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Infinitely many solutions for nonlinear elliptic equations with degenerate coercivity

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Abstract. In this article, we consider the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div} \dfrac{\nabla u}{(1+u)^{2\theta}} - \dfrac{\gamma\theta|\nabla u|^2}{(1+u)^{1+2\theta}} = u^p, \ x \in \mathbb{R}^N, \\ u > 0, \ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases}$$

where $\theta > 0$, $\gamma \in \mathbb{R}$, and $p > \frac{N+2}{N-2}$. We first perform a change of variables, which is more general because it generalizes the study of the cases $\gamma = 1$ and $\gamma = 0$ to all real γ. Then, by using the Pohozaev identity in the entire space, we show that the equation does not admit solutions in $H^1(\mathbb{R}^N)$ in some cases, which leads us to find solutions in a larger space. Lastly, we introduce a parameter and apply a perturbation method to get a continuum of solutions with decay $O\big(|x|^{-\frac{2}{p-1}}\big)$ at infinity.

Keywords: degenerate coercivity, quasilinear elliptic equations, perturbation methods.

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1 Introduction

The equations studied in this paper are quasilinear elliptic equations of the form

$$\begin{cases} -\operatorname{div}\left(g^2(u)\nabla u\right) + \gamma g(u)g'(u)|\nabla u|^2 = h(x,u), \ x \in \mathbb{R}^N, \\ u > 0, \ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{1.1}$$

In the study of second-order elliptic equations, a crucial requirement is the uniform ellipticity condition. Specifically, for (1.1), this means that there exists a constant $c_0 > 0$ such that $g^2(s) \geq c_0$ holds for all $s \in \mathbb{R}$. Considering the principal symbol of (1.1):

$$\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 \mathrm{d}x \geq c_0 \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x,$$

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one observes that the energy of the principal part is coercive in the space $H^1(\mathbb{R}^N)$, which facilitates the use of variational methods, Fourier transform, and energy estimates. A natural question then arises: if we only have $g^2(s) > 0$ and thus (1.1) is degenerate coercive, what can be said about the existence of solutions?

In this paper, we consider the following model problem with degenerate coercivity:

$$\begin{cases} -\operatorname{div} \dfrac{\nabla u}{(1+u)^{2\theta}} - \dfrac{\gamma\theta|\nabla u|^2}{(1+u)^{1+2\theta}} = u^p, \ x \in \mathbb{R}^N, \\ u > 0, \ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases} \tag{1.2}$$

where $\theta > 0$, $\gamma \in \mathbb{R}$, $p > 2^* - 1 = \frac{N+2}{N-2}$ and $N \geq 3$. Here $2^* = \frac{2N}{N-2}$ denotes the H^1-critical exponent. Problems arising from the degeneracy of the principal part as u tends to infinity have received considerable attention in recent years. To the best of our knowledge, the first significant existence result was given by Arcoya et al. in [2]. They considered the Dirichlet problem on a bounded domain Ω:

$$-\operatorname{div} \frac{\nabla u}{(b(x)+|u|)^{2\theta}} - \frac{\theta u|\nabla u|^2}{(b(x)+|u|)^{1+2\theta}|u|} = |u|^{p-1}u, \quad x \in \Omega. \tag{1.3}$$

To overcome the degenerate coercivity of (1.3) in $H_0^1(\Omega)$, they proved the existence of nontrivial solutions in the larger function space $W_0^{1,q}(\Omega)$ with $1 < q < 2$, provided that $b(x)$ is bounded and uniformly positive, and the exponents satisfy $1 < p < 2^*(1-\theta) - 1$ and $0 < \theta < \frac{N}{2N-2}$. Moreover, they pointed out that if Ω is star-shaped, $b(x) \equiv 1$, and $p > 2^*$, then the Pohozaev technique shows that (1.3) admits no positive solution in $H_0^1(\Omega) \cap L^\infty(\Omega) \cap H^2(\Omega)$. In [4], the authors introduced the change of variables

$$\Psi(\xi) := \int_0^\xi (1+|t|)^{-\theta} \mathrm{d}t = \frac{\operatorname{sgn}(\xi)}{1-\theta}\left((1+|\xi|)^{1-\theta} - 1\right), \quad \xi \in \mathbb{R},$$

thereby extending the study of the symmetry of minimizers by Girão and Weth [9] to the degenerate coercive setting. Further related investigations can be found in [5,12].

The case $\gamma = 0$ in equation (1.2) has also been an active research topic. In this situation the equation is no longer the Euler–Lagrange equation of a variational problem. Mercaldo and Peral [10] studied the corresponding problem on a bounded domain:

$$\begin{cases} -\operatorname{div} \dfrac{\nabla u}{(1+u)^{2\theta}} = u^p, & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

By a suitable change of variables they constructed a variational framework with a mountain pass geometry and proved the existence of a positive solution to (1.4) when $0 < \theta < \frac{1}{2}$ and $1 < p < (1-2\theta)(2^*-1)$. For more details on related work, we refer to [1,3].

Generally speaking, existing investigations of equations of the form (1.2) have mostly focused on the cases $\gamma = 1$ or $\gamma = 0$, while other values of γ remain relatively unexplored. From the viewpoint of variational methods, when $\gamma = 1$, equation (1.2) corresponds to the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x, \tag{1.5}$$

and two main difficulties arise. First, the functional is degenerate and non-uniformly elliptic even on bounded domains, and it cannot be defined on $H^1(\mathbb{R}^N)$. Second, the exponent p of interest is supercritical $\big(p > \frac{N+2}{N-2}\big)$, which renders many classical variational methods ineffective due to the failure of the Sobolev embedding theorem. Consequently, compared with subcritical and critical problems, results for supercritical problems are still relatively scarce. For the subcritical and critical regimes we refer to [12]. However, the existence results in [12] still rely on a potential function $V(x)$ that does not vanish at infinity, a feature absent from our equation (1.2). When $\gamma = 0$, although the change of variables

$$v = \int_0^u \frac{\mathrm{d}s}{(1+s)^{2\theta}}$$

adopted in [10] transforms (1.2) into a semilinear equation, the lack of compactness and the absence of the Sobolev embedding theorem still pose serious obstacles to a direct variational treatment.

In this paper we investigate the solvability of equation (1.2) for H^1-supercritical exponents p and for all real $\gamma \in \mathbb{R}$. In contrast to the aforementioned studies, we will employ a perturbation technique in weighted spaces to construct a continuum of solutions, thereby overcoming the difficulties related to the variational structure and degenerate coercivity, and we will describe the properties of these solutions.

In [12], the authors, by means of two different changes of variables, discussed in detail the two cases of their equation: $\lambda = 1$ and $\lambda = 0$. This motivated us to consider whether the analysis can be extended to the general case $\lambda \in \mathbb{R}$ (where λ corresponds to γ in the present paper). In this paper we will employ a more general change of variables, which not only simplifies the exposition but also allows us to treat the general situation.

Let $g(s) = (1+s)^{-\theta}$, then equation (1.2) is equivalent to

$$-g^2(u)\Delta u + (\gamma - 2)g(u)g'(u)\left|\nabla u\right|^2 = u^p. \tag{1.6}$$

After change of variables $v = G(u) = \int_0^u g^{2-\gamma}(t)\mathrm{d}t$ and $u = G^{-1}(v)$. And we have

$$\begin{aligned}
&\nabla v = G'(u)\nabla u = g^{2-\gamma}(u)\nabla u,\\
&\Delta v = \nabla g^{2-\gamma}(u)\cdot\nabla u + g^{2-\gamma}(u)\Delta u = g^{2-\gamma}(u)\Delta u + (2-\gamma)g^{1-\gamma}(u)g'(u)\left|\nabla u\right|^2.
\end{aligned}$$

Then we can convert (1.6) to

$$\begin{cases} \Delta v + f(v) = 0, \ x \in \mathbb{R}^N; \\ v > 0 \text{ and } \lim\limits_{|x|\to+\infty} v(x) = 0, \end{cases} \tag{1.7}$$

where $f(v) = G^{-1}(v)^p g(G^{-1}(v))^{-\gamma}$. Thus, v solves (1.7) if and only if $u = G^{-1}(v)$ is a solution of (1.2).

We restrict to the radially symmetric case, that is, $v(x) = v(r)$, where $r = |x|$, and first consider the problem in the entire space

$$\begin{cases} \Delta u + u^p = 0, \ x \in \mathbb{R}^N; \\ u(0) = 1. \end{cases} \tag{1.8}$$

It is well known that this problem possesses a unique positive symmetric solution $w(|x|)$ whenever $p > \frac{N+2}{N-2}$. If the definite solution condition at the origin is not considered, we see

$w_\lambda(r) = \lambda^{\frac{2}{p-1}} w(\lambda r)$ still solves (1.8). And, at a main order, one has

$$w(r) = C_{p,N} r^{-\frac{2}{p-1}} + o(1) \quad \text{as } r = |x| \to +\infty,$$

which implies that this behavior is common to all solutions $w_\lambda(r)$.

For the purpose of asymptotic analysis, for $\lambda > 0$, we replace the variables v in (1.7) by $\lambda^{\frac{2}{p-1}} v(\lambda|x|)$ and then a parameter appears in this equation. Thus, the problem (1.7) can be written as

$$\begin{cases} \Delta v + \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} v\right) = 0, \ r = |x| \in (0, \infty); \\ v > 0, \ \text{and} \ \lim_{r \to +\infty} v(r) = 0. \end{cases} \tag{1.9}$$

Then, a direct calculation shows that $G(0) = 0$, $G'(0) = g(0) = 1$ and $G''(0) = g'(0) = (\gamma - 2)\theta$, and we get the expansions $G^{-1}(s) = s + \frac{(\gamma-2)\theta}{2} s^2 + o(s^2)$ as $s \to 0$. Now we calculate the expansion of $f(s)$ at zero:

$$\begin{aligned} f(s) &= G^{-1}(s)^p \left(1 + G^{-1}(s)\right)^{-\gamma\theta} \\ &= \left(s + \frac{(\gamma-2)\theta}{2} s^2 + o\left(s^2\right)\right)^p \left(1 + s + \frac{(\gamma-2)\theta}{2} s^2 + o\left(s^2\right)\right)^{-\gamma\theta} \\ &= \left(s^p + \frac{(\gamma-2)p\theta}{2} s^{p+1} + o\left(s^{p+1}\right)\right) (1 - \gamma\theta s + o(s)) \\ &= s^p + \left(\frac{(\gamma-2)p\theta}{2} - \gamma\theta\right) s^{p+1} + o\left(s^{p+1}\right). \end{aligned}$$

So we conclude that $\lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} s\right) \to s^p$ as $\lambda \to 0^+$. Thus, (1.9) can be regarded as a small perturbation of the problem

$$\Delta v + v^p = 0$$

when $\lambda > 0$ is sufficiently small.

The rest of the paper will be organized as follows: First we use the Pohozaev identity to reach the following result, which indicates that the problem (1.2) does not admit positive solutions in $H^1(\mathbb{R}^N)$ when it is supercritical.

Theorem 1.1. *There exist no H^1 solutions to the problem (1.2) for $\gamma = 1$.*

We then turn to looking for solutions to (1.2) which are not in $H^1(\mathbb{R}^N)$. After change of variables, we construct solutions for (1.2) by asymptotic analysis and Liapunov–Schmidt reduction method introduced by Davila, del Pino, Musso and Wei [6–8].

Theorem 1.2. *There exist infinitely many positive solutions to (1.2) with decay $O\left(|x|^{-\frac{2}{p-1}}\right)$ at infinity.*

2 Proof of Theorem 1.1

In this section, we prove the nonexistence of the solution using the following Pohozaev identity, the proof of which can be found mainly in [11].

Lemma 2.1 (Pohozaev identity). *Suppose $F(x, u, r) \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfies*

$$\operatorname{div} F_r(x, u, \nabla u) = F_u(x, u, \nabla u), \tag{2.1}$$

where

$$F_r(x,u,r) = (F_{r_1}(x,u,r),\ F_{r_2}(x,u,r), \dots,\ F_{r_N}(x,u,r)), \quad r = (r_1,\ r_2, \dots,\ r_N),$$
$$F_{r_i}(x,u,r) = \frac{\partial F(x,u,r)}{\partial r_i}, \quad i = 1,\ 2, \dots,\ N$$

and

$$F_u(x,u,r) = \frac{\partial F(x,u,r)}{\partial u}.$$

Then, if $F(x,u,\nabla u)$, $x \cdot F_x(x,u,\nabla u)$ and $F_r(x,u,\nabla u) \cdot \nabla u \in L^1(\mathbb{R}^N)$, the following identity holds:

$$N \int_{\mathbb{R}^N} F(x,u,\nabla u)\mathrm{d}x + \int_{\mathbb{R}^N} x \cdot F_x(x,u,\nabla u)\mathrm{d}x - \int_{\mathbb{R}^N} F_r(x,u,\nabla u) \cdot \nabla u \mathrm{d}x = 0. \tag{2.2}$$

As a consequence, we reach the following lemma based on Lemma 2.1.

Lemma 2.2. *Suppose that $u \in H^1(\mathbb{R}^N)$ is a solution of (1.2). Then*

$$\frac{N-2}{2} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x = \frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x. \tag{2.3}$$

Proof. We take

$$F(x,u,r) = \frac{|r|^2}{2(1+u)^{2\theta}} - \frac{u^{p+1}}{p+1}.$$

Then we find

$$\begin{aligned} F_r(x,u,\nabla u) &= \frac{\nabla u}{(1+u)^{2\theta}}, \\ \operatorname{div} F_r(x,u,\nabla u) &= \operatorname{div} \frac{\nabla u}{(1+u)^{2\theta}}, \\ F_u(x,u,\nabla u) &= -\frac{\theta |\nabla u|^2}{(1+u)^{1+2\theta}} - u^p, \\ F_x(x,u,\nabla u) &= 0 \end{aligned}$$

and

$$F_r(x,u,\nabla u) \cdot \nabla u = \frac{|\nabla u|^2}{(1+u)^{2\theta}}.$$

So if u solves (1.2) when $\gamma = 1$, then we have (2.1), and the following identity holds by Lemma 2.1:

$$N \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^2}{2(1+u)^{2\theta}} - \frac{u^{p+1}}{p+1} \right) \mathrm{d}x - \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x = 0,$$

or

$$\frac{N-2}{2} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x = \frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x,$$

as desired. □

For any $\varphi \in H^1(\mathbb{R}^N)$, the solution u of equation (1.2) for $\gamma = 1$ satisfies

$$\int_{\mathbb{R}^N} -\left(\operatorname{div} \frac{\nabla u}{(1+u)^{2\theta}} \right) \varphi \mathrm{d}x - \int_{\mathbb{R}^N} \frac{\theta |\nabla u|^2 \varphi}{(1+u)^{1+2\theta}} \mathrm{d}x = \int_{\mathbb{R}^N} u^p \varphi \mathrm{d}x.$$

That is
$$\int_{\mathbb{R}^N} \frac{\nabla u \nabla \varphi}{(1+u)^{2\theta}} \mathrm{d}x - \theta \int_{\mathbb{R}^N} \frac{|\nabla u|^2 \varphi}{(1+u)^{1+2\theta}} \mathrm{d}x = \int_{\mathbb{R}^N} u^p \varphi \mathrm{d}x.$$
By taking $\varphi = u$, we achieve
$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x - \theta \int_{\mathbb{R}^N} \frac{|\nabla u|^2 u}{(1+u)^{1+2\theta}} \mathrm{d}x = \int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x. \tag{2.4}$$

Now we combine (2.3) and (2.4) to cancel $\int_{\mathbb{R}^N} u^{p+1} \mathrm{d}x$, to obtain
$$\left(1 - \frac{(p+1)(N-2)}{2N}\right) \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(1+u)^{2\theta}} \mathrm{d}x = \theta \int_{\mathbb{R}^N} \frac{|\nabla u|^2 u}{(1+u)^{1+2\theta}} \mathrm{d}x.$$
Since u is positive, we have $1 - \frac{(p+1)(N-2)}{2N} > 0$, which leads to the conclusion that $p < \frac{N+2}{N-2}$. This proves Theorem 1.1.

3 Proof of Theorem 1.2

Let
$$X = \left\{ \phi \in C^2(\mathbb{R}^N) \;\middle|\; \|\phi\|_* = \sup_{|x|\le 1} |x|^{\sigma} |\phi(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}} |\phi(x)| < \infty \right\},$$
$$Y = \left\{ h \in C(\mathbb{R}^N) \;\middle|\; \|h\|_{**} = \sup_{|x|\le 1} |x|^{2+\sigma} |h(x)| + \sup_{|x|\ge 1} |x|^{2+\frac{2}{p-1}} |h(x)| < \infty \right\}$$
be weighted spaces with $\sigma > 0$ and $L(\phi) := \phi'' + \frac{N-1}{r}\phi' + pw^{p-1}\phi$ is a linear operator from X to Y. The following lemma which is Lemma A. 1 proved by Dávila et al. [6] gives the solvability of the linear problem
$$\begin{cases} L(\phi) = h, \ r \in (0, +\infty); \\ \lim\limits_{r\to+\infty} \phi(r) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume $0 < \sigma < N-2$ and $p > \frac{N+2}{N-2}$. Then there exists a constant $C > 0$ such that for any $h \in Y$, the equation (3.1) has a solution $\phi = \mathcal{T}(h)$ such that $\mathcal{T}$ defines a linear map and*
$$\|\phi\|_* = \|\mathcal{T}(h)\|_* \le \|h\|_{**}.$$

We look for a solution to (1.7) of the form $v = w + \phi$ by the perturbation method for a small perturbation ϕ. By putting $v = w + \phi$ into (1.9), we get the following equation for $\phi = \phi(r)$
$$\begin{cases} L(\phi) = S(w) + N(\phi), \ r \in (0, +\infty); \\ \lim\limits_{r\to+\infty} \phi(r) = 0, \end{cases} \tag{3.2}$$
where
$$S(w) = -\Delta w - \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right),$$
$$N(\phi) = \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right) + pw^{p-1}\phi - \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} (w+\phi)\right).$$

First, we will estimate the error $\|S(w)\|_{**}$ of the approximate solution. The fact

$$\lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right) = w^p + \left(\frac{(\gamma-2)p\theta}{2} - \gamma\theta\right) \lambda^{\frac{2}{p-1}} w^{p+1} + o\left(\lambda^{\frac{2}{p-1}} w^{p+1}\right), \quad \lambda \to 0$$

and the equation $\Delta w + w^p = 0$ show that

$$\begin{aligned}
|S(w)| &= \left|\Delta w + \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right)\right| \\
&= \left|\Delta w + w^p + \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right) - w^p\right| \\
&= \left|\lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right) - w^p\right| \\
&= \left|\left(\frac{(\gamma-2)p\theta}{2} - \gamma\theta\right) \lambda^{\frac{2}{p-1}} w^{p+1} + o\left(\lambda^{\frac{2}{p-1}} w^{p+1}\right)\right| \\
&\leq C\lambda^{\frac{2}{p-1}} |w|^{p+1}.
\end{aligned}$$

Thus, we have

$$\sup_{|x|\leq 1} |x|^{2+\sigma} |S(w)| \leq C\lambda^{\frac{2}{p-1}} \|w\|_\infty^{p+1} \sup_{|x|\leq 1} |x|^{2+\sigma} \leq C\lambda^{\frac{2}{p-1}}. \tag{3.3}$$

On the other hand, noting that $w(x) \leq C\,(1+|x|)^{-\frac{2}{p-1}}$ for $x \in \mathbb{R}^N$, we achieve

$$\begin{aligned}
\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} |S(w)| &\leq C\lambda^{\frac{2}{p-1}} \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} w^{p+1} \\
&\leq C\lambda^{\frac{2}{p-1}} \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} (1+|x|)^{-\frac{2(p+1)}{p-1}} \\
&\leq C\lambda^{\frac{2}{p-1}} \sup_{|x|\geq 1} \left(\frac{|x|}{1+|x|}\right)^{\frac{2(p+1)}{p-1}} \\
&\leq C\lambda^{\frac{2}{p-1}}.
\end{aligned} \tag{3.4}$$

So, we have

$$\|S(w)\|_{**} = \sup_{|x|\leq 1} |x|^{2+\sigma} |S(w)| + \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} |S(w)| \leq C\lambda^{\frac{2}{p-1}} \tag{3.5}$$

by (3.3) and (3.4).

Next, our proof is based on the contraction mapping theorem. We know that ϕ is a solution of (3.2) if and only if ϕ is a fixed point for the operator

$$\mathcal{A}(\phi) := \mathcal{T}(S(w) + N(\phi)),$$

where $\mathcal{T}$ is given in Lemma 3.1. We define

$$\Sigma = \left\{\phi :\ \mathbb{R}^N \to \mathbb{R} \,\middle|\, \|\phi\|_* \leq C\lambda^{\frac{2}{p-1}}\right\}$$

and we shall prove that $\mathcal{A}$ has a fixed point in Σ.

Note that

$$\begin{aligned}
N(\phi) &= \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} w\right) + pw^{p-1}\phi - \lambda^{-\frac{2p}{p-1}} f\left(\lambda^{\frac{2}{p-1}} (w+\phi)\right) \\
&= w^p + pw^{p-1}\phi - (w+\phi)^p + o(1),\ \lambda \to 0.
\end{aligned}$$

By the argument given in [6], it follows that

$$\|N(\phi)\|_{**} \leq C[\|\phi\|_*^2 + \|\phi\|_*^p] \tag{3.6}$$

for any $\phi \in \Sigma$ and $\sigma \in \left(0, \min\left\{2, \frac{2}{p-1}\right\}\right)$. So, we have

$$\begin{aligned} \|\mathcal{A}(\phi)\|_* &\leq C[\|S(W)\|_{**} + \|N(\phi)\|_{**}] \\ &\leq C[\lambda^{\frac{2}{p-1}} + \lambda^{\frac{4}{p-1}} + \lambda^{\frac{2p}{p-1}}] \\ &\leq C\lambda^{\frac{2}{p-1}} \end{aligned} \tag{3.7}$$

by (3.5), (3.6) and Lemma 3.1. Therefore, we conclude that $\mathcal{A}(\Sigma) \subset \Sigma$.

Now we will prove that $\mathcal{A}$ is a contraction mapping in Σ. By taking $\phi_1, \phi_2 \in \Sigma$, we achieve

$$\begin{aligned} \|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* &= \|\mathcal{T}(S(w) + N(\phi_1)) - \mathcal{T}(S(w) + N(\phi_2))\|_* \\ &= \|\mathcal{T}(N(\phi_1) - N(\phi_2))\|_* \\ &\leq C\|N(\phi_1) - N(\phi_2)\|_{**}. \end{aligned}$$

Then, combining this with the fact

$$\begin{aligned} |N(\phi_1) - N(\phi_2)| &\leq C|pw^{p-1}(\phi_1 - \phi_2) - (w + \phi_1)^p + (w + \phi_2)^p| \\ &\leq C(w^{p-2}(|\phi_1| + |\phi_2|) + |\phi_1|^{p-1} + |\phi_2|^{p-1})|\phi_1 - \phi_2|, \quad \lambda \to 0, \end{aligned}$$

it follows that

$$\begin{aligned} \|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* &\leq C\|N(\phi_1) - N(\phi_2)\|_{**} \\ &\leq C\left[\|\phi_1\|_*^{\min\{1,p-1\}} + \|\phi_2\|_*^{\min\{1,p-1\}}\right]\|\phi_1 - \phi_2\|_* \\ &\leq \frac{1}{2}\|\phi_1 - \phi_2\|_* \end{aligned}$$

if λ is small. So $\mathcal{A} : \Sigma \to \Sigma$ is a contraction mapping and hence has a fixed point $\phi \in \Sigma$. This means that $v_\lambda(|x|) = \lambda^{\frac{2}{p-1}}(w(\lambda|x|) + \phi(\lambda|x|))$ is a continuum of solutions to (1.7) with the decay $O(|x|^{-\frac{2}{p-1}})$ at infinity. Then $u_\lambda(|x|) = G^{-1}(v_\lambda(|x|))$ is our desired solution satisfying (1.2). This completes the proof of Theorem 1.2.

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