Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **41**, 1–16; https://doi.org/10.14232/ejqtde.2026.1.41 www.math.u-szeged.hu/ejqtde/

On positive solutions for ordinary differential equations of the Kirchhoff type

Ricardo Lima Alves✉[1] and **Síglia Souza Oliveira**[2]

[1]Centro de Ciências e Tecnologia, Universidade Federal do Cariri – UFCA, CEP: 63048–080, Juazeiro do Norte – CE, Brazil
[2]Centro de Ciências Exatas e Tecnológicas, Universidade Federal do Acre – UFAC, CEP: 69920–900, Rio Branco – AC, Brazil

Received 19 March 2026, appeared 13 August 2026
Communicated by Gennaro Infante

Abstract. In this paper, we study ordinary differential equations of the Kirchhoff type. These equations exhibit a nonlinearity that is monotonically increasing and depends on one positive parameter. Using a fixed-point theorem and the sub-supersolution method, we prove the existence of positive solutions. In some cases, we also prove the non-existence of positive solutions.

Keywords: fixed point, Kirchhoff's equations, positive solutions, sub-supersolution.

2020 Mathematics Subject Classification: 34B08, 34B18, 35A16.

1 Introduction

The hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \left(1 + \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = g, \qquad u(0,t) = u(L,t) = 0,$$

was introduced by Kirchhoff in [17] to describe the small transversal oscillations of an elastic clamped string which is not homogeneous. After this work, many researchers studied the existence of stationary solutions for this type of problem (for domains in dimension $N \geq 1$ as well), see [1–3, 5–9, 11–16, 18, 19] and the references therein. In these references, the authors used variational methods or topological methods to prove their results.

Motivated by the papers mentioned above, in this paper we deal with the existence and nonexistence of positive solutions of the following Kirchhoff type problems:

$$\begin{cases} -a\left(\int_0^1 (u'(s))^2\, ds\right) u''(t) = \lambda f(u(t)) & \text{in } (0,1), \\ u(0) = u(1) = 0, \\ u > 0 & \text{in } (0,1), \end{cases} \tag{P_λ}$$

✉Corresponding author. Email: ricardoalveslima8@gmail.com

where $0 < \lambda$ is a parameter, $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_0^+$ ($\mathbb{R}_0^+ = (0, \infty)$) are continuous functions. Specifically, we introduce the following assumptions:

(a_1) $a(t) \geq a_0$ for all $t \geq 0$ and some constant $a_0 > 0$;

(a_2) $a(t) > 0$ for all $t \in [0, 1), a(1) = 0$, $a(t) < 0$ for all $t > 1$ and a is decreasing for $t \geq 0$;

(f_1) f is monotone increasing, $f(t) > 0$ for all $t \in \mathbb{R}$ and $f(t) > t$ for all $t \geq 0$.

We will work in the Sobolev space $X := H_0^1(0, 1)$ with scalar product and norm given by

$$(u, v) = \int_0^1 u'(t)v'(t)\, dt, \quad \|u\| = \left(\int_0^1 (u'(t))^2\, dt \right)^{1/2}.$$

We say that a function $u \in X$ is a solution of (P_λ) if $u > 0$ in $(0, 1)$ and

$$-a(\|u\|^2) \int_0^1 u'(t)\varphi'(t)\, dt = \lambda \int_0^1 f(u(t))\varphi(t)\, dt,$$

for all $\varphi \in X$.

Now, our main results in this paper can be stated as follows.

Theorem 1.1. *Assume that (a_1) and (f_1) hold. Then there exists $\Lambda_{a_0}^* > 0$ such that Problem (P_λ) has at least one solution for all $0 < \lambda < \Lambda_{a_0}^*$.*

Theorem 1.2. *Assume that (a_1) and (f_1) hold. Moreover, assume that there exists a constant $\beta > 0$ such that $a(t) \leq \beta$ for all $t \geq 0$. Then there exists $\Lambda_\beta^* > 0$ such that Problem (P_λ) has at least one solution for $0 < \lambda < \Lambda_{a_0}^*$ and has no solution for $\lambda > \Lambda_\beta^*$. Here, $\Lambda_{a_0}^*$ is given by Theorem 1.1.*

Theorem 1.3. *Assume that (a_2) and (f_1) hold. Then there exists $\Lambda > 0$ such that Problem (P_λ) has at least one solution for $0 < \lambda \leq \Lambda$ and has no solution for $\lambda > \Lambda$.*

To prove our results we will use the sub-supersolution method and a fixed-point theorem that can be found in [10]. Theorem 1.1 presents a local result; that is, it shows the existence of solutions for sufficiently small parameters. However, assumptions (a_1) and (f_1) are not sufficient to show the non-existence of solutions, which leads us to add the boundedness assumption on the function a and obtain both existence and non-existence of solutions in Theorem 1.2. In the presence of assumptions (a_2) and (f_1), we obtain a global result; that is, we find the parameter $\Lambda > 0$ such that Problem (P_λ) admits a solution if and only if $\lambda \in (0, \Lambda]$. To prove Theorem 1.3, we develop a new theorem of supersolutions and work on a new convex set $\mathcal{C}$ (see Theorem 2.7) to apply Theorem 7.2.11 of [10]. To prove the preliminary results of Theorem 1.3, we must be careful to work within the unit ball B_1 of X, since there is no solution to Problem (P_λ) outside this ball (see Lemma 5.1). After a bibliographic review, we did not find results equal to ours for problems of type (P_λ) under assumptions (a_1), (a_2) and (f_1), and therefore our results are new in the literature.

Now, we will give examples of functions that satisfy the assumptions of our main results.

a) The function $a(t) = 1 + t$ satisfies assumption (a_1).

b) The function $a(t) = 1 + \frac{t}{1+|t|}$ satisfies the assumption of Theorems 1.1 and 1.2.

c) The function $a(t) = 1 - t$ satisfies assumption (a_2).

d) The function $f(t) = e^t$ satisfies assumption (f_1).

Notation. Throughout this paper, we make use of the following notations:

- $L^q(0,1)$, for $1 \le q \le \infty$, denotes the Lebesgue space with usual norm denoted by $\|u\|_q$.
- $X = H_0^1(0,1)$ denotes the Sobolev space endowed with inner product

$$(u,v) = \int_0^1 u'(t)v'(t)\,dt, \qquad \forall u,v \in X.$$

  The norm associated with this inner product will be denoted by $\|\ \ \|$.

- Let us consider the space $C([0,1],\mathbb{R}) = \{u : [0,1] \to \mathbb{R} : u \text{ is continuous in } [0,1]\}$ equipped with the norm $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$.
- $B_R \subset X$ denotes the ball centered at $0 \in X$ with radius $R > 0$.
- $k, c, c_1, c_2, \ldots$ and C denote (possibly different from line to line) positive constants.
- The arrow $\rightharpoonup$ (respectively, $\to$) denotes weak (respectively, strong) convergence.

2 Preliminaries

In this section we will give some results that we will need throughout the paper. Throughout this section we will assume that (a_1) and (f_1) hold. Let us start with the following theorems.

Theorem 2.1 ([10, Theorem 7.2.11])**.** *If E is a normed space, $K \subset E$ is a nonempty, convex (not necessarily closed) set and $\mathcal{T} : K \to K$ is a continuous map, such that*

$$\overline{\mathcal{T}(K)} \subset E \ \text{ is compact},$$

then $\mathcal{T}$ has at least one fixed point.

Theorem 2.2 (Comparison principle)**.** *Assume that $u, v \in C^2([0,1],\mathbb{R})$ satisfy*

$$\begin{cases} -u''(t) \le -v''(t) & \text{in } (0,1), \\ u(0) = v(0) = u(1) = v(1) = 0, \end{cases}$$

then $u(t) \le v(t)$ for all $t \in [0,1]$.

Proof. The function $u - v$ assumes its maximum at a point $t_M \in (0,1)$. If $0 \le t \le t_M$, then

$$0 \le \int_t^{t_M} -(v-u)''(s)\,ds = (v-u)'(t),$$

that is, $v - u$ is increasing for $t \in [0, t_M]$. Since $(v-u)(0) \le (v-u)(t)$ for $t \in [0, t_M]$, we have $u(t) \le v(t)$ for $t \in [0, t_M]$.

If $t_M \le t \le 1$, then

$$0 \le \int_{t_M}^{t} -(v-u)''(s)\,ds = -(v-u)'(t),$$

namely, $v - u$ is decreasing for $t \in [t_M, 1]$. Thus, $(v-u)(1) \le (v-u)(t)$ which means that $u(t) \le v(t)$ for $t \in [t_M, 1]$. In conclusion we find that $u(t) \le v(t)$ for $t \in [0,1]$, and this completes the proof. □

From [4, Theorem 8.8] we know that the embedding $X \hookrightarrow C([0,1],\mathbb{R})$ is compact. Then, given $u \in X$, we apply the Lax–Milgram theorem with the bilinear form $b(w,v) = \int_0^1 w'v'\,dt$ and the linear functional $\varphi : X \to \mathbb{R}$ given by $\varphi(v) = \frac{\lambda}{a(\|u\|^2)}\int_0^1 f(u)v\,dt$, to obtain a unique function $T_\lambda(u) \in X$ satisfying

$$b(T_\lambda(u),v) = \frac{\lambda}{a(\|u\|^2)}\int_0^1 f(u)v\,dt, \quad \text{for all } v \in X.$$

Since $\frac{\lambda}{a(\|u\|^2)}f(u) \in C([0,1],\mathbb{R})$, by regularity theory (see [4, Proposition 8.16]), one has $T_\lambda(u) \in C^2([0,1],\mathbb{R}) \cap C^1([0,1],\mathbb{R})$ and that $T_\lambda(u)$ is a classical solution as well.

Therefore, the operator $T_\lambda : X \to X$, where $T_\lambda(u)$ is the unique weak solution of

$$\begin{cases} -(T_\lambda(u))''(t) = \frac{\lambda}{a(\|u\|^2)}f(u(t)) & \text{in } (0,1), \\ T_\lambda(u)(0) = T_\lambda(u)u(1) = 0, \end{cases} \tag{S_λ}$$

is well defined. From this it follows that

$$T_\lambda(u)(t) = \frac{\lambda}{a(\|u\|^2)}\int_0^1 G(t,s)f(u(s))\,ds, \quad \text{for all } t \in (0,1),$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s < t \le 1, \\ t(1-s), & 0 \le t \le s \le 1 \end{cases}$$

is the Green's function in dimension 1. In particular, it holds that $T_\lambda(u)(t) > 0$ for all $t \in (0,1)$.

According to the previous arguments, (P_λ) can be translated into the operator equation $u = T_\lambda(u)$, with $u \in X$.

We will now highlight some more properties of the operator T_λ in the following lemmas.

Lemma 2.3. *Assume that (a_1) and (f_1) hold. Then T_λ is continuous.*

Proof. Let $u_n \in X$ be such that $u_n \to u$ in X. Then $\|u_n\|^2 \to \|u\|^2$, and consequently

$$\frac{\lambda}{a(\|u_n\|^2)} \to \frac{\lambda}{a(\|u\|^2)}.$$

On the other hand, by using the compact embedding of X into $C([0,1],\mathbb{R})$, we deduce that, up to a subsequence, u_n converges strongly to u in $C([0,1],\mathbb{R})$. This implies that $u_n(t) \to u(t)$ for all $t \in [0,1]$ and that there exists a constant $c > 0$ such that $\|u_n\|_\infty < c$ for all n. Hence,

$$\left[\frac{\lambda f(u_n(t)))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 \to 0 \quad \text{in } [0,1]$$

and

$$\begin{aligned}\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 &\le \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} + \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 \\ &\le \left[\frac{\lambda f(c)}{a_0} + \frac{\lambda f(c)}{a_0}\right]^2 \\ &= \frac{4\lambda^2 f^2(c)}{a_0^2}.\end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\frac{\lambda f(u_n)}{a(\|u_n\|^2)} \to \frac{\lambda f(u)}{a(\|u\|^2)} \quad \text{in } L^2(0,1). \tag{2.1}$$

Now, taking $T_\lambda(u_n) - T_\lambda(u)$ as a test function in (S_λ) and using Hölder's inequality and the Sobolev embedding theorem we also get

$$\begin{aligned}\|T_\lambda(u_n) - T_\lambda(u)\|^2 &= \int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right][T_\lambda(u_n)(t) - T_\lambda(u)(t)]\,dt \\ &\le C\left(\int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 dt\right)^{1/2} \|T_\lambda(u_n) - T_\lambda(u)\|,\end{aligned}$$

which, by (2.1), implies that $T_\lambda(u_n) \to T_\lambda(u)$ in X. Therefore, T_λ is continuous. This completes the proof. □

Lemma 2.4. *Assume that* (a_1) *and* (f_1) *hold. Then* T_λ *is a compact operator.*

Proof. Let $\{u_n\} \subset X$ be a bounded sequence. We are going to show that $\{T_\lambda(u_n)\}$ has a convergent subsequence. By the Sobolev embedding theorem, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in X$ and, so $u_n \to u$ in $C([0,1],\mathbb{R})$. Moreover, we can assume that $\|u_n\|^2 \to \sigma \in [0,\infty)$. Let $\tilde{u} \in X$ be the unique solution of the problem:

$$\begin{cases} -\tilde{u}''(t) = \dfrac{\lambda f(u(t))}{a(\sigma)} & \text{in } (0,1), \\ \tilde{u}(0) = \tilde{u}(1) = 0, & \\ \tilde{u} > 0 & \text{in } (0,1). \end{cases} \tag{2.2}$$

Let us show that $T_\lambda(u_n) \to \tilde{u}$ in X. In order to do this, observe that

$$\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\sigma)}\right]^2 \to 0 \quad \text{in } [0,1]$$

and

$$\begin{aligned}\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\sigma)}\right]^2 &\le \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} + \frac{\lambda f(u(t))}{a(\sigma)}\right]^2 \\ &\le \left[\frac{\lambda f(c)}{a_0} + \frac{\lambda f(c)}{a(\sigma)}\right]^2,\end{aligned}$$

where $\max\{\|u_n\|, \|u\|\} < c$ for all $n \in \mathbb{N}$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\frac{\lambda f(u_n)}{a(\|u_n\|^2)} \to \frac{\lambda f(u)}{a(\sigma)} \quad \text{in } L^2(0,1). \tag{2.3}$$

On the other hand, using Hölder's inequality and the Sobolev embedding theorem

$$\begin{aligned}\|T_\lambda(u_n) - \tilde{u}\|^2 &\le \int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\sigma)}\right][T_\lambda(u_n)(t) - \tilde{u}(t)]\,dt \\ &\le C\left(\int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\sigma)}\right]^2 dt\right)^{1/2} \|T_\lambda(u_n) - \tilde{u}\|,\end{aligned}$$

and this together with (2.3) implies that $T_\lambda(u_n) \to \tilde{u}$ in X. This completes the proof. □

We will now establish our definition of subsolution and supersolution for Problem (P_λ).

Definition 2.5. We say that $\underline{u}$ is a subsolution of Problem (P_λ) if $\underline{u} \in C^2([0,1],\mathbb{R}), \underline{u}(t) > 0$ for $t \in (0,1)$, and

$$\begin{cases} -a(\|u\|^2)\underline{u}''(t) \leq \lambda f(\underline{u}(t)) & \text{in } (0,1), \\ \underline{u}(0) = \underline{u}(1) = 0. \end{cases}$$

Similarly, $\overline{u} \in C^2([0,1],\mathbb{R}), \overline{u}(t) > 0$ for $t \in (0,1)$, is a supersolution of (P_λ) if

$$\begin{cases} -a(\|u\|^2)\overline{u}''(t) \geq \lambda f(\overline{u}(t)) & \text{in } (0,1), \\ \overline{u}(0) = \overline{u}(1) = 0. \end{cases}$$

Now, we can prove the following result.

Lemma 2.6. *Assume that (a_1) and (f_1) hold. Let $\underline{u}$ and $\overline{u}$ be, respectively, a subsolution and a supersolution of Problem (P_λ) such that $\underline{u}(t) \leq \overline{u}(t)$ for all $t \in [0,1]$. Let $K = \{u \in X : \underline{u} \leqslant u \leqslant \overline{u} \text{ in } (0,1)\}$. Then the set $\overline{T_\lambda(K)}$ is compact in X.*

Proof. We will first show that for all $w \in \overline{T_\lambda(K)}$, we have:

$$|w'(t)| \leq \frac{\lambda}{a_0}\|e_1\|_\infty f(\|\overline{u}\|_\infty) \tag{2.4}$$

and

$$|w'(t) - w'(s)| \leq \frac{\lambda}{a_0} f(\|\overline{u}\|_\infty)|t-s| \quad \text{for all } t,s \in [0,1], \tag{2.5}$$

where $e_1(t) = \int_0^1 G(t,s)\, ds$.

Indeed, if $w \in \overline{T_\lambda(K)}$, then there exists $\{u_n\} \subset K$ such that $T_\lambda(u_n) \to w$ in X. Since $\int_0^1 (T'_\lambda(u_n)(t) - w'(t))^2\, dt \to 0$, one gets $T'_\lambda(u_n) \to w'$ in $[0,1]$. Also $T_\lambda(u_n) \to w$ in $C([0,1],\mathbb{R})$, and thus

$$\begin{aligned} |w(t)| = \lim_{n\to\infty} |T_\lambda(u_n)(t)| &= \lim_{n\to\infty} \left| \frac{\lambda}{a(\|u_n\|)} \int_0^1 G(t,s) f(u_n(s))\, ds \right| \\ &\leq \frac{\lambda e_1(t)}{a_0} f(\|\overline{u}\|_\infty) \\ &\leq \frac{\lambda}{a_0}\|e_1\|_\infty f(\|\overline{u}\|_\infty), \end{aligned}$$

which shows (2.4).

We now take $\bar{t}_n \in [0,1]$ such that $(T_\lambda(u_n))'(\bar{t}_n) = 0$. Then, for all $t \in [0,1]$,

$$\begin{aligned} (T_\lambda(u_n))'(t) &= \int_t^{\bar{t}_n} -(T_\lambda(u_n))''(\tau)\, d\tau \\ &= \frac{\lambda}{a(\|u_n\|^2)} \int_t^{\bar{t}_n} f(u_n(\tau))\, d\tau, \end{aligned}$$

and consequently,

$$\begin{aligned} |(T_\lambda(u_n))'(t) - (T_\lambda(u_n))'(s)| &= \frac{\lambda}{a(\|u_n\|^2)} \left| \int_t^s f(u_n(\tau))\, d\tau \right| \\ &\leq \frac{\lambda}{a_0} f(\|\overline{u}\|_\infty)|t-s| \end{aligned}$$

for all $s, t \in [0,1]$. Therefore,

$$|w'(t)| = \lim_{n\to\infty} |(T_\lambda(u_n))'(t)| = \lim_{n\to\infty} \frac{\lambda}{a(\|u_n\|^2)} \left| \int_t^s f(u_n(\tau))\, d\tau \right| \le \frac{\lambda}{a_0} f(\|\overline{u}\|_\infty)$$

and

$$|w'(t) - w'(s)| = \lim_{n\to\infty} |(T_\lambda(u_n))'(t) - (T_\lambda(u_n))'(s)| \le \frac{\lambda}{a_0} f(\|\overline{u}\|_\infty)|t-s|,$$

for all $s, t \in [0,1]$, which shows (2.4)–(2.5).

Let $\{w_n\} \subset \overline{T_\lambda(K)}$. From (2.4)–(2.5) it follows that the sequence $\{w_n'\}$ is bounded and equicontinuous in $C([0,1], \mathbb{R})$, and by the Ascoli–Arzelà theorem, there exists $w \in C([0,1], \mathbb{R})$ such that $w_n' \to w$ in $C([0,1], \mathbb{R})$. In particular, one has also that $w_n' \to w$ in $L^2(0,1)$. Moreover, up to a subsequence, there exists $z \in X$ such that $w_n \rightharpoonup z$ in X and $w_n \to z$ in $C([0,1], \mathbb{R})$. Since the embedding of $L^2(0,1)$ into $L^1(0,1)$ is continuous and $w_n' \to w$ in $L^2(0,1)$, one deduces that

$$z(t) = \lim_{n\to\infty} w_n(t) = \lim_{n\to\infty} \int_0^t w_n'(\tau)\, d\tau = \int_0^t w(\tau)\, d\tau,$$

namely $w(t) = z'(t)$ for all $t \in [0,1]$. Therefore, we conclude that $w_n \to z$ in X. This shows that $\overline{T_\lambda(K)}$ is compact in X, completing the proof. □

Now, we will consider the problem

$$\begin{cases} -\alpha u''(t) = \lambda f(u(t)) & \text{in } (0,1), \\ u(0) = u(1) = 0, & \\ u > 0 & \text{in } (0,1), \end{cases} \tag{P_λ^α}$$

where $\alpha > 0$ is a constant. In this case, the operator T_λ takes the form

$$T_\lambda(u)(t) = \frac{\lambda}{\alpha} \int_0^1 G(t,s) f(u(s))\, ds, \quad \text{for all } t \in (0,1),$$

and satisfies

$$\begin{cases} -\alpha(T_\lambda(u))''(t) = \lambda f(u(t)) & \text{in } (0,1), \\ T_\lambda(u)(0) = T_\lambda(u)(1) = 0, & \\ T_\lambda(u) > 0 & \text{in } (0,1). \end{cases}$$

We start with the following result.

Theorem 2.7 (Sub-supersolution)**.** *Assume that (f_1) holds. Let $\underline{u}$ and $\overline{u}$ be, respectively, a subsolution and a supersolution of Problem (P_λ^α) such that $\underline{u}(t) \le \overline{u}(t)$ for all $t \in [0,1]$. Then there exists a solution $u \in X$ of (P_λ^α) such that $\underline{u}(t) \le u(t) \le \overline{u}(t)$ for all $t \in [0,1]$.*

Proof. We set $K = \{u \in X : \underline{u} \leqslant u \leqslant \overline{u} \quad \text{in } (0,1)\}$. We have that K is a closed convex set of X. From Lemma 2.4 the set $\overline{T_\lambda(K)}$ is compact in X. Now, we are going to show that $T_\lambda(K) \subset K$. Indeed, for all $u \in K$, we obtain

$$-\alpha(T_\lambda(u))''(t) = \lambda f(u(t)) \le \lambda f(\overline{u}(t)) \le -\alpha \overline{u}''(t)$$

and

$$-\alpha \underline{u}''(t) \le \lambda f(\underline{u}(t)) \le \lambda f(u(t)) = -\alpha(T_\lambda(u))''(t),$$

and by Theorem 2.2 we arrive at $\underline{u}(t) \le u(t) \le \overline{u}(t)$ for all $t \in (0,1)$. This shows that $T_\lambda(K) \subset K$.

Finally, applying Theorem 2.1, we deduce the existence of $u \in K$ such that $u = T_\lambda(u)$; that is, u is a solution of (P_λ^α). The proof of the theorem is complete. □

For $R > 0$ let
$$\overline{B}_R = \{u \in X : \|u\| \leq R\}$$
be the closed ball in X with center at the origin and radius R. Recall that $\overline{B}_R$ is a closed and convex subset of the Banach space X. Now we have the following lemma.

Lemma 2.8. *Assume that (f_1) holds. For any $R > 0$ there exists $\tilde{\lambda} > 0$ such that $T_\lambda(\overline{B}_R) \subset \overline{B}_R$ for all $0 < \lambda \leq \tilde{\lambda}$. In particular, Problem (P_λ^α) has a solution for all $0 < \lambda \leq \tilde{\lambda}$. Here, $\tilde{\lambda}$ depends on R.*

Proof. We are going to use Theorem 2.1 with $K = \overline{B}_R$. For this purpose, let $u \in \overline{B}_R$ and $\bar{t} \in (0,1)$ such that $(T_\lambda(u))'(\bar{t}) = 0$. Integrating (P_λ^α) it follows that
$$(T_\lambda(u))'(t) = \int_t^{\bar{t}} -(T_\lambda(u))''(\tau)\, d\tau = \int_t^{\bar{t}} \frac{\lambda}{\alpha} f(u(\tau))\, d\tau.$$
Using the embedding $X \hookrightarrow C([0,1])$ there exists a constant $c > 0$ such that $\|u\|_\infty \leq c\|u\|$ for all $u \in X$. Thus,
$$|(T_\lambda(u))'(t)| \leq \frac{\lambda}{\alpha} f(\|u\|_\infty)|t - \bar{t}| \leq \frac{\lambda}{\alpha} f(cR)$$
for all $u \in \overline{B}_R$, which implies that
$$\|T_\lambda(u)\| \leq \frac{\lambda}{\alpha} f(cR),$$
for all $u \in \overline{B}_R$. As a consequence, we deduce that $\|T_\lambda(u)\| \leq R$ for all $u \in \overline{B}_R$ and for every $0 < \lambda \leq \tilde{\lambda}$, where $\tilde{\lambda} = R\alpha(f(cR))^{-1}$. Namely, $T_\lambda(\overline{B}_R) \subset \overline{B}_R$ and, since T_λ is compact the set $\overline{T_\lambda(\overline{B}_R)}$ is compact. So, we can apply Theorem 2.1 to deduce the existence of $u \in \overline{B}_R$ such that $T_\lambda(u) = u$, that is, u is a solution of (P_λ^α). This completes the proof. □

Let $\lambda_1 > 0$ be the first eigenvalue of
$$\begin{cases} -u''(t) = \lambda u(t) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.6}$$
and $\varphi_1(t) > 0$, for all $t \in (0,1)$, the first eigenfunction normalized in $C([0,1], \mathbb{R})$. Now we have the following result.

Lemma 2.9. *Assume that (f_1) holds. Then Problem (P_λ^α) has no solution for $\lambda > \alpha\lambda_1$.*

Proof. Assume that Problem (P_λ^α) has a solution u. Thus, taking as a test function in the weak formulation of (P_λ^α) φ_1 and using (f_1), we get
$$\alpha\lambda_1 \int_0^1 u\varphi_1\, dt = \alpha \int_0^1 u'\varphi_1'\, dt = \lambda \int_0^1 f(u)\varphi_1 dt > \lambda \int_0^1 u\varphi_1\, dt,$$
which implies that $\alpha\lambda_1 > \lambda$, and the proof is complete. □

We set
$$\Lambda_\alpha^* = \sup \mathcal{L}_\alpha,$$
where
$$\mathcal{L}_\alpha = \{\lambda > 0 : \text{such that Problem } (P_\lambda^\alpha) \text{ admits a solution}\}.$$
We conclude this section with the following theorem.

Theorem 2.10. *Assume that* (f_1) *holds. Then Problem* (P^{α}_{λ}) *has at least one solution for* $0 < \lambda < \Lambda^*_{\alpha}$ *and has no solution for* $\lambda > \Lambda^*_{\alpha}$.

Proof. Let $0 < \lambda < \Lambda_{\alpha}$. Then there exists μ, with $\lambda < \mu < \Lambda_{\alpha}$, such that (P^{α}_{μ}) has a solution u_{μ}, which is a supersolution for (P^{α}_{λ}) for all $\lambda < \mu$. Since $\underline{u} = 0$ is a subsolution of (P^{α}_{λ}) with $\underline{u} \le u_{\mu}$, from Theorem 2.7, we find a solution of (P^{α}_{λ}). The proof of the theorem is complete. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1. According to Theorem 2.10, if $0 < \lambda < \Lambda^*_{a_0}$, then $(P^{a_0}_{\lambda})$ has a solution $\overline{u}$ such that

$$\lambda f(\overline{u}(t)) = -a_0\overline{u}''(t) \le -a(\|\overline{u}\|^2)\overline{u}''(t);$$

that is, $\overline{u}$ is a supersolution of (P_{λ}). Let us remark that $\underline{u} = 0$ is a subsolution of (P_{λ}) with $\underline{u} \le \overline{u}$ in $[0,1]$. We set $K = \{u \in X : \underline{u} \leqslant u \leqslant \overline{u} \quad \text{in } (0,1)\}$. For $u \in K$ one has

$$-a_0(T_{\lambda}(u)(t))''(t) \le -a(\|u\|^2)(T_{\lambda}(u)(t))''(t) = \lambda f(u(t)) \le \lambda f(\overline{u}(t)) = -a_0\overline{u}''(t).$$

Repeating the arguments carried out in Theorem 2.7 it follows that $T_{\lambda}(K) \subset K$ and by Lemma 2.6 $\overline{T_{\lambda}(K)}$ is compact. Thus, from Theorem 2.1 there exists $u \in K$ such that $T_{\lambda}(u) = u$, and u is a solution of (P_{λ}). This completes the proof. □

4 Proof of Theorem 1.2

We will need the following lemma.

Lemma 4.1. *Assume that* (a_1) *and* (f_1) *hold. Moreover, assume that there exists a constant* $\beta > 0$ *such that* $a(s) \le \beta$ *for all* $s \ge 0$. *Then Problem* (P_{λ}) *has no solution for* $\lambda > \Lambda^*_{\beta}$.

Proof. Let Λ^*_{β} be given by Theorem 2.10. For contradiction, let us suppose that there exists $\lambda > \Lambda^*_{\beta}$ such that (P_{λ}) has a solution $\overline{u}$. Then,

$$\lambda f(\overline{u}(t)) = -a(\|\overline{u}\|^2)\overline{u}''(t) \le -\beta\overline{u}''(t);$$

that is, $\overline{u}$ is a supersolution of (P^{β}_{λ}). On the other hand, $\underline{u} = 0$ is a subsolution of (P^{β}_{λ}) with $\underline{u} \le \overline{u}$, and so, by Theorem 2.7, (P^{β}_{λ}) has a solution. We deduce then that $0 < \lambda < \Lambda^*_{\beta}$, a contradiction. This completes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\Lambda := \Lambda^*_{\beta}$ be defined as in Lemma 4.1. For any fixed $0 < \lambda < \Lambda^*_{a_0}$, there exists a solution $\overline{u}$ of $(P^{a_0}_{\lambda})$. We define the set $K = \{u \in X : 0 \leqslant u \leqslant \overline{u} \quad \text{in } (0,1)\}$. For $u \in K$ one deduces that

$$\begin{aligned}
0 \le T_{\lambda}(u)(t) &= \frac{\lambda}{a(\|u\|^2)} \int_0^1 G(t,s) f(u(s))\, ds \\
&\le \frac{\lambda}{a(\|u\|^2)} \int_0^1 G(t,s) f(\overline{u}(s))\, ds \\
&\le \frac{\lambda}{a_0} \int_0^1 G(t,s) f(\overline{u}(s))\, ds \\
&= \overline{u}(t),
\end{aligned}$$

for all $t \in (0,1)$. In other words, $T_\lambda(K) \subset K$.

Since

$$f(\overline{u}(t)) = -a_0\overline{u}''(t) \leq a(\|\overline{u}\|^2)\overline{u}''(t),$$

from Lemma 2.6 it follows that $\overline{T_\lambda(K)}$ is compact. Thus, from Theorem 2.1 there exists $u \in K$ such that $T_\lambda(u) = u$; that is, u is a solution of (P_λ).

Finally, by Lemma 4.1 we deduce that Problem (P_λ) has no solution for $\lambda > \Lambda^*_\beta$, and this completes the proof. □

5 Proof of Theorem 1.3

Throughout this section, we will assume (a_2) and (f_1). First, we will prove the following estimates.

Lemma 5.1. *Assume that (a_2) and (f_1) hold. Moreover, assume that u is a solution of (P_λ). Then $\int_0^1 (u'(s))^2\, ds < 1$ and $0 < \lambda < M\lambda_1$. Here, λ_1 is the first eigenvalue of (2.6) and $M = \max_{s\in\mathbb{R}} a(s) > 0$.*

Proof. Indeed,

$$a(\|u\|^2)\int_0^1 (u'(t))^2\, dt = \lambda \int_0^1 f(u(t))u(t)\, dt > 0,$$

and using (a_2) one deduces that $\displaystyle\int_0^1 (u'(t))^2\, dt < 1$.

Now, using φ_1 as test function in (P_λ) we obtain

$$\begin{aligned} a(\|u\|^2)\lambda_1 \int_0^1 u(t)\varphi_1(t)\, dt &= a(\|u\|^2)\int_0^1 u'(t)\varphi_1'(t)\, dt \\ &= \lambda \int_0^1 f(u(t))\varphi_1(t)\, dt \\ &> \lambda \int_0^1 u(t)\varphi_1(t)\, dt, \end{aligned}$$

which implies

$$M\lambda_1 \int_0^1 u(t)\varphi_1(t)\, dt > \lambda \int_0^1 u(t)\varphi_1(t)\, dt,$$

and we conclude that $0 < \lambda < M\lambda_1$. The proof is complete. □

Now, take $B_R = \{u \in X : \|u\| < R\}$ for $R > 0$ and consider the operator $S_\lambda : B_1 \to X$ given by

$$\begin{cases} -(S_\lambda(u))''(t) = \frac{\lambda f(u(t))}{a(\|u\|^2)} & \text{in } (0,1), \\ u(0) = u(1) = 0, & \\ u > 0 & \text{in } (0,1). \end{cases} \tag{5.1}$$

Note that the operator S_λ is not well-defined on $\partial B_1 = \{u \in X : \|u\| = 1\}$ and that a changes sign. Therefore, the arguments used in the previous sections do not work here and new arguments must be introduced.

First, we have the following lemma.

Lemma 5.2. *Assume that* (a_2) *and* (f_1) *hold. The following properties are satisfied:*

a) $S_\lambda|_{\overline{B}_R}$ *is continuous for* $0 < R < 1$.

b) $S_\lambda|_{\overline{B}_R}$ *is compact for* $0 < R < 1$.

c) there exists $\lambda_{**} > 0$ *such that* $S_\lambda(\overline{B}_{1/2}) \subset \overline{B}_{1/2}$ *for all* $0 < \lambda < \lambda_{**}$*. Here,* λ_{**} *depends on* m, λ_1 *and* f.

Proof. $a)$ Let $\{u_n\} \subset \overline{B}_R$ be such that $u_n \to u \in \overline{B}_R$ in X. Then $\|u_n\|^2 \to \|u\|^2$, and consequently

$$\frac{\lambda}{a(\|u_n\|^2)} \to \frac{\lambda}{a(\|u\|^2)}.$$

On the other hand, by using the embedding of X into $C([0,1],\mathbb{R})$, we deduce that u_n converges strongly to u in $C([0,1],\mathbb{R})$. This implies that $u_n(t) \to u(t)$ for all $t \in [0,1]$ and there exists a constant $c > 0$ such that $\|u_n\|_\infty < c$ for all n. Hence,

$$\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 \to 0 \quad \text{in } [0,1]$$

and

$$\begin{aligned}\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 &\le \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} + \frac{\lambda f(u(t))}{a(\|u\|^2)}\right]^2 \\ &\le \left[\frac{\lambda f(c)}{m} + \frac{\lambda f(c)}{m}\right]^2 \\ &= \frac{4\lambda^2 f^2(c)}{m^2},\end{aligned}$$

where $m = \inf_{0\le s\le R} a(s) > 0$.

Thus, by the Lebesgue dominated convergence theorem one finds

$$\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} \to \frac{\lambda f(u(t))}{a(\|u\|^2)} \quad \text{in } L^2(0,1). \tag{5.2}$$

Now, taking $S_\lambda(u_n) - S_\lambda(u)$ as a test function in (P_λ) and using Hölder's inequality and the Sobolev embedding theorem we also get

$$\begin{aligned}\|S_\lambda(u_n) - S_\lambda(u)\|^2 &\le \int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\right][S_\lambda(u_n)(t) - S_\lambda(u)(t)]\, dt \\ &\le C\left(\int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\|u\|^2)}\, dt\right]^2\right)^{1/2} \|S_\lambda(u_n) - S_\lambda(u)\|,\end{aligned}$$

which, by (5.2), implies that $S_\lambda(u_n) \to S_\lambda(u)$ in X. Therefore, $S_\lambda|_{\overline{B}_R}$ is continuous for $0 < R < 1$.

$b)$ Let $\{u_n\} \subset B_{1/2}$ be a bounded sequence. We are going to show that $\{S_\lambda(u_n)\}$ has a convergent subsequence. By the Sobolev embedding theorem, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in X$ and, so $u_n \to u$ in $C([0,1],\mathbb{R})$. Moreover, we can assume that $\|u_n\|^2 \to \theta \in [0, \frac{1}{4}]$. Let $\tilde{u} \in X$ be the unique solution of the problem:

$$\begin{cases} -\tilde{u}''(t) = \dfrac{\lambda f(u(t))}{a(\theta)} & \text{in } (0,1), \\ u(0) = u(1) = 0, & \\ u > 0 & \text{in } (0,1). \end{cases} \tag{5.3}$$

Let us show that $S_\lambda(u_n) \to \tilde{u}$ in X. In order to do this, observe that

$$\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\theta)}\right]^2 \to 0 \quad \text{in } [0,1]$$

and

$$\begin{aligned}\left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\theta)}\right]^2 &\leq \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} + \frac{\lambda f(u(t))}{a(\theta)}\right]^2 \\ &\leq \left[\frac{\lambda f(c)}{m} + \frac{\lambda f(c)}{a(\theta)}\right]^2,\end{aligned}$$

where $m = \inf_{0\leq s\leq 1/4} a(s) > 0$ and $\max\{\|u_n\|, \|u\|\} < c$ for all $n \in \mathbb{N}$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} \to \frac{\lambda f(u(t))}{a(\theta)} \quad \text{in } L^2(0,1). \tag{5.4}$$

On the other hand, using Hölder's inequality and the Sobolev embedding theorem

$$\begin{aligned}\|S_\lambda(u_n) - \tilde{u}\|^2 &\leq \int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\theta)}\right] [S_\lambda(u_n)(t) - \tilde{u}(t)]\ dt \\ &\leq C\left(\int_0^1 \left[\frac{\lambda f(u_n(t))}{a(\|u_n\|^2)} - \frac{\lambda f(u(t))}{a(\theta)}\right]^2 dt\right)^{1/2} \|S_\lambda(u_n) - \tilde{u}\|,\end{aligned}$$

and this together with (5.4) implies that $S_\lambda(u_n) \to \tilde{u}$ in X.

c) If $u \in \overline{B}_{1/2}$, then, by the embedding $X \hookrightarrow C([0,1],\mathbb{R})$, there exists a constant $c > 0$ such that

$$\|u\|_\infty \leq 2c\|u\| \leq \frac{2c}{2} = c.$$

Let $m = \inf\limits_{0\leq s\leq 1/4} a(s) > 0$. Since

$$\begin{aligned}\|S_\lambda(u)\|^2 &= \frac{\lambda}{a(\|u\|^2)} \int_0^1 f(u(t))S_\lambda(u)(t)\ dt \\ &\leq \frac{\lambda}{m} \int_0^1 f(u(t))S_\lambda(u)(t)\ dt \\ &\leq \frac{\lambda}{m} \left(\int_0^1 f^2(u(t))\ dt\right)^{1/2} \left(\int_0^1 S_\lambda^2(u(t))\ dt\right)^{1/2} \\ &\leq \frac{\lambda f(c)}{m\sqrt{\lambda_1}}\|S_\lambda(u)\|,\end{aligned}$$

it follows that $\|S_\lambda(u)\| \leq \frac{1}{2}$ provided $\frac{\lambda f(c)}{m\sqrt{\lambda_1}} \leq \frac{1}{2}$. So choosing $\lambda_{**} = \frac{m\sqrt{\lambda_1}}{2f(c)}$, we deduce that $S_\lambda(\overline{B}_{1/2}) \subset \overline{B}_{1/2}$ for all $0 < \lambda < \lambda_{**}$. This completes the proof. $\square$

We set

$$\Lambda = \sup \mathcal{L},$$

where

$$\mathcal{L} = \{\lambda > 0 : \text{such that Problem } (P_\lambda) \text{ admits a solution}\}.$$

With the help of Lemma 5.2, we will prove in the next result that $\mathcal{L} \neq \varnothing$.

Lemma 5.3. *Assume that* (a_2) *and* (f_1) *hold. It holds that* $(0, \lambda_{**}) \subset \mathcal{L}$. *In particular, one has* $\mathcal{L} \neq \varnothing$.

Proof. Let $K = \overline{B}_{1/2}$. Then K is nonempty, closed, bounded and convex. From Lemma 5.2 we have that $S_\lambda(K) \subset K$, for $0 < \lambda < \lambda_{**}$, and that $\overline{S_\lambda(K)}$ is compact. Thus, we can apply Theorem 2.1 to obtain a $u \in X$ such that $u = S_\lambda(u)$; that is, u is a solution of (P_λ) for $\lambda \in (0, \lambda_{**})$. The proof is complete. □

We will now present a supersolution theorem in the presence of assumption (a_2).

Theorem 5.4 (Supersolution)**.** *Assume that* (a_2) *and* (f_1) *hold. If* (P_λ) *has a supersolution* $\overline{u}$, *then it has a solution* u *with* $0 \leq u \leq \overline{u}$ *in* $[0, 1]$.

Proof. Let

$$K = \left\{ u \in X : 0 \leq u(t) \leq \overline{u}(t) \text{ for all } t \in [0,1] \text{ and } \int_0^1 (u'(t))^2 \, dt \leq \int_0^1 (\overline{u}'(t))^2 \, dt \right\}.$$

It is worthwhile to observe that, in view of

$$0 < \lambda f(\overline{u}(t)) \leq -a(\|\overline{u}\|^2)\overline{u}''(t) \quad \text{in } (0,1),$$

one finds that $\|\overline{u}\| < 1$.

The proof of the theorem will be divided into two steps.

Step 1. The set K is nonempty, closed, bounded and convex.

First, observe that $\overline{u} \in K$ and, so $K \neq \varnothing$. Now we are going to show that K is closed. Indeed, if $\{u_n\} \subset K$ and $u_n \to u$ in X, one finds $u_n \to u$ in $[0, 1]$,

$$0 \leq u(t) = \lim_{n\to\infty} u_n(t) \leq \overline{u}(t) \ \text{ and } \ \int_0^1 (u'(t))^2 \, dt = \lim_{n\to\infty} \lim \int_0^1 (u_n'(t))^2 \, dt \leq \int_0^1 (\overline{u}'(t))^2 \, dt.$$

That is, K is closed. Moreover, it is also clear that $\|u\| \leq \|\overline{u}\|$ for all $u \in K$, which means that K is bounded.

It remains to show that K is convex. Let $\alpha \in [0, 1]$ and $u, w \in K$. Obviously one has that $(1 - \alpha)u + \alpha w \leq \overline{u}$ in $[0, 1]$. On the other hand, since the function $g(x) = x^2$ is convex, one gets

$$\begin{aligned} \int_0^1 ((1-\alpha)u'(t) + \alpha w'(t))^2 \, dt &\leq (1-\alpha) \int_0^1 (u'(t))^2 \, dt + \alpha \int_0^1 (w'(t))^2 \, dt \\ &\leq \int_0^1 (\overline{u}'(t))^2 \, dt. \end{aligned}$$

In other words, $(1 - \alpha)u + \alpha w \in K$ and K is convex.

Step 2. $S_\lambda(K) \subset K$.

Let $u \in K$. Since, from (a_2),

$$a(\|\overline{u}\|^2) \leq a(\|u\|^2),$$

then, by (f_1) one has

$$-(S_\lambda(u))''(t) = \frac{\lambda f(u(t))}{a(\|u\|^2)} \leq \frac{\lambda f(\overline{u}(t))}{a(\|\overline{u}\|^2)} \leq -\overline{u}''(t) \quad \text{in } (0,1).$$

Hence, by the comparison principle one finds $S_\lambda(u) \le \overline{u}$ in $[0,1]$. Moreover, this inequality implies that

$$\begin{aligned}\|S_\lambda(u)\|^2 &\le \int_0^1 \overline{u}'(t)S_\lambda'(u)(t)\, dt \\ &\le \left(\int_0^1 (\overline{u}'(t))^2\, dt\right)^{1/2} \left(\int_0^1 (S_\lambda'(u)(t))^2\, dt\right)^{1/2},\end{aligned}$$

and thus $\int_0^1 (S_\lambda'(u)(t))^2\, dt \le \int_0^1 (\overline{u}'(t))^2\, dt$ holds. This completes the proof of step 2.

Finally, since $\overline{S_\lambda(K)}$ is compact, by Theorem 2.1 there exists $u \in K$ such that $u = S_\lambda(u)$; that is, u is a solution of (P_λ). The proof of the theorem is complete. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $0 < \lambda < \Lambda$. Then there exists μ, with $\lambda < \mu$, such that (P_μ) has a solution u_μ, which is a supersolution for (P_λ) for all $\lambda < \mu$. According to Theorem 5.4, Problem (P_λ) has a solution u_λ with $0 < u_\lambda < u_\mu$ in $[0,1]$. Thus, Problem (P_λ) has a solution for all $0 < \lambda < \Lambda$.

It remains to show that Problem (P_λ) has a solution for $\lambda = \Lambda$. Let $\{\lambda_n\} \subset \mathcal{L}$ be an increasing sequence such that $\lambda_n \uparrow \Lambda$ and let u_n be a solution of (P_λ) for $\lambda = \lambda_n$. Then, from Lemma 5.1 $\|u_n\| < 1$, and, up to a subsequence, $u_n \rightharpoonup u$ in X, for some $u \in X$, $u_n \to u$ in $C([0,1], \mathbb{R})$ and $\|u_n\|^2 \to \theta$. Moreover, setting $\sup\{\|u_n\|\} = c$, it follows that

$$f(u_n(t))\varphi(t) \to f(u(t))\varphi(t) \quad \text{in } [0,1] \quad \text{and} \quad |f(u_n(t))\varphi(t)| \le f(c)|\varphi(t)| \quad \text{in } [0,1],$$

for all $\varphi \in X$.

Therefore, by the Lebesgue dominated convergence theorem we yield

$$\begin{aligned}a(\theta)\int_0^1 u'(t)\varphi'(t)\, dt &= \lim_{n\to\infty} a(\|u_n\|^2)\int_0^1 u_n'(t)\varphi'(t)\, dt \\ &= \lim_{n\to\infty} \lambda_n \int_0^1 f(u_n(t))\varphi(t)\, dt \\ &= \Lambda \int_0^1 f(u(t))\varphi(t)\, dt,\end{aligned}$$

for all $\varphi \in X$. In particular, since $f(t) > 0$ for $t \in \mathbb{R}$, one deduces that $0 \le \theta < 1$ and $u(t) > 0$ for all $t \in (0,1)$.

Because u_n is a weak solution of (P_{λ_n}), we have

$$\begin{aligned}\limsup_{n\to\infty} \int_0^1 u_n'(u_n'(t) - u'(t))\, dt &= \limsup_{n\to\infty} \frac{\lambda_n}{a(\|u_n\|^2)} \int_0^1 f(u_n(t))(u_n(t) - u(t))\, dt \\ &= 0,\end{aligned}$$

and this implies $u_n \to u$ in X. As a consequence we get

$$\begin{aligned}a(\|u\|^2)\int_0^1 u'(t)\varphi'(t)dt &= \lim_{n\to\infty} a(\|u_n\|^2)\int_0^1 u_n'(t)\varphi'(t)\, dt \\ &= \lim_{n\to\infty} \lambda_n \int_0^1 f(u_n(t))\varphi(t)\, dt \\ &= \Lambda \int_0^1 f(u(t))\varphi(t)\, dt,\end{aligned}$$

for all $\varphi \in X$. In other words, u is a solution for (P_Λ).

Finally, by the definition of Λ, Problem (P_λ) has no solution for $\lambda > \Lambda$. This completes the proof of Theorem 1.3. $\square$

Acknowledgements

We would like to thank the anonymous referee for his/her careful readings of our manuscript and the useful comments.

Ricardo Lima Alves was partially supported by Universidade Federal do Cariri (UFCA), Brazil.

Síglia Souza Oliveira was partially supported by Universidade Federal do Acre (UFAC), PIVIC/UFAC Edital 016/2024.

References

[1] A. Ambrosetti, D. Arcoya, Positive solutions of elliptic Kirchhoff equations, *Adv. Nonlinear Stud.* **17**(2017), No. 1, 3–15. https://doi.org/10.1515/ans-2016-6004; MR3604942.

[2] C. O. Alves, F. J. S. A. Corrêa, A sub-supersolution approach for a quasilinear Kirchhoff equation, *J. Math. Phys.* **56**(2015), No. 5, 051501, 12 pp. https://doi.org/10.1063/1.4919670; MR3390975; Zbl 1318.35040.

[3] S. Boulaaras, Y. Bouizem, R. Guefaifia, Further results of existence of positive solutions of elliptic Kirchhoff equation with general nonlinearity of source terms, *Math. Methods Appl. Sci.* **43**(2020), No. 15, 9195–9205. https://doi.org/10.1002/mma.6613; MR4151400; Zbl 1454.35132.

[4] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.

[5] N. T. Chung, An existence result for a class of Kirchhoff-type systems via sub- and supersolutions method, *Appl. Math. Lett.* **35**(2014), 95–101. https://doi.org/10.1016/j.aml.2013.11.005; MR3212854; Zbl 1318.35024.

[6] F. J. S. A. Corrêa, S. D. B. Menezes, J. Ferreira, On a class of problems involving a nonlocal operator, *Appl. Math. Comput.* **147**(2004), No. 2, 475–489. https://doi.org/10.1016/S0096-3003(02)00740-3; MR2012587.

[7] G. M. Figueiredo, A. Suárez, Some remarks on the comparison principle in Kirchhoff equations, *Rev. Mat. Iberoam.* **34**(2018), No. 2, 609–620. https://doi.org/10.4171/RMI/997; Zbl 1400.35104.

[8] G. M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.* **401**(2013), No. 2, 706–713. https://doi.org/10.1016/j.jmaa.2012.12.053; MR3018020; Zbl 1307.35110.

[9] J. García-Melián, L. Iturriaga, Some counterexamples related to the stationary Kirchhoff equation, *Proc. Amer. Math. Soc.* **144**(2016), No. 8, 3405–3411. https://doi.org/10.1090/proc/12971; Zbl 1341.35030.

[10] L. Gasiński, N. S. Papageorgiou, *Nonlinear analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[11] C. S. Goodrich, A one-dimensional Kirchhoff equation with generalized convolution coefficients, *J. Fixed Point Theory Appl.* **23**(2021), No. 4, Paper No. 73, 23 pp. https://doi.org/10.1007/s11784-021-00910-z; MR4336000; Zbl 1504.34052.

[12] C. S. Goodrich, Nonlocal differential equations with convolution coefficients and applications to fractional calculus, *Adv. Nonlinear Stud.* **21**(2021), No. 4, 767–787. https://doi.org/10.1515/ans-2021-2145; Zbl 1504.34051.

[13] C. S. Goodrich, Nonexistence and parameter range estimates for convolution differential equations, *Proc. Amer. Math. Soc. Ser. B* **9**(2022), No. 24, 254–265. https://doi.org/10.1090/bproc/130; Zbl 1517.34034.

[14] C. S. Goodrich, Nonexistence of nontrivial solutions to Kirchhoff-like equations, *Proc. Amer. Math. Soc. Ser. B* **11**(2024), No. 28, 304–314. https://doi.org/10.1090/bproc/224; Zbl 1554.34033.

[15] G. Infante, Nonzero positive solutions of nonlocal elliptic systems with functional BCs, *J. Elliptic Parabol. Equ.* **5**(2019), No. 2, 493–505. https://doi.org/10.1007/s41808-019-00049-6; Zbl 1433.35072.

[16] G. Infante, Nontrivial solutions of systems of perturbed Hammerstein integral equations with functional terms, *Mathematics* **9**(2021), No. 4, Paper No. 330, 9 pp. https://doi.org/10.3390/math9040330.

[17] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[18] Q. Song, X. Hao, Positive solutions for nonlocal differential equations with concave and convex coefficients, *Positivity* **28**(2024), No. 5, Paper No. 68, 25 pp. https://doi.org/10.1007/s11117-024-01086-9; Zbl 1556.34035.

[19] Y. Wang, W. Wei, Optimal control problem governed by a kind of Kirchhoff-type equation, *Chaos Solitons Fractals* **187**(2024), Paper No. 115422. https://doi.org/10.1016/j.chaos.2024.115422.