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On Edwards curves in differential equations

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Abstract. We study ordinary differential equations related with the Edwards curves $x^2 + y^2 = 1 + \delta\, x^2y^2$ over the field of real numbers with negative δ. We show that the Edwards curves can be parameterized by means of solutions of a linear-cubic oscillator depending upon the parameter δ, and the solutions can be expressed in terms of the Jacobian elliptic functions. The parameterizations obtained allow us to interpret the Edwards addition law on the Edwards curves similarly as the addition of the classical circle group can be interpreted using the addition formulas for trigonometric sine and cosine functions. We pay attention to the Edwards curve with $\delta = -2$, which can be parameterized also by means of solutions of a quintic oscillator; these solutions also can be expressed in terms of the Jacobian elliptic functions. This parameterization allows us to define the point addition on the Edwards curve with $\delta = -2$ that differs from the Edwards addition law.

Keywords: Edwards curves, Edwards addition law, linear-cubic oscillator, quintic oscillator, Jacobian elliptic functions, parameterization.

2020 Mathematics Subject Classification: 34C25, 14H52, 33E05, 34B15.

1 Introduction

H. Edwards in [9] introduced the curves

$$x^2 + y^2 = c^2(1 + x^2y^2) \tag{1.1}$$

and showed that every elliptic curve over an algebraic number field K could be transformed to the form (1.1) over an extension of K. The addition law on elliptic curves takes the form

$$(x_1, y_1) + (x_2, y_2) \mapsto \left(\frac{x_1y_2 + x_2y_1}{c(1 + x_1y_1x_2y_2)}, \frac{y_1y_2 - x_1x_2}{c(1 - x_1y_1x_2y_2)} \right), \tag{1.2}$$

with $(0, c)$ as neutral element. H. Edwards built his observations on L. Euler and C. Gauss's works. L. Euler in [11] considered "addition formula" for the curve

$$x^2 + x^2y^2 + y^2 = 1. \tag{1.3}$$

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C. Gauss was very interested in this curve and he introduced in [13] the functions sl (sinum lemniscaticum) and cl (cosinum lemniscaticum) and proved that these functions parameterize the curve (1.3),

$$\operatorname{sl}^2 t + \operatorname{sl}^2 t \operatorname{cl}^2 t + \operatorname{cl}^2 t = 1.$$

C. Gauss stated also the addition formulas

$$\operatorname{sl}(t_1 + t_2) = \frac{\operatorname{sl} t_1 \operatorname{cl} t_2 + \operatorname{sl} t_2 \operatorname{cl} t_1}{1 - \operatorname{sl} t_1 \operatorname{cl} t_1 \operatorname{sl} t_2 \operatorname{cl} t_2}, \qquad \operatorname{cl}(t_1 + t_2) = \frac{\operatorname{cl} t_1 \operatorname{cl} t_2 - \operatorname{sl} t_1 \operatorname{sl} t_2}{1 + \operatorname{sl} t_1 \operatorname{cl} t_1 \operatorname{sl} t_2 \operatorname{cl} t_2}. \tag{1.4}$$

As was mentioned in [9], formula (1.4) is the case $c = \sqrt{i}$ and

$$\begin{aligned} (x_j, y_j) &= \big(\sqrt{i} \operatorname{sl} t_j, \sqrt{i} \operatorname{cl} t_j\big) \ (j = 1, 2), \\ (x_3, y_3) &= \big(\sqrt{i} \operatorname{sl}(t_1 + t_2), \sqrt{i} \operatorname{cl}(t_1 + t_2)\big) \end{aligned}$$

of (1.2).

The same year the article [9] came out, D. J. Bernstein and T. Lange provided in [5] a generalization of the recently introduced curves (1.1) for cryptographic applications. They introduced, see also [21], *the Edwards curves*

$$E_\delta : \ x^2 + y^2 = 1 + \delta\, x^2 y^2, \tag{1.5}$$

over a field K of characteristic different from 2, where $\delta \in K \setminus \{0, 1\}$, and considered *the Edwards addition law*

$$(x_1, y_1) + (x_2, y_2) \mapsto \left(\frac{x_1 y_2 + y_1 x_2}{1 + \delta\, x_1 y_1 x_2 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - \delta\, x_1 y_1 x_2 y_2} \right) \tag{1.6}$$

on (1.5) over K. If δ is not a square in K, then the Edwards addition law (1.5) is complete, see [5, 21], that is, $1 \pm \delta\, x_1 y_1 x_2 y_2 \neq 0$ for all points (x_1, y_1) and (x_2, y_2) on (1.5) over K. If the Edwards addition law (1.5) is complete, then, see [21], the set of points of E_δ forms an Abelian group under the addition (1.6). The point $(0, 1)$ on (1.5) is a neutral element of (1.6); the inverse of a point (x, y) on (1.5) is $(-x, y)$.

In 2008, D. J. Bernstein et al. introduced in [3] the twisted Edwards curves $ax^2 + y^2 = 1 + \delta\, x^2 y^2$ over a field K of characteristic different from 2, where the coefficients $a, \delta \in K$ are distinct and nonzero. The Edwards curves (1.5) are the case $a = 1$ of the twisted Edwards curves. Nowadays the twisted Edwards curves over finite fields are widely used in elliptic curve cryptography. The reader may consult, for instance, [4, 12, 21] and references therein, for more information about applications of the twisted Edwards curves in elliptic curve cryptography.

The relation of the Edwards curves with differential equations has a historical background. L. Euler in [10, §8] showed that (1.3) is a particular integral of the differential equation

$$\frac{dx}{\sqrt{1 - x^4}} = \frac{dy}{\sqrt{1 - y^4}}. \tag{1.7}$$

He demonstrated in [10, §16] also that the complete integral of the differential equation

$$\frac{dx}{\sqrt{1 + mx^2 + nx^4}} = \frac{dy}{\sqrt{1 + my^2 + ny^4}} \tag{1.8}$$

is

$$0 = c^2 - x^2 - y^2 + nc^2 x^2 y^2 + 2xy\sqrt{1 + mc^2 + nc^4}, \tag{1.9}$$

where c is an arbitrary constant. Suppose that $m = -(\delta+1)$ and $n = \delta$. Then (1.8) takes the form

$$\frac{dx}{\sqrt{1-(1+\delta)x^2+\delta\,x^4}} = \frac{dy}{\sqrt{1-(1+\delta)y^2+\delta\,y^4}}. \tag{1.10}$$

Formula (1.7) is contained in (1.10), setting $\delta = -1$. For the initial conditions $x(t_0) = 0$ and $y(t_0) = \pm 1$, it follows from (1.9) that $c^2 = 1$ and (1.9) is of the form

$$0 = 1 - x^2 - y^2 + \delta\,x^2y^2 + 2xy\sqrt{1-(1+\delta)+\delta},$$

which is equivalent to

$$0 = 1 - x^2 - y^2 + \delta\,x^2y^2.$$

Hence, the Edwards curve E_δ over the field of real numbers $\mathbb{R}$ is a particular integral of the differential equation (1.10). This fact seems to be unnoted in the existing literature. In the following, we will pay special attention to the curve E_{-2} defined as $0 = 1 - x^2 - y^2 - 2x^2y^2$.

L. Euler proved in [10, §36] also that the complete integral of the differential equation

$$\frac{dx}{\sqrt{f+gx^6}} = \frac{dy}{\sqrt{f+gy^6}} \tag{1.11}$$

is

$$g^2(x^4+y^4) - 4g^2cx^4y^4 - 4fgc^2x^2y^2(x^2+y^2) - 2g^2x^2y^2 - 2fgc(x^2+y^2) + f^2c^2 \;=\; 0, \tag{1.12}$$

where c is an arbitrary constant. Suppose that $f = 1$ and $g = -1$. Then (1.11) takes the form

$$\frac{dx}{\sqrt{1-x^6}} = \frac{dy}{\sqrt{1-y^6}}. \tag{1.13}$$

For the initial conditions $x(t_0) = 0$ and $y(t_0) = \pm 1$, it follows from (1.12) that $c = -1$ and (1.12) is of the form

$$x^4 + y^4 + 4x^4y^4 + 4x^2y^2(x^2+y^2) - 2x^2y^2 - 2(x^2+y^2) + 1 = 0,$$

or

$$(x^2 + 2x^2y^2 + y^2 - 1)^2 = 0.$$

Therefore, the Edwards curve E_{-2} over $\mathbb{R}$ is a particular integral of the differential equation (1.13). Surprisingly, the curve E_{-2} is a shared particular integral for (1.10) and (1.13).

A mixture of geometrical objects (the Edwards curves), addition formulas, and differential equations was an object of investigation by prominent mathematicians and found many applications. H. Edwards in [9] mentioned: “I have not been able to find the following generalization of (1.4) in the literature. If it is not new, it is certainly not as well known as it deserves to be.” Probably, the role of differential equations in this story is not known completely. We would like to state some useful relations and to show differential equations that play important role in this chapter of classical mathematics.

In what follows, we will study the Edwards curves E_δ over $\mathbb{R}$ and we will assume that δ is negative, unless otherwise indicated.

In Section 2, we consider a Hamiltonian system

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}, \\ y' = -\dfrac{\partial H}{\partial x} \end{cases} \tag{1.14}$$

with the Hamiltonian function

$$H(x,y) = \frac{1}{2}(x^2 - \delta\, x^2y^2 + y^2). \tag{1.15}$$

The level sets $H(x,y) = \frac{1}{2}$ are exactly the Edwards curves. The key role in our considerations plays the maximal solution (x,y) of the system (1.14) passing the neutral element $(0,1)$ of the Edwards addition law on E_δ,

$$x(0) = 0, \qquad y(0) = 1. \tag{1.16}$$

The maximal interval of existence for (x,y) is the open interval $(-\infty,\infty)$. We express x and y in terms of the Jacobian elliptic functions solving the linear-cubic oscillator $w'' = -(1+\delta)w + 2\delta\, w^3$ with the initial conditions $x(0) = 0$, $x'(0) = 1$ and $y(0) = 1$, $y'(0) = 0$, respectively. The functions x and y are sin and cos like functions, respectively.

The functions x and y allow us to construct parameterizations of the Edwards curves and to discuss completeness of the Edwards addition law. If δ is negative, then δ is not a square in the field $\mathbb{R}$, and, as was already mentioned above, the Edwards addition law is complete. For $\delta < 0$, we prove the addition formulas

$$x(t_1+t_2) = \frac{x(t_1)y(t_2) + x(t_2)y(t_1)}{1 + \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}, \tag{1.17}$$

$$y(t_1+t_2) = \frac{y(t_1)y(t_2) - x(t_1)x(t_2)}{1 - \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}. \tag{1.18}$$

If $\delta = -1$, then $x = \mathrm{sl}$, $y = \mathrm{cl}$, and (1.4) is the case of (1.17), (1.18).

In Section 3, we introduce functions x and y that are solutions of the nonlinear quintic oscillator $w'' = -3w^5$ with the initial conditions $x(0) = 0$, $x'(0) = 1$ and $y(0) = 1$, $y'(0) = 0$, respectively, in the interval $(-\infty,\infty)$. The functions x and y can be expressed in terms of the Jacobian elliptic functions sharing the third singular value $k = \frac{\sqrt{3}-1}{\sqrt{8}}$ as elliptic modulus. We prove that x and y ensure a parameterization of E_{-2} and solve the planar system

$$\begin{cases} x' = y\sqrt{(1+2x^2)(1+x^2+x^4)}, \\ y' = -x\sqrt{(1+2y^2)(1+y^2+y^4)}; \end{cases} \tag{1.19}$$

see also [15]. The parameterization of E_{-2} by the functions x and y obtained allows us to define the point addition on E_{-2} that is different from the Edwards addition law.

The concluding remarks completes our paper.

2 The Edwards curves and solutions of a linear-cubic oscillator

For a negative number δ, it follows from (1.5) that $|x| \le 1$ and $|y| \le 1$ for every $(x,y) \in E_\delta$, and thus the Edwards curve E_δ is bounded. In Figure 2.1, the Edwards curves E_δ for some values of δ and the unit circle are depicted.

2.1 Linear-cubic oscillator

Consider the system of nonlinear differential equations

$$\begin{cases} x' = y(1-\delta\, x^2), \\ y' = -x(1-\delta\, y^2), \end{cases} \tag{2.1}$$

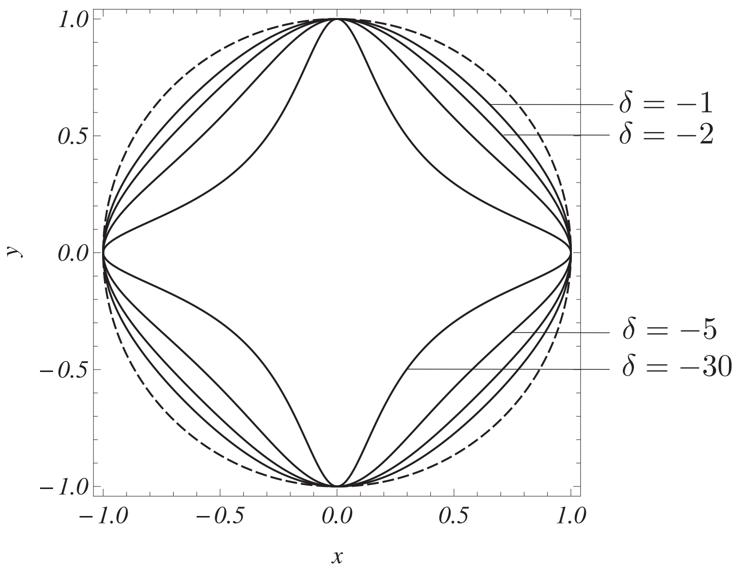


Figure 2.1: The Edwards curves E_δ for $\delta = -30, -15, -2, -1$ and the unit circle (dashed).

satisfying the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0. \tag{2.2}$$

The system (2.1) is a Hamiltonian system (1.14) with the Hamiltonian function (1.15). Since δ is negative, the system (2.1) or (1.14) has a unique equilibrium $(0, 0)$, which is a center.

The right hand sides in (2.1) have continuous partial derivatives on $\mathbb{R}^2$, and thus, see [20, p. 354], for every $(x_0, y_0) \in \mathbb{R}^2$, the Cauchy problem (2.1), (2.2) has a unique solution (x, y) in an open interval J containing t_0.

Proposition 2.1. *Consider $(x_0, y_0) \in \mathbb{R}^2$. A solution (x, y) of (2.1), (2.2) in an open interval $J \ni t_0$ is located on the Edwards curve E_δ, that is,*

$$x^2(t) + y^2(t) = 1 + \delta\, x^2(t) y^2(t), \quad \forall t \in J, \tag{2.3}$$

if and only if $(x_0, y_0) \in E_\delta$.

Proof. Let (x, y) be a solution of (2.1), (2.2) in an open interval $J \ni t_0$. If (2.3) is fulfilled, then $(x_0, y_0) \in E_\delta$. Now, suppose that $(x_0, y_0) \in E_\delta$. Then $H(x_0, y_0) = \frac{1}{2}$. In accordance with [17, p. 209], the Hamiltonian function H is constant along every solution of (2.1). Therefore, $H\big(x(t), y(t)\big) = \frac{1}{2}$ for every $t \in J$, which, in its turn, gives (2.3). □

Proposition 2.2. *If (x, y) is a solution of the Cauchy problem (2.1), (1.16) in an open interval $J \ni 0$, then x and y are solutions of the Cauchy problems*

$$x'' = -(1+\delta)x + 2\delta\, x^3, \quad x(0) = 0, \quad x'(0) = 1, \tag{2.4}$$

$$y'' = -(1+\delta)y + 2\delta\, y^3, \quad y(0) = 1, \quad y'(0) = 0, \tag{2.5}$$

respectively, in the interval J.

Proof. Let (x, y) be a solution of the Cauchy problem (2.1), (1.16) in an open interval $J \ni 0$. Since $(0, 1) \in E_\delta$, it follows from Proposition 2.1 that (2.3) is fulfilled. Taking into account (2.1) and (2.3), we have

$$
\begin{aligned}
x'' &= y'(1-\delta x^2) - y \cdot 2\delta xx' = -x(1-\delta x^2)(1-\delta y^2) - 2\delta xy^2(1-\delta x^2) \\
&= -x(1-\delta) - 2\delta x(1-x^2) = -(1+\delta)x + 2\delta x^3, \\
y'' &= -x'(1-\delta y^2) + x \cdot 2\delta yy' = -y(1-\delta x^2)(1-\delta y^2) - 2\delta yx^2(1-\delta y^2) \\
&= -y(1-\delta) - 2\delta y(1-y^2) = -(1+\delta)y + 2\delta y^3.
\end{aligned}
$$

It follows from (2.1) and (1.16) that $x'(0) = 1$ and $y'(0) = 0$. The statement is proved. □

If (x, y) is a solution of the Cauchy problem (2.1), (1.16) in an open interval $J \ni 0$, then x and y are solutions of the same linear-cubic oscillator

$$
w'' = -(1+\delta)w + 2\delta w^3 =: f(w) \tag{2.6}
$$

with the potential energy $U(w) = -\int_0^w f(s)ds = \frac{1+\delta}{2} w^2 - \frac{\delta}{2} w^4$ and the total energy

$$
\frac{(w')^2}{2} + U(w) = \frac{1}{2}. \tag{2.7}
$$

It appears that x and y are sine and cosine like functions, defined in a nonlinear manner. Recall that sine and cosine can be defined as solutions of the same harmonic oscillator, but with different initial conditions.

2.2 Phase plain analysis

If $-1 \le \delta < 0$, then the potential energy U has only one critical point $w = 0$ (the global minimum) and $U(0) = 0$. If $\delta < -1$, then U has three critical points: $w = 0$ (a local maximum point), $w_{1,2} = \pm\sqrt{\frac{1+\delta}{4\delta}}$ (local minima points) and $U(0) = 0$, $U(w_{1,2}) = \frac{(1+\delta)^2}{\delta} < 0$.

Let (x, y) be a solution of the Cauchy problem (2.1), (1.16) in an open interval $J \ni 0$. It follows from Proposition 2.2 that x and y are solutions of (2.6). The value of U at any of its critical points is less than the total energy $\frac{1}{2} = U(\pm 1)$, and thus (2.7) defines, see [1, p. 145], the closed phase curve $(w')^2 = 1 - (1+\delta)w^2 + \delta w^4$, $-1 \le w \le 1$, which is matched by the both solutions x and y of (2.6). Hence, see [1, p. 147], the functions x and y are periodic with the same amplitude equal to one. Since the phase curve is symmetric with respect to the vertical and horizontal axes in the phase plane (w, w'), the functions x and y determine the clockwise motion of a point on the phase curve in a such way that $(0, 1) \to (1, 0) \to (0, -1) \to (-1, 0) \to (0, 1)$ and each motion from a point to the next point takes the same positive time A. Taking into account the initial conditions in (2.4) and (2.5), the solutions x and y determine the motions along the phase curve starting at the time $t = 0$ at the points $(0, 1)$ and $(1, 0)$, respectively. Hence, the functions x and y have the same minimal positive period $T = 4A$. Arguing in a similar way as in [23], the functions x and y have the same symmetries with respect to $A = \frac{T}{4}$ as $\sin t$ and $\cos t$ with respect to $\frac{\pi}{2}$.

2.3 Solutions of Cauchy problems

In the following considerations the Jacobian elliptic functions sn, cn, dn, and others are used. For more information on the Jacobian elliptic functions, see, for example, [24, Chapter 22].

Proposition 2.3. *Consider $b = \sqrt{1-\delta}$ and $k = \sqrt{\frac{-\delta}{1-\delta}}$. The functions*

$$x(t) = \frac{1}{b}\,\mathrm{sd}(bt,k), \qquad y(t) = \mathrm{cn}(bt,k) \tag{2.8}$$

are solutions of the Cauchy problems (2.4) and (2.5), respectively, in the interval $(-\infty,\infty)$. The pair (x,y) is a solution of the Cauchy problem (2.1), (1.16) in the interval $(-\infty,\infty)$.

Proof. Find $k' = \sqrt{1-k^2} = \frac{1}{\sqrt{1-\delta}}$ and

$$k^2 - (k')^2 = -\frac{1+\delta}{1-\delta}, \qquad k^2(k')^2 = -\frac{\delta}{(1-\delta)^2}. \tag{2.9}$$

Let t be an arbitrary real number. Taking into account (2.9) and [24, Formulas 22.13.14, 22.13.17], we have

$$\begin{aligned}
x''(t) &= b\,\left(\frac{d^2}{du^2}\,\mathrm{sd}(u,k)\right)\bigg|_{u=bt} = b\Big[\big(k^2-(k')^2\big)\,\mathrm{sd}(bt,k) - 2k^2(k')^2\,\mathrm{sd}^3(bt,k)\Big]\\
&= b\left[-\frac{1+\delta}{b^2}\,\mathrm{sd}(bt,k) + \frac{2\,\delta}{b^4}\,\mathrm{sd}^3(bt,k)\right] = -(1+\delta)x(t) + 2\delta\,x^3(t),\\
y''(t) &= b^2\,\left(\frac{d^2}{du^2}\,\mathrm{cn}(u,k)\right)\bigg|_{u=bt} = b^2\Big[\big(k^2-(k')^2\big)\,\mathrm{cn}(bt,k) - 2k^2\,\mathrm{cn}^3(bt,k)\Big]\\
&= b^2\left[-\frac{1+\delta}{b^2}\,\mathrm{cn}(bt,k) + \frac{2\,\delta}{b^2}\,\mathrm{cn}^3(bt,k)\right] = -(1+\delta)y(t) + 2\delta\,y^3(t).
\end{aligned}$$

Note that $x(0)=0$ and $y(0)=1$. On account of [24, §22.13(i)], we obtain

$$x'(t) = \frac{\mathrm{cn}(bt,k)}{\mathrm{dn}^2(bt,k)}, \qquad y'(t) = -b\,\mathrm{sn}(bt,k)\,\mathrm{dn}(bt,k), \tag{2.10}$$

and thus $x'(0)=1$ and $y'(0)=0$. Hence, x and y are solutions of the Cauchy problems (2.4) and (2.5), respectively, in the interval $(-\infty,\infty)$.

In view of (2.10) and the equalities

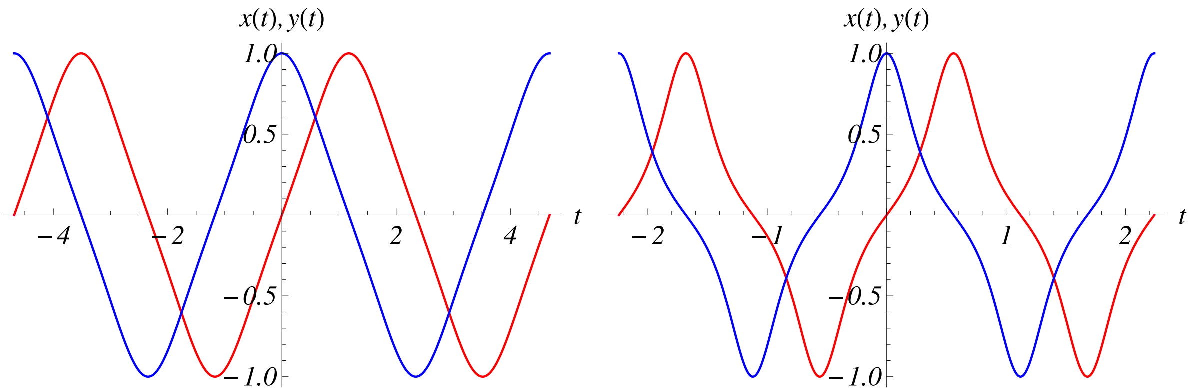
$$k^2\,\mathrm{sn}^2\,(bt,k) + \mathrm{dn}^2\,(bt,k) = 1, \qquad (k')^2 + k^2\,\mathrm{cn}^2\,(bt,k) = \mathrm{dn}^2\,(bt,k)\,,$$

see [24, Formulas 22.6.1], we obtain

$$\begin{aligned}
y(t)\left[1-\delta\,x_1^2(t)\right] &= \mathrm{cn}(bt,k)\left[1+k^2\,\mathrm{sd}^2(bt,k)\right]\\
&= \frac{\mathrm{cn}(bt,k)}{\mathrm{dn}^2(bt,k)}\left[\,\mathrm{dn}^2(bt,k) + k^2\,\mathrm{sn}^2(bt,k)\right] = x'(t),\\
-x(t)\left[1-\delta\,y^2(t)\right] &= -\frac{1}{\beta}\,\mathrm{sd}(bt,k)\left[1-d\,\mathrm{cn}^2(bt,k)\right]\\
&= -\beta\,\frac{\mathrm{sn}(bt,k)}{\mathrm{dn}(u,k)}\left[(k')^2 + k^2\,\mathrm{cn}^2(bt,k)\right] = y'(t).
\end{aligned}$$

Hence, (x,y) solves the Cauchy problem (2.1), (1.16) in the interval $(-\infty,\infty)$. □

In Figure 2.2, the graphs of the functions x un y defined by (2.8) are depicted for some values of δ.



(a) $\delta = -2$.   (b) $\delta = -30$.

Figure 2.2: The graphs of the functions x (in red) and y (in blue) defined by (2.8) for $\delta = -2$ and $\delta = -30$.

Proposition 2.4. *The functions* (2.8) *have the following properties.*

1. *The functions x and y are periodic with the minimal positive quarter-period*

$$A = \frac{K}{\sqrt{1-\delta}}, \tag{2.11}$$

where $K = K(k) = \int_0^1 \frac{du}{\sqrt{(1-k^2u^2)(1-u^2)}}$ is the Legendre's complete integral of the first kind with the modulus $k = \sqrt{\frac{-\delta}{1-\delta}}$.

2. *The functions x and y are odd and even, respectively, besides*

$$x(t+A) = y(t), \quad y(t+A) = -x(t), \quad \forall t \in \mathbb{R}. \tag{2.12}$$

3. *The function x is strictly increasing in the set $(0,A) \cup (3A,4A)$ and strictly decreasing in the interval $(A,3A)$, besides*

$$\begin{array}{llll} x(0) = 0, & x(A) = 1, & x(2A) = 0, & x(3A) = -1, \\ x'(0) = 1, & x'(A) = 0, & x'(2A) = -1, & x'(3A) = 0. \end{array}$$

4. *The function y is strictly decreasing in the interval $(0,2A)$ and strictly increasing in the interval $(2A,4A)$, besides*

$$\begin{array}{llll} y(0) = 1, & y(A) = 0, & y(2A) = -1, & y(3A) = 0, \\ y'(0) = 0, & y'(A) = -1, & y'(2A) = 0, & y'(3A) = 1. \end{array}$$

Proof. (1) The Jacobian elliptic functions $\mathrm{sd}(u,k)$ and $\mathrm{cn}(u,k)$ are periodic, see [24, Table 22.4.1], with the minimal positive period $4K$. Taking into account (2.8), the functions x and y are periodic with the minimal positive period $4A$, where A is given by (2.11).

Actually, the statements (2)-(4) are valid in view of the phase plane analysis had made in Section 2.2. The proof can be provided also using properties of the Jacobian elliptic functions, see [23, 24]. $\square$

2.4 Parameterization of the Edwards curves

Definition 2.5. Consider sets $\varnothing \neq M \subset \mathbb{R}$ and $S \subset \mathbb{R}^2$. A continuous mapping $\varphi : M \to \mathbb{R}^2$ is called a *parameterization* of S if $\varphi(M) = S$.

Definition 2.6. A subset $S \subset \mathbb{R}^2$ is called a *Jordan curve* if there exists a parameterization $\varphi : [a,b] \to \mathbb{R}^2$ $(a < b)$ of S such that $\varphi(a) = \varphi(b)$ and the mapping φ, restricted to $[a,b)$, is injective.

Theorem 2.7. *For a given negative δ, the Edwards curve E_δ is a Jordan curve with a parameterization $\varphi : [0,4A] \to \mathbb{R}^2$ such that $\varphi(t) = \big(x(t), y(t)\big)$ for every $t \in [0,4A]$, where x and y are defined by* (2.8).

Proof. Since $(0,1) \in E_\delta$, it follows from Propositions 2.1, 2.3, and 2.4 that $\varphi(0) = (0,1) = \varphi(4A)$ and

$$\varphi\big([0,4A]\big) \subset E_\delta. \tag{2.13}$$

Let us prove that for an arbitrary $(x^*, y^*) \in E_\delta$ there exists a unique $t^* \in [0,4A)$ such that $\varphi(t^*) = \big(x(t^*), y(t^*)\big) = (x^*, y^*)$. Let (x^*, y^*) be a point on E_δ. Then $|x^*| \le 1$, $|y^*| \le 1$. Consider the case when $0 < x^*, y^* < 1$. It follows from Proposition 2.4 that there exist only two $t_1^*, t_2^* \in [0,4A)$ such that $y(t_1^*) = y(t_2^*) = y^*$, where $t_1^* \in (0,A)$ and $t_2^* \in (3A,4A)$. By Proposition 2.4, $x(t_1^*) > 0$ and $x(t_2^*) < 0$, and thus the argument t_2^* does not fit. Let $t^* = t_1^*$. In view of (2.13), we have $x^2(t^*) + y^2(t^*) = 1 + \delta\, x^2(t^*) y^2(t^*)$, and thus

$$x(t^*) = \sqrt{\frac{1 - y^2(t^*)}{1 - \delta\, y^2(t^*)}}\,. \tag{2.14}$$

Since $(x^*, y^*) \in E_\delta$, we have $(x^*)^2 + (y^*)^2 = 1 + \delta\,(x^*)^2(y^*)^2$, and thus

$$x^* = \sqrt{\frac{1 - (y^*)^2}{1 - \delta\,(y^*)^2}}\,. \tag{2.15}$$

It follows from (2.14) and (2.15) that $x(t^*) = x^*$. The other cases can be considered similarly. Thereby, $\varphi\big([0,4A)\big) = E_\delta$ and the restriction $\varphi|_{[0,4A)}$ is injective. Since $\varphi(4A) = (0,1) \in E_\delta$, we have $\varphi\big([0,4A]\big) = E_\delta$. Hence, E_δ is a Jordan curve. $\square$

2.5 Addition formulas

Since negative δ is not a square in $\mathbb{R}$, as already mentioned in Introduction, the Edwards addition law is complete, that is, for every $(x_1, y_1), (x_2, y_2) \in E_\delta$ we have $1 \pm \delta\, x_1 y_1 x_2 y_2 \neq 0$, and the Edwards curve E_δ is an Abelian group with respect to the Edwards addition law. Let us show that the completeness of the Edwards addition law on E_δ is a consequence of Theorem 2.7 and properties of the Jacobian elliptic functions. Consider arbitrary $(x_1, y_1), (x_2, y_2) \in E_\delta$. It follows from Theorem 2.7 that there exist $t_1, t_2 \in \mathbb{R}$ such that

$$(x_1, y_1) = \big(x(t_1), y(t_1)\big), \qquad (x_2, y_2) = \big(x(t_2), y(t_2)\big), \tag{2.16}$$

where x and y are defined by (2.8). Denote bt_1 and bt_2 by u and v, respectively. In accordance with [24, Formula 22.8.3], for the common modulus $k = \sqrt{\frac{-\delta}{1-\delta}}$ suppressed, we have

$$\operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} > 0. \tag{2.17}$$

In view of (2.8), (2.17), and the equalities $k^2b^2 = \frac{-\delta}{1-\delta}(1-\delta) = -\delta$, we obtain

$$\begin{aligned} 0 \neq 1 \mp k^2 \operatorname{sd} u \operatorname{cn} u \operatorname{sd} v \operatorname{cn} v &= 1 \mp k^2b^2 \frac{\operatorname{sd} u}{b} \operatorname{cn} u \frac{\operatorname{sd} v}{b} \operatorname{cn} v \\ &= 1 \pm \delta\, x(t_1)y(t_1)x(t_2)y(t_2) \\ &= 1 \pm \delta\, x_1y_1x_2y_2. \end{aligned} \tag{2.18}$$

Hence, the Edwards addition law on E_δ is complete.

Theorem 2.8. *The functions x and y defined by* (2.8) *satisfy the addition formulas* (1.17), (1.18) *for every $t_1, t_2 \in \mathbb{R}$.*

Proof. For arbitrary $t_1, t_2 \in \mathbb{R}$, denote bt_1 and bt_2 by u and v, respectively. By [24, Formulas 2.8.1, 2.8.3], for the common modulus $k = \sqrt{\frac{-\delta}{1-\delta}}$ suppressed, we have

$$\operatorname{sd}(u+v) = \frac{\operatorname{sd} u \operatorname{cn} v + \operatorname{sd} v \operatorname{cn} u}{1 - k^2 \operatorname{sd} u \operatorname{cn} u \operatorname{sd} v \operatorname{cn} v}. \tag{2.19}$$

Note that the denominator in the right-hand side of (2.19) is positive. Combining (2.8), (2.12), (2.18), and (2.19), we obtain

$$\begin{aligned} x(t_1+t_2) &= \frac{1}{b} \operatorname{sd}\big(b(t_1+t_2)\big) \\ &= \frac{1}{b} \operatorname{sd}(u+v) \\ &= \frac{\left(\frac{1}{b}\operatorname{sd} u\right)\operatorname{cn} v + \left(\frac{1}{b}\operatorname{sd} v\right)\operatorname{cn} u}{1 - k^2b^2\left(\frac{1}{b}\operatorname{sd} u\right)\operatorname{cn} u\left(\frac{1}{b}\operatorname{sd} v\right)\operatorname{cn} v} \\ &= \frac{x(t_1)y(t_2) + x(t_2)y(t_1)}{1 + \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}, \end{aligned}$$

$$\begin{aligned} y(t_1+t_2) &= x\big(t_1 + (t_2 + A)\big) \\ &= \frac{x(t_1)y(t_2+A) + x(t_2+A)y(t_1)}{1 + \delta x(t_1)y(t_1)x(t_2+A)y(t_2+A)} \\ &= \frac{x(t_1)\big(-x(t_2)\big) + y(t_2)y(t_1)}{1 + \delta x(t_1)y(t_1)y(t_2)\big(-x(t_2)\big)} \\ &= \frac{y(t_1)y(t_2) - x(t_1)x(t_2)}{1 - \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}. \end{aligned}$$ □

Remark 2.9. Consider arbitrary $(x_1,y_1), (x_2,y_2) \in E_\delta$. It follows from Theorem 2.7 that there exist $t_1, t_2 \in \mathbb{R}$ such that (2.16) is fulfilled, where x and y are defined by (2.8). In view of (1.6), (1.17), and (1.18), we have

$$\begin{aligned} (x_1,y_1) + (x_2,y_2) &= \left(\frac{x_1y_2 + x_2y_1}{1 + \delta\, x_1y_1x_2y_2}, \frac{y_1y_2 - x_1x_2}{1 - \delta\, x_1y_1x_2y_2}\right) \\ &= \left(\frac{x(t_1)y(t_2) + x(t_2)y(t_1)}{1 + \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}, \frac{y(t_1)y(t_2) - x(t_1)x(t_2)}{1 - \delta\, x(t_1)y(t_1)x(t_2)y(t_2)}\right) \\ &= \big(x(t_1+t_2), y(t_1+t_2)\big), \end{aligned}$$

and thus

$$(x_1, y_1) + (x_2, y_2) = \big(x(t_1 + t_2), y(t_1 + t_2)\big). \tag{2.20}$$

The last formula provides an another way to prove that $(E_\delta, +)$ is an Abelian group. On account of Theorem 2.7 and the periodicity of x and y, the point addition law (2.20) on E_δ does not depend on t_1 and t_2 at which (2.16) holds. It follows from (2.20), taking into account Propositions 2.1 and 2.3, that $(x_1, y_1) + (x_2, y_2) \in E_\delta$. The law (2.20) is commutative and associative because the addition of real numbers has these properties. The initial conditions (1.16) yields $(0, 1) = \big(x(0), y(0)\big)$. In accordance with (2.20),

$$(x_1, y_1) + (0, 1) = \big(x(t_1 + 0), y(t_1 + 0)\big) = (x_1, y_1).$$

Since x and y are odd and even functions, respectively, we obtain

$$(-x_1, y_1) = \big(x(-t_1), y(-t_1)\big).$$

It follows from (2.20) that

$$(x_1, y_1) + (-x_1, y_1) = \big(x(t_1 - t_1), y(t_1 - t_1)\big) = (0, 1).$$

2.6 Limiting and particular cases

Consider the functions x un y defined by (2.8).

If $\delta \to 0-$, then $k = \sqrt{\frac{-\delta}{1-\delta}} \to 0+$, $b = \sqrt{1-\delta} \to 1-$, and, in view of [24, §22.5(ii)], we have $x(t) \to \sin t$, $y(t) \to \cos t$, and $A = \frac{K}{\sqrt{1-\delta}} \to \frac{\pi}{2}$. If $\delta \to 0-$, then (1.5), (1.17), and (1.18) lead to the well-known formulas $\sin^2 t + \cos^2 t = 1$, $\sin(t_1 + t_2) = \sin t_1 \cos t_2 + \sin t_2 \cos t_1$, and $\cos(t_1 + t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2$, respectively.

If $\delta = -1$, then x and y coincide with the lemniscate sine sl and cosine cl, respectively,

$$\operatorname{sl} t = \frac{1}{\sqrt{2}} \operatorname{sd}\left(t\sqrt{2}, \frac{1}{\sqrt{2}}\right), \qquad \operatorname{cl} t = \operatorname{cn}\left(t\sqrt{2}, \frac{1}{\sqrt{2}}\right), \tag{2.21}$$

and satisfy (1.3) and (1.4).

3 The Edwards curve with $\delta = -2$ and solutions of a quintic oscillator

3.1 Solutions of Cauchy problems

In accordance with [2, 15], the Cauchy problem

$$y'' = -3y^5, \quad y(0) = 1, \quad y'(0) = 0 \tag{3.1}$$

has the solution

$$y(t) = \frac{\operatorname{cd}(\alpha t, k)}{\sqrt{\operatorname{cd}^2(\alpha t, k) + \alpha^2 \operatorname{sn}^2(\alpha t, k)}} \tag{3.2}$$

in the interval $(-\infty, \infty)$, where $\alpha = \sqrt[4]{3}$ and

$$k = \sin\frac{\pi}{12} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{3}-1}{\sqrt{8}} = 0.2588. \tag{3.3}$$

The function y is even and periodic, see [2], with the minimal positive quarter period

$$A = \frac{1}{\alpha} K(k). \tag{3.4}$$

Note that $A = \int_0^1 \frac{dt}{\sqrt{1-t^6}} = 1.2143$, see [15].

Definition 3.1 ([6]). Let N be a positive integer and let $k \in (0,1)$. The number k is *the Nth singular value* of $K(k)$ if $\frac{K(k')}{K(k)} = \sqrt{N}$, where k' is the complementary to k modulus.

Remark 3.2. The elliptic modulus $k = \frac{1}{\sqrt{2}}$ of the Jacobian elliptic functions in expressions (2.21) of the lemniscate sine and cosine, is the first singular value of $K(k)$, see [6, p. 26], since $\frac{K(k')}{K(k)} = 1$. The elliptic modulus (3.3) is the third singular value of $K(k)$, see [6, p. 26], since $\frac{K(k')}{K(k)} = \sqrt{3}$. The last equality was obtained by A. M. Legendre, see [25, p. 525]. Although the elliptic modulus $k = \sqrt{2}-1$ is not used in this paper, we note that it is the second singular value of $K(k)$, see [6, p. 27], since $\frac{K(k')}{K(k)} = \sqrt{2}$.

In view of [24, §22.4(iii), §22.5(i), §22.13(i)], the function

$$x(t) = y(t-A) = \frac{\operatorname{sn}(\alpha t, k)}{\sqrt{\operatorname{sn}^2(\alpha t, k) + \alpha^2 \operatorname{cd}^2(\alpha t, k)}} \tag{3.5}$$

solves the Cauchy problem

$$x'' = -3x^5, \quad x(0) = 0, \quad x'(0) = 1 \tag{3.6}$$

in the interval $(-\infty, \infty)$, see also [15], where k is defined by (3.3). The functions x and y have the same properties specified in Proposition 2.4 with A defined by (3.4) and k defined by (3.3). Integrating $w'' = -3w^5$ and taking into account the initial conditions in (3.6) and (3.1), we obtain

$$(x')^2 + x^6 = 1, \qquad (y')^2 + y^6 = 1. \tag{3.7}$$

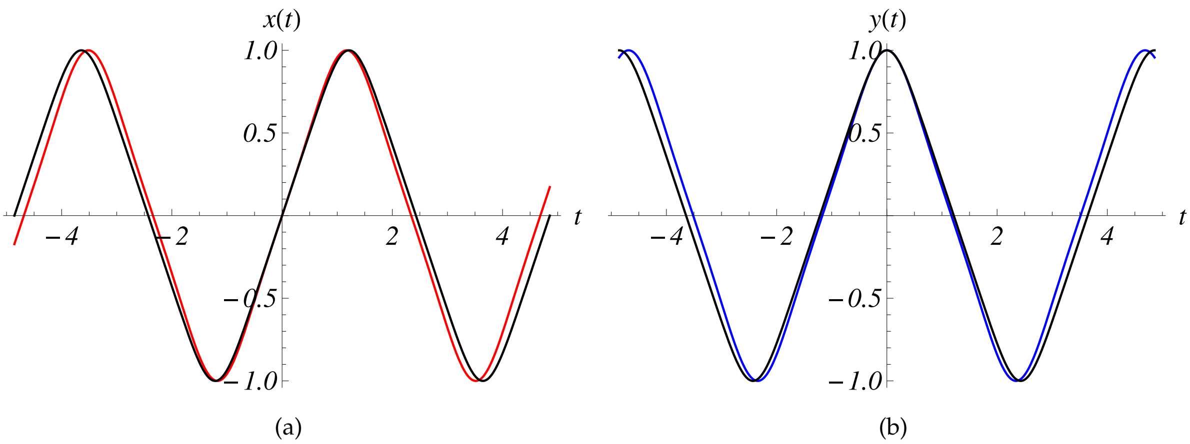


Figure 3.1: In (a), the graphs of the functions x defined by (2.8) with $\delta = -2$ (in red) and by (3.5) (in black). In (b), the graphs of the functions y defined by (2.8) with $\delta = -2$ (in blue) and by (3.2) (in black).

Proposition 3.3. *If x and y are defined by (3.5) and (3.2), respectively, then*

$$(1+2x^2)(1+2y^2)=3, \tag{3.8}$$
$$x^2+y^2=1-2x^2y^2, \tag{3.9}$$
$$\frac{1+x^2+x^4}{1+y^2+y^4}=\frac{3}{(1+2y^2)^2}=\frac{(1+2x^2)^2}{3}=\frac{1+2x^2}{1+2y^2}, \tag{3.10}$$
$$\left[1-(\alpha^2+1)x^2\right]\left[1+(\alpha^2-1)y^2\right]=\alpha^2(y^2-x^2). \tag{3.11}$$

Proof. For k defined by (3.3) and $t\in\mathbb{R}$, introduce the functions

$$p(t)=\operatorname{sn}^2(\alpha t,k)+\alpha^2\operatorname{cd}^2(\alpha t,k), \qquad q(t)=\operatorname{cd}^2(\alpha t,k)+\alpha^2\operatorname{sn}^2(\alpha t,k), \tag{3.12}$$

which are positive on $\mathbb{R}$. Taking into account (3.2) and (3.5), we have

$$\frac{p(t)}{q(t)}=\frac{1+2y^2(t)}{\alpha^2}, \qquad \frac{q(t)}{p(t)}=\frac{1+2x^2(t)}{\alpha^2}. \tag{3.13}$$

Multiplying the last two equalities, we obtain (3.8) and (3.9).

In view of (3.9), we find

$$1+x^2+x^4=1+\frac{1-y^2}{1+2y^2}+\left(\frac{1-y^2}{1+2y^2}\right)^2=\frac{3(1+y^2+y^4)}{(1+2y^2)^2}.$$

Hence,

$$\frac{1+x^2+x^4}{1+y^2+y^4}=\frac{3}{(1+2y^2)^2}. \tag{3.14}$$

On account of (3.8), we have

$$\frac{3}{(1+2y^2)^2}=\frac{1+2x^2}{1+2y^2}=\frac{(1+2x^2)^2}{3}. \tag{3.15}$$

Thereby, it follows from (3.14) and (3.15) that (3.10) is fulfilled.

By (3.9), we obtain

$$\begin{aligned}\left[1-(\alpha^2+1)x^2)\right]\left[1+(\alpha^2-1)y^2)\right]&=\left[1-x^2-\alpha^2x^2\right]\left[1-y^2+\alpha^2y^2\right]\\&=\left[y^2(1+2x^2)-\alpha^2x^2\right]\left[x^2(1+2y^2)+\alpha^2y^2\right]\\&=\alpha^2y^4(1+2x^2)-\alpha^2x^4(1+2y^2)\\&=\alpha^2y^2(1-x^2)-\alpha^2x^2(1-y^2)\\&=\alpha^2(y^2-x^2).\end{aligned}$$

Hence, (3.11) is fulfilled. □

Remark 3.4. The same arguments used in the proof of Theorem 2.7, lead us to a conclusion that the Edwards curve E_{-2} is a Jordan curve with a parameterization $\psi:[0,4A]\to\mathbb{R}^2$ such that $\psi(t)=\big(x(t),y(t)\big)$ for every $t\in[0,4A]$, where A is given by (3.4) and the functions x and y are defined by (3.5) and (3.2), respectively.

3.2 Planar system of differential equations

Proposition 3.5. *If x and y are defined by* (3.5) *and* (3.2)*, respectively, then x and y solve the Cauchy problem* (1.19), (1.16).

Proof. Since x and y are solutions of the Cauchy problems (3.6) and (3.1), respectively, we have $x(0) = 0$ and $y(0) = 1$. It follows from (3.7) and (3.9) that

$$\begin{aligned}(x')^2 = 1 - x^6 &= (x^2 + 2x^2y^2 + y^2)^2 - x^4(1 - 2x^2y^2 - y^2)\\ &= y^2(1 + 2x^2)(1 + x^2 + x^4),\end{aligned}$$

and thus, $g(t)h(t) = 0$ for every real t, where

$$\begin{aligned}g(t) &:= x'(t) - y(t)\sqrt{\left[1 + 2x^2(t)\right]\left[1 + x^2(t) + x^4(t)\right]},\\ h(t) &:= x'(t) + y(t)\sqrt{\left[1 + 2x^2(t)\right]\left[1 + x^2(t) + x^4(t)\right]}.\end{aligned}$$

Since x and y satisfy Proposition 2.4 with A given by (3.4), we see that $h(t) \neq 0$ for every $t \in J := (0, A) \cup (A, 3A) \cup (3A, 4A)$. Hence, $g(t) = 0$ for every $t \in J$. On account of Proposition 2.4, $g(t) = 0$ if $t \in \{0, A, 3A, 4A\}$. We conclude that $g(t) = 0$ for every $t \in [0, 4A]$. Taking into account that x and y are periodic functions with the the minimal positive period $4A$, the functions x and y satisfy the first equation in (1.19) for every $t \in \mathbb{R}$. Similarly, one can prove that x and y satisfy the second equation in (1.19) for every $t \in \mathbb{R}$. □

3.3 Addition formulas

Theorem 3.6. *Consider the functions x and y defined by* (3.5) *and* (3.2)*, respectively. For every $t_1, t_2 \in \mathbb{R}$, the values $x(t_1 + t_2)$ and $y(t_1 + t_2)$ can be expressed through $x(t_1)$, $y(t_1)$, $x(t_2)$, and $y(t_2)$ using addition, subtraction, multiplication, division, and taking square root.*

Proof. Consider k defined by (3.3). For arbitrary real t_1 and t_2, denote αt_1 and αt_2 by u and v, respectively. In accordance with [24, Formulas 2.8.1, 2.8.4], for the common modulus k suppressed, we have

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{3.16}$$

$$\operatorname{cd}(u + v) = \frac{\operatorname{cd} u \operatorname{cd} v - k'^2 \operatorname{sd} u \operatorname{nd} u \operatorname{sd} v \operatorname{nd} v}{1 + k^2k'^2 \operatorname{sd}^2 u \operatorname{sd}^2 v}, \tag{3.17}$$

where $k' = \sqrt{1 - k^2}$. It follows from (3.2), (3.5), (3.16), and (3.17) that

$$x(t_1 + t_2) = \frac{\operatorname{sn}(u + v)}{\sqrt{\operatorname{sn}^2(u + v) + \alpha^2 \operatorname{cd}^2(u + v)}}, \tag{3.18}$$

$$y(t_1 + t_2) = \frac{\operatorname{cd}(u + v)}{\sqrt{\operatorname{cd}^2(u + v) + \alpha^2 \operatorname{sn}^2(u + v)}}. \tag{3.19}$$

Note that the denominators in the right-hand sides of (3.16)-(3.19) are positive. To prove the statement, in view of (3.16)-(3.19), it is sufficient to express $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ through $x(t_1)$ and $y(t_1)$ using the operations operations listed in the formulation of the theorem. For convenience, we will use t instead t_1; then, $u = \alpha t$.

Taking into account $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$ and $\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1$, we obtain

$$(1 - k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u)^2 = \operatorname{cd}^4 u + \alpha^2 \operatorname{sn}^2 u \operatorname{cd}^2 u + \operatorname{sn}^4 u, \tag{3.20}$$

$$1 + k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u = \operatorname{cd}^2 u + \operatorname{sn}^2 u. \tag{3.21}$$

Combining (3.2), (3.5), (3.9)-(3.12), (3.20), and (3.21), we have

$$\begin{aligned}
\left(\frac{1 - k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u}{p(t)}\right)^2 &= \frac{\operatorname{cd}^4 u}{q^2(t)\left(\frac{\alpha^2}{1+2x^2(t)}\right)^2} + \alpha^2 \frac{\operatorname{sn}^2 u}{p(t)} \frac{\operatorname{cd}^2 u}{q(t)\frac{\alpha^2}{1+2x^2(t)}} + \frac{\operatorname{sn}^4 u}{p^2(t)} \\
&= \frac{y^4(t)\left[1 + 2x^2(t)\right]^2}{\alpha^4} + x^2(t) y^2(t)\left[1 + 2x^2(t)\right] + x^4(t) \\
&= \frac{\left[1 - x^2(t)\right]^2}{3} + x^2(t)\left[1 - x^2(t)\right] + x^4(t) \\
&= \frac{1 + x^2(t) + x^4(t)}{\alpha^4}, \\
\frac{1 + k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u}{p(t)} &= \frac{\operatorname{cd}^2 u}{q(t)\frac{\alpha^2}{1+2x^2(t)}} + \frac{\operatorname{sn}^2 u}{p(t)} = \frac{y^2(t)\left[1 + 2x^2(t)\right]}{\alpha^2} + x^2(t) \\
&= \frac{1 - x^2(t)}{\alpha^2} + x^2(t) \\
&= \frac{1 + (\alpha^2 - 1)x^2(t)}{\alpha^2}.
\end{aligned}$$

Note that $1 - k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u > 0$. Adding the equalities

$$\frac{1 - k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u}{p(t)} = \frac{\sqrt{1 + x^2(t) + x^4(t)}}{\alpha^2},$$

$$\frac{1 + k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u}{p(t)} = \frac{1 + (\alpha^2 - 1)x^2(t)}{\alpha^2},$$

we obtain

$$p(t) = \frac{2\alpha^2}{1 + (\alpha^2 - 1)x^2(t) + \sqrt{1 + x^2(t) + x^4(t)}}. \tag{3.22}$$

On account of (3.13) and (3.22), we have

$$q(t) = \frac{2\left(1 + 2x^2(t)\right)^2}{1 + (\alpha^2 - 1)x^2(t) + \sqrt{1 + x^2(t) + x^4(t)}}. \tag{3.23}$$

It follows from (3.2), (3.5), and (3.12) that

$$\operatorname{sn} u = x(t)\sqrt{p(t)}, \quad \operatorname{cd} u = y(t)\sqrt{q(t)} \tag{3.24}$$

The proof follows from (3.22)–(3.24) and the identities $\operatorname{dn} u = \sqrt{1 - k^2 \operatorname{sn}^2 u}$, $\operatorname{cn} u = \operatorname{cd} u \operatorname{dn} u$. □

Consider arbitrary points (x_1, y_1) and (x_2, y_2) on E_{-2}. By Remark 3.4, there exist real t_1 and t_2 such that (2.16) holds, where the functions x and y are defined by (3.5) and (3.2), respectively. Let us define the point addition on E_{-2} as follows:

$$(x_1, y_1) \oplus (x_2, y_2) := \big(x(t_1 + t_2), y(t_1 + t_2)\big). \tag{3.25}$$

Reasoning similarly to Remark 2.9, we can conclude that $(E_{-2}, \oplus)$ is an Abelian group in which the point $(0, 1)$ on E_{-2} is a neutral element and the inverse of a point (x, y) on E_{-2} is $(-x, y)$.

On account of Remark 3.4 and the proof of Theorem 3.6, there exist continuous functions $\Phi, \Psi : E_{-2} \times E_{-2} \to \mathbb{R}$ such that

$$x(t_1 + t_2) = \Phi\Big(\big(x(t_1), y(t_1)\big), \big(x(t_2), y(t_2)\big)\Big), \tag{3.26}$$

$$y(t_1 + t_2) = \Psi\Big(\big(x(t_1), y(t_1)\big), \big(x(t_2), y(t_2)\big)\Big) \tag{3.27}$$

for every $t_1, t_2 \in \mathbb{R}$ and such that the right-hand sides in (3.26) and (3.27) can be expressed through $x(t_1)$, $y(t_1)$, $x(t_2)$, and $y(t_2)$ using the operations listed in the formulation of Theorem 3.6. Consequently, the point addition law (3.25) on E_{-2} can be written as follows: $(x_1, y_1) \oplus (x_2, y_2) = \Big(\Phi\big((x_1, y_1), (x_2, y_2)\big), \Psi\big((x_1, y_1), (x_2, y_2)\big)\Big)$ for every points (x_1, y_1) and (x_2, y_2) on E_{-2}. It follows from the proof of Theorem 3.6 that this addition law on E_{-2} differs from the Edwards addition law on E_{-2}.

Concluding remarks

This article investigates the Edwards curves $E_\delta : \; x^2 + y^2 = 1 + \delta\, x^2 y^2$ over the field of real numbers with negative δ from the point of view of ordinary differential equations. The Edwards curves are closely related with a planar cubic Hamiltonian system depending upon the parameter δ. Actually, for a given δ the Edwards curve E_δ is the level set of the corresponding Hamiltonian function. We have obtained parameterizations of the Edwards curves in terms of the Jacobian elliptic functions solving a linear-cubic oscillator. Negative δ is not a square in $\mathbb{R}$ and, as was already known, in this case the Edwards addition law is complete. We provide addition formulas for the functions parameterizing E_δ what allows us to interpret the Edwards addition law similarly as the point addition on the unit circle can be interpreted using the addition formulas for the trigonometric sine and cosine functions.

We pay attention to the curve E_{-2} and show that E_{-2} has an another parameterization by means of solutions of a quintic oscillator. We prove that these solutions solve also a planar autonomous system of ordinary differential equations and obey addition formulas that are different from the addition formulas for the solutions of the linear-cubic oscillator. Moreover, these functions can be effectively used for investigation of non-autonomous the second order quintic differential equations as the lemniscate sine and cosine functions were used in [16] for investigation of the second order non-autonomous ordinary differential equations with cubic nonlinearities (see also [18, 19, 22]), or as the generalized sine and cosine functions were used, see [7, 8], in the theory of half-linear differential equations. More generally, the relation between geometrical objects like Edwards curves, their parameterization, and nonlinear oscillators still can be the research field for extensive investigations.

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