposition 2.5(iv) also implies that it is coercive. In the following we recall some properties of the functional Φ' (see Fan–Zhang [9, Theorem 3.1]).

Proposition 2.6. *Let (H) be satisfied. Then Φ' is continuous, bounded, strictly monotone, a homeomorphism and it satisfies the (S_+)-property, that is*

$$\text{if } u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad \limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \le 0, \quad \text{then } u_n \to u \text{ in } X.$$

Assumptions (f_1)–(f_2), Theorem 2.2 and Remark 2.4 ensure that the functional Ψ has compact derivative

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)v\,\mathrm{d}x \quad \text{for all } u, v \in X.$$

We introduce the so-called energy functional related to problem (1.1) by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \quad \text{for all } u \in X, \lambda > 0,$$

and we observe that the critical points of I_λ are the weak solutions of problem (1.1), i.e.

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\,\mathrm{d}x = \lambda \int_\Omega f(x,u(x))\,v(x)\,\mathrm{d}x,$$

for each $v \in X$. In order to obtain weak solutions for problem (1.1), we rely on an abstract critical point theorem established by Bonanno and D'Aguì [2, Theorem 2.1 and Remark 2.2], that we recall in the following together with the definition of Cerami condition.

Definition 2.7. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $J : X \to \mathbb{R}$ a Gateaux differentiable functional. We say that J satisfies the Cerami condition (in short (C)-condition) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

(C1) $J(u_n)$ is bounded in $\mathbb{R}$,

(C2) $\lim_{n\to\infty}(1 + \|u_n\|)\|J'(u_n)\|_{X^*} = 0$,

has a strongly convergent subsequence in X.

Theorem 2.8. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that Φ is coercive and there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.3}$$

and for all $\lambda \in \Lambda_r = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (C)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3 Main result

In this section, we present our main result on the existence of two nontrivial weak solutions for problem (1.1), obtained through critical point theory. In particular, we verify that the functionals Φ, Ψ and I_λ satisfy the assumptions of Theorem 2.8, which ensures the existence of two critical points of the energy functional, i.e. weak solutions of our problem. First, we provide the following lemma.

Lemma 3.1. *Let* (H) *and* (H_f) *be satisfied. Then, for all* $\lambda > 0$ *the functional* I_λ *satisfies the* (C)*-condition.*

Proof. Fix $\lambda > 0$, suppose that $\beta^- = \min\{\beta^-, \theta^-\}$ (see (f_3)) and let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a sequence such that (C1) and (C2) hold. Condition (C1) implies that there exists $M_1 \geq 0$ such that

$$|I_\lambda(u_n)| \leq M_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Moreover, by (C2) we have that there exists $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, with $\varepsilon_n \to 0^+$, such that

$$\left|\langle I'_\lambda(u_n), v\rangle\right| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|} \quad \text{for all } n \in \mathbb{N}, \text{for all } v \in X. \tag{3.2}$$

Choosing $v = u_n$, (3.2) reads as

$$\left|\rho_{p(\cdot)}(|\nabla u_n|) - \lambda \int_\Omega f(x, u_n)u_n \,\mathrm{d}x\right| \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

We provide the proof in three steps.

Step 1. $\{u_n\}$ *is bounded in* $L^{\beta^-}(\Omega)$.

Standard computations show that from (f_3) it follows that

$$f(x,t)t - p^+F(x,t) \geq M_2|t|^{\beta^-} - M_3 \quad \text{for a.a. } x \in \Omega, \text{ for all } t \in \mathbb{R}, \tag{3.4}$$

with $M_2, M_3 > 0$. Furthermore, from (3.1) we obtain

$$\rho_{p(\cdot)}(|\nabla u_n|) - \lambda \int_\Omega p^+F(x, u_n)\mathrm{d}x \leq M_1 p^+ \quad \text{for all } n \in \mathbb{N},$$

and summing the previous inequality with (3.3) we get

$$\lambda \int_\Omega \left(f(x, u_n)u_n - p^+F(x, u_n)\right) \mathrm{d}x \leq \varepsilon_n + M_4 \quad \text{for all } n \in \mathbb{N},$$

for some $M_4 > 0$. Combining the latter together with (3.4) one has

$$\|u_n\|^{\beta^-}_{\beta^-} \leq M_5 \quad \text{for all } n \in \mathbb{N},$$

for some $M_5 \geq 0$, and this proves that $\{u_n\}$ is bounded in $L^{\beta^-}(\Omega)$.

Step 2. $\{u_n\}$ *is bounded in* X.

Assume that $\|u_n\| > 1$, otherwise the claim is true. By (f_2) and (f_3) we know that

$$\beta^- < \alpha^+ < (p^-)^*.$$

Hence, there exists $\tau \in (0,1)$ such that

$$\frac{1}{\alpha^+} = \frac{\tau}{(p^-)^*} + \frac{1-\tau}{\beta^-}. \tag{3.5}$$

By the interpolation inequality (see Papageorgiou–Winkert [13, Proposition 2.3.17]) we obtain

$$\|u_n\|_{\alpha^+} \leq \|u_n\|^{\tau}_{(p^-)^*} \|u_n\|^{1-\tau}_{\beta^-} \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the boundedness of $\{u_n\}$ in $L^{\beta^-}(\Omega)$ (Step 1), we have

$$\|u_n\|_{\alpha^+} \leq M_6 \|u_n\|^{\tau}_{(p^-)^*} \quad \text{for all } n \in \mathbb{N}, \tag{3.6}$$

for some $M_6 > 0$. From (3.3), using (f_2) and (3.6), one has

$$\begin{aligned}
\rho_{p(\cdot)}(|\nabla u_n|) &\leq \varepsilon_n + \lambda \int_\Omega f(x,u_n)u_n \mathrm{d}x \\
&\leq \varepsilon_n + \lambda \left(a_1 \|u_n\|_1 + a_2 \|u_n\|^{\alpha^+}_{\alpha^+} \right) \\
&\leq \varepsilon_n + M_7 \left(1 + \|u_n\|^{\alpha^+ \tau}_{(p^-)^*} \right),
\end{aligned}$$

for all $n \in \mathbb{N}$. By the embedding $X \hookrightarrow L^{(p^-)^*}(\Omega)$ and Proposition 2.1(iii) it follows that

$$\|u_n\|^{p^-} \leq \varepsilon_n + M_8 \left(1 + \|u_n\|^{\alpha^+ \tau} \right) \quad \text{for all } n \in \mathbb{N},$$

with $M_8 > 0$. By using (3.5), the definition of $(p^-)^*$ and assumption (f_3) it holds that

$$\begin{aligned}
\alpha^+ \tau &= \frac{(p^-)^*(\alpha^+ - \beta^-)}{(p^-)^* - \beta^-} = \frac{Np^-(\alpha^+ - \beta^-)}{Np^- - N\beta^- + p^-\beta^-} \\
&< \frac{Np^-(\alpha^+ - \beta^-)}{Np^- - N\beta^- + p^-(\alpha^+ - p^-)\frac{N}{p^-}} = p^-.
\end{aligned}$$

and we proved that $\{u_n\}$ is bounded in X.

Step 3. $u_n \to u$ in X up to a subsequence.

Since u_n is bounded in X and X is a reflexive space, there exists a subsequence, not relabeled, that converges weakly in X and strongly in $L^{\alpha(\cdot)}(\Omega)$, that is,

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad u_n \to u \text{ in } L^{\alpha(\cdot)}(\Omega).$$

Moreover, from (C2) we know that

$$\langle I'_\lambda(u_n), v \rangle \longrightarrow 0 \quad \text{for all } v \in X.$$

Choosing $v = u_n - u$ we have

$$\langle I'_\lambda(u_n), u_n - u \rangle = \langle \Phi'(u_n), u_n - u \rangle - \lambda \int_\Omega f(x,u_n)(u_n - u)\,\mathrm{d}x \longrightarrow 0.$$

Using hypothesis (f_2), Hölder's inequality (2.1) and $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$, we obtain that

$$\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x \le 2\tilde{a_1}\|u_n-u\|_{\alpha(\cdot)} + 2\tilde{a_2}\|u_n-u\|_{\alpha(\cdot)},$$

for some $\tilde{a_1},\tilde{a_2}>0$. Since $u_n \to u$ in $L^{\alpha(\cdot)}(\Omega)$ it follows that

$$\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x \longrightarrow 0.$$

Therefore $\langle \Phi'(u_n), u_n-u\rangle \longrightarrow 0$ and thanks to (S_+)-property $u_n \to u$ in X. □

We next recall the following lemma, which yields information concerning the sign of the solutions. In particular, we assume that

$$f(x,t) = f(x,0) \quad \text{for all } x\in\Omega \text{ and } t<0.$$

Lemma 3.2. *Let $u \in X$ be a weak solution of problem (1.1). If $f(x,0)\ge 0$ for a.a. $x\in\Omega$, then u is nonnegative in Ω.*

Proof. Let u be a weak solution of problem (1.1), which means that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x = \lambda\int_\Omega f(x,u)v\,\mathrm{d}x \qquad \text{for all } v\in X.$$

Suppose, by contradiction, that $\Omega_- = \{x\in\Omega : u(x)<0\}\neq\emptyset$ and choose as test function $v=u^-=\max\{-u,0\}$. Then we get

$$0\le\|u^-\| = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla u^-\,\mathrm{d}x = \lambda\int_{\Omega_-} f(x,0)u^-\,\mathrm{d}x \le 0.$$

So it must be $\|u^-\|=0$, which implies $u^-\equiv 0$, and hence $u\ge 0$, completing the proof. □

Now, we state our main result. Put

$$R := \sup_{x\in\Omega}\operatorname{dist}(x,\partial\Omega),$$

then there exist $x_0\in\Omega$ such that $B(x_0,R)\subseteq\Omega$ and we denote by

$$\omega_R := \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}R^N$$

the Lebesgue measure of the N-dimensional ball of radius R, where $\Gamma(\cdot)$ is the Gamma function. For $\bar{\mu}\in]0,1[$ one has $B(x_0,\bar{\mu}R)\subset B(x_0,R)$.

For the reader's convenience, we set

$$K = \frac{1}{p^-}\frac{1-\bar{\mu}^N}{(1-\bar{\mu})^{p^+}}\frac{\omega_R}{[R]_p},$$

and for any $r,d>0$ we put

$$A(r) = a_1k_1\left[rp^+\right]^{\frac{1}{p}} + \frac{a_2}{\alpha^-}\left[k_\alpha\right]^\alpha \max\left\{(rp^+)^{\frac{\alpha^+}{p^-}},(rp^+)^{\frac{\alpha^-}{p^+}}\right\},$$

$$B(d) = \int_{B(x_0,R)\setminus B(x_0,\bar{\mu}R)} F\left(x,\frac{d}{R(1-\bar{\mu})}(R-|x-x_0|)\right)\mathrm{d}x + \int_{B(x_0,\bar{\mu}R)} F(x,d)\,\mathrm{d}x.$$

Theorem 3.3. *Let* (H) *and* (H_f) *be satisfied. Assume that there exist* $r, d > 0$ *such that*

(h_1) $K[d]^p < r$,

(h_2) $A(r) < B(d)$.

Then, for each $\lambda \in \Lambda^{r,d}$, *with*

$$\Lambda^{r,d} = \left]K\frac{[d]^p}{B(d)}, \frac{r}{A(r)}\right[,$$

problem (1.1) *admits at least two nontrivial weak solutions with opposite energy sign.*

Proof. Our aim is to apply Theorem 2.8. Hypothesis (f_4) and Lemma 3.1 ensure that for all $\lambda > 0$ I_λ is unbounded from below and it satisfies (C)-condition, respectively. Therefore, we only need to verify condition (2.3). Consider $\tilde{u} \in X$ defined by

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \dfrac{d}{R(1-\overline{\mu})}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B(x_0, \overline{\mu}R), \\ d & \text{if } x \in B(x_0, \overline{\mu}R). \end{cases}$$

Hypothesis (h_1) guarantees that $0 < \Phi(\tilde{u}) < r$. Indeed,

$$\begin{aligned} 0 < \Phi(\tilde{u}) &= \int_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} \frac{1}{p(x)}\left(\frac{d}{R(1-\overline{\mu})}\right)^{p(x)} \mathrm{d}x \\ &\leq \frac{1}{p^-}\left[\frac{d}{R(1-\overline{\mu})}\right]^p (1-\overline{\mu}^N)\omega_R \\ &\leq \frac{1}{p^-}\frac{[d]^p}{[R]_p(1-\bar{\mu})^{p^+}}(1-\overline{\mu}^N)\omega_R \\ &= K[d]^p < r. \end{aligned}$$

Moreover,

$$\Psi(\tilde{u}) = \int_{B(x_0,R)\setminus B(x_0,\bar{\mu}R)} F\left(x, \frac{d}{R(1-\bar{\mu})}(R - |x - x_0|)\right) \mathrm{d}x + \int_{B(x_0,\bar{\mu}R)} F(x, d)\, \mathrm{d}x.$$

We note that from Proposition 2.5 it follows that

$$\frac{1}{p^+}\left[\|u\|\right]_p \leq \Phi(u) \leq \frac{1}{p^-}\left[\|u\|\right]^p \quad \text{for all } u \in X.$$

In particular, for each $u \in \Phi^{-1}(]-\infty, r])$ one has

$$\frac{1}{p^+}\left[\|u\|\right]_p \leq \Phi(u) \leq r,$$

which means

$$\|u\| \leq [rp^+]^{\frac{1}{p}}. \tag{3.7}$$

Then, hypothesis (f_2), Remark 2.4, Proposition 2.1 and (3.7) imply that, for each $u \in \Phi^{-1}(]-\infty, r])$ one has

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(\int_\Omega a_1|u|\,\mathrm{d}x + \int_\Omega a_2\frac{|u|^{\alpha(x)}}{\alpha(x)}\,\mathrm{d}x\right)\\ &\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(a_1\|u\|_1 + \frac{a_2}{\alpha^-}\rho_{\alpha(\cdot)}(u)\right)\\ &\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(a_1k_1\|u\|_1 + \frac{a_2}{\alpha^-}\left[\|u\|_{\alpha(\cdot)}\right]^\alpha\right)\\ &\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(a_1k_1\|u\| + \frac{a_2}{\alpha^-}\left[k_\alpha\right]^\alpha\left[\|u\|\right]^\alpha\right)\\ &\leq a_1k_1\left[rp^+\right]^{\frac{1}{p}} + \frac{a_2}{\alpha^-}\left[k_\alpha\right]^\alpha \max\left\{(rp^+)^{\frac{\alpha^+}{p^-}}, (rp^+)^{\frac{\alpha^-}{p^+}}\right\}.\end{aligned}$$

where we recall that k_1 and k_α denote the best constants associated with the embeddings $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$, respectively. Hence,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{A(r)}{K[d]^p} < \frac{B(d)}{\Phi(\tilde{u})} = \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

namely hypothesis (2.3) is satisfied. In addition, the latter implies that $\Lambda^{r,d}$ is nonempty and $\Lambda^{r,d} \subset \Lambda_r$. Therefore, Theorem 2.8 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$ and the proof is complete. Assuming additionally that $f(x,0) \geq 0$ for almost every $x \in \Omega$, Lemma 3.2 ensures that the weak solutions are nonnegative. □

Furthermore, we consider two specific cases. The first is when the nonlinear term satisfies an additional sign condition (see ($\tilde{h}_3$) below), which enables us to deduce a consequence of Theorem 3.3. The second is when the nonlinear term is nonnegative and exhibits specific behaviour near zero.

Corollary 3.4. *Let* (H) *and* (H_f) *be satisfied. Assume that there exist* $r, d > 0$ *verifying* (h_1) *and such that*

($\tilde{h}_2$) $A(r) < \int_{B(x_0,\overline{\mu}R)} F(x,d)\,\mathrm{d}x$,

($\tilde{h}_3$) $F(x,t) > 0$ *for all* $t \in [0,d]$.

Then, for each $\lambda \in \Lambda^{r,d}$, *with*

$$\Lambda^{r,d} = \left]\frac{K[d]^p}{\int_{B(x_0,\overline{\mu}R)} F(x,d)\,\mathrm{d}x}, \frac{r}{A(r)}\right[,$$

problem (1.1) *admits at least two nontrivial weak solutions with opposite energy sign.*

Theorem 3.5. *Let* (H) *and* (H_f) *be satisfied. Suppose that* f *is nonnegative and*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p^-}} = +\infty. \tag{3.8}$$

Then, for each $\lambda \in]0, \lambda^*[$, *with*

$$\lambda^* = \sup_{r>0} \frac{r}{A(r)},$$

problem (1.1) *admits at least two nontrivial and nonnegative weak solutions with opposite energy sign.*

Proof. Fix $\lambda \in]0, \lambda^*[$. Then, there exists $r > 0$ such that

$$\lambda < \frac{r}{a_1 k_1 \left[rp^+\right]^{\frac{1}{p}} + \frac{a_2}{\alpha^-}\left[k_\alpha\right]^\alpha \max\left\{(rp^+)^{\frac{\alpha^+}{p^-}}, (rp^+)^{\frac{\alpha^-}{p^+}}\right\}}. \tag{3.9}$$

Moreover, assumption (3.8) implies that

$$\begin{aligned}\limsup_{d\to 0^+} B(d) &\geq \limsup_{d\to 0^+} \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)\,\mathrm{d}x}{K[d]^p} \\ &\geq \limsup_{d\to 0^+} \frac{\overline{\mu}^N \omega_R}{K} \frac{\inf\limits_{x\in\Omega} F(x,d)\,\mathrm{d}x}{d^{p^-}} = +\infty.\end{aligned}$$

Therefore, there exists $d > 0$ small enough such that

$$\frac{\overline{\mu}^N \omega_R}{K} \frac{\inf_{x\in\Omega} F(x,d)\,\mathrm{d}x}{d^{p^-}} > \frac{1}{\lambda},$$

which together with (3.9) yields $A(r) < B(d)$. Then, the conclusion follows from Theorem 3.3 and Lemma 3.2. □

As an application of the previous result, we present an example of a nonlinearity that satisfies all the assumptions of Theorem 3.5.

Example 3.6. Let $N = 2$, consider the domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{4}\right\},$$

and let

$$p(x,y) = x^2 + y^2 + \frac{4}{3}, \qquad (x,y) \in \Omega.$$

Then

$$p^- = \frac{4}{3}, \qquad p^+ = \frac{19}{12}.$$

Moreover, let

$$s(x,y) = x^2 + y^2 + 2,$$

and consider the nonlinearity

$$f((x,y),t) = 1 + s(x,y)|t|^{s(x,y)-1}.$$

We show that f satisfies the assumptions of Theorem 3.5. Condition (f_1) is immediate. Moreover, assumption (f_2) holds with

$$a_1 = 1, \qquad a_2 = \frac{9}{4}, \qquad \alpha(x,y) = s(x,y).$$

Indeed,

$$\alpha^+ = \frac{9}{4} < (p^-)^* = 4.$$

Furthermore,

$$\frac{(\alpha^+ - p^-)N}{p^-} = \frac{11}{8} < 2 \leq s(x,y) = \alpha(x,y),$$

for every $(x,y) \in \Omega$. Hence, choosing $\beta(x,y) = \theta(x,y) \equiv c \in \left(\frac{11}{8}, 2\right)$, we have

$$\beta^- = \theta^- = c \in \left(\frac{(\alpha^+ - p^-)N}{p^-}, \alpha^+\right),$$

and, since $c < 2 \leq s(x,y)$ for every $(x,y) \in \Omega$,

$$\beta(x,y) = \theta(x,y) = c < s(x,y) = \alpha(x,y), \qquad \text{for every } (x,y) \in \Omega.$$

Since $F((x,y),t) = t + t^{s(x,y)}$, $t \geq 0$ and $s(x,y) > p^+$, assumptions (f_3) and (f_4) are fulfilled. Moreover,

$$F((x,y),t) = t + t^{s(x,y)} \geq t.$$

Therefore,

$$\frac{\inf_{(x,y)\in\Omega} F((x,y),t)}{t^{p^-}} \geq \frac{t}{t^{p^-}} = t^{1-p^-}.$$

Since $p^- = \frac{4}{3}$, it follows that

$$t^{1-p^-} = t^{-1/3} \longrightarrow +\infty \quad \text{as } t \to 0^+.$$

Hence, condition (3.8) is satisfied, and all the assumptions of Theorem 3.5 are fulfilled.

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