Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **59**, 1–22; https://doi.org/10.14232/ejqtde.2026.1.59 www.math.u-szeged.hu/ejqtde/

The existence of normalized solutions for a fractional Kirchhoff equation with Choquard-type nonlinearities

Dongqing Wang and **Yanbin Sang**✉

School of Mathematics, North University of China, Taiyuan, Shanxi 030051, PR China

Received 20 March 2026, appeared 6 November 2026
Communicated by Paul Eloe

Abstract. We consider the fractional Kirchhoff equation with Choquard-type nonlinearities in $\mathbb{R}^3$:

$$\left(a+b\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)(-\Delta)^s u=\lambda u+\mu\left(I_\alpha*|u|^q\right)|u|^{q-2}u+\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u,$$

subject to the constraint

$$\int_{\mathbb{R}^3}|u|^2dx=c^2,$$

where $a,b,c>0$, $\alpha\in(0,3)$, $s\in(\frac{3-\alpha}{4},1)$, $\mu>0$, $\frac{3+\alpha}{3}<q<\frac{3+\alpha+2s}{3}$, and I_α denotes the Riesz potential. We tackle the challenges of nonlocality and critical exponent associated with the above equation by employing Pohožaev manifold decomposition, fibering map analysis, and compactness arguments for Palais–Smale sequences. Specifically, we establish the existence of a positive, radially decreasing normalized solution to the above equation, and further characterize its asymptotic behavior as $\mu\to0^+$.

Keywords: fractional Kirchhoff, normalized solutions, Pohožaev manifold, fibering map.

2020 Mathematics Subject Classification: 35B33, 35R11, 35J20.

1 Introduction

In this paper, we study the existence and asymptotic behavior of positive normalized solutions for the following fractional Kirchhoff equation with Choquard-type nonlinearities:

$$\left(a+b\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)(-\Delta)^s u=\lambda u+\mu\left(I_\alpha*|u|^q\right)|u|^{q-2}u+\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u, \quad (1.1)$$

with prescribed L^2-norm

$$\int_{\mathbb{R}^3}|u|^2\,dx=c^2, \quad (1.2)$$

✉Corresponding author. Email: syb6662004@163.com

where $a, b, c > 0$, $\alpha \in (0,3)$, $s \in \left(\frac{3-\alpha}{4}, 1\right)$, μ is the positive parameter, $\frac{3+\alpha}{3} < q < 2^*_{\alpha,s}$, $\frac{3+\alpha}{3}$ and $2^*_{\alpha,s} = \frac{3+\alpha}{3-2s}$ are the lower and upper Hardy–Littlewood–Sobolev critical exponents, respectively. $I_\alpha : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)}.$$

The fractional Laplacian $(-\Delta)^s$ admits the definition of a pseudo-differential operator, see [9,11] and references therein,

$$(-\Delta)^s u(x) = C_s \text{P.V.} \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy = C_s \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy, \quad \forall u \in \mathcal{S}(\mathbb{R}^3), \quad (1.3)$$

where $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^3$, P.V. stands for the principal value of the integral, and

$$C_s = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\zeta_1)}{|\zeta|^{3+2s}} d\zeta \right)^{-1} = \pi^{-\left(2s+\frac{3}{2}\right)} \frac{\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(-s)}.$$

Furthermore, if u is smooth enough, by [8, Lemma 3.2], the definition (1.3) is equivalent to

$$(-\Delta)^s u(x) = -\frac{1}{2} C_s \int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}} dy, \quad \forall x \in \mathbb{R}^3.$$

We define the Hilbert space $H^s(\mathbb{R}^3)$ via

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^3) \right\},$$

endowed with the inner product and associated norm

$$\langle u, v \rangle_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv \right) dx, \quad \|u\|_{H^s(\mathbb{R}^3)} = \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|u\|_2^2 \right)^{\frac{1}{2}},$$

where $\|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \frac{C_s}{2} \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy$ and $\|u\|_p^p = \int_{\mathbb{R}^3} |u|^p dx$ for simplification.

The problem (1.1) stands in close relation to the equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u = f(x,u),$$

corresponding to the stationary analog of the equation

$$u_{tt} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u = f(x,u), \quad (1.4)$$

where $f(x,u)$ stands for a general nonlinearity, u designates the displacement, the nonlinear term f denotes the external force, and a is the initial tension while b is related to the intrinsic properties of the string. The problem (1.4) is introduced by Kirchhoff [15] as an extension of the classical D'Alembert's wave equation to free vibration of elastic strings. In addition, comparing with other equations, it is much more challenging and interesting to study problem (1.1) due to the nonlocal term $\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx (-\Delta)^s u$ and the Choquard term.

Since the pioneering contributions of Lions [20], the study of nonlocal systems with fixed frequencies λ and $V(x)$ has been extensively developed over the past few decades, see [13,

23, 25, 26, 32] and references therein. Under such circumstances, it is well established that solutions of (1.1) correspond to critical points of the following energy functional:

$$\Phi_\lambda(u) = \frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2\,dx - \frac{\mu}{2q}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q dx - \frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}dx.$$

As an alternative approach, one may seek solutions to (1.1) that possess the prescribed mass given by (1.2). From a physical perspective, the prescribed L^2-norm (1.2) corresponds to the particle count of the component in mean-field models of Bose–Einstein condensation for further details, see [1, 12] and references therein. The standing wave solutions of problem (1.1) is associated with the form $\Psi(t,x) = e^{-i\lambda t}u(x)$. In this case, u satisfies (1.1) subject to the constraint

$$\int_{\mathbb{R}^3}|\Psi(t,x)|^2dx = \int_{\mathbb{R}^3}|u|^2dx = c^2.$$

Investigating the prescribed L^2-norm solution or normalized solutions of the Kirchhoff problem is particularly compelling, as these solutions correspond to a fixed total mass; for additional physical motivations pertaining to the fractional Kirchhoff model, we refer the reader to [7]. These types of solutions can be obtained as the constrained critical points of the following functional:

$$E_\mu(u) = \frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4 - \frac{\mu}{2q}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q dx - \frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}dx,$$

on the L^2-sphere $S_c = \left\{u \in H^s(\mathbb{R}^3) : \|u\|_2 = c\right\}$.

When $s = 1$, i.e., in the context of the classical Laplacian, numerous studies have addressed this problem. For instance, if $a = 1, b = 0$ and no Choquard terms, problem (1.1) simplifies to the classical Schrödinger equation,

$$-\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u,\ x \in \mathbb{R}^N,$$

where $N \geq 1$, $p \in (2, 2^*)$, and $2^* = \frac{2N}{N-2}$. The L^2-critical exponent $2 + \frac{4}{N}$ is widely regarded as a key component in the Schrödinger equation. Cazenave et al. [4] and Shibata [28] established the existence of normalized solutions using minimization techniques. Recently, Soave [30] investigated normalized solutions for Sobolev critical growth and mixed nonlinearities $\mu \in \mathbb{R}$, $2 < q \leq 2 + \frac{4}{N} \leq p < 2^*$. His work fully categorized the problem by L^2-subcritical, critical, and supercritical regimes, analyzed interactions between L^2-subcritical and L^2-supercritical terms, and verified the multiplicity of solutions. Subsequently, in [29], Soave also tackled the critical case. Owing to the lack of compactness in the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, this problem is more challenging. Drawing on techniques from [3, 14], he derived a constrained Palais–Smale sequence with additional properties by analyzing the geometry of the corresponding Pohožaev manifold, and proved the compactness of this sequence under specific energy levels. If the Choquard term is present, problem (1.1) simplifies to the Choquard equation with Hardy–Littlewood–Sobolev exponent. Of course, this class of equations has also been extensively studied in the literature. Chen [6] proved the existence of normalized ground states for a Hardy–Littlewood–Sobolev upper critical Schrödinger equation with double Choquard-type nonlinear terms via minimization methods. In the same year, Shang et

al. [27] addressed the two cases $\frac{N+\alpha}{N} < q < \frac{N+\alpha+2}{N}$ and $\frac{N+\alpha+2}{N} \le q < 2_\alpha^* = \frac{2N}{N-2}$ by using the mountain pass lemma, they verified the existence of positive normalized solutions to the nonlinear Choquard equations with the Hardy–Littlewood–Sobolev upper critical exponent, as well as the asymptotic behavior of such solutions as $\mu \to 0^+$.

In recent years, Kirchhoff equations with different exponents have attracted the extensive attention of scholars. If $a, b > 0$ are arbitrary fixed constants, problem (1.1) reduces to the Kirchhoff equation, whose corresponding L^2-critical exponent is $2 + \frac{8}{N}$. Results for the purely nonlinear case can be found in [33] and the references therein. Recently, for $N = 3$ with $2 < q < \frac{14}{3} < p \le 6$ or $\frac{14}{3} < q < p \le 6$, Li et al. [18] proved the existence and asymptotic properties of solutions to Kirchhoff equation; for the case $\frac{14}{3} \le q < p = 6$, Zhu et al. [34] investigated the existence and asymptotic behavior of mixed critical and purely L^2-supercritical solutions for the Choquard equation with Kirchhoff perturbation via variational methods.

Inspired by these studies, many researchers have extended this theory to the fractional Laplacian setting and relevant results for fractional equations (including Schrödinger, Choquard, Kirchhoff, and their mixed-term variants) can be found in [16, 17, 22, 24]. However, to the best of our knowledge, the existence of normalized solutions for the fractional Kirchhoff equation with a Choquard term (i.e., $\frac{3+\alpha}{3} < q < 2_{\alpha,s}^*$) remains rare in the current literature.

To fill this research gap, we propose dedicated analytical strategies to overcome the technical challenges brought by the three core components of the model. For the fractional Kirchhoff structure, the quartic fractional gradient term generates degenerate Pohožaev branches and distorts the limiting equations of Palais–Smale sequences, so we introduce the threshold constant $\bar{C}$ to eliminate degenerate solutions and reconstruct the analytical framework of fibering maps. For mixed subcritical and Hardy–Littlewood–Sobolev critical Choquard convolution, subcritical and Hardy–Littlewood–Sobolev critical nonlinear terms exhibit entirely different compactness properties under weak convergence, and the critical term gives rise to the residual mass parameter ℓ, based on which we establish the dichotomy compactness lemma. For the prescribed L^2 mass constraint, the energy functional is unbounded from below on the mass sphere S_c, thus we construct local minimizers on Pohožaev manifolds and prove a novel asymptotic result that the energy and fractional gradient semi-norm vanish as $\mu \to 0^+$.

It should be noted that for the fractional Kirchhoff problem, via the inequality in Lemma 2.1 or a simple scaling argument (see (2.6)), the L^2-critical exponent is derived as $\frac{3+\alpha+4s}{3}$, which determines whether the energy functional is bounded below.

Now, we state our main results. For convenience, let $\delta_{s,q} = \frac{3q-3-\alpha}{2sq}$. For $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, we deduce

$$\bar{C} := \frac{q(2_{\alpha,s}^*-2)b}{2(2_{\alpha,s}^*-q\delta_{s,q})C_{s,q}} \left(\frac{2_{\alpha,s}^*(2-q\delta_{s,q})bS_{\alpha,s}^{2_{\alpha,s}^*}}{2(2_{\alpha,s}^*-q\delta_{s,q})} \right)^{\frac{2-q\delta_{s,q}}{2_{\alpha,s}^*-2}}, \tag{1.5}$$

$$\tilde{C} := \frac{2qA_{s,q}}{C_{s,q}} \left(\frac{a}{2} \left(\frac{2_{\alpha,s}^* bS_{\alpha,s}^{2_{\alpha,s}^*}}{2} \right)^{\frac{1-q\delta_{s,q}}{2_{\alpha,s}^*-2}} + \left(\frac{b}{4} \right)^{\frac{2_{\alpha,s}^*-q\delta_{s,q}}{2_{\alpha,s}^*-2}} \left(22_{\alpha,s}^* S_{\alpha,s}^{2_{\alpha,s}^*} \right)^{\frac{2-q\delta_{s,q}}{2_{\alpha,s}^*-2}} \right), \tag{1.6}$$

$$\hat{C} := \frac{1}{2(2_{\alpha,s}^*-q\delta_{s,q})C_{s,q}} \left(\frac{(22_{\alpha,s}^*-4)b}{\delta_{s,q}} \right)^{\frac{q\delta_{s,q}}{2}} \left(\frac{8q2_{\alpha,s}^* c^*}{2-q\delta_{s,q}} \right)^{1-\frac{q\delta_{s,q}}{2}}, \tag{1.7}$$

where

$$c^* = \left(\frac{1}{2} - \frac{1}{22_{\alpha,s}^*} \right) aS_{\alpha,s}T^2 + \left(\frac{1}{4} - \frac{1}{22_{\alpha,s}^*} \right) bS_{\alpha,s}^2 T^4,$$

$$A_{s,q} := \left(\frac{2(2-q\delta_{s,q})}{2^*_{\alpha,s}(2^*_{\alpha,s}-1)(2^*_{\alpha,s}-q\delta_{s,q})}\right)^{\frac{2-q\delta_{s,q}}{2^*_{\alpha,s}-2}} - \left(\frac{2(2-q\delta_{s,q})}{2^*_{\alpha,s}(2^*_{\alpha,s}-1)(2^*_{\alpha,s}-q\delta_{s,q})}\right)^{\frac{2^*_{\alpha,s}-q\delta_{s,q}}{2^*_{\alpha,s}-2}} > 0,$$

$T > 0$ is the unique global maximum point of the function $K(t) = \frac{a}{2}S_{\alpha,s}t^2 + \frac{b}{4}S^2_{\alpha,s}t^4 - \frac{1}{22^*_{\alpha,s}}t^{22^*_{\alpha,s}}$, and $C_{s,q}$ and $S_{\alpha,s}$ denote the best constant in the (2.1) and (2.2) inequality, respectively (see Sec. II for details).

Theorem 1.1. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\bar{C}, \tilde{C}, \hat{C}\}$. Then, we deduce the following conclusions.*

1. *The restriction $E_\mu|_{S_c}$ admits a critical point u_μ at the energy level $m(c,\mu) < 0$, which serves as a local minimizer of E_μ over the set $A_{R_0} := \left\{u \in S_c : \left\|(-\Delta)^{\frac{s}{2}}u\right\|_2 < R_0\right\}$ for an appropriate $R_0 = R_0(c,\mu) > 0$. Furthermore, u_μ constitutes a ground state of $E_\mu|_{S_c}$, and every ground state of $E_\mu|_{S_c}$ acts as a local minimizer of E_μ on A_{R_0}.*
2. *u_μ is a solution to equation (1.1) corresponding to some $\lambda_\mu < 0$. Furthermore, u_μ is positive and radially decreasing.*
3. *$m(c,\mu) \to 0^-$ and $\left\|(-\Delta)^{\frac{s}{2}}u_\mu\right\|_2 \to 0$ as $\mu \to 0^+$.*

Remark 1.2. We clarify the analytical necessity of all parameter restrictions and threshold constants imposed in Theorem 1.1:

1. Fractional order constraint $s \in \left(\frac{3-\alpha}{4}, 1\right)$. The lower bound $s > \frac{3-\alpha}{4}$ ensures the exponent interval for q is non-empty, while the full domain $s \in (0,1)$ complies with the validity range of the fractional regularity lemma given in [10].
2. Exponent range $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$. The lower bound arises from the Hardy–Littlewood–Sobolev inequality, which yields compact embedding of the subcritical Choquard term on radial spaces and serves as a prerequisite for constructing Palais–Smale sequences. The upper bound guarantees that the one-dimensional radial energy function $h(t)$ admits negative local minimizers alongside a positive global maximum, which allows us to define the local minimizing set A_{R_0} on the mass sphere S_c. If q exceeds this upper threshold, the energy functional becomes unbounded from below, and the fibering-map analysis fails entirely.
3. Threshold constants defined in (1.5)–(1.7) and the combined smallness condition

$$0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\overline{C}, \tilde{C}, \hat{C}\}.$$

   - $\overline{C}$: Utilized in Lemma 4.1 to eliminate the degenerate Pohožaev branch $\mathcal{P}^0_{c,\mu}$ and split the Pohožaev manifold into two disjoint smooth branches $\mathcal{P}^{\pm}_{c,\mu}$;
   - $\tilde{C}$: Applied in Lemma 4.3 to produce two positive cut-off values $0 < R_0 < R_1$ for $h(t)$, which characterize the bounded region containing ground-state minimizers;
   - $\hat{C}$: Deployed in the contradiction argument for Theorem 1.1 to rule out the mass-loss alternative in Proposition 3.1, thus avoiding the contradictory inequality $0 > m(c,\mu) \geq c^*$.

The combined constraint $0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\overline{C}, \tilde{C}, \hat{C}\}$ synthesizes three independent geometric restrictions and is indispensable for every step of our existence proof.

Remark 1.3. Compared to the findings in [18, 24, 29, 30, 33]. Theorem 1.1 may be viewed as an extension of these results, though we employ distinct arguments here. First, we extend the results in [24, 29, 33] for the general Schrödinger and Choquard equations to their fractional counterparts. Second, we further generalize the findings in [6, 17, 27, 34] for fractional Schrödinger and Choquard equations to fractional Kirchhoff equations of Schrödinger or Choquard-type.

The structure of this paper is outlined as follows. In Section 2, we provide some preliminary concepts and lemmas. Section 3 is dedicated to the establishment of a compactness lemma. The proof of Theorem 1.1 is presented in Section 4.

Throughout the present paper, we adopt the following notations:

- $o_n(1)$ denotes an arbitrary quantity that converges to zero as $n \to \infty$.
- $H_r^s(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) : u(x) = u(|x|)\}$ and $S_{c,r} := \{u \in H_r^s(\mathbb{R}^3) : \|u\|_2^2 = c^2\}$.
- $\to$ and $\rightharpoonup$ stand for strong convergence and weak convergence in the corresponding function space, respectively.

2 Preliminaries

In this section, we present several lemmas that will be utilized in the subsequent parts of this paper. To begin with, we state the sharp Gagliardo–Nirenberg type inequality, where $\mathcal{D}^{s,2}(\mathbb{R}^3)$ refers to the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the semi-norm $\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2$.

Lemma 2.1 ([10]). *Let $u \in H^s(\mathbb{R}^3)$ and $\frac{3+\alpha}{3} < q < 2_{\alpha,s}^*$. Then, there exists a best constant $C_{s,q}$ in the following sharp Gagliardo–Nirenberg inequality,*

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u|^q\right)|u|^q\,dx \le C_{s,q}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{2q\delta_{s,q}}\|u\|_2^{2q(1-\delta_{s,q})}. \tag{2.1}$$

In the special case of (2.1), denoting that $S_{\alpha,s} = C_{s,2_{\alpha,s}^*}^{-\frac{1}{2_{\alpha,s}^*}}$, if $q = 2_{\alpha,s}^*$, we obtain the following inequality

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u|^{2_{\alpha,s}^*}\right)|u|^{2_{\alpha,s}^*}\,dx \le S_{\alpha,s}^{-2_{\alpha,s}^*}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{22_{\alpha,s}^*}. \tag{2.2}$$

Lemma 2.2 ([5]). *Let $u \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a weak solution of*

$$(-\Delta)^s u = g(u)$$

and $G(u) \in L^1(\mathbb{R}^3)$. Then, it holds that

$$\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx = \frac{6}{3-2s}\int_{\mathbb{R}^3} G(u)dx,$$

where $G(u) = \int_0^u g(t)dt$.

Remark 2.3. In Lemma 2.2, $g(u) = \lambda u + \mu\left(I_\alpha * |u|^q\right)|u|^{q-2}u + \left(I_\alpha * |u|^{2_{\alpha,s}^*}\right)|u|^{2_{\alpha,s}^*-2}u$. Given that $u \in H^s(\mathbb{R}^3)$, and leveraging the continuity of the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\gamma(\mathbb{R}^3)$ for $\gamma \in [2, 2_s^*]$ (see [21]), it is straightforward to verify that $u \in L^\gamma(\mathbb{R}^3)$. Consequently, $G(u) = \frac{\lambda}{2}|u|^2 + \frac{\mu}{2q}\frac{3+\alpha}{3}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q\,dx + \frac{1}{22_{\alpha,s}^*}\frac{3+\alpha}{3}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{2_{\alpha,s}^*}\right)|u|^{2_{\alpha,s}^*}\,dx \in L^1(\mathbb{R}^3)$. Therefore, we may adapt the reasoning from the proof in [5, Proposition 4.1] to conclude that $u \in L^\infty(\mathbb{R}^3)$, which then enables us to apply the above Pohožaev identity to our equation.

Lemma 2.4. *Let $a,b>0$, $q\in(\frac{3+\alpha}{3},2^*_{\alpha,s})$, and $\mu,\lambda\in\mathbb{R}$. If $u\in H^s(\mathbb{R}^3)$ is a weak solution of (1.1), then the Pohožaev identity*

$$\begin{aligned}P_\mu(u)=a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4-\mu\delta_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx\\-\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}\,dx=0\end{aligned}\tag{2.3}$$

holds.

Proof. Let $M=\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx$. Then, $u\in H^s(\mathbb{R}^3)$ is a weak solution of the following problem:

$$(-\Delta)^s u=\frac{1}{a+bM}\left(\mu\left(I_\alpha*|u|^q\right)|u|^{q-2}u+\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u+\lambda u\right).$$

By similar arguments as in [31, Lemma 2.8], we may infer that $u\in C^{2,\eta}(\mathbb{R}^N)$ for some $\eta\in(0,1)$. As $u\in H^s(\mathbb{R}^3)$ is a weak solution of (1.1), we derive

$$a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4=\mu\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx+\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}\,dx+\lambda\|u\|_2^2.\tag{2.4}$$

From Lemma 2.2, we deduce that

$$\begin{aligned}a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4=\frac{6}{3-2s}\left(\frac{\mu}{2q}\frac{3+\alpha}{3}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx\right.\\\left.+\frac{1}{22^*_{\alpha,s}}\frac{3+\alpha}{3}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}\,dx+\frac{\lambda}{2}\|u\|_2^2\right).\end{aligned}\tag{2.5}$$

Combining (2.4) and (2.5), we deduce that the lemma holds. □

Following [14], the below Pohožaev manifold will play a special role in the proof:

$$\mathcal{P}_{c,\mu}=\{u\in S_c:P_\mu(u)=0\}.$$

In light of Lemma 2.4, every solution to (1.1) lies in $\mathcal{P}_{c,\mu}$. For any $t\in\mathbb{R}$ and $u\in H^s(\mathbb{R}^3)$, we set

$$(t\star u)(x):=e^{\frac{3t}{2}}u(e^tx).\tag{2.6}$$

Then, $\|u\|_2=\|t\star u\|_2$ for all $t\in\mathbb{R}$. Consequently, for any $u\in S_c$, it follows that $t\star u\in S_c$, which naturally motivates the study of the fiber map

$$\begin{aligned}\Psi^\mu_u(t):=&E_\mu(t\star u)\\=&\frac{ae^{2st}}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+\frac{be^{4st}}{4}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\&-\frac{\mu}{2q}e^{2q\delta_{s,q}st}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q dx-\frac{1}{22^*_{\alpha,s}}e^{22^*_{\alpha,s}st}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}dx.\end{aligned}$$

It is straightforward to verify that $(\Psi^\mu_u)'(t)=sP_\mu(t\star u)$ and $\mathcal{P}_{c,\mu}=\{u\in S_c:(\Psi^\mu_u)'(0)=0\}$. On the basis of these results, we obtain the following conclusion.

Lemma 2.5. *Let $u\in S_c$. Then, $t\in\mathbb{R}$ is a critical point for Ψ^μ_u if and only if $t\star u\in\mathcal{P}_{c,\mu}$.*

In this sense, $\mathcal{P}_{c,\mu}$ can be split into the disjoint union $\mathcal{P}_{c,\mu} = \mathcal{P}^+_{c,\mu} \cup \mathcal{P}^0_{c,\mu} \cup \mathcal{P}^-_{c,\mu}$, where

$$\mathcal{P}^+_{c,\mu} := \{u \in \mathcal{P}_{c,\mu} : (\Psi^\mu_u)''(0) > 0\},$$

$$\mathcal{P}^0_{c,\mu} := \{u \in \mathcal{P}_{c,\mu} : (\Psi^\mu_u)''(0) = 0\},$$

$$\mathcal{P}^-_{c,\mu} := \{u \in \mathcal{P}_{c,\mu} : (\Psi^\mu_u)''(0) < 0\},$$

where

$$\begin{aligned}(\Psi^\mu_u)''(0) := 2a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + 4b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4 - 2\mu q\delta^2_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q dx \\ - 22^*_{\alpha,s}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}} dx.\end{aligned}$$

To conclude, we state the subsequent technical lemma.

Lemma 2.6 ([18]). *Let $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{q} \in (0, +\infty)$ and $f(t) := \tilde{a}t^2 + \tilde{b}t^4 - \tilde{c}t^{\tilde{q}} - \tilde{d}t^{22^*_{\alpha,s}}$ for $t \geq 0$. If $\tilde{q} \in (0, 2)$ and*

$$\begin{aligned}&\left[\left(\frac{2(4-\tilde{q})}{2^*_{\alpha,s}(2^*_{\alpha,s}-1)(22^*_{\alpha,s}-\tilde{q})}\right)^{\frac{4-\tilde{q}}{22^*_{\alpha,s}-4}} - \left(\frac{2(4-\tilde{q})}{2^*_{\alpha,s}(2^*_{\alpha,s}-1)(22^*_{\alpha,s}-\tilde{q})}\right)^{\frac{22^*_{\alpha,s}-\tilde{q}}{22^*_{\alpha,s}-4}}\right] \\ &\cdot\left[\frac{\tilde{a}}{\tilde{c}}\left(\frac{\tilde{b}}{\tilde{d}}\right)^{\frac{2-\tilde{q}}{22^*_{\alpha,s}-4}} + \frac{1}{\tilde{c}}\frac{\tilde{b}^{\frac{22^*_{\alpha,s}-\tilde{q}}{22^*_{\alpha,s}-4}}}{\tilde{d}^{\frac{4-\tilde{q}}{22^*_{\alpha,s}-4}}}\right] > 1,\end{aligned}$$

then conclude that $f(t)$ admits a local strict minimum at a negative level as well as a global strict maximum at a positive level over $[0, +\infty)$.

3 Compactness result of Palais–Smale sequences

In this section, we present the compactness result for Palais–Smale sequences, a key component in the proof of our main theorem. Drawing inspiration from [18, 29], we develop a perturbed Pohožaev constraint method that builds on the splitting properties of the Brézis–Lieb lemma.

Proposition 3.1. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\mu > 0$, and $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$. Let $\{u_n\} \subset S_{c,r}$ be a Palais–Smale sequence for $E_\mu|_{S_c}$ at energy level $m \neq 0$, with $P_\mu(u_n) \to 0$ as $n \to \infty$, and*

$$m < c^*, \quad c^* := \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right)aS_{\alpha,s}T^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)bS^2_{\alpha,s}T^4,$$

where $T > 0$ denotes the unique global maximum point of the function $K(t) = \frac{a}{2}S_{\alpha,s}t^2 + \frac{b}{4}S^2_{\alpha,s}t^4 - \frac{1}{22^*_{\alpha,s}}t^{22^*_{\alpha,s}}$ for $t \geq 0$. Passing to a subsequence, one of the following alternatives holds:

1. *$u_n \rightharpoonup u \not\equiv 0$ in $H^s(\mathbb{R}^3)$, where u is a solution of the following equation:*

$$(a + Bb)(-\Delta)^s u = \lambda u + \mu\left(I_\alpha * |u|^q\right)|u|^{q-2}u + \left(I_\alpha * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u, \quad x \in \mathbb{R}^3, \tag{3.1}$$

for some $\lambda < 0$ *and*

$$m - c^* \geq I_\mu(u) := \left(\frac{a}{2} + \frac{Bb}{4}\right) \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 - \frac{\mu}{2q} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^q\right) |u|^q \, dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{2^*_{\alpha,s}}\right) |u|^{2^*_{\alpha,s}} \, dx,$$

for $B := \lim_{n\to\infty} \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 > 0$.

2. $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, $E_\mu(u) = m$, *and* u *satisfies equations* (1.1) *and* (1.2) *for some* $\lambda < 0$.

Proof. We divide the proof into four main steps. To begin with, we apply the Pohožaev identity combined with the exponent range of the equation to carry out energy estimates, which proves that all Palais–Smale sequences are uniformly bounded in the fractional Sobolev space $H^s(\mathbb{R}^3)$ and lays a foundation for extracting weakly convergent subsequences. Relying on the compact embedding from the radial subspace $H_r^s(\mathbb{R}^3)$ to the subcritical L^q space, we extract a weakly convergent subsequence $u_n \rightharpoonup u$ from the bounded sequence, and then utilize the Lagrange multiplier rule to verify that the multiplier sequence $\{\lambda_n\}$ admits a convergent subsequence with limit multiplier λ. Next, we adopt a proof by contradiction: supposing the limit function satisfies $u \equiv 0$ (equivalently $\lambda = 0$), we deduce a lower energy bound $m \geq c^*$ via the critical Hardy–Littlewood–Sobolev inequality, which contradicts the given condition $m < c^*$. This contradiction implies that the limit function is nontrivial and $\lambda < 0$, and we further derive a crucial inequality for the critical remainder term ℓ. Finally, the Brézis–Lieb lemma is employed to decompose the functional associated with the two types of Choquard convolution terms separately. The integral remainder of the subcritical term vanishes as the sequence tends to the limit, while only the critical term retains a residual mass parameter ℓ. Based on the monotonicity of the single-variable function $K(t)$, there exist two mutually exclusive cases: either $\ell \geq T$ corresponding to weak convergence with mass loss, or $\ell = 0$ corresponding to strong convergence without mass loss. This dichotomy conclusion for the convergence branches is therefore established.

Step 1. $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. From $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$ and $P_\mu(u_n) \to 0$, we deduce

$$\left(\frac{a}{2} - \frac{a}{22^*_{\alpha,s}}\right) \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + \left(\frac{b}{4} - \frac{b}{22^*_{\alpha,s}}\right) \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^4 \leq m + 1 + \frac{\mu}{2q} \left(1 - \frac{q\delta_{s,q}}{2^*_{\alpha,s}}\right) C_{s,q} \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^{2q\delta_{s,q}} c^{2q(1-\delta_{s,q})}$$

which implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Step 2. There exists some $\lambda \in \mathbb{R}$ such that, up to a subsequence, $\lambda_n \to \lambda$ as $n \to \infty$. By virtue of the compact embedding $H_r^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, there exists $u \in H_r^s(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$, $u_n \to u$ in $L^q(\mathbb{R}^3)$, $u_n \to u$ a.e. in $\mathbb{R}^3$, and $\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^q\right) |u_n|^q \, dx \to \int_{\mathbb{R}^3} \left(I_\alpha * |u|^q\right) |u|^q \, dx$. Since $\{u_n\}$ is a bounded Palais–Smale sequence of $E_\mu|_{S_c}$, by the Lagrange multiplier's rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$\begin{aligned} &\left(a + b \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2\right) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx \\ &\quad - \int_{\mathbb{R}^3} \left(\mu \left(I_\alpha * |u_n|^q\right) |u_n|^{q-2} u_n + \left(I_\alpha * |u_n|^{2^*_{\alpha,s}}\right) |u_n|^{2^*_{\alpha,s}-2} u_n - \lambda_n u_n\right) \varphi dx \\ &\quad = o_n(1), \end{aligned} \tag{3.2}$$

for every $\varphi \in H^s(\mathbb{R}^3)$. In particular, choosing $\varphi = u_n$, we obtain

$$\lambda_n c^2 = a\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + b\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^4 - \mu\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^q\right)|u_n|^q\,dx - \int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}\,dx + o_n(1).$$

That the sequence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ implies that $\{\lambda_n\}$ is also bounded. As a result, there exists $\lambda \in \mathbb{R}$ for which, up to a subsequence, $\lambda_n \to \lambda$ as $n \to \infty$.

Step 3. $\lambda < 0$ and $u \not\equiv 0$. Using again $P_\mu(u_n) \to 0$, we derive

$$\lambda c^2 = \lim_{n\to\infty}\lambda_n c^2 = \lim_{n\to\infty}\mu(\delta_{s,q}-1)\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^q\right)|u_n|^q\,dx = \mu(\delta_{s,q}-1)\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q\,dx.$$

By virtue of $\mu > 0$ and $0 < \delta_{s,q} < 1$, we conclude that $\lambda \le 0$, with equality $\Longleftrightarrow u \equiv 0$. If $\lambda_n \to 0$, the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$ ensures that, passing to a subsequence,

$$\lim_{n\to\infty}\left(a\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + b\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^4\right) = \lim_{n\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}\,dx = \ell^{22^*_{\alpha,s}}. \tag{3.3}$$

This consequently implies that $\lim_{n\to\infty}\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 = \sqrt{\frac{\ell^{22^*_{\alpha,s}}}{b} + \frac{a^2}{4b^2}} - \frac{a}{2b}$. It is worth noting that

$$\begin{aligned} m &= \lim_{n\to\infty} E_\mu(u_n) \\ &= \lim_{n\to\infty}\left(\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^4 - \frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}\,dx\right) \\ &= \lim_{n\to\infty}\left(\frac{a}{4}\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}\,dx\right) \\ &= \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)\ell^{22^*_{\alpha,s}} + \frac{a}{4}\sqrt{\frac{\ell^{22^*_{\alpha,s}}}{b} + \frac{a^2}{4b^2}} - \frac{a^2}{8b}, \end{aligned} \tag{3.4}$$

and $m \neq 0$. We obtain $\ell \neq 0$. From (3.3) and (2.2), we get

$$\ell^{22^*_{\alpha,s}} \ge aS_{\alpha,s}\ell^2 + bS^2_{\alpha,s}\ell^4. \tag{3.5}$$

Since $T > 0$ is the unique global maximum point of the function $K(t)$, it is easy to check that

$$aS_{\alpha,s}T^2 + bS^2_{\alpha,s}T^4 = T^{22^*_{\alpha,s}}. \tag{3.6}$$

From (3.5), we obtain $K'(\ell) \le 0$, which implies that $\ell \ge T$. By (3.4) and (3.6), we derive

$$\begin{aligned} m &\ge \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)T^{22^*_{\alpha,s}} + \frac{a}{4}\sqrt{\frac{T^{22^*_{\alpha,s}}}{b} + \frac{a^2}{4b^2}} - \frac{a^2}{8b} \\ &= \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)T^{22^*_{\alpha,s}} + \frac{a}{4}S_{\alpha,s}T^2 \\ &= \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right)aS_{\alpha,s}T^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right)bS^2_{\alpha,s}T^4 = c^*, \end{aligned}$$

which contradicts our initial assumption that $m < c^*$. We conclude that $\lambda < 0$ and $u \not\equiv 0$.

Step 4. One of the alternatives (i) or (ii) holds. From $u_n \rightharpoonup u \not\equiv 0$ in $H^s(\mathbb{R}^3)$, we have $B := \lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 \geq \|(-\Delta)^{\frac{s}{2}} u\|_2^2 > 0$. Subsequently, by letting $n \to \infty$ in (3.2), we obtain

$$\begin{aligned}(a+Bb)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi\,dx-\mu\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^{q-2}u\varphi\,dx\\ -\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u\varphi\,dx-\lambda\int_{\mathbb{R}^3}u\varphi\,dx=0,\end{aligned} \tag{3.7}$$

for every $\varphi \in H^s(\mathbb{R}^3)$, which implies that u is a solution to the following equation:

$$(a+Bb)(-\Delta)^s u=\lambda u+\mu\left(I_\alpha*|u|^q\right)|u|^{q-2}u+\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u.$$

By Lemma 2.4, it follows that

$$Q_\mu(u):=(a+Bb)\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2-\mu\delta_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx-\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}\,dx=0.$$

Let $v_n = u_n - u$. It follows that $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$, $v_n \to 0$ in $L^q(\mathbb{R}^3)$, $v_n \to 0$ a.e. in $\mathbb{R}^3$, and

$$\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^q\right)|v_n|^q\,dx\to 0,\quad \left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2=\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2+o_n(1).$$

By the Brézis–Lieb lemma in [2, Theorem 1 and 2], we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^q\right)|u_n|^q dx&=\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q dx+\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^q\right)|v_n|^q dx+o_n(1),\\ \int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}dx&=\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}}dx+\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^{2^*_{\alpha,s}}\right)|v_n|^{2^*_{\alpha,s}}dx+o_n(1).\end{aligned}$$

Note that $\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^q\right)|v_n|^q\,dx\to 0$ in $L^q(\mathbb{R}^3)$. Then,

$$\int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^q\right)|u_n|^q\,dx=\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx+o_n(1).$$

Therefore, we are able to rewrite $P_\mu(u_n) = o_n(1)$ as

$$\begin{aligned}P_\mu(u_n)=(a+Bb)\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2-\mu\delta_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^q\right)|u|^q\,dx\\ -\int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}}\,dx+o(1).\end{aligned}$$

From $Q_\mu(u) = 0$, we obtain that up to a subsequence,

$$\begin{aligned}\ell^{22^*_{\alpha,s}}&=\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^{2^*_{\alpha,s}}\right)|v_n|^{2^*_{\alpha,s}}\,dx=\lim_{n\to\infty}(a+Bb)\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2\\ &\geq\lim_{n\to\infty}\left(a\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2+b\|(-\Delta)^{\frac{s}{2}}v_n\|_2^4\right).\end{aligned} \tag{3.8}$$

From (2.2), we derive $\ell^{22^*_{\alpha,s}} \geq aS_{\alpha,s}\ell^2 + bS^2_{\alpha,s}\ell^4$ and

$$\begin{aligned}\lim_{n\to\infty}\left(a\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2+b\|(-\Delta)^{\frac{s}{2}}v_n\|_2^4\right)&\leq\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha*|v_n|^{2^*_{\alpha,s}}\right)|v_n|^{2^*_{\alpha,s}}\,dx\\ &\leq S_{\alpha,s}^{-2^*_{\alpha,s}}\lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}v_n\|_2^{22^*_{\alpha,s}}.\end{aligned}$$

This implies that

$$\text{either} \quad \ell \geq T \quad \text{and} \quad \lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \geq S_{\alpha,s} T^2 \quad \text{or} \quad \ell = 0 = \lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2.$$

First, suppose $\ell \geq T$ and $\lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \geq S_{\alpha,s} T^2$. Then by (3.6) and (3.8), we obtain

$$\begin{aligned}
m &= \lim_{n\to\infty} E_\mu(u_n) \\
&= I_\mu(u) + \lim_{n\to\infty} \left(\frac{a}{2} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 + \frac{Bb}{4} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{2^*_{\alpha,s}} \right) |v_n|^{2^*_{\alpha,s}} \, dx \right) \\
&= I_\mu(u) + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}} \right) \ell^{22^*_{\alpha,s}} + \lim_{n\to\infty} \frac{a}{4} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \\
&\geq I_\mu(u) + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}} \right) T^{22^*_{\alpha,s}} + \frac{a}{4} S_{\alpha,s} T^2 \\
&= I_\mu(u) + \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}} \right) a S_{\alpha,s} T^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}} \right) b S_{\alpha,s}^2 T^4,
\end{aligned}$$

where

$$I_\mu(u) := \left(\frac{a}{2} + \frac{Bb}{4} \right) \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 - \frac{\mu}{2q} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^q \right) |u|^q \, dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{2^*_{\alpha,s}} \right) |u|^{2^*_{\alpha,s}} \, dx.$$

Hence, alternative (i) of the proposition holds. If instead $\ell = 0 = \lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2$, we get $u_n \to u$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ and $L^{2^*_{\alpha,s}}(\mathbb{R}^3)$. To show that $u_n \to u$ in $L^2(\mathbb{R}^3)$, we test (3.2) and (3.7) with $\varphi = u_n - u$, we obtain

$$(a + Bb) \|(-\Delta)^{\frac{s}{2}} (u_n - u)\|_2^2 - \lambda \|u_n - u\|_2^2 \to 0,$$

which implies that alternative (ii) of the proposition holds. The proof is thus completed. □

4 The proof of Theorem 1.1

In this section, we investigate the existence and asymptotic behavior of normalized ground states for (1.1) and (1.2), and we present the proof of Theorem 1.1. We first analyze the structure of the Pohožaev manifold $\mathcal{P}_{c,\mu}$.

Lemma 4.1. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \bar{C}$. It then follows that $\mathcal{P}^0_{c,\mu} = \emptyset$ and $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$. Here, $\bar{C}$ is given in (1.5).*

Proof. Assume that there exists $u \in \mathcal{P}^0_{c,\mu}$. Let

$$\begin{aligned}
W(t) &= t \left(\Upsilon^\mu_u \right)'(0) - \left(\Upsilon^\mu_u \right)''(0) \\
&= (t-2) a \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + (t-4) b \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^4 \\
&\quad - \mu \left(t - 2q\delta_{s,q} \right) \delta_{s,q} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^q \right) |u|^q dx - \left(t - 22^*_{\alpha,s} \right) \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{2^*_{\alpha,s}} \right) |u|^{2^*_{\alpha,s}} dx \\
&= 0.
\end{aligned}$$

On the one hand, $W(2q\delta_{s,q}) = 0$ implies that

$$\begin{aligned}(2-2q\delta_{s,q})a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+(4-2q\delta_{s,q})b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\ \leq (22^*_{\alpha,s}-2q\delta_{s,q})S_{\alpha,s}^{-2^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{22^*_{\alpha,s}}.\end{aligned}$$

It follows that

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2 \geq \left(\frac{(2-q\delta_{s,q})bS_{\alpha,s}^{2^*_{\alpha,s}}}{2^*_{\alpha,s}-q\delta_{s,q}}\right)^{\frac{1}{22^*_{\alpha,s}-4}}. \tag{4.1}$$

On the other hand, from $W(22^*_{\alpha,s}) = 0$, we obtain

$$\begin{aligned}(22^*_{\alpha,s}-2)a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+(22^*_{\alpha,s}-4)b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\ \leq \mu\delta_{s,q}(22^*_{\alpha,s}-2q\delta_{s,q})C_{s,q}c^{2q(1-\delta_{s,q})}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{2q\delta_{s,q}},\end{aligned}$$

implies that

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2 \leq \left(\frac{\mu\delta_{s,q}(2^*_{\alpha,s}-q\delta_{s,q})C_{s,q}c^{2q(1-\delta_{s,q})}}{(2^*_{\alpha,s}-2)b}\right)^{\frac{1}{4-2q\delta_{s,q}}}. \tag{4.2}$$

From (4.1) and (4.2), we can infer that

$$\mu c^{2q(1-\delta_{s,q})} \geq \frac{(2^*_{\alpha,s}-2)b}{\delta_{s,q}(2^*_{\alpha,s}-q\delta_{s,q})C_{s,q}}\left(\frac{(2-q\delta_{s,q})bS_{\alpha,s}^{2^*_{\alpha,s}}}{2^*_{\alpha,s}-q\delta_{s,q}}\right)^{\frac{2-q\delta_{s,q}}{2^*_{\alpha,s}-2}} > \bar{C},$$

where we get the fact that

$$\left(\frac{2^*_{\alpha,s}}{2}\right)^{4-2q\delta_{s,q}}\left(\frac{q\delta_{s,q}}{2}\right)^{22^*_{\alpha,s}-4} < 1,$$

and this result can be established by virtue of the monotonically decreasing property of $\frac{\ln x}{x-1}$. We then arrive at a contradiction. This shows that $\mathcal{P}^0_{c,\mu} = \emptyset$. The remainder of the proof is essentially analogous to the argument in [30, Lemma 5.2], and thus we omit the details here. □

Lemma 4.2. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \bar{C}$. If $u \in \mathcal{P}_{c,\mu}$ is a critical point of the functional E_μ restricted to $\mathcal{P}_{c,\mu}$, then u is also a critical point of E_μ restricted to S_c. Here, $\bar{C}$ is given in (1.5).*

Proof. Assume $u \in \mathcal{P}_{c,\mu}$ is a critical point for $E_\mu|_{\mathcal{P}_{c,\mu}}$. By the Lagrange multiplier's rule, there exist $\lambda, \nu \in \mathbb{R}$ such that for all $\varphi \in H^s(\mathbb{R}^3)$,

$$\left\langle (E_\mu)'(u), \varphi\right\rangle = \lambda\int_{\mathbb{R}^3} u\varphi dx + \nu\left\langle P'_\mu(u), \varphi\right\rangle.$$

This implies that u is a solution to the equation:

$$\begin{aligned}&\left((1-2\nu)a+(1-4\nu)b\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)(-\Delta)^s u\\ &\quad=\lambda u+\mu\left(1-2q\nu\delta_{s,q}\right)\left(I_\alpha * |u|^q\right)|u|^{q-2}u+\left(1-22^*_{\alpha,s}\nu\right)\left(I_\alpha * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u.\end{aligned}$$

Therefore, we can get the corresponding Pohožaev identity,

$$\begin{aligned}&(1-2\nu)a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+(1-4\nu)b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\&\quad=\mu\delta_{s,q}\left(1-2q\nu\delta_{s,q}\right)\int_{\mathbb{R}^3}(I_\alpha*|u|^q)|u|^q dx+\left(1-22^*_{\alpha,s}\nu\right)\int_{\mathbb{R}^3}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx.\end{aligned}\tag{4.3}$$

From (4.3) and $u\in\mathcal{P}_{c,\mu}$, we obtain

$$\begin{aligned}\nu\Bigg(2a\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+4b\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4-2\mu\delta_{s,q}^2q\int_{\mathbb{R}^3}(I_\alpha*|u|^q)|u|^q dx\\-22^*_{\alpha,s}\int_{\mathbb{R}^3}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx\Bigg)=0.\end{aligned}\tag{4.4}$$

By Lemma 4.1, $\mathcal{P}^0_{c,\mu}=\emptyset$, and thus $\nu=0$. The proof is complete.

From (2.1) and (2.2), we deduce

$$\begin{aligned}E_\mu(u)\geq&\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+\frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\&-\frac{\mu}{2q}C_{s,q}c^{2q(1-\delta_{s,q})}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{2q\delta_{s,q}}-\frac{S_{\alpha,s}^{-2^*_{\alpha,s}}}{22^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{22^*_{\alpha,s}},\end{aligned}\tag{4.5}$$

for all $u\in S_c$. To analyze the geometric properties of $E_\mu|_{S_c}$, we define the function $h:\mathbb{R}^+\to\mathbb{R}$,

$$h(t)=\frac{a}{2}t^2+\frac{b}{4}t^4-\frac{\mu}{2q}C_{s,q}c^{2q(1-\delta_{s,q})}t^{2q\delta_{s,q}}-\frac{S_{\alpha,s}^{-2^*_{\alpha,s}}}{22^*_{\alpha,s}}t^{22^*_{\alpha,s}},\quad t>0.$$

It holds that $h(0^+)=0^-$ and $h(+\infty)=-\infty$. □

Lemma 4.3. *Let $s\in\left(\frac{3-\alpha}{4},1\right)$, $a,b,c>0$, $\frac{3+\alpha}{3}<q<\frac{3+\alpha+2s}{3}$, and $0<\mu c^{2q(1-\delta_{s,q})}<\tilde{C}$. Then, the function h attains a local strict minimum at a negative level and a global strict maximum at a positive level. Furthermore, there exist constants $0<R_0<R_1$, which depend only on c and μ, such that $h(R_0)=0=h(R_1)$ and $h(t)>0$ if and only if $t\in(R_0,R_1)$. Here, $\tilde{C}$ is given in (1.6).*

Proof. Take $\tilde{a}=\frac{a}{2}$, $\tilde{b}=\frac{b}{4}$, $\tilde{c}=\frac{\mu}{2q}C_{s,q}c^{2q(1-\delta_{s,q})}$, $\tilde{d}=\frac{S_{\alpha,s}^{-2^*_{\alpha,s}}}{22^*_{\alpha,s}}$, and $\tilde{q}=2q\delta_{s,q}$ in Lemma 2.6. Then, it suffices to show that

$$\begin{aligned}&\left[\left(\frac{2(4-\tilde{q})}{2^*_{\alpha,s}\left(2^*_{\alpha,s}-1\right)\left(22^*_{\alpha,s}-\tilde{q}\right)}\right)^{\frac{4-\tilde{q}}{22^*_{\alpha,s}-4}}-\left(\frac{2(4-\tilde{q})}{2^*_{\alpha,s}\left(2^*_{\alpha,s}-1\right)\left(22^*_{\alpha,s}-\tilde{q}\right)}\right)^{\frac{22^*_{\alpha,s}-\tilde{q}}{22^*_{\alpha,s}-4}}\right]\\&\cdot\left(\frac{\tilde{a}}{\tilde{c}}\left(\frac{\tilde{b}}{\tilde{d}}\right)^{\frac{2-\tilde{q}}{22^*_{\alpha,s}-4}}+\frac{1}{\tilde{c}}\frac{\tilde{b}^{\frac{22^*_{\alpha,s}-\tilde{q}}{22^*_{\alpha,s}-4}}}{\tilde{d}^{\frac{4-\tilde{q}}{22^*_{\alpha,s}-4}}}\right)>1.\end{aligned}$$

Indeed, direct computation reveals that the above inequality is equivalent to

$$\mu c^{2q(1-\delta_{s,q})}<\frac{2qA_{s,q}}{C_{s,q}}\left(\frac{a}{2}\left(\frac{2^*_{\alpha,s}bS_{\alpha,s}^{2^*_{\alpha,s}}}{2}\right)^{\frac{1-q\delta_{s,q}}{2^*_{\alpha,s}-2}}+\left(\frac{b}{4}\right)^{\frac{2^*_{\alpha,s}-q\delta_{s,q}}{2^*_{\alpha,s}-2}}\left(22^*_{\alpha,s}S_{\alpha,s}^{2^*_{\alpha,s}}\right)^{\frac{2-q\delta_{s,q}}{2^*_{\alpha,s}-2}}\right)=:\tilde{C}.$$

Hence, the conclusion holds, provided that $0<\mu c^{2q(1-\delta_{s,q})}<\tilde{C}$. □

Lemma 4.4. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\bar{C}, \tilde{C}\}$. For every $u \in S_c$, the functional Ψ_u^μ admits exactly two critical points $s_u < t_u \in \mathbb{R}$ and two zeros $c_u < d_u \in \mathbb{R}$, satisfying the strict ordering $s_u < c_u < t_u < d_u$. Furthermore, we obtain the following.*

1. *$s_u \star u \in \mathcal{P}_{c,\mu}^+$ and $t_u \star u \in \mathcal{P}_{c,\mu}^-$, and if $t \star u \in \mathcal{P}_{c,\mu}$, then either $t = s_u$ or $t = t_u$.*
2. *$\left\|(-\Delta)^{\frac{s}{2}}(t \star u)\right\|_2 \leq R_0$ for every $t \leq c_u$, and*
$$E_\mu(s_u \star u) = \min\left\{E_\mu(t \star u) : t \in \mathbb{R} \text{ and } \left\|(-\Delta)^{\frac{s}{2}}(t \star u)\right\|_2 < R_0\right\} < 0.$$
3. $$E_\mu(t_u \star u) = \max_{t \in \mathbb{R}} E_\mu(t \star u) > 0,$$
and Ψ_u^μ is strictly decreasing and concave on $(t_u, +\infty)$.
4. *The mappings $u \mapsto s_u$ and $u \mapsto t_u$ from S_c to $\mathbb{R}$ are both of class C^1.*

Proof. From (4.5), we deduce that

$$\Psi_u^\mu(t) = E_\mu(t \star u) \geq h\left(\left\|(-\Delta)^{\frac{s}{2}}(t \star u)\right\|_2\right) = h\left(e^{st}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2\right),$$

and consequently,

$$\Psi_u^\mu(t) > 0, \ \forall t \in \left(\log \frac{R_0}{\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2}, \log \frac{R_1}{\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2}\right).$$

Recalling that $\Psi_u^\mu(-\infty) = 0^-$ and $\Psi_u^\mu(+\infty) = -\infty$, it follows that Ψ_u^μ possess at least two critical points $s_u < t_u$, where s_u is a local minimum point on $\left(-\infty, \log \frac{R_0}{\|(-\Delta)^{\frac{s}{2}}u\|_2}\right)$ at a negative level and t_u is a global maximum point at a positive level. Next, we establish that $\Psi_u^\mu(t)$ has at most two critical points in $\mathbb{R}$. Indeed, $(\Psi_u^\mu)'(t) = 0$ entails that $\varphi(s) = \mu\delta_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q\,dx$, where

$$\begin{aligned}\varphi(s) = {} & ae^{(2-2q\delta_{s,q})st}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + be^{(4-2q\delta_{s,q})st}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4 \\ & - e^{(22^*_{\alpha,s}-2q\delta_{s,q})st}\int_{\mathbb{R}^3}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx\,.\end{aligned}$$

We observe that φ admits a unique global maximum point, and as such, the equation $\varphi(s) = \mu\delta_{s,q}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^q\right)|u|^q\,dx$ has at most two solutions. Therefore, we may conclude that $\Psi_u^\mu(t)$ has exactly two critical points s_u and t_u. Since $(\Psi_u^\mu)'(t) = sP_\mu(t \star u)$, it follows that $t \star u \in \mathcal{P}_{c,\mu}$ implies $t = s_u$ or t_u. Furthermore, noting that $(\Psi_u^\mu)''(s_u) \geq 0$, $(\Psi_u^\mu)''(t_u) \leq 0$, and $\mathcal{P}_{c,\mu}^0 = \emptyset$, we infer that $s_u \star u \in \mathcal{P}_{c,\mu}^+$ and $t_u \star u \in \mathcal{P}_{c,\mu}^-$.

By virtue of monotonicity, Ψ_u^μ has exactly two zero-points c_u and d_u, satisfying $s_u < c_u < t_u < d_u$. It remains only to show that the mappings $u \mapsto t_u$ and $u \mapsto s_u$ are of class C^1. We apply the implicit function theorem to $\Phi(s, u) := (\Psi_u^\mu)'(s)$. Utilizing the fact that

$$\begin{aligned}\Phi(s_u, u) = \Phi(t_u, u) &= 0, \\ \partial_s\Phi(s_u, u) = (\Psi_u^\mu)''(s_u) &> 0, \\ \partial_s\Phi(t_u, u) = (\Psi_u^\mu)''(t_u) &< 0,\end{aligned}$$

and $\mathcal{P}_{c,\mu}^0 = \emptyset$, we can establish that $u \mapsto t_u$ and $u \mapsto s_u$ are of class C^1. The proof is therefore complete. □

For $k > 0$, we set

$$A_k := \left\{ u \in S_c : \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2 < k \right\}$$

and

$$m(c,\mu) := \inf_{u \in A_{R_0}} E_\mu(u).$$

From Lemma 4.4, it follows immediately that we get the following corollary.

Corollary 4.5. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\bar{C}, \tilde{C}\}$. Then, the set $\mathcal{P}^+_{c,\mu}$ is contained in $A_{R_0} = \left\{ u \in S_c : \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2 < R_0 \right\}$ and $\sup_{\mathcal{P}^+_{c,\mu}} E_\mu \leq 0 \leq \inf_{\mathcal{P}^-_{c,\mu}} E_\mu$.*

Lemma 4.6. *Let $s \in \left(\frac{3-\alpha}{4}, 1\right)$, $a, b, c > 0$, $\frac{3+\alpha}{3} < q < \frac{3+\alpha+2s}{3}$, and $0 < \mu c^{2q(1-\delta_{s,q})} < \min\{\bar{C}, \tilde{C}\}$. It follows that $m(c,\mu) \in (-\infty, 0)$, and there exists a sufficiently small constant ρ such that*

$$m(c,\mu) = \inf_{\mathcal{P}_{c,\mu}} E_\mu = \inf_{\mathcal{P}^+_{c,\mu}} E_\mu \quad \text{and} \quad m(c,\mu) < \inf_{\overline{A_{R_0}} \setminus A_{R_0-\rho}} E_\mu.$$

Proof. For $u \in A_{R_0}$, it holds that

$$E_\mu(u) \geq h\left(\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2 \right) \geq \min_{t \in [0, R_0]} h(t) > -\infty,$$

and consequently, $m(c,\mu) > -\infty$. Furthermore, for any $u \in S_c$, we derive $\left\| (-\Delta)^{\frac{s}{2}} (t \star u) \right\|_2 < R_0$ and $E_\mu(t \star u) < 0$ for $t \ll -1$, from which it follows that $m(c,\mu) < 0$.

By virtue of Corollary 4.5 and the inclusion $\mathcal{P}^+_{c,\mu} \subset A_{R_0}$, we deduce that $m(c,\mu) \leq \inf_{\mathcal{P}^+_{c,\mu}} E_\mu$. On the other hand, if $u \in A_{R_0}$, we obtain $t_u \star u \in \mathcal{P}^+_{c,\mu} \subset A_{R_0}$ and

$$E_\mu(t_u \star u) = \min\left\{ E_\mu(t \star u) : t \in \mathbb{R} \text{ and } \left\| (-\Delta)^{\frac{s}{2}} (t \star u) \right\|_2 < R_0 \right\} \leq E_\mu(u).$$

This implies that $\inf_{\mathcal{P}^+_{c,\mu}} E_\mu \leq m(c,\mu)$. Recalling that $E_\mu \geq 0$ on $\mathcal{P}^-_{c,\mu}$, we infer that $\inf_{\mathcal{P}^+_{c,\mu}} E_\mu = \inf_{\mathcal{P}_{c,\mu}} E_\mu$. Finally, by the continuity of h, there exists a constant $\rho > 0$ (independent of c and μ) such that $h(t) \geq \frac{m(c,\mu)}{2}$ for all $t \in [R_0 - \rho, R_0]$. Hence,

$$E_\mu(u) \geq h\left(\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2 \right) \geq \frac{m(c,\mu)}{2} > m(c,\mu),$$

for every $u \in \overline{A_{R_0}} \setminus A_{R_0-\rho}$. □

Proof of Theorem 1.1-(i), (ii). Let $\{v_n\}$ be a minimizing sequence for $\inf_{A_{R_0}} E_\mu$. By virtue of the results in [19, Lemma 7.17] and [17], we have $E_\mu(|v_n|^*) \leq E_\mu(v_n)$ for all n, where $|v_n|^*$ stands for the Schwarz symmetric nonincreasing rearrangement of $|v_n|$. This rearrangement cannot raise the value of the energy functional. Consequently, each term of the minimizing sequence can be replaced by its radially symmetric nonincreasing rearrangement without changing the infimum of the energy. For this reason, we assume without loss of generality that $0 \leq v_n \in S_c$ is radially nonincreasing for all n. By virtue of Lemma 4.4 and Corollary 4.5, we deduce that $s_{v_n} \star v_n \in \mathcal{P}^+_{c,\mu}$, $\left\| (-\Delta)^{\frac{s}{2}} (s_{v_n} \star v_n) \right\|_2 < R_0$, and

$$E_\mu(s_{v_n} \star v_n) = \min\left\{ E_\mu(t \star v_n) : t \in \mathbb{R} \text{ and } \left\| (-\Delta)^{\frac{s}{2}} (t \star v_n) \right\|_2 < R_0 \right\} \leq E_\mu(v_n).$$

Thus, we are able to obtain a new minimizing sequence $\{\omega_n = s_{v_n} \star v_n\} \subset S_{c,r} \cap \mathcal{P}^+_{c,\mu}$. From Lemma 4.6, we infer that $\left\| (-\Delta)^{\frac{s}{2}} \omega_n \right\|_2 < R_0 - \rho$. By virtue of the Ekeland's variational

principle, there exists a new minimizing sequence $\{u_n\} \subset A_{R_0}$ corresponding to $m(c,\mu) < 0$, such that $\|u_n - \omega_n\|_{H^s(\mathbb{R}^3)} \to 0$, as $n \to \infty$. This sequence is also a Palais–Smale sequence for E_μ on S_c. Given the boundedness of $\{\omega_n\}$ and the Brézis–Lieb lemma, we derive

$$P_\mu(u_n) = P_\mu(\omega_n) + o_n(1) \to 0,$$

as $n \to \infty$. Hence, one of the alternatives stated in Proposition 3.1 is valid. Our goal is to establish that alternative (ii) is the case. Adopting a proof by contradiction, suppose that $u_n \rightharpoonup u_\mu \not\equiv 0$ in $H^s(\mathbb{R}^3)$, where u_μ is a solution to (3.1) corresponding to some $\lambda < 0$, and

$$\begin{aligned} I_\mu(u_\mu) := & \left(\frac{a}{2} + \frac{Bb}{4}\right) \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 - \frac{\mu}{2q} \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^q\right) |u_\mu|^q \, dx \\ & - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{2^*_{\alpha,s}}\right) |u_\mu|^{2^*_{\alpha,s}} \, dx \le m(c,\mu) - c^*, \end{aligned}$$

where $B := \lim_{n\to\infty} \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 \ge \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 > 0$. Since u_μ solves (3.1), we can obtain the Pohožaev identity

$$\begin{aligned} Q_\mu(u_\mu) := & (a + Bb)\|(-\Delta)^{\frac{s}{2}} u_\mu\|_2^2 - \mu\delta_{s,q} \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^q\right) |u_\mu|^q \, dx \\ & - \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{2^*_{\alpha,s}}\right) |u_\mu|^{2^*_{\alpha,s}} \, dx = 0, \end{aligned}$$

which implies that

$$\begin{aligned} I_\mu(u_\mu) = & \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right) a \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right) Bb \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 \\ & - \mu \left(\frac{1}{2q} - \frac{\delta_{s,q}}{22^*_{\alpha,s}}\right) \int_{\mathbb{R}^3} (I_\alpha * |u_\mu|^q)|u_\mu|^q dx. \end{aligned}$$

By virtue of the weakly lower semi-continuity of the norm, we obtain $\|u_\mu\|_2 \le c$. Therefore,

$$\begin{aligned} m(c,\mu) \ge & \, c^* + \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right) a \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right) Bb \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 \\ & - \mu \left(\frac{1}{2q} - \frac{\delta_{s,q}}{22^*_{\alpha,s}}\right) \int_{\mathbb{R}^3} (I_\alpha * |u_\mu|^q)|u_\mu|^q dx \\ \ge & \, c^* + \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right) b \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^4 - \mu \left(\frac{1}{2q} - \frac{\delta_{s,q}}{22^*_{\alpha,s}}\right) C_{s,q} c^{2q(1-\delta_{s,q})} \left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^{2q\delta_{s,q}}. \end{aligned} \tag{4.6}$$

Consider the function

$$g(t) = \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right) bt^4 - \mu \left(\frac{1}{2q} - \frac{\delta_{s,q}}{22^*_{\alpha,s}}\right) C_{s,q} c^{2q(1-\delta_{s,q})} t^{2q\delta_{s,q}} \quad \text{for } t > 0.$$

From $0 < \mu c^{2q(1-\delta_{s,q})} < \hat{C}$, we deduce

$$\min_{t\ge 0} g(t) = -\left(\frac{2}{q\delta_{s,q}} - 1\right) \left(\frac{1}{4} - \frac{1}{22^*_{\alpha,s}}\right) bt_0^4 > -c^*,$$

for $t_0 = \left(\frac{\mu\delta_{s,q}(2^*_{\alpha,s} - q\delta_{s,q}) C_{s,q} c^{2q(1-\delta_{s,q})}}{(2^*_{\alpha,s} - 2)b}\right)^{\frac{1}{4-2q\delta_{s,q}}}$. From (4.6), we deduce that

$$0 > m(c,\mu) \ge c^* + g\left(\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2\right) \ge c^* + \min_{t\ge 0} g(t) > 0,$$

and this is a contradiction. Thus, $u_n \to u_\mu$ in $H^s(\mathbb{R}^3)$, $E_\mu(u_\mu) = m(c,\mu)$, and u_μ solves (1.1) for some $\lambda_\mu < 0$, and it is non-negative and radially nonincreasing.

Next, we aim to demonstrate that $u_\mu > 0$. By means of reasoning similar to that in [31, Lemma 2.8], we obtain that $u_\mu \in C^{2,\eta}(\mathbb{R}^3)$ for some $\eta \in (0,1)$. It then follows from [8, Lemma 3.2] that

$$(-\Delta)^s u_\mu(x) = -\frac{1}{2}C_s \int_{\mathbb{R}^3} \frac{u_\mu(x+y) + u_\mu(x-y) - 2u_\mu(x)}{|y|^{3+2s}} dy, \quad \forall x \in \mathbb{R}^3.$$

In view of the regularity property of u_μ, we can assume that there exists $x_0 \in \mathbb{R}^3$ such that $u_\mu(x_0) = 0$. Then, from $u_\mu \geq 0$ and $u_\mu \not\equiv 0$, we obtain

$$(-\Delta)^s u_\mu(x_0) = -\frac{1}{2}C_s \int_{\mathbb{R}^3} \frac{u_\mu(x_0+y) + u_\mu(x_0-y)}{|y|^{3+2s}} dy < 0.$$

However, as u_μ solves (1.1) with some $\lambda_\mu < 0$, we also get

$$(-\Delta)^s u_\mu(x_0) = \frac{\lambda_\mu u_\mu(x_0) + \mu(I_\alpha * |u_\mu|^q)|u_\mu|^{q-2}u_\mu(x_0) + (I_\alpha * |u_\mu|^{2^*_{\alpha,s}})|u_\mu|^{2^*_{\alpha,s}-2}u_\mu(x_0)}{\left(a + b\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_\mu(x_0)\right|^2 dx\right)} = 0.$$

This leads to a contradiction, which means our assumption that u_μ vanishes at some point is false. Since $u_\mu \geq 0$ from rearrangement, we finally obtain the strict positivity $u_\mu > 0$ everywhere on $\mathbb{R}^3$.

In fact, u_μ is a ground state for $E_\mu|_{S_c}$. Since any critical point of $E_\mu|_{S_c}$ is contained in $\mathcal{P}_{c,\mu}$ and $E_\mu(u_\mu) = \inf_{\mathcal{P}_{c,\mu}} E_\mu$, it only remains to prove that any other ground state is a local minimizer of E_μ on A_{R_0}. Let u be a critical point of $E_\mu|_{S_c}$ at the energy level $m(c,\mu)$. Since $E_\mu(u) < 0$, then $u \in \mathcal{P}^+_{c,\mu}$. From Corollary 4.5 we deduce that $\mathcal{P}^+_{c,\mu} \subset A_{R_0}$. Thus, from Lemma 4.4, we can conclude that u is a local minimizer for $E_\mu|_{A_{R_0}}$. □

Remark 4.7. We explain the dichotomy principle and Brézis–Lieb splitting for different Choquard nonlinearities concisely.

1. Let $\{u_n\} \in S_c$ be a radial Palais–Smale sequence with weak limit u and remainder $v_n = u_n - u$. In Step 4 of Proposition 3.1, we first employ the Brézis–Lieb decomposition to split the subcritical and critical Choquard convolution integrals. Combined with the Pohožaev identity Q_μ satisfied by the weak limit u and equation (3.8), we obtain the crucial lower bound for $\ell = \lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}} v_n\|_2$. Together with the critical Hardy–Littlewood–Sobolev inequality (2.2), we finally deduce that only two mutually exclusive convergence behaviours are available for the sequence:

   - $\ell \geq T$: mass loss occurs, and the weak limit satisfies the modified Kirchhoff equation;
   - $\ell = 0$: the fractional gradient norm of the remainder vanishes, and the sequence converges strongly to a solution of the original normalized equation.

2. The Brézis–Lieb decomposition applied to the two classes of Choquard nonlinearities is presented as follows:

   - Subcritical exponent q: The compact embedding $H^s \hookrightarrow L^q$ forces the convolution remainder integral $\int_{\mathbb{R}^3} (I_\alpha * |v_n|^q)|v_n|^q dx \to 0$;

- Critical exponent $2^*_{\alpha,s}$: Compactness fails at the Hardy–Littlewood–Sobolev critical exponent, and the remainder integral satisfies

$$\int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{2^*_{\alpha,s}}\right)|v_n|^{2^*_{\alpha,s}} dx = \ell^{22^*_{\alpha,s}}.$$

Proof of Theorem 1.1-(iii). We recall that u_μ acts as an interior local minimizer of E_μ in A_{R_0}, where $R_0(c,\mu)$ is given by Lemma 4.3. It is clear that $R_0(c,\mu) \to 0$ as $\mu \to 0^+$, and then, $\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2 < R_0(c,\mu) \to 0$ as well. Furthermore, from the (2.1) and (2.2), we obtain

$$\begin{aligned} 0 > m(c,\mu) &= E_\mu(u_\mu) \\ &\geq \frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^2 + \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^4 - \frac{\mu C_{s,q}}{2q}\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^{2q\delta_{s,q}} c^{2q(1-\delta_{s,q})} \\ &\quad - \frac{S_{\alpha,s}^{-2^*_{\alpha,s}}}{22^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}} u_\mu\right\|_2^{22^*_{\alpha,s}} \to 0, \end{aligned}$$

as $\mu \to 0^+$, and this implies that $m(c,\mu) \to 0$. □

Acknowledgements

This work was supported by the Fundamental Research Program of Shanxi Province (20210302 1224198).

References

[1] N. Bigagli, W. Yuan, S. Zhang, B. Bulatovic, T. Karman, I. Stevenson, S. Will, Observation of Bose–Einstein condensation of dipolar molecules, *Nature* **631**(2024), No. 8020, 289–293. https://doi.org/10.1038/s41586-024-07492-z

[2] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**(1983), No. 3, 486–490. https://doi.org/10.1090/S0002-9939-1983-0699419-3; MR0699419; Zbl 0526.46037

[3] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**(1983), No. 4, 437–477. https://doi.org/10.1002/cpa.3160360405; MR0709644; Zbl 0541.35029

[4] T. Cazenave, P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.* **85**(1982), No. 4, 549–561. https://doi.org/10.1007/BF01403504; MR0677997; Zbl 0513.35007

[5] X. J. Chang, Z. Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity* **26**(2013), No. 2, 479–494. https://doi.org/10.1088/0951-7715/26/2/479; MR3007900; Zbl 1276.35080

[6] J. Q. Chen, Z. W. Chen, Normalized ground states for a Hardy–Littlewood–Sobolev upper critical Schrödinger equation with double Choquard type nonlinear terms, *Appl. Math. Lett.* **138**(2023), Paper No. 108521, 8 pp. https://doi.org/10.1016/j.aml.2022.108521; MR4522948; Zbl 1505.35114

[7] M. Chipot, B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.* **30**(1997), No. 7, 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7; MR1603446; Zbl 0894.35119

[8] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**(2012), No. 5, 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004; MR2944369; Zbl 1252.46023

[9] S. Dipierro, E. P. Lippi, C. Sportelli, E. Valdinoci, The Neumann condition for the superposition of fractional Laplacians, *ESAIM Control Optim. Calc. Var.* **31**(2025), Paper No. 25, 45 pp. https://doi.org/10.1051/cocv/2025015; MR4882759; Zbl 1562.35491

[10] B. H. Feng, H. H. Zhang, Stability of standing waves for the fractional Schrödinger–Choquard equation, *Comput. Math. Appl.* **75**(2018), No. 7, 2499–2507. https://doi.org/10.1016/j.camwa.2017.12.025; MR3777115; Zbl 1409.35213

[11] A. Fernandez, J. E. Restrepo, J.-D. Djida, On the fractional Laplacian of a function with respect to another function, *Math. Methods Appl. Sci.* **47**(2024), No. 18, 14079–14110. https://doi.org/10.1002/mma.10256; MR4827349; Zbl 07996580

[12] D. J. Frantzeskakis, Dark solitons in atomic Bose–Einstein condensates: from theory to experiments, *J. Phys. A* **43**(2010), No. 21, Paper No. 213001, 68 pp. https://doi.org/10.1088/1751-8113/43/21/213001; MR2644602; Zbl 1192.82033

[13] Q. H. He, Z. Y. Lv, Z. W. Tang, The existence of normalized solutions to the Kirchhoff equation with potential and Sobolev critical nonlinearities, *J. Geom. Anal.* **33**(2023), No. 7, Paper No. 236, 30 pp. https://doi.org/10.1007/s12220-023-01298-7; MR4587616; Zbl 1518.35363

[14] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.* **28**(1997), No. 10, 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1; MR1430506; Zbl 0877.35091

[15] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[16] L. Z. Kong, H. B. Chen, Normalized ground states for fractional Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities, *J. Math. Phys.* **64**(2023), No. 6, Paper No. 061501, 13 pp. https://doi.org/10.1063/5.0098126; MR4597174; Zbl 1516.35465

[17] J. L. Lan, X. M. He, Y. X. Meng, Normalized solutions for a critical fractional Choquard equation with a nonlocal perturbation, *Adv. Nonlinear Anal.* **12**(2023), No. 1, Paper No. 20230112, 40 pp. https://doi.org/10.1515/anona-2023-0112; MR4668062; Zbl 1529.35218

[18] G. B. Li, X. Luo, T. Yang, Normalized solutions to a class of Kirchhoff equations with Sobolev critical exponent, *Ann. Fenn. Math.* **47**(2022), No. 2, 895–925. https://doi.org/10.54330/afm.120247; MR4443235; Zbl 1497.35232

[19] E. H. Lieb, M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, Vol. 14, Amer. Math. Soc., Providence, RI, 2001. https://doi.org/10.1090/gsm/014; MR1817225; Zbl 0966.26002

[20] J. L. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.* **30**(1978), 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3; MR0519648; Zbl 0404.35002

[21] P. L. Lions, Symétrie et compacité dans les espaces de Sobolev, *J. Funct. Anal.* **49**(1982), No. 3, 315–334. https://doi.org/10.1016/0022-1236(82)90072-6; MR0683027; Zbl 0501.46032

[22] L. T. Liu, H. B. Chen, J. Yang, Normalized solutions to the fractional Kirchhoff equations with a perturbation, *Appl. Anal.* **102**(2023), No. 4, 1229–1249. https://doi.org/10.1080/00036811.2021.1979222; MR4572315; Zbl 1512.35543

[23] T. J. Luo, H. Hajaiej, Normalized solutions for a class of scalar field equations involving mixed fractional Laplacians, *Adv. Nonlinear Stud.* **22**(2022), No. 1, 228–247. https://doi.org/10.1515/ans-2022-0013; MR4445936; Zbl 1497.35219

[24] H. J. Luo, Z. T. Zhang, Normalized solutions to the fractional Schrödinger equations with combined nonlinearities, *Calc. Var. Partial Differential Equations* **59**(2020), No. 4, Paper No. 143, 35 pp. https://doi.org/10.1007/s00526-020-01814-5; MR4135640; Zbl 1445.35307

[25] K. Perera, Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations* **221**(2006), No. 1, 246–255. https://doi.org/10.1016/j.jde.2005.03.006; MR2193850; Zbl 1357.35131

[26] S. Secchi, On fractional Schrödinger equations in $\mathbb{R}^N$ without the Ambrosetti–Rabinowitz condition, *Topol. Methods Nonlinear Anal.* **47**(2016), No. 1, 19–41. https://doi.org/10.12775/TMNA.2015.090; MR3469045; Zbl 1362.35290

[27] X. D. Shang, P. Ma, Normalized solutions to the nonlinear Choquard equations with Hardy–Littlewood–Sobolev upper critical exponent, *J. Math. Anal. Appl.* **521**(2023), No. 2, Paper No. 126916, 29 pp. https://doi.org/10.1016/j.jmaa.2022.126916; MR4523475; Zbl 1511.35165

[28] M. Shibata, Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term, *Manuscripta Math.* **143**(2014), No. 1–2, 221–237. https://doi.org/10.1007/s00229-013-0627-9; MR3147450; Zbl 1290.35252

[29] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, *J. Funct. Anal.* **279**(2020), No. 6, Paper No. 108610, 43 pp. https://doi.org/10.1016/j.jfa.2020.108610; MR4096725; Zbl 1440.35311

[30] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differential Equations* **269**(2020), No. 9, 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016; MR4107073; Zbl 1440.35312

[31] K. M. Teng, Ground state solutions for the non-linear fractional Schrödinger–Poisson system, *Appl. Anal.* **98**(2019), No. 11, 1959–1996. https://doi.org/10.1080/00036811.2018.1441998; MR3982764; Zbl 1447.35363

[32] L. P. Xu, H. B. Chen, Ground states of Kirchhoff equations via Pohožaev–Nehari manifold: existence, concentration and nonexistence, *Ann. Funct. Anal.* **14**(2023), No. 1,

Paper No. 23, 31 pp. https://doi.org/10.1007/s43034-022-00245-x; MR4531708; Zbl 1505.35202

[33] H. Y. Ye, The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.* **67**(2016), No. 2, Paper No. 29, 16 pp. https://doi.org/10.1007/s00033-016-0624-4; MR3483878; Zbl 1341.35059

[34] S. N. Zhu, G. F. Che, H. B. Chen, Existence and asymptotic behavior of normalized solutions for Choquard equations with Kirchhoff perturbation, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **119**(2025), No. 1, Paper No. 2, 25 pp. https://doi.org/10.1007/s13398-024-01668-y; MR4808381; Zbl 1555.35156