

Oscillatory solutions and period-doubling bifurcations of functional differential equations

 Michal Fečkan ^{1, 2} and Július Pačuta¹

¹Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

²Institute of Mathematics, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

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Abstract. In our work, we analyze a first order differential equation that contains a piece-wise constant term. Such equations admit periodic solutions and also, some types of bifurcations may occur. We establish sufficient conditions for local and global stability of the zero solution, existence of nontrivial periodic solutions, and period-doubling bifurcations.

Keywords: functional differential equations, periodic solutions, period-doubling bifurcation.

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1 Introduction

In our paper, we analyze solutions of functional differential equation

$$x'(t) = p(t - [t])f(x(t)) + x([t])g(t - [t]), \quad t \geq 0 \quad (1.1)$$

where $[\cdot] : [0, \infty) \rightarrow \mathbb{N} \cup \{0\}$ is the standard floor function, i.e. $[t] = n$ if $t \in [n, n + 1)$ for some $n \in \mathbb{N} \cup \{0\}$. We will assume that the function f is negative for positive x , positive for x negative, and equals zero at zero, p is positive over the interval $[0, 1]$, and g is negative over $[0, 1]$ (see conditions I–IV below for more precise assumptions).

In this work, we use the comparison principle for differential equations to analyze sufficient conditions for a local stability and the global stability of the zero solution, the existence of 2-periodic solutions of the equation (1.1) under certain assumptions on the functions f and g . Using the general bifurcation theory, we will prove that under a change of the function g , a period-doubling bifurcation occurs in (1.1); see Theorem 3.11 for more details.

A similar problem to (1.1) was studied in [7] where $p \equiv 1$ and $g > 0$ were considered. The results of [7] may be used as a numeric model that describes temperature changes in a space bounded by outside environment. The space is simultaneously warmed by a heating agent

 Corresponding author. Email: Michal.Feckan@fmph.uniba.sk

inside and cooled by cold environment from outside. The problem of height stabilization of drone flight was reduced to analysis of equation with such delay in the paper [2]. Appearing delays of this sort in Marchuk's model in treatment of infection diseases is discussed in Section 6 of the paper [5]. In [7], the existence, uniqueness and stability of 1-periodic solution was proven and such solution represents a stable change of the room temperature. However, 1-periodic solutions do not occur in the case of the problem (1.1) under the assumptions I)–IV). Also any results concerning the uniqueness and stability of periodic solutions are not known.

Equations analogous to (1.1) were studied by many authors; we refer to e.g. [7] and references therein. Other related results are as follows: Smooth stable invariant manifolds are constructed in [1] for a class of delay equations with a piece-wise constant delay, for any sufficiently small perturbation of a nonuniform exponential dichotomy. The global exponential stability and periodicity of the Cohen–Grossberg neural network model with generalized piece-wise constant delay is investigated in [3]. Pseudo S -asymptotically ω -periodic solutions were studied in [4] for certain ordinary differential equations with piece-wise constant arguments and possessing exponential dichotomies. A computational technique for solving linear delay differential equations with a piece-wise constant delay function is presented in [9].

2 Preliminaries

Throughout the whole paper, for the functions f, g in the equation (1.1), we will assume the following conditions:

- I) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, $f(0) = 0$ and $xf(x) < 0$ for $x \neq 0$,
- II) $p : [0, 1] \rightarrow (0, \infty)$ and $g : [0, 1] \rightarrow (-\infty, 0)$ are continuous,
- III) $\frac{f(x)}{x}$ is decreasing over $(0, \infty)$ and increasing over $(-\infty, 0)$,
- IV) $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = -\infty$ and $f'(0) = 0$.

We say that a continuous function $x : [0, \infty) \rightarrow \mathbb{R}$ is a solution of the equation (1.1) if for every $n \in \mathbb{N} \cup \{0\}$, the function x is continuous over $[n, n+1]$, continuously differentiable over $(n, n+1)$, and the equation (1.1) is satisfied for $t \in (n, n+1)$. It is clear that (1.1) is actually valid also for $t = n^+$ and $t = (n+1)^-$.

We say that the solution x of (1.1) is eventually positive (negative) if $x(t) > 0$ ($x(t) < 0$) for every $t \geq 0$. The solution x is non-oscillatory if there is $t_0 > 0$ such that $x(t_0) = 0$ and $x(t) \neq 0$ for $t > t_0$. The solution $x \not\equiv 0$ is oscillatory if there is a sequence $\{t_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $x(t_n) = 0$. We will actually prove that if x is oscillatory, then under the conditions I)–IV), it holds that $x(n)x(n-1) < 0$ for every $n \in \mathbb{N}$.

In order to study solutions of (1.1), we will consider the equation

$$x'(t) = p(t)f(x(t)) + x(0)g(t), \quad t \in [0, 1]. \quad (2.1)$$

Note that the solution x is actually differentiable at $t = 0^+$ and $t = 1^-$.

To study local stability of the zero solution of (1.1), we define a map as

$$q : \mathbb{R} \rightarrow \mathbb{R}, \quad q(x(0)) = x(1) \quad (2.2)$$

where x is a solution of (2.1). Using the Implicit function theorem, one can verify that if f is C^r -smooth for $r = 1, 2, 3, \dots$, then q is C^r -smooth. The next result is known [10], but we present its proof for a reader convenience.

Lemma 2.1. *Let q be a C^1 -smooth function defined by (2.2) and $q(0) = 0$. If $|q'(0)| < 1$, then the zero solution of (1.1) is locally asymptotically stable. If $|q'(0)| > 1$, then the zero solution of (1.1) is locally asymptotically unstable.*

Proof. By the assumption, we have $q(z) = z(q'(0) + \omega(z))$, where $\lim_{z \rightarrow 0} \omega(z) = 0$. If $|q'(0)| < 1$, then for some $\delta > 0$, it holds $|q(z)| < |z|$ for $0 < |z| < \delta$. Thus for $|x(0)| < \delta$, we get $|x(1)| < |x(0)|$ and consequently, $|x(n+1)| < |x(n)|$ for $n \in \mathbb{N}$, since $|x(n)| < \delta$. It holds that $\lim_{n \rightarrow \infty} x(n) = 0$. Otherwise, if $x(n)$ (or some subsequence) converges to $\alpha \neq 0$, then $|\alpha| < \delta$ a the solution y starting with $y(0) = \alpha$ would be a nontrivial 1-periodic solution. However, it holds $|\alpha| = |y(1)| = |q(y(0))| < |\alpha|$ what is a contradiction.

If $|q'(0)| > 1$, then for some $\delta > 0$, it holds $|q(z)| > |z|$ for $0 < |z| < \delta$. Thus for every $|x(0)| < \delta$, we get $|x(1)| > |x(0)|$ and the instability of zero solution follows. $\square$

To study bifurcations that may occur in the equation (1.1) (see Theorem 3.11 for precise formulations), we consider the following parametrized equation

$$x'(\lambda, z, t) = p(t - \lfloor t \rfloor) f(x(\lambda, z, t)) + x(\lambda, z, \lfloor t \rfloor) g(t - \lfloor t \rfloor, \lambda), \quad t \geq 0, \quad x(\lambda, z, 0) = z, \quad z \in \mathbb{R} \quad (2.3)$$

where $\lambda \in \mathbb{R}$ is a parameter and for every fixed λ , the function $g(\cdot, \lambda)$ satisfies the condition II). For a solution $x = x(\lambda, z, t)$ of (2.3), we define a corresponding map

$$q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad q(z, \lambda) = x(\lambda, z, 1). \quad (2.4)$$

If f is C^r -smooth and g is C^r -smooth with respect to λ for $r = 1, 2, 3, \dots$, then q is C^r -smooth.

Lemma 2.2 (Period-doubling bifurcation). *Let $r > 0$ and $q : \mathbb{R} \times (-r, r) \rightarrow \mathbb{R}$ be a C^4 -smooth function, $q = q(z, \lambda)$. Assume that*

$$q(0, \lambda) = 0 \quad \text{and} \quad \partial_z q(0, 0) = -1.$$

Denote

$$\alpha = \partial_{z\lambda}^2 q(0, 0), \quad \beta = \frac{1}{2} \partial_z^2 q(0, 0), \quad \gamma = \frac{1}{6} \partial_z^3 q(0, 0).$$

If $\alpha(\beta^2 + \gamma) < 0$ ($\alpha(\beta^2 + \gamma) > 0$), then for $\lambda > 0$ ($\lambda < 0$) the diffeomorphism q has one fixed point $z(\lambda)$ and two 2-periodic points $z_1(\lambda)$, $z_2(\lambda)$ and for $\lambda < 0$ ($\lambda > 0$) it has just one fixed point.

Lemma 2.2 is a basic result from the bifurcation theory and its proof can be found in e.g. [10]. If $z_1 = z_1(\lambda)$ is a 2-periodic point of q , i.e. $q(q(z_1, \lambda), \lambda) = z_1$ and $q(z_1, \lambda) \neq z_1$, then due to the definition (2.4), we have

$$x(\lambda, x(\lambda, z_1, 1), 1) = x(\lambda, z_1, 2) = z_1.$$

This means that $x(\lambda, z_1, t)$ is a 2-periodic solution of (2.3).

3 Results

Lemma 3.1. *Assume that I)–III) hold.*

1. *Let x be a solution of the equation (2.1) such that $x(0) > 0$ and $x(t_0) = 0$ for some $t_0 \in (0, 1]$. Then every solution y of (2.1) starting with $y(0) > x(0)$ crosses zero at some $t_1 \in (0, t_0)$ and it holds $y(1) < x(1)$.*

2. Let x be a solution of the equation (2.1) such that $x(0) < 0$ and $x(t_0) = 0$ for some $t_0 \in (0, 1]$. Then every solution y of (2.1) starting with $y(0) > x(0)$ crosses zero at some $t_1 \in (0, t_0)$ and it holds $y(1) > x(1)$.

Proof. First, we prove the part 1. Let $\alpha = \frac{y(0)}{x(0)} > 1$ and denote $z(t) = \alpha x(t)$ for $t \in [0, 1]$. Since the function $\frac{f(x)}{x}$ is decreasing over $(0, \infty)$, it holds

$$\frac{f(\alpha \xi)}{\alpha \xi} < \frac{f(\xi)}{\xi} \quad \text{for } \xi > 0$$

and so $\alpha f(x(t)) > f(\alpha x(t))$ as long as $x(t) > 0$, i.e. for $t \in (0, t_0)$. Thus, we obtain

$$z'(t) = \alpha(p(t)f(x(t)) + x(0)g(t)) > p(t)f(z(t)) + z(0)g(t).$$

Since $y(0) = z(0) = \alpha x(0)$, it holds $y(t) < z(t)$ over $[0, 1]$ whenever $y(t) > 0$. The function y crosses zero at some point $t_1 \in (0, t_0]$. Note that necessarily, $y(t) < z(t)$ over $(0, t_1]$ so $t_1 < t_0$ and $y(t) < 0$ for $t > t_1$. Thus, $y(t_0) < x(t_0) = 0$ and by the comparison principle, we have $y(1) < x(1)$.

Next, we move to the proof of the part 2. Let α and z be as in the first part. If $\xi < 0$, then $\alpha \xi < \xi$, hence it holds

$$\frac{f(\alpha \xi)}{\alpha \xi} < \frac{f(\xi)}{\xi}$$

and $\alpha f(x(t)) < f(\alpha x(t))$ if $x(t) < 0$. Thus z is a supersolution of (2.1) and using similar arguments as in the first part, the assertion follows. $\square$

Lemma 3.2. Assume that the conditions I–IV are fulfilled. For $\alpha \neq 0$, let x be the corresponding solution of the equation (2.1) such that $x(0) = \alpha$ and $t_\alpha \in (0, 1]$ be such that $x(t_\alpha) = 0$. Then $t_\alpha \rightarrow 0^+$ as $\alpha \rightarrow \infty$ or $\alpha \rightarrow -\infty$ and t_α depends on α continuously.

Proof. Denote $G(t) = \int_0^t g(s) ds$ and $P(t) = \int_0^t p(s) ds$. Due to Lemma 3.1, we know that t_α is decreasing with respect to α . We prove the case that $t_\alpha \rightarrow 0^+$ as $\alpha \rightarrow \infty$. Assume by a contradiction that $t_\alpha \geq 2\delta$ for some $\delta > 0$. Integrating the equation (2.1) over the interval $[0, t_\alpha]$, we have

$$x(t_\alpha) - x(0) = \int_0^{t_\alpha} p(t)f(x(t)) dt + \alpha \int_0^{t_\alpha} g(t) dt. \quad (3.1)$$

The conditions I) and II) imply that $x'(t) \leq \alpha g(t)$ for $t \in [0, t_\alpha]$ hence $x(t) \geq \alpha \int_t^{t_\alpha} g(s) ds = \alpha(G(t) - G(t_\alpha))$.

Let $t \in [0, \delta]$. For every $K > 0$, it holds $f(\xi) < -K\xi$ for $\xi > 0$ sufficiently large. Note that $G(t) - G(t_\alpha)$ is bounded from below by a positive constant η and if we set $\xi = \alpha(G(t) - G(t_\alpha))$, then such ξ may be arbitrarily large if α is arbitrarily large. Since f is decreasing over $[0, \infty)$, it holds

$$f(x(t)) \leq f(\alpha(G(t) - G(t_\alpha))) < -K\alpha(G(t) - G(t_\alpha)) \leq -K\alpha\eta. \quad (3.2)$$

Combining (3.1) and (3.2), we obtain

$$-\alpha \leq \int_0^\delta p(t)f(x(t)) dt + \alpha G(t_\alpha) < \alpha(-K\eta P(\delta) + G(2\delta)),$$

a contradiction if K is chosen large enough what implies that α is large enough. The case $\alpha \rightarrow -\infty$ would be treated similarly.

Now we prove that t_α is continuously dependent on α . Let $\alpha_1 > 0$ be fixed and $\alpha_2 > 0$ is close to α_1 . We can assume that $\alpha_1 > \alpha_2$. Then due to Lemma 3.1, it holds that $t_{\alpha_1} < t_{\alpha_2}$. Denote x_1, x_2 to be solutions of (2.1) such that $x_i(0) = \alpha_i, i = 1, 2$. Note that $x_{1,2}$ are bounded and f is Lipschitz continuous on bounded sets with Lipschitz constant $C = C(f, \alpha_1)$. For $t \in [0, 1]$, we estimate

$$|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)| \int_0^t |1 + g(s)| \, ds + C \int_0^t p(s) |x_1(s) - x_2(s)| \, ds$$

and due to the Gronwall inequality, it follows that

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq |x_1(0) - x_2(0)| \left(\int_0^t |1 + g(s)| \, ds \right. \\ &\quad \left. + C(f, \alpha_1) \int_0^t p(s) \int_0^s |1 + g(r)| \, dr e^{\int_s^t C(f, \alpha_1) p(r) \, dr} \, ds \right) \\ &\leq C(f, g, p, \alpha_1) |x_1(0) - x_2(0)|. \end{aligned} \quad (3.3)$$

Hence it holds that $0 < x_2(t_{\alpha_1}) \leq C(\alpha_1 - \alpha_2)$ and since $x_2'(t) < \alpha_2 g(t) < 0$ for $t \in [t_{\alpha_1}, t_{\alpha_2}]$, we conclude that t_{α_2} is close to t_{α_1} . $\square$

Lemma 3.3. Assume that the conditions I–IV are fulfilled. Let x be a solution of the equation (2.1). Then there exist constants $c_1 \leq 0 \leq c_2$ dependent on f, g and p such that for every solution x of (1.1), it holds $c_1 \leq \liminf_{t \rightarrow \infty} x(t) \leq \limsup_{t \rightarrow \infty} x(t) \leq c_2$.

Proof. We will assume the case $x(0) > 0$, since the case $x(0) < 0$ would be treated similarly. Denote $p_0 := \min_{t \in [0,1]} p(t) > 0$ and $g_0 := \min_{t \in [0,1]} g(t) < 0$. It is clear that $x'(t) < 0$ for t as long as $x(t) \geq 0$. Then either $x(1) > 0$, $x(1) = 0$, or $x(1) < 0$. If $x(1) > 0$, then $x(n) > 0$ for $n \in \mathbb{N}$ due to Lemma 3.1 and so, $x(t) > 0$ for $t \geq 0$. If $x(1) = 0$, then $x(t) = 0$ for $t \geq 1$.

Assume that $x(1) < 0$. Then there is some $t_0 \in (0, 1)$ such that $x(t_0) = 0$. Since $x'(t_0) < 0$, one can deduce that $x(t) < 0$ for $t \in (t_0, 1]$. Let y be the solution of autonomous problem

$$y'(t) = p_0 f(y(t)) + x(0)g_0, \quad y(t_0) = 0. \quad (3.4)$$

Clearly, $y'(t_0) < 0$ and necessarily, $y(t) < 0$ for $t > t_0$. Since $p_0 f(\xi) \leq p(t)f(\xi)$ for $\xi \leq 0$ and $x(0)g_0 \leq x(0)g(t)$, it follows that $y(t) \leq x(t)$ for $t \in [t_0, 1]$. Note that the equation (3.4) is autonomous hence y is bounded from below by the constant solution $x_1 := f^{-1}(-\frac{g_0}{p_0}x(0))$ of (3.4) and consequently, $x(1) > x_1$. The value x_1 is a lower bound for solution x over $[0, 1]$.

Next, assume that $t \in [1, 2]$ and x is the solution of

$$x'(t) = p(t)f(x(t)) + x(1)g(t)$$

where $x(1) < 0$. If $x(2) \leq 0$, then $x(t) \leq 0$ for $t \geq 1$ as in the previous part of the proof. If $x(2) > 0$, then $x(t_1) = 0$ for some $t_1 \in (1, 2)$ and $x(t) > 0$ for $t \in (t_1, 2]$. If we denote y to be the solution of

$$y'(t) = p_0 f(y(t)) + x_1 g_0, \quad y(t_1) = 0,$$

then similar arguments as in the previous part lead to $x(2) \leq y(2) < x_2 := f^{-1}(-\frac{g_0}{p_0}x_1)$. Hence x_2 is an upper bound for x over $[1, 2]$.

Let x be an oscillatory solution of (1.1). Due the previous arguments, there exists a sequence of local bounds $\{x_n\}_{n \in \mathbb{N}}$ of x . The value x_{2n} is an upper bound for x over $[2n - 1, 2n]$. Note that $x'(t) \leq 0$ as long as $x(t) \geq 0$ hence x_{2n} bounds x over $[2n, 2n + 1]$. Analogously,

The value x_{2n-1} is a lower bound for x over $[2n-2, 2n]$. The numbers x_n can be expressed by recurrence relation $x_n := f^{-1}\left(-\frac{g_0}{p_0}x_{n-1}\right)$, $n \in \mathbb{N}$ with $x_0 = x(0)$. Consequently, $x_{n+2} = F(x_n)$ where

$$F(\xi) = f^{-1}\left(-\frac{g_0}{p_0}f^{-1}\left(-\frac{g_0}{p_0}\xi\right)\right).$$

Denote M to be the set of fixed points of F . Observe that $F : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function and $F(0) = 0$, thus, M is nonempty. Simple arguments show that $\frac{F(\xi)}{\xi} \rightarrow 0$ as $\xi \rightarrow \pm\infty$. Thus, $0 < F(\xi) < \xi$ for $\xi > 0$ sufficiently large and $\xi < F(\xi) < 0$ for $\xi < 0$ large in absolute value and so M is closed and bounded. We choose c_1 to be a lower bound of M and c_2 to be an upper bound of M .

If $|x(0)|$ is large enough, then the sequences x_{2n} , x_{2n-1} converge to some of fixed points of F . Let $x(0) > \alpha$ where $\alpha = \sup M$. Since F is continuous and increasing, it follows that $F((\alpha, \infty)) = (\alpha, \infty)$. Using properties of F , we get $0 < x_{2n+2} = F(x_{2n}) < x_{2n}$ and $x_{2n} > \alpha$ for $n \in \mathbb{N}$. Clearly, x_{2n} converges to an element of M and thus to α . Thus, $\limsup_{t \rightarrow \infty} x(t) \leq c_2$. Similar arguments show that if $x(0) \in (0, \alpha)$, then x_{2n} converges to some element of M . Analogously, the sequence x_{2n-1} converges to a fixed point of F and so $\liminf_{t \rightarrow \infty} x(t) \geq c_1$. The case $x(0) < 0$ is similar.

Let x be non-oscillatory. If $x(n_0) > 0$, then $x(n_0 + 1) < x(n_0)$, since x is positive and therefore decreasing over $[n_0, n_0 + 1]$. Lemma 3.1 implies that $x(t) > 0$ for $t \geq n_0$. If $x(n_0) < 0$, then $x(t) < 0$ for $t \geq n_0$. In both cases, the sequence $\{x(n)\}_{n \geq n_0}$ is obviously strictly monotone and $|x(n+1)| < |x(n)|$ for $n \geq n_0$. As in the proof of Lemma 2.1, it holds $\lim_{n \rightarrow \infty} x(n) = 0$ and we can choose $c_1 = c_2 = 0$. $\square$

Lemma 3.4. Assume that the conditions I)–IV) are fulfilled. Let x be a solution of the equation (1.1).

1. If $\int_0^1 g(t) dt > -1$, then there exists $\delta > 0$ such that for every $0 < |x(0)| < \delta$, the solution x does not cross zero in $[0, \infty)$.
2. If $\int_0^1 g(t) dt \geq -1$, then $\lim_{t \rightarrow \infty} x(t) = 0$ for every solution x .

Proof. We will prove the lemma in the case of $x(0) > 0$, since the proof in the other case is similar.

Assume that $\int_0^1 g(t) dt > -1$. Due to our assumption, it holds $f(\xi) > -\varepsilon\xi$, $\xi \in (0, \delta)$ for some $\varepsilon > 0$ and $\delta > 0$ sufficiently small that depends on ε . Hence

$$x'(t) = p(t)f(x(t)) + x(0)g(t) \geq -\varepsilon p(t)x(t) + x(0)g(t)$$

if $x(t) \in (0, \delta)$. We choose $x(0) \in (0, \delta)$ and let y be the solution of problem

$$y'(t) = -\varepsilon p(t)y(t) + x(0)g(t), \quad y(0) = x(0), \quad t \in [0, 1].$$

Then $x(t) > y(t)$ over $(0, 1]$ by the comparison principle. The solution y is of the form

$$y(t) = e^{-\varepsilon \int_0^t p(s) ds} \left(1 + \int_0^t e^{\varepsilon \int_0^\tau p(s) ds} g(\tau) d\tau \right) x(0). \quad (3.5)$$

Denote $A = e^{-\varepsilon \int_0^1 p(s) ds} \left(1 + \int_0^1 e^{\varepsilon \int_0^t p(s) ds} g(t) dt \right)$ and obviously, $A < 1$. Since $\int_0^1 g(t) dt > -1$, then $\int_0^1 e^{\varepsilon \int_0^t p(s) ds} g(t) dt > -1$ for $\varepsilon > 0$ small enough. Thus, $A > 0$ if δ is small and hence $x > y > 0$ over $[0, 1]$. Since $x(1) < x(0) < \delta$, we repeat the same argument and deduce that x is positive over $[0, \infty)$.

If $\int_0^1 g(t) dt \geq -1$ and $x(0) > 0$, then $x'(t) \geq x(0)g(t)$ for $t \in [0, 1]$ as long as $x(t) \leq 0$, hence $x(t) > x(0) \int_0^t g(s) ds$. Thus, $x(1) > -x(0)$. Since the inequality $x(1) < x(0)$ is obvious, we obtain that $|x(1)| < |x(0)|$. It holds that $|x(n)| < |x(n-1)|$ for $n \in \mathbb{N}$ and we conclude that $\lim_{t \rightarrow \infty} x(t) = 0$. $\square$

Theorem 3.5. Assume that the conditions I)–IV) hold.

1. If $\int_0^1 g(t) dt > -1$, then either there exists $n \in \mathbb{N}$ such that $x(t) = 0$ for $t \in [n, \infty)$, or x crosses zero at most finitely many times.
2. If $\int_0^1 g(t) dt \leq -1$, then x is oscillatory.

Proof. If there exists $n \in \mathbb{N}$ such that $x(n) = 0$, then $x(t) = 0$ for $t \in [n, \infty)$. If $x(n) \neq 0$ for every $n \in \mathbb{N}$, then due to the assertion 1 of Lemma 3.4, there exists some $\delta > 0$ such that if $0 < |x(0)| < \delta$, then x does not cross zero in $[0, \infty)$. If $\int_0^1 g(t) dt > -1$, then the assertion 2 of Lemma 3.4 implies that $\lim_{t \rightarrow \infty} x(t) = 0$. Hence there exists some $n \in \mathbb{N}$ so that $|x(n)| < \delta$. Thus, x does not cross zero in $[0, \infty)$.

Assume that $\int_0^1 g(t) dt \leq -1$. Let $x(0) > 0$ and y be the solution of problem

$$y'(t) = g(t)x(0), \quad y(0) = x(0), \quad t \in [0, 1].$$

Thus, $y(1) = y(0) \left(1 + \int_0^1 g(t) dt\right) \leq 0$ and y crosses zero at some $t_0 \in (0, 1]$. Since $x(t) < y(t)$ for some $t > 0$ and close to zero, then necessarily $x(t) < y(t)$ as long as $x(t) \geq 0$. Consequently, $x(t_1) = 0$ for some $t_1 < t_0$. Note that $x'(t_1) = x(0)g(t_1) < 0$ and we deduce that x crosses zero at exactly one point hence $x(1) < 0$. If $x(0) < 0$, we use similar arguments to obtain $x(1) > 0$ and therefore, x is oscillatory. $\square$

Theorem 3.6. Assume that the conditions I)–IV) hold. Let x be a solution of the equation (1.1).

1. If $-2 < \int_0^1 g(t) dt < 0$, then the zero solution of (1.1) is asymptotically stable.
2. If $-2 > \int_0^1 g(t) dt$, then the zero solution of (1.1) is asymptotically unstable.

Proof. We set $x(z, t)$ to be the unique solution of

$$x'(z, t) = p(t)f(x(z, t)) + zg(t), \quad t \in [0, 1], \quad x(z, 0) = z \quad (3.6)$$

and let q be defined by (2.2). Clearly, $q(z) = x(z, 1)$. Then (3.6) implies

$$\partial_z x'(z, t) = p(t)f'(x(z, t))\partial_z x(z, t) + g(t), \quad \partial_z x(z, 0) = 1.$$

Note that $x(0, t) = 0$ for $t \in [0, 1]$, since the corresponding solution of (3.6) for $x(0) = 0$ is the zero solution. Due to the assumption $f'(0) = 0$, we obtain $\partial_z x'(0, t) = g(t)$ and consequently, $\partial_z x(0, 1) = 1 + \int_0^1 g(t) dt$, since $\partial_z x(0, 0) = 1$. Hence

$$q'(0) = \partial_z x(0, 1) = 1 + \int_0^1 g(t) dt < 1.$$

So far we proved that if $-2 < \int_0^1 g(t) dt$, then $-1 < q'(0) < 1$ and if $\int_0^1 g(t) dt < -2$, then $q'(0) < -1$. The assertion follows from Lemma 2.1. $\square$

Remark 3.7. Since $q'(0) < -1$ in the assertion 2 of Theorem 3.6, 0 is a repeller of $q(z)$. Thus, any local non-zero iteration $\{q^n(z)\}_{n=0}^\infty$ exhibits a growth $|q^{n+1}(z)| > \frac{1-q'(0)}{2}|q^n(z)|$ as well as the property $q^{n+1}(z)q^n(z) < 0$. Then the corresponding solution $x(t)$, $x(0) = z$ is oscillatory with increasing amplitude, provided it starts in the neighborhood of 0, which it eventually leaves in an oscillatory manner after a finite time. Consequently, the assertion 2 of Theorem 3.6 is meant locally, i.e. there exist $\delta_2 > \delta_1 > 0$ such that for every $|x(0)| < \delta_1$, it holds $|x(n+1)| > \frac{1-q'(0)}{2}|x(n)|$ until $|x(t)| < \delta_2$.

Corollary 3.8. Assume that the conditions I–IV hold. If $-2 > \int_0^1 g(t) dt$, then there exist at least two 2-periodic solutions $x_{1,2}$ of (1.1) and $x_1(0) > 0 > x_2(0)$.

Proof. We only prove that there is a 2-periodic solution x of (1.1) starting at $x(0) > 0$. Lemma 3.3 implies that if we take a solution x with $x(0) > 0$ large enough, then $0 < \limsup_{t \rightarrow \infty} x(t) < x(0)$. Necessarily, $0 < x(2) < x(0)$, otherwise, if $x(2) > x(0)$, then Lemma 3.1 would imply that $x(2n+2) > x(2n)$ for every $n \in \mathbb{N}$, a contradiction. Thus, the sequence $\{x(2n)\}_{n \in \mathbb{N}}$ is decreasing and $\lim_{n \rightarrow \infty} x(2n) = \alpha \geq 0$. Consequently, $\alpha > 0$, since the zero solution is locally unstable due to Theorem 3.6. For $n \in \mathbb{N}$, we define y_n to be solutions of (1.1) over the interval $[0, 2]$ with $y_n(0) = x(2n)$. Clearly, y_n converge to a solution y of (1.1) with $y(0) = y(2) = \alpha$ and y is a desired 2-periodic solution. $\square$

Remark 3.9. In the linear case for constant coefficient, the assertion 1 of Theorem 3.6 was obtained as necessary and sufficient condition in Theorem 3.4 of the paper [6].

Remark 3.10. In the case where the period is not fixed as 1, and the period ω of the delay can be chosen, there exists the infinite number of the intervals such that equation will be exponentially stable if this period is situated in one of these intervals; see Remark 3.7 and Corollary 3.1 in [6].

If $\int_0^1 g(t) dt = -2$, then $q'(0) = -1$, and a double period bifurcation occurs.

Theorem 3.11. Assume that the conditions I–IV are fulfilled, f is C^4 -smooth in a neighborhood of $x = 0$, $f'''(0) \neq 0$. Consider a continuous perturbation $g : [0, 1] \times [-r, r]$ for some $r > 0$ such that $\int_0^1 g(t, 0) dt = -2$ and $g(t, \lambda)$ is C^1 -smooth with respect to λ . If

$$\int_0^1 \partial_\lambda g(t, 0) dt \int_0^1 p(t) f'''(0) \left(1 + \int_0^t g(s, 0) ds\right)^3 dt < 0 (> 0)$$

then the function $q(z, \lambda)$ defined by (2.4) has a local unique fixed point $z = 0$ for $\lambda < 0 (> 0)$, and a local unique fixed point $z = 0$ and two small 2-periodic points $z_1(\lambda) > 0 > z_2(\lambda)$ of (1.1) for $\lambda > 0 (< 0)$.

Proof. It is clear that $q(0, \lambda) = 0$. Then we compute

$$\begin{aligned} \partial_z x'(\lambda, z, t) &= p(t) f'(x(\lambda, z, t)) \partial_z x(\lambda, z, t) + g(t, \lambda), \\ \partial_z x(\lambda, z, 0) &= 1, \\ \partial_z^2 x'(\lambda, z, t) &= p(t) f'(x(\lambda, z, t)) \partial_z^2 x(\lambda, z, t) + p(t) f''(x(\lambda, z, t)) (\partial_z x(\lambda, z, t))^2, \\ \partial_z^2 x(\lambda, z, 0) &= 0, \\ \partial_z^3 x'(\lambda, z, t) &= p(t) f'(x(\lambda, z, t)) \partial_z^3 x(\lambda, z, t) + p(t) f''(x(\lambda, z, t)) \partial_z x(\lambda, z, t) \partial_z^2 x(\lambda, z, t) \\ &\quad + p(t) f'''(x(\lambda, z, t)) (\partial_z x(\lambda, z, t))^3 + 2p(t) f''(x(\lambda, z, t)) \partial_z x(\lambda, z, t) \partial_z^2 x(\lambda, z, t), \\ \partial_z^3 x(\lambda, z, 0) &= 0. \end{aligned}$$

Since $x(\lambda, 0, t) = 0$ and $f'(0) = 0$, we obtain $\partial_z x'(\lambda, 0, t) = g(t, \lambda)$. By integration we get $\partial_z x(\lambda, 0, s) = 1 + \int_0^s g(t, \lambda) dt$, $s \in [0, 1]$. Hence

$$\partial_z q(0, \lambda) = \partial_z x(\lambda, 0, 1) = 1 + \int_0^1 g(t, \lambda) dt.$$

So we have

$$q(0, 0) = 0, \quad \partial_z q(0, 0) = 1 + \int_0^1 g(t, 0) dt = -1, \quad \partial_{z\lambda}^2 q(0, 0) = \int_0^1 \partial_\lambda g(t, 0) dt.$$

Note that $f''(0) = 0$ due to the conditions I) and IV). Then we deduce that

$$\begin{aligned} \partial_z^2 x'(0, 0, t) &= 0, \quad \partial_z^2 x(0, 0, 0) = 0, \\ \partial_z^3 x'(0, 0, t) &= p(t)f'''(0) \left(1 + \int_0^t g(s, \lambda) ds\right)^3, \quad \partial_z^3 x(0, 0, 0) = 0, \end{aligned}$$

which implies

$$\begin{aligned} \partial_z^2 q(0, 0) &= \partial_z^2 x(0, 0, 1) = 0, \\ \partial_z^3 q(0, 0) &= \partial_z^3 x(0, 0, 1) = \int_0^1 p(t)f'''(0) \left(1 + \int_0^t g(s, 0) ds\right)^3 dt. \end{aligned}$$

The assertion now follows from Lemma 2.2. □

4 Example

Consider the equation

$$x'(t, \lambda) = -x^3(t, \lambda) + x(\lfloor t \rfloor, \lambda)g(t - \lfloor t \rfloor, \lambda), \quad t \geq 0 \quad (4.1)$$

where $g(s, \lambda) = (-0.2s - 1.9 + \lambda)$, $s \in [0, 1]$ and $\lambda \in (-\infty, 1.9)$. Hence $\int_0^1 g(s, \lambda) ds = -2 + \lambda$. It is clear that the functions $f(x) = -x^3$, $p(t) \equiv 1$ and g satisfy the conditions I)-IV).

If $\lambda \in [1, 1.9)$, then the zero solution is globally asymptotically stable due to Lemma 3.4. Moreover, if $\lambda \in (1, 1.9)$, then for $x(0)$ close enough to zero, the corresponding solution of (4.1) does not change sign. Using Lemmas 3.1 and 3.2, one can deduce that there is a unique $y(0) > 0$ such that the corresponding solution y crosses zero at $t = 1$ and hence becomes identical to zero over $[1, \infty)$. Any solution x with $x(0) > y(0)$ necessarily crosses zero within $(0, 1)$ hence changes sign at least once. Any solution x with $0 < x(0) < y(0)$ does not cross zero. The situation is illustrated in Figure 4.1 where $y(0) \approx 1.34$ is the positive initial value for which the corresponding solution becomes zero.

If λ passes zero, then we can apply Theorem 3.11. In fact, we compute

$$\int_0^1 \partial_\lambda g(t, 0) dt \int_0^1 p(t)f'''(0) \left(1 + \int_0^t g(s, 0) ds\right)^3 dt = -6 \int_0^1 (1 - 0.1t^2 - 1.9t)^3 dt < 0.$$

Hence for λ close to zero, a period-doubling bifurcation occurs: if $\lambda > 0$, then the zero solution is locally stable and two small unstable 2-periodic solutions exist; if $\lambda < 0$, then the zero solution is locally unstable and unique. From Corollary 3.8, we know that there exist at least two 2-periodic solutions of (4.1) for every $\lambda < 0$. They are not bifurcating from the zero solution. Numeric calculations show that for $\lambda = -0.5$, a 2-periodic solution satisfies an initial condition that is close to 1.5253; see Figure 4.2. Since f is an odd function, there is also a periodic solution starting at a value close to -1.5253 .

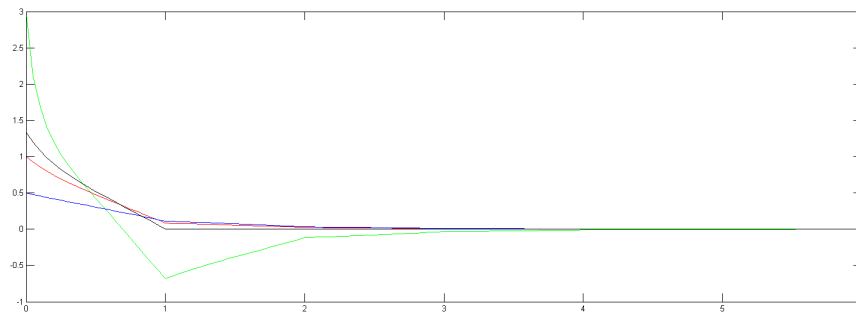


Figure 4.1: Comparison of solutions of (4.1) for $\lambda = 1.3$ with $x(0) = 3, 1.34, 1$ and 0.5 .

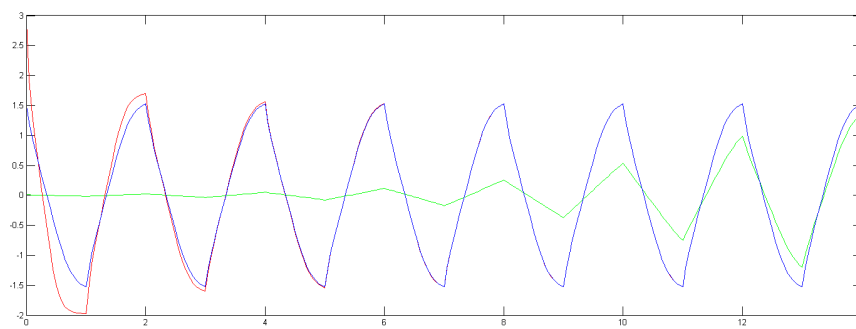


Figure 4.2: Comparison of solutions of (4.1) for $\lambda = -0.5$ with $x(0) = 3, 1.5253$ and 0.01 .

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