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Qualitative analysis and iterative approximation of the Laplace equation with exponential Robin boundary conditions

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Abstract. This paper investigates an elliptic boundary value problem involving the Laplace equation on the unit disk in two-dimensional space, subject to a nonlinear exponential Robin boundary condition on a portion of the boundary. This model is physically motivated by the study of corrosion localized on metallic structures, where the flux law exhibits an exponential dependence on the electrochemical potential.

We establish the existence and uniqueness of the solution in an appropriate Hilbert space under suitable smallness assumptions on the boundary data. Our mathematical approach is constructive: we define an iterative sequence of linearized problems and prove its strong convergence toward the solution of the original nonlinear system. A key technical contribution of this work is the derivation of explicit Sobolev embedding constants for the periodic setting on the unit disk, providing refined estimates necessary to control the exponential nonlinearity.

Keywords: Laplace equation, nonlinear Robin boundary condition, exponential nonlinearity, existence and uniqueness of solution, Sobolev spaces, iterative method.

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1 Introduction

We study an elliptic problem posed on the open unit disk $\Omega = D(0,1) \subset \mathbb{R}^2$, subject to a nonlinear exponential boundary condition prescribed on a portion of the boundary. The problem is formulated as

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = \phi(x), & \text{on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi(x)\left(e^{\alpha u} - e^{-(1-\alpha)u}\right) = g(x), & \text{on } \Gamma_R. \end{cases} \tag{S}$$

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The boundary $\Gamma = \partial\Omega = S(0,1)$ is decomposed into three open subsets Γ_D, Γ_N, and Γ_R, up to a negligible set, satisfying

$$\begin{cases} \Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_R}, \\ \Gamma_D \cap \Gamma_N = \Gamma_D \cap \Gamma_R = \Gamma_N \cap \Gamma_R = \emptyset, \\ \sigma(\Gamma_D) > 0,\ \sigma(\Gamma_N) > 0,\ \sigma(\Gamma_R) > 0, \end{cases} \tag{$H1$}$$

where σ denotes the length on $\partial\Omega$.

The data of the problem are assumed to satisfy

$$\phi \in L^2(\Gamma_N), \quad \phi \not\equiv 0, \quad g \in L^2(\Gamma_R), \quad \varphi \in C^0(\overline{\Gamma_R}), \quad \alpha \in (0,1).$$

It is well known that elliptic problems with Robin boundary conditions arise in the modeling of various physical and electrochemical phenomena. In particular, such models can describe the damage caused by corrosion on a portion of the boundary of a metallic structure.

The mathematical model considered here, described by the system of equations (S), was introduced by Vogelius et al. [10, 12] in the context of corrosion. In this framework, the corrosion effect on the boundary is represented through a nonlinear flux law exhibiting an exponential dependence on the electrochemical potential. More precisely, the nonlinear boundary condition is given by

$$\frac{\partial u}{\partial n} + \varphi(x)\left(e^{\alpha u} - e^{-(1-\alpha)u}\right) = g(x) \quad \text{on } \Gamma_R.$$

A first approximation of problem (S) is obtained by linearizing this nonlinear boundary condition on Γ_R wich leads to the linear Robin condition

$$\frac{\partial u}{\partial n} + \varphi(x)u = g(x) \quad \text{on } \Gamma_R.$$

The linear version of this problem has been extensively studied in the literature. In particular, S. Chaabane et al. [5, 6] established existence and uniqueness results by applying the Lax–Milgram theorem to the variational formulation associated with the problem. Furthermore, numerical results were provided for both the direct problem and the inverse problem of identifying the Robin coefficient. In the case where the boundary condition on Γ_R is quadratic, specifically:

$$\frac{\partial u}{\partial n} + \varphi(x)u + \psi(x)u^2 = g(x) \quad \text{on } \Gamma_R,$$

J. Benameur et al. [3] established the existence of solutions in an appropriate Hilbert space V. Uniqueness was proved within a closed ball of V. These results were obtained under suitable assumptions on the functions φ and ψ. The proofs rely on an iterative method based on the construction of a sequence of linear boundary value problems, which converges to the solution of the original problem with the quadratic nonlinear boundary condition. A generalization of this study was recently proposed by C. Elhechmi [8]. The author investigated an elliptic partial differential equation subject to a polynomial Robin boundary condition, established existence and uniqueness results for the associated solutions, and developed an iterative scheme for their construction.

In this work, we investigate problem (S), which is characterized by an exponential-type boundary condition on Γ_R. To establish existence and uniqueness results for solutions to problem (S), we employ an iterative scheme by constructing a sequence of linear problems $(S_k)_{k\in\mathbb{N}}$ designed to approximate (S).

We prove that each linearized problem (S_k) admits a unique solution u_k, and that the sequence (u_k) converges to the unique solution of the original nonlinear problem (S). The analysis relies on the application of suitable Sobolev embedding results in the periodic setting, together with the development of specific technical estimates. It should be noted that the uniqueness result is local, established within a specific closed ball of the Hilbert space V.

For $\alpha \in (0,1)$, we introduce the function f_α defined by

$$f_\alpha(r) = \begin{cases} \dfrac{e^{\alpha r} - e^{-(1-\alpha)r}}{r} & \text{if } r \neq 0, \\ 1 & \text{if } r = 0, \end{cases}$$

then, problem (S) can be rewritten as

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = \phi(x), & \text{on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi(x) f_\alpha(u) u = g(x), & \text{on } \Gamma_R. \end{cases} \tag{S}$$

Remark 1.1. The function f_α satisfies the following properties that will play a key role in the analysis of problem (S):

(P1) $\forall\, r \in \mathbb{R}: \quad f_\alpha(r) > 0.$

(P2) $\forall\, r \in \mathbb{R}: \quad 0 \leq f_\alpha(r) r^2 \leq 2 \sum_{k=1}^{\infty} \dfrac{\delta^k}{k!} |r|^{k+1},$ where $\delta = \max(\alpha, 1-\alpha)$.

(P3) $\forall\, r \in \mathbb{R} \setminus \{0\}: \; 0 \; < f_\alpha(r) \leq 2\delta(1 + \delta |r| e^{\delta |r|}).$

The proofs of properties **(P1)** and **(P2)** are obtained by direct calculation, while the proof of **(P3)** is given in Appendix A.

The paper is organized as follows. Section 2 presents several preliminary results. In Section 3, we state the main result and introduce the sequence of linear problems $(S_k)_{k\in\mathbb{N}}$, for which we prove existence and uniqueness of solutions. In Subsection 3.2, we analyze the convergence of the sequence of solutions (u_k) to a limit u in the Hilbert space V. Subsections 3.3 and 3.4 are devoted to proving the existence and uniqueness of a solution to problem (S), respectively. Finally, Section 4 provides explicit estimates of the Sobolev embedding constants in the periodic setting.

2 Notations and preliminary results

In this section, we recall several functional spaces and operators that will be used throughout the paper. The trace operator is defined by

$$\tau_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega), \qquad u \mapsto u_{|\partial\Omega}.$$

This operator is linear and continuous from $(H^1(\Omega), \|\cdot\|_{H^1})$ into $(L^2(\partial\Omega), \|\cdot\|_{L^2})$. The range of τ_0 is the space $H^{1/2}(\partial\Omega)$, which is endowed with the norm

$$\|\psi\|_{H^{1/2}(\partial\Omega)} = \inf\Big\{\|u\|_{H^1(\Omega)}\ ;\ u \in H^1(\Omega),\ u_{|\partial\Omega} = \psi\Big\}.$$

Accordingly, we define the operator

$$\tilde{\tau}_0 : (H^1(\Omega), \|\cdot\|_{H^1}) \longrightarrow (H^{1/2}(\partial\Omega), \|\cdot\|_{H^{1/2}}), \qquad u \mapsto u_{|\partial\Omega}.$$

Clearly, $\tilde{\tau}_0$ is continuous and satisfies $\|\tilde{\tau}_0\| \leq 1$. Using this definition of the space $H^{1/2}(\partial\Omega)$, we obtain the following continuous embedding.

Lemma 2.1 ([7]). *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitzian domain. Then, for $p \in [2, +\infty)$ we have*

$$H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega).$$

Precisely, there is a constant $C_{\Omega,p} > 0$, depending only on p and domain Ω, such that

$$\|u\|_{L^p(\partial\Omega)} \leq C_{\Omega,p}\|u\|_{H^{1/2}(\partial\Omega)}, \qquad \forall u \in H^1(\Omega). \tag{2.1}$$

and

$$\|f\|_{L^p(\partial\Omega)} \leq C_{\Omega,p}\|f\|_{H^1(\Omega)}, \qquad \forall f \in H^1(\Omega). \tag{2.2}$$

In the present study, we extend the right-hand side of equation (2.1) to the fractional Sobolev space $H^s(\Gamma)$ for $s \in (0,1)$, where Γ is a nonempty open subset of $S(0,1)$, and derive a refined estimate for the constant $C_{\Omega,p}$. To obtain precise inequalities with explicit constants, we provide upper bounds for the embedding constant in the specific case where the domain is the unit disk, $\Omega = D(0,1)$. Specifically, we establish the following result.

Proposition 2.2. *Let $2 \leq p < \infty$. The embedding*

$$H^{1/2}(S(0,1)) \hookrightarrow L^p(S(0,1))$$

is continuous. In particular, for every $f \in H^{1/2}(S(0,1))$, we have

$$\|f\|_{L^p(S(0,1))} \leq \lambda_p\|f\|_{H^{1/2}(S(0,1))}, \tag{2.3}$$

where the embedding constant λ_p is given by

$$\lambda_p = \left(C_1\frac{p}{p-2}\,C_2^{p-2}\,p^{\frac{p-2}{2}}\right)^{\frac{1}{p}}, \qquad \forall p > 2, \tag{2.4}$$

and C_1, C_2 are universal constants.

Remark 2.3.

1. If $p \geq 3$, one may take
$$\lambda_p = \left(\mathcal{C}^{p-2}p^{\frac{p-2}{2}}\right)^{\frac{1}{p}},$$
where $\mathcal{C} = 2\max(C_1, C_2)$.

2. A detailed proof of Proposition 2.2 is provided in Appendix C.

3. The proof of Proposition 2.2 proceeds in two steps:

   (a) First, we prove the embedding

$$H^s(S(0,1)) \hookrightarrow L^p(S(0,1)), \quad \text{with} \quad \frac{1}{p} + s = \frac{1}{2}.$$

   (b) Next, we prove the inequality

$$\| \cdot \|_{H^s(S(0,1))} \leq \| \cdot \|_{H^{1/2}(S(0,1))}, \qquad \forall s \in (0, \tfrac{1}{2}).$$

Now we introduce the Sobolev space V adapted to our setting. Let

$$V = \{v \in H^1(\Omega) : \ v_{|\Gamma_D} = 0\}.$$

The space V is a Hilbert space when endowed with the inner product

$$\langle u, v \rangle_V = \int_\Omega \nabla u . \nabla v dx,$$

and the associated norm

$$\|u\|_V = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}.$$

Let τ denote the trace operator defined by

$$\begin{aligned} \tau : V &\to H^{1/2}(\Gamma_R) \\ u &\mapsto u_{|\Gamma_R}. \end{aligned}$$

The linear operator τ is continuous when $H^{1/2}(\Gamma_R)$ is equipped with the norm $\| \cdot \|_{L^p(\Gamma_R)}$ for $2 \leq p < \infty$. Moreover, we define the constant

$$\beta_p = \inf \Big\{ C > 0 \, ; \, \|u\|_{L^p(\Gamma_R)} \leq C \|u\|_V, \quad \forall u \in V \Big\} \in (0, \infty).$$

Remark 2.4. Proposition 2.2 plays a crucial role in the analysis of the nonlinear exponential boundary condition on Γ_R. Indeed, the embedding

$$H^{1/2}\big(S(0,1)\big) \hookrightarrow L^p\big(S(0,1)\big), \qquad 2 \leq p < \infty,$$

together with the continuity of the trace operator $\tau : V \to H^{1/2}(\Gamma_R)$, implies that for every $u \in V$ and every $p \geq 2$,

$$\|u\|_{L^p(\Gamma_R)} \leq \beta_p \, \|u\|_V,$$

where the constant β_p can be explicitly controlled in terms of λ_p(given in Proposition 2.2). This estimate allows us to control polynomial growth terms on the boundary. More importantly, it provides a fundamental tool for handling the exponential nonlinearity appearing in the Robin boundary condition: $\varphi(x)(e^{\alpha u} - e^{-(1-\alpha)u})$.

Dependence of the constant β_p on λ_p: Recall that, for $p \in [2, \infty)$, the constant β_p is defined by

$$\beta_p = \inf \Big\{ C > 0 \ : \ \|u\|_{L^p(\Gamma_R)} \leq C \|u\|_V, \ \forall u \in V \Big\}.$$

The continuity of the operator

$$\tau : V \longrightarrow H^{1/2}(\Gamma_R), \quad u \longmapsto u_{/\Gamma_R},$$

and

$$H^{1/2}(\Gamma_R) \hookrightarrow L^p(\Gamma_R),$$

imply that,

$$\|u\|_{L^p(\Gamma_R)} \leq \lambda_p \|u\|_{H^{1/2}(\Gamma_R)} \leq \lambda_p \|\tau\| \|u\|_V. \tag{2.5}$$

Therefore, the constant β_p satisfies the estimate

$$\beta_p \leq \lambda_p \mathcal{T}. \tag{2.6}$$

Here

- $\mathcal{T} = \|\tau\|$ denotes the norm of the trace operator $\tau : V \to H^{1/2}(\Gamma_R)$.
  By the definition of the $H^{1/2}$ norm, we have $\|\tau\| \leq 1$, and therefore

$$\beta_p \leq \lambda_p.$$

- λ_p is the embedding constant given explicitly in Proposition 2.2.

We now introduce the basic notation and assumptions used throughout the paper:

Assumptions 2.5.

- $M_0 = \beta_2(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)})$.

- We denote by Φ_{ad} the set of admissible coefficient:

$$\Phi_{ad} = \left\{ h \in C^0(\overline{\Gamma_R});\ \ 0 \leq h \leq \xi\Lambda \right\}, \tag{2.7}$$

  where the parameter $\xi \in (0,1)$, and

$$\Lambda = \left[\beta_3^3 M_0 + \beta_4^4 M_0^2 + (CM_0)^3 R(CM_0) \right]^{-1}. \tag{2.8}$$

  The constant C and the function R will be defined in the proof of Lemma 3.5.

- In all that follows, we assume that $\varphi \in \Phi_{ad}$.

3 Main result

Before stating the main result, we specify the notion of solution associated with problem (S).

Definition 3.1. A weak solution of problem (S) is a function $u \in V$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi(x) f_\alpha(u) u v \, d\sigma = \int_{\Gamma_N} \phi v \, d\sigma + \int_{\Gamma_R} g v \, d\sigma,$$

for every test function $v \in V$.

We are now ready to state the main result of this paper.

Theorem 3.2. *Let $\varphi \in \Phi_{ad}$ be defined as in (2.7), and M_0 as above. Then, problem (S) admits a solution $u \in V$. Moreover, this solution is unique in the closed ball*

$$\overline{B_V(0, M_0)} := \{v \in V \ : \ \|v\|_V \leq M_0\}.$$

The existence part of the proof is established by means of an iterative scheme (S_k) adapted to our nonlinear problem. At each step, we prove the existence and uniqueness of a solution u_k, together with a uniform estimate in the V-norm. We then show that the sequence $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space V. Finally, we prove that its limit

$$u = \lim_{k\to\infty} u_k$$

is a solution of the original problem (S).

3.1 Existence of a solution

To approximate the solution of (S), we introduce the following iterative sequence of linear problems. Let $u_0 = 0$, and for each $k \geq 1$, consider

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = \phi(x), & \text{on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi(x) f_\alpha(u_{k-1})u = g(x), & \text{on } \Gamma_R. \end{cases} \tag{S_k}$$

From [5], we have the following result.

Lemma 3.3. *The first linear problem (S_1) admits a unique solution $u_1 \in V$. Moreover, this solution satisfies the uniform bound*

$$\|u_1\|_V \leq M_0, \tag{3.1}$$

where

$$M_0 = \beta_2\Big(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)}\Big).$$

The next step is to extend this result by induction to all $k \geq 1$ and then to prove that the sequence (u_k) converges in V to the solution of problem (S).

In what follows, we state the main result of this section. More precisely, the following proposition establishes the well-posedness of the iterative scheme and the uniform boundedness of its solutions.

Proposition 3.4. *For each $k \in \mathbb{N}$, the linear problem (S_k) admits a unique solution $u_k \in V$. Moreover, the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in V, namely*

$$\|u_k\|_V \leq M_0, \qquad \forall k \in \mathbb{N}, \tag{3.2}$$

where M_0 is the constant defined in Lemma 3.3.

Proof. The proof relies on the variational formulation of problem (S_k) and the application of the Lax–Milgram theorem.

(i) Existence of u_k. The problem (S_k) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ B_{\alpha,k}(u,v) = L(v), \qquad \forall v \in V, \end{cases} \tag{$\tilde{S}_k$}$$

where

$$B_{\alpha,k}(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi(x) f_\alpha(u_{k-1}) \, uv \, d\sigma,$$

and

$$L(v) = \int_{\Gamma_N} \phi v \, d\sigma + \int_{\Gamma_R} gv \, d\sigma.$$

To apply the Lax–Milgram theorem, it suffices to show that the bilinear form $B_{\alpha,k}$ is coercive and continuous on $V \times V$, and that L is continuous on V.

(ii) Coercivity of $B_{\alpha,k}$. For $u \in V$ we have

$$B_{\alpha,k}(u,u) = \int_\Omega |\nabla u|^2 \, dx + \int_{\Gamma_R} \varphi(x) f_\alpha(u_{k-1}) u^2 \, d\sigma.$$

Since φ and f_α are non–negative functions, it follows that

$$B_{\alpha,k}(u,u) \geq \|u\|_V^2.$$

Hence $B_{\alpha,k}$ is coercive on V.

(iii) Continuity of $B_{\alpha,k}$. Let $(u,v) \in V^2$. Using the Cauchy-Schwarz inequality and the L^∞ bound on φ, we obtain

$$|B_{\alpha,k}(u,v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\varphi\|_{L^\infty(\Gamma_R)} \int_{\Gamma_R} f_\alpha(u_{k-1}) |u||v| \, d\sigma.$$

Using property **(P3)**, we obtain

$$|B_{\alpha,k}(u,v)| \leq \|u\|_V \|v\|_V + 2\delta \|\varphi\|_{L^\infty(\Gamma_R)} \int_{\Gamma_R} \left(1 + \delta |u_{k-1}| e^{\delta |u_{k-1}|}\right) |u||v| \, d\sigma.$$

Applying Hölder's inequality yields

$$\begin{aligned} |B_{\alpha,k}(u,v)| \leq \|u\|_V \|v\|_V + 2\delta \|\varphi\|_{L^\infty(\Gamma_R)} \Big[& \|u\|_{L^2(\Gamma_R)} \|v\|_{L^2(\Gamma_R)} \\ & + \|u\|_{L^3(\Gamma_R)} \|v\|_{L^3(\Gamma_R)} \Big(\int_{\Gamma_R} (\delta |u_{k-1}|)^3 e^{3\delta |u_{k-1}|} \, d\sigma \Big)^{1/3} \Big]. \end{aligned}$$

Let

$$I_k = \int_{\Gamma_R} |u_{k-1}|^3 e^{3\delta |u_{k-1}|} \, d\sigma.$$

Using the power series expansion of the exponential function, we obtain

$$I_k = \sum_{n=0}^{\infty} \frac{(3\delta)^n}{n!} \|u_{k-1}\|_{L^{3+n}(\Gamma_R)}^{3+n}.$$

Using the inequalities (2.5)–(2.6) together with the uniform bound $\|u_{k-1}\|_V \leq M_0$, we obtain

$$\|u_{k-1}\|_{L^{3+n}(\Gamma_R)}^{3+n} \leq \lambda_{3+n}^{3+n} \mathcal{T}^{3+n} M_0^{3+n}.$$

Since $\mathcal{T} \leq 1$, it follows that

$$I_k \leq \sum_{n=0}^{\infty} \frac{(3\delta)^n}{n!} \lambda_{3+n}^{3+n} M_0^{3+n}.$$

Using the explicit expression of λ_{3+n} we obtain

$$I_k \leq (CM_0)^3 \sum_{n=0}^{\infty} (3\delta CM_0)^n a_n,$$

where

$$a_n = \frac{(n+3)^{\frac{n+1}{2}}}{n!}.$$

Using Stirling's formula we deduce that the series converges, hence

$$I_k \leq r_0 < \infty.$$

Therefore,

$$|B_{\alpha,k}(u,v)| \leq \left(1 + 2\delta\xi\Lambda(\beta_2^2 + \beta_3^2 r_0^{1/3})\right) \|u\|_V \|v\|_V,$$

which proves the continuity of $B_{\alpha,k}$.

(iv) Continuity of L. For $v \in V$, using the Cauchy–Schwarz inequality and the trace estimate we obtain

$$\begin{aligned} |L(v)| &= \left| \int_{\Gamma_N} \phi v \, d\sigma + \int_{\Gamma_R} g v \, d\sigma \right| \\ &\leq \|\phi\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \|v\|_{L^2(\Gamma_R)} \\ &\leq \beta_2 \left(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \right) \|v\|_V. \end{aligned}$$

Thus L is continuous on V.

Conclusion. By the Lax–Milgram theorem, the variational problem ($\tilde{S}_k$) admits a unique solution $u_k \in V$. Consequently, the problem (S_k) has a unique solution $u_k \in V$.

Proof of estimate (3.2). Taking $v = u_k$ in the variational formulation gives

$$B_{\alpha,k}(u_k, u_k) = L(u_k).$$

Using the positivity of f_α and φ, we obtain

$$\|\nabla u_k\|_{L^2(\Omega)}^2 \leq \beta_2 \left(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \right) \|u_k\|_V.$$

Hence

$$\|u_k\|_V \leq M_0.$$

This completes the proof of Proposition 3.4. □

3.2 Convergence result

The purpose of this section is to prove that the sequence of solutions $(u_k)_{k\geq 1}$ of the problems $(S_k)_{k\geq 1}$ converges to a solution of the original problem (S).

For each $k \in \mathbb{N}$, let u_k be the unique solution of (S_k) given by Proposition 3.4, with the initial iteration $u_0 = 0$. Let $w_k := u_{k+1} - u_k$, where u_{k+1} and u_k are respectively the solutions of the problems (S_{k+1}) and (S_k). Then w_k satisfies

$$\begin{cases} \Delta w_k = 0, & \text{in } \Omega, \\ w_k = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial w_k}{\partial n} = 0, & \text{on } \Gamma_N, \\ \dfrac{\partial w_k}{\partial n} + \varphi(x)\left[f_\alpha(u_k)u_{k+1} - f_\alpha(u_{k-1})u_k\right] = 0, & \text{on } \Gamma_R. \end{cases}$$

Using Green's formula, we obtain

$$\int_\Omega \nabla w_k \nabla v - \int_{\Gamma_R} \frac{\partial w_k}{\partial n} v = 0 \qquad \forall v \in V.$$

Choosing $v = w_k$, it follows that

$$\|w_k\|_V^2 + \int_{\Gamma_R} \varphi(x)\left[f_\alpha(u_k)u_{k+1} - f_\alpha(u_{k-1})u_k\right] w_k = 0.$$

Replacing u_{k+1} by $w_k + u_k$, we obtain

$$\|w_k\|_V^2 + \int_{\Gamma_R} \varphi(x)\left[f_\alpha(u_k)w_k + (f_\alpha(u_k) - f_\alpha(u_{k-1}))u_k\right] w_k = 0.$$

Hence

$$\|w_k\|_V^2 + \int_{\Gamma_R} \varphi(x) f_\alpha(u_k) w_k^2 + \int_{\Gamma_R} \varphi(x)\left[f_\alpha(u_k) - f_\alpha(u_{k-1})\right] u_k w_k = 0.$$

Since $\varphi(x) \geq 0$ for all $x \in \Gamma_R$, we deduce

$$\|w_k\|_V^2 \leq \int_{\Gamma_R} \varphi(x)\left|f_\alpha(u_k) - f_\alpha(u_{k-1})\right| |u_k|\, |w_k|\, d\sigma. \tag{3.3}$$

The Taylor expansion of $f_\alpha(x)$ yields

$$f_\alpha(x) = \frac{1}{x}\left(\sum_{m=0}^{\infty} \frac{(\alpha x)^m}{m!} - \sum_{m=0}^{\infty} (-1)^m \frac{(1-\alpha)^m x^m}{m!}\right) = 1 + \sum_{m\geq 1} b_{m+1}(\alpha) x^m,$$

where

$$b_m(\alpha) = \frac{\alpha^m - (-1)^m (1-\alpha)^m}{m!}.$$

Consequently,

$$f_\alpha(u_k) - f_\alpha(u_{k-1}) = b_2(\alpha) w_{k-1} + b_3(\alpha)(u_k + u_{k-1}) w_{k-1} + \sum_{m\geq 3} b_{m+1}(\alpha)(u_k^m - u_{k-1}^m).$$

Using the elementary inequality

$$|x^m - y^m| \leq m|x-y|(|x|^{m-1} + |y|^{m-1}) \qquad \forall x, y \in \mathbb{R},\ \forall m \in \mathbb{N}, \tag{3.4}$$

we obtain

$$
\begin{aligned}
|f_\alpha(u_k) - f_\alpha(u_{k-1})| &\le |b_2(\alpha)||w_{k-1}| + |b_3(\alpha)|(|u_k| + |u_{k-1}|)|w_{k-1}| \\
&\quad + \sum_{m\ge 3} |b_{m+1}(\alpha)|\, m \left(|u_k|^{m-1} + |u_{k-1}|^{m-1}\right) |w_{k-1}|.
\end{aligned} \tag{3.5}
$$

Lemma 3.5 (Contraction estimate)**.** *There exists a constant $\mathcal{K} \in (0,1)$ such that the sequence (w_k) satisfies the contraction estimate*

$$
\|w_k\|_V \le \mathcal{K}\|w_{k-1}\|_V, \qquad \forall k \in \mathbb{N}.
$$

Proof. From inequalities (3.3) and (3.5), we deduce

$$
\begin{aligned}
\|w_k\|_V^2 &\le \|\varphi\|_{L^\infty(\Gamma_R)} \int_{\Gamma_R} |f_\alpha(u_k) - f_\alpha(u_{k-1})||w_k||u_k|\, d\sigma \\
&\le \|\varphi\|_{L^\infty(\Gamma_R)} \big(J_{k,1} + J_{k,2} + J_{k,3}\big),
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
J_{k,1} &= \int_{\Gamma_R} |b_2(\alpha)||w_{k-1}||w_k||u_k|\, d\sigma, \\
J_{k,2} &= \int_{\Gamma_R} |b_3(\alpha)||w_{k-1}|\big(|u_k| + |u_{k-1}|\big)|w_k||u_k|\, d\sigma, \\
J_{k,3} &= \sum_{m\ge 3} |b_{m+1}(\alpha)| m \int_{\Gamma_R} \big(|u_k|^{m-1} + |u_{k-1}|^{m-1}\big)|w_{k-1}||w_k||u_k|\, d\sigma.
\end{aligned}
$$

Since $\alpha \in (0,1)$, we have

$$
|b_m(\alpha)| = \left| \frac{\alpha^m - (-1)^m(1-\alpha)^m}{m!} \right| \le \frac{2}{m!}, \qquad \forall m \in \mathbb{N}.
$$

- **Estimation of $J_{k,1}$.**

Applying Hölder's inequality on Γ_R with exponents $(3,3,3)$, we obtain

$$
\begin{aligned}
J_{k,1} &\le |b_2(\alpha)|\|w_{k-1}\|_{L^3(\Gamma_R)}\|w_k\|_{L^3(\Gamma_R)}\|u_k\|_{L^3(\Gamma_R)} && (3.7)\\
&\le |b_2(\alpha)|\beta_3^3 M_0 \|w_{k-1}\|_V \|w_k\|_V && (3.8)\\
&\le \beta_3^3 M_0 \|w_{k-1}\|_V \|w_k\|_V, && (3.9)
\end{aligned}
$$

where we used the uniform bound $\|u_k\|_V \le M_0$ and the estimate $|b_2(\alpha)| \le 1$.

- **Estimation of $J_{k,2}$.**

Applying Hölder's inequality on Γ_R with exponents $(4,4,4,4)$ together with the trace embedding, we obtain

$$
\begin{aligned}
J_{k,2} &\le |b_3(\alpha)|\|w_{k-1}\|_{L^4(\Gamma_R)}\|w_k\|_{L^4(\Gamma_R)}(\|u_k\|_{L^4(\Gamma_R)} + \|u_{k-1}\|_{L^4(\Gamma_R)})\|u_k\|_{L^4(\Gamma_R)} && (3.10)\\
&\le 2|b_3(\alpha)|\beta_4^4 M_0^2 \|w_{k-1}\|_V \|w_k\|_V && (3.11)\\
&\le \beta_4^4 M_0^2 \|w_{k-1}\|_V \|w_k\|_V. && (3.12)
\end{aligned}
$$

- **Estimation of $J_{k,3}$.**

Applying Hölder's inequality on Γ_R with exponents $(m+2, m+2, m+2)$, we obtain

$$
\begin{aligned}
J_{k,3} &\le \sum_{m\ge 3} |b_{m+1}(\alpha)| m \Big(\int_{\Gamma_R} |u_k|^m |w_k||w_{k-1}| + \int_{\Gamma_R} |u_{k-1}|^{m-1} |u_k||w_k||w_{k-1}| \Big) \\
&\le \sum_{m\ge 3} |b_{m+1}(\alpha)| m \Big(\|u_k\|^m_{L^{m+2}(\Gamma_R)} \|w_{k-1}\|_{L^{m+2}(\Gamma_R)} \|w_k\|_{L^{m+2}(\Gamma_R)} \\
&\quad + \|u_{k-1}\|^{m-1}_{L^{m+2}(\Gamma_R)} \|u_k\|_{L^{m+2}(\Gamma_R)} \|w_{k-1}\|_{L^{m+2}(\Gamma_R)} \|w_k\|_{L^{m+2}(\Gamma_R)} \Big).
\end{aligned}
$$

Using the trace embedding and the uniform bound $\|u_k\|_V \le M_0$, we obtain

$$
\begin{aligned}
J_{k,3} &\le \sum_{m\ge 3} |b_{m+1}(\alpha)| m \big(\beta^{m+2}_{m+2} M_0^m + \beta^{m+2}_{m+2} M_0^m \big) \|w_{k-1}\|_V \|w_k\|_V \\
&\le \sum_{m\ge 3} 2 |b_{m+1}(\alpha)| m \beta^{m+2}_{m+2} M_0^m \|w_{k-1}\|_V \|w_k\|_V.
\end{aligned}
$$

Since

$$
|b_m(\alpha)| \le \frac{2}{m!}, \quad \mathcal{T} \le 1 \quad \text{and} \quad \beta_{m+2} \le \lambda_{m+2}
$$

then by using the inequalities (2.5)–(2.6), we obtain

$$
\begin{aligned}
J_{k,3} &\le \left(\sum_{m\ge 3} \frac{4}{(m+1)!} m \lambda^{m+2}_{m+2} M_0^m \mathcal{T}^m \right) \|w_{k-1}\|_V \|w_k\|_V \\
&\le \left(\sum_{m\ge 3} \frac{4}{(m+1)!} (m+2)^{\frac{m+2}{2}} (CM_0)^m \right) \|w_{k-1}\|_V \|w_k\|_V.
\end{aligned}
$$

Hence

$$
J_{k,3} \le (CM_0)^3 \left(\sum_{m\ge 3} q_m (CM_0)^{m-3} \right) \|w_{k-1}\|_V \|w_k\|_V, \tag{3.13}
$$

where

$$
q_m = \frac{4}{(m+1)!} (m+2)^{\frac{m+2}{2}}.
$$

Let R the entire function defined by (see Appendix B)

$$
R(z) = \sum_{m=3}^{\infty} q_m z^{m-3}, \qquad z \in \mathbb{C}.
$$

Thus

$$
J_{k,3} \le (CM_0)^3 R(CM_0) \|w_{k-1}\|_V \|w_k\|_V. \tag{3.14}
$$

Combining (3.6), (3.7), (3.10), and (3.14), we obtain

$$
\|w_k\|_V^2 \le \|\varphi\|_{L^\infty(\Gamma_R)} \big(\beta_3^3 M_0 + \beta_4^4 M_0^2 + (CM_0)^3 R(CM_0) \big) \|w_{k-1}\|_V \|w_k\|_V.
$$

Therefore

$$
\|w_k\|_V \le \|\varphi\|_{L^\infty(\Gamma_R)} \big(\beta_3^3 M_0 + \beta_4^4 M_0^2 + (CM_0)^3 R(CM_0) \big) \|w_{k-1}\|_V.
$$

By the definition of the admissible set Φ_{ad} in (2.7), we have

$$\|\varphi\|_{L^\infty(\Gamma_R)} < \xi\Lambda, \qquad \xi \in (0,1),$$

where Λ is defined in (2.8) by

$$\Lambda = \left(\beta_3^3 M_0 + \beta_4^4 M_0^2 + (CM_0)^3 R(CM_0)\right)^{-1}.$$

Consequently,

$$\|w_k\|_V \leq \xi \|w_{k-1}\|_V.$$

Setting $\mathcal{K} = \xi \in (0,1)$, we obtain

$$\|w_k\|_V \leq K\|w_{k-1}\|_V, \qquad k \in \mathbb{N}.$$

Thus the sequence $(w_k)_{k\in\mathbb{N}}$ is a contraction in V, which completes the proof of Lemma 3.5. $\square$

We now complete the proof of the convergence result. Since the sequence $(w_k)_{k\geq 1}$ satisfies a contraction estimate in the Banach space V, the series $\sum_{k\geq 1} w_k$ converges absolutely in V. We By writing

$$u_k = u_1 + \sum_{j=1}^{k-1} w_j,$$

we deduce that the sequence $(u_k)_{k\geq 1}$ is Cauchy in V. Consequently, $(u_k)_{k\geq 1}$ converges strongly in V to some limit $u \in V$.

3.3 Existence of a solution of problem (S)

In this section, we prove that the limit

$$u = \lim_{k\to\infty} u_k \in V,$$

obtained in the previous subsection, is a solution of problem (S).

Since $u_k \to u$ strongly in V, and each u_k solves the linear problem (S_k), we can pass to the limit inside the domain Ω and on the boundary parts Γ_D and Γ_N. More precisely, we have

$$\left(\forall k \geq 1: \ \Delta u_k = 0 \text{ in } \Omega\right) \Longrightarrow \Delta u = 0 \text{ in } \Omega,$$
$$\left(\forall k \geq 1: \ u_k = 0 \text{ on } \Gamma_D\right) \Longrightarrow u = 0 \text{ on } \Gamma_D,$$
$$\left(\forall k \geq 1: \ \frac{\partial u_k}{\partial n} = 0 \text{ on } \Gamma_N\right) \Longrightarrow \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N.$$

It remains to identify the limit in the nonlinear Robin boundary condition. To this end, it suffices to prove that

$$\lim_{k\to\infty} \|f_\alpha(u_{k-1})u_k - f_\alpha(u)u\|_{L^1(\Gamma_R)} = 0.$$

For $k \geq 1$ set $\tilde{w}_k = u_k - u$. Then

$$\begin{aligned}\|f_\alpha(u_{k-1})u_k - f_\alpha(u)u\|_{L^1(\Gamma_R)} &\leq \|f_\alpha(u_{k-1})\tilde{w}_k + (f_\alpha(u_{k-1}) - f_\alpha(u))u\|_{L^1(\Gamma_R)} \\ &\leq \underbrace{\|f_\alpha(u_{k-1})\tilde{w}_k\|_{L^1(\Gamma_R)}}_{X_k} + \underbrace{\|(f_\alpha(u_{k-1}) - f_\alpha(u))u\|_{L^1(\Gamma_R)}}_{Y_k}.\end{aligned}$$

- **Estimate of X_k.**

  Using the series expansion of f_α and Hölder's inequality, we obtain

$$
\begin{aligned}
X_k &\leq \int_{\Gamma_R} |\tilde{w}_k| + \sum_{m=2}^{\infty} |b_{m+1}(\alpha)| \int_{\Gamma_R} |u_{k-1}|^m |\tilde{w}_k| \\
&\leq (\sigma(\Gamma_R))^{1/2} \|\tilde{w}_k\|_{L^2(\Gamma_R)} + \sum_{m=2}^{\infty} |b_{m+1}(\alpha)| \|u_{k-1}\|^m_{L^{m+1}(\Gamma_R)} \|\tilde{w}_k\|_{L^{m+1}(\Gamma_R)} \\
&\leq (\sigma(\Gamma_R))^{1/2} \beta_2 \|\tilde{w}_k\|_V + \sum_{m=2}^{\infty} |b_{m+1}(\alpha)| \beta_{m+1}^{m+1} \|u_{k-1}\|_V^m \|\tilde{w}_k\|_V \\
&\leq \left((\sigma(\Gamma_R))^{1/2} \beta_2 + \sum_{m=2}^{\infty} |b_{m+1}(\alpha)| \lambda_{m+1}^{m+1} M_0^m \right) \|\tilde{w}_k\|_V.
\end{aligned}
$$

  Since the series

$$
\sum_{m \geq 2} |b_{m+1}(\alpha) \lambda_{m+1}^{m+1} M_0^m|
$$

  is convergent and $\|\tilde{w}_k\|_V \to 0$, we conclude that

$$
\lim_{k \to \infty} X_k = 0.
$$

- **Estimate of Y_k.**

  Combining the elementary inequality (3.4) with Hölder's inequality, we obtain

$$
\begin{aligned}
Y_k &\leq \sum_{m=2}^{\infty} |b_{m+1}(\alpha)| \int_{\Gamma_R} |u_{k-1}^m - u^m| |u| \\
&\leq \sum_{m=2}^{\infty} m |b_{m+1}(\alpha)| \int_{\Gamma_R} (|u_{k-1}|^{m-1} + |u|^{m-1}) |\tilde{w}_{k-1}| |u| \\
&\leq \sum_{m=2}^{\infty} m |b_{m+1}(\alpha)| \Big(\|u_{k-1}\|^{m-1}_{L^{m+1}(\Gamma_R)} \|\tilde{w}_{k-1}\|_{L^{m+1}(\Gamma_R)} \|u\|_{L^{m+1}(\Gamma_R)} \\
&\qquad + \|u\|^m_{L^{m+1}(\Gamma_R)} \|\tilde{w}_{k-1}\|_{L^{m+1}(\Gamma_R)} \Big).
\end{aligned}
$$

  Using the Sobolev embedding and the boundedness of $\|u_{k-1}\|_V$ and $\|u\|_V$, we obtain

$$
\begin{aligned}
Y_k &\leq \sum_{m=2}^{\infty} m |b_{m+1}(\alpha)| \lambda_{m+1}^{m+1} \left(\|u_{k-1}\|_V^{m-1} \|u\|_V + \|u\|_V^m \right) \|\tilde{w}_{k-1}\|_V \\
&\leq \left(\sum_{m=2}^{\infty} 2m |b_{m+1}(\alpha)| \lambda_{m+1}^{m+1} M_0^m \right) \|\tilde{w}_{k-1}\|_V.
\end{aligned}
$$

  Since the series

$$
\sum_{m \geq 2} 2m |b_{m+1}(\alpha)| \lambda_{m+1}^{m+1} M_0^m
$$

  is convergent and $\|\tilde{w}_{k-1}\|_V \to 0$, we deduce that

$$
\lim_{k \to \infty} Y_k = 0.
$$

Combining the estimates for X_k and Y_k, we obtain

$$\lim_{k\to\infty} \|f_\alpha(u_{k-1})u_k - f_\alpha(u)u\|_{L^1(\Gamma_R)} = 0.$$

Hence, passing to the limit in the nonlinear Robin boundary condition yields

$$\frac{\partial u}{\partial n} + \varphi(x) f_\alpha(u) u = g(x) \quad \text{on } \Gamma_R,$$

Finaly, since the iterative sequence $(u_k)_{k\geq 1}$ is defined as the sequence of weak solutions of the linearized problems (S_k):

$$\int_\Omega \nabla u_k \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi(x) f_\alpha(u_{k-1}) u_k v \, d\sigma = \int_{\Gamma_N} \phi v \, d\sigma + \int_{\Gamma_R} g v \, d\sigma, \qquad \forall v \in V.$$

Passing to the limit in the variational formulation of (S_k), we obtain that the limit $u \in V$ satisfies

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi(x) f_\alpha(u) u v \, d\sigma = \int_{\Gamma_N} \phi v \, d\sigma + \int_{\Gamma_R} g v \, d\sigma, \qquad \forall v \in V.$$

Hence, by Definition 3.1, u is a weak solution of problem (S), which shows that u is a solution of problem (S).

3.4 Uniqueness of the solution in $\overline{B_V(0, M_0)}$

Let $v_1, v_2 \in V$ such that

$$\|v_1\|_V, \|v_2\|_V \leq M_0$$

be two solutions of problem (S).

Set $w = v_1 - v_2$. Using Green's formula, we obtain

$$\|w\|_V^2 \leq \int_{\Gamma_R} \varphi(x) |f_\alpha(v_1)v_1 - f_\alpha(v_2)v_2| \, |w| \leq \Lambda \int_{\Gamma_R} |f_\alpha(v_1)v_1 - f_\alpha(v_2)v_2| \, |w|.$$

Using the decomposition

$$f_\alpha(v_1)v_1 - f_\alpha(v_2)v_2 = f_\alpha(v_1)w + (f_\alpha(v_1) - f_\alpha(v_2))v_2,$$

we obtain

$$\|w\|_V \leq \Lambda(I + J),$$

where

$$I = \int_{\Gamma_R} |f_\alpha(v_1)| \, |w|^2, \qquad J = \int_{\Gamma_R} |f_\alpha(v_1) - f_\alpha(v_2)| \, |v_2| \, |w|.$$

Applying the same estimates used for the convergence of (u_k), we obtain the existence of a constant $\tilde{K} < 1$ such that

$$\|w\|_V^2 \leq \tilde{K} \|w\|_V^2.$$

Hence $w = 0$, which implies $v_1 = v_2$.

This completes the proof of Theorem 3.2.

4 Appendix

4.1 Appendix A. Proof of property (P3)

For any $r \in \mathbb{R} \setminus \{0\}$, we can write $f_\alpha(r)$ as the series

$$\begin{aligned}
0 \le f_\alpha(r) &\le \sum_{n=1}^{\infty} \frac{|\alpha^n - (-1)^n(1-\alpha)^n|}{n!}|r|^{n-1} \\
&\le \sum_{n=1}^{\infty} \frac{2\delta^n}{n!}|r|^{n-1} \\
&\le 2\delta\left(1 + \sum_{n=2}^{\infty} \frac{(\delta|r|)^{n-1}}{n!}\right) \\
&\le 2\delta\left(1 + \delta|r| \sum_{n=2}^{\infty} \frac{(\delta|r|)^{n-2}}{n!}\right) \\
&\le 2\delta\left(1 + \delta|r| \sum_{n=2}^{\infty} \frac{(\delta|r|)^{n-2}}{(n-2)!}\right) \\
&\le 2\delta(1 + \delta|r|e^{\delta|r|}).
\end{aligned}$$

This completes the proof.

4.2 Appendix B

To prove that the function

$$R(z) = \sum_{m=3}^{\infty} q_m z^{m-3}, \qquad q_m = \frac{4}{(m+1)!}(m+2)^{\frac{m+2}{2}},$$

is entire, it suffices to show that its radius of convergence is infinite.

Using the ratio test, we compute

$$\frac{q_{m+1}}{q_m} = \frac{(m+3)^{\frac{m+3}{2}}}{(m+2)(m+2)^{\frac{m+2}{2}}} = \frac{\sqrt{m+3}}{m+2}\left(\frac{m+3}{m+2}\right)^{\frac{m+2}{2}}.$$

Since

$$\left(\frac{m+3}{m+2}\right)^{\frac{m+2}{2}} \longrightarrow e^{1/2}.$$

We have

$$\frac{q_{m+1}}{q_m} \sim \frac{\sqrt{e}\,\sqrt{m+3}}{m+2} \longrightarrow 0.$$

By the ratio test, the series converges for every $z \in \mathbb{C}$. Therefore, its radius of convergence is infinite, and R is entire.

4.3 Appendix C. Sobolev constants in Fourier series

In this appendix, we recall some necessary definitions and results to establish an improvement of Lemma 2.1.

We begin by proving a technical lemma which we use to demonstrate Sobolev's inequality in the periodic case.

Lemma 4.1. *For $\theta \in (0,1)$, we have $\sum_{k=1}^{N} k^{-\theta} \leq 1 + \frac{N^{1-\theta}}{1-\theta} \leq 2\frac{N^{1-\theta}}{1-\theta}$, $\forall N \in \mathbb{N}$.*

Proof. It suffices to compare $\sum_{i=1}^{N}(i+1)^{-\theta}$ to the integral $\int_1^N t^{-\theta} dt$. □

We now introduce the periodic Lebesgue spaces and the periodic Sobolev spaces, which will be used to estimate the Sobolev constants in the context of Fourier series.

Definition 4.2 ([11]).

(i) For $p \in [1, \infty]$, the periodic Lebesgue spaces are defined by

$$L^p_{2\pi} = \left\{ f : \mathbb{R} \to \mathbb{C} \text{ measurable and } 2\pi\text{-periodic} : \int_0^{2\pi} |f(t)|^p dt < \infty \right\} / \sim$$

for $p < \infty$, and

$$L^\infty_{2\pi} = \left\{ f : \mathbb{R} \to \mathbb{C} \text{ measurable and } 2\pi\text{-periodic} : \exists\, C \geq 0 \text{ such that } |f(t)| \leq C \text{ a.e.} \right\} / \sim,$$

where

$$f \sim g \quad \Longleftrightarrow \quad f = g \text{ a.e.}$$

The associated norm is defined by

$$\|f\|_{L^p_{2\pi}} = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(t)|^p dt \right)^{1/p}, & p \neq \infty, \\ \inf \{ C \geq 0 : \ |f(t)| \leq C \text{ a.e.} \}, & p = \infty. \end{cases}$$

(ii)
$$C^\infty_{2\pi} = \{ f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty(\mathbb{R}),\ f \text{ is } 2\pi\text{-periodic} \}.$$

(iii) Let $f \in L^1_{2\pi}$. The Fourier coefficients of f are defined by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt, \qquad k \in \mathbb{Z}.$$

(iv)
$$C_0(\mathbb{Z}) = \{ (z_k)_{k \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \lim_{|k| \to \infty} z_k = 0 \}.$$

Remark 4.3 ([11]).

1. $C^\infty_{2\pi}$ is dense in $L^p_{2\pi}$ for all $p \in [1, \infty)$.
2. The operator
$$\mathcal{F} : L^1_{2\pi} \to C_0(\mathbb{Z}), \qquad f \mapsto (\widehat{f}(k))_{k \in \mathbb{Z}},$$
is well defined and injective.
3. The operator
$$\mathcal{F} : L^2_{2\pi} \to \ell^2(\mathbb{Z}), \qquad f \mapsto (\widehat{f}(k))_{k \in \mathbb{Z}},$$
is well defined and unitary.

In the following, we define the periodic Sobolev spaces and state some of their properties. For more details and proofs, see [1,4,9].

Definition 4.4. For $s \geq 0$, the periodic Sobolev space of index s is defined by

$$H^s_{2\pi} = \left\{ f \in L^2_{2\pi} : \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\widehat{f}(k)|^2 < \infty \right\}.$$

The space $H^s_{2\pi}$ is equipped with the inner product

$$\langle f, g\rangle_{H^s_{2\pi}} = \sum_{k\in\mathbb{Z}} (1+|k|^2)^s \widehat{f}(k)\overline{\widehat{g}(k)},$$

and the associated norm

$$\|f\|_{H^s_{2\pi}} = \left(\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\widehat{f}(k)|^2 \right)^{1/2}.$$

Proposition 4.5. *The following properties hold:*

1. *For $s \geq 0$, $(H^s_{2\pi}, \langle \cdot, \cdot \rangle_{H^s_{2\pi}})$ is a Hilbert space.*
2. *For $s \geq 0$, $C^\infty_{2\pi}$ is dense in $H^s_{2\pi}$.*
3. *If $s_2 \geq s_1 \geq 0$, then*

$$H^{s_2}_{2\pi} \hookrightarrow H^{s_1}_{2\pi}.$$

More precisely,

$$\|f\|_{H^{s_1}_{2\pi}} \leq \|f\|_{H^{s_2}_{2\pi}}, \qquad \forall f \in H^{s_2}_{2\pi}.$$

Now we state the main result of this section.

Proposition 4.6. *Let $2 < p < \infty$ and $s \in (0, 1/2)$ such that*

$$\frac{1}{p} + s = \frac{1}{2}.$$

Then

$$H^s_{2\pi} \hookrightarrow L^p_{2\pi}.$$

More precisely,

$$\|f\|_{L^p_{2\pi}} \leq 2\widetilde{R}_p \|f\|_{H^s_{2\pi}}, \tag{4.1}$$

where

$$\widetilde{R}^p_p = C_1 \frac{p}{p-2} C_2^{p-2} p^{\frac{p-2}{2}}, \tag{4.2}$$

and C_1, C_2 are universal constants.

Remark 4.7. Combining this result with Remark 2.3, we obtain that for $p \in (2, \infty)$,

$$\|f\|_{L^p_{2\pi}} \leq \lambda_p \|f\|_{H^{1/2}_{2\pi}}, \qquad \forall f \in H^{1/2}_{2\pi},$$

where $\lambda_p = 2\widetilde{R}_p$.

Proof of Proposition 4.6. The proof is inspired by the classical argument in [2]. We consider three cases.

First case: Let $f \in C^\infty_{2\pi}$ with $\widehat{f}(0) = 0$.

We have

$$2\pi \|f\|^p_{L^p_{2\pi}} = p \int_0^\infty t^{p-1} \big|\{x \in [0, 2\pi] : |f(x)| > t\}\big| dt.$$

For $N \in \mathbb{N} \cup \{0\}$, we split f into low and high frequencies:

$$f = f_{1,N} + f_{2,N}, \quad f_{1,N} = \mathcal{F}^{-1}(\mathbf{1}_{\{|k|<N\}} \widehat{f}(k)), \quad f_{2,N} = \mathcal{F}^{-1}(\mathbf{1}_{\{|k|\geq N\}} \widehat{f}(k)).$$

For $N \geq 1$, by the Cauchy–Schwarz inequality we have

$$\|f_{1,N}\|_{L^\infty_{2\pi}} \leq \left(\sum_{k=-(N-1)}^{N-1} (1+k^2)^{-s} \right)^{1/2} \|f\|_{H^s_{2\pi}}.$$

Setting

$$S_N = \sum_{k=-(N-1)}^{N-1} (1+k^2)^{-s} \leq 2 \sum_{k=0}^{N-1} (1+k^2)^{-s} \leq 8 \frac{N^{1-2s}}{1-2s},$$

we get

$$\|f_{1,N}\|_{L^\infty_{2\pi}} \leq 2\sqrt{2}(1-2s)^{-1/2} N^{1/2-s} \|f\|_{H^s_{2\pi}}. \tag{4.3}$$

For any $t > 0$, choose N_t such that

$$N_t \leq \left(\frac{t(1-2s)^{1/2}}{4\|f\|_{H^s_{2\pi}}} \right)^{1/(1/2-s)} < N_t + 1.$$

Then $\|f_{1,N_t}\|_{L^\infty_{2\pi}} \leq t/4$ by (4.3), and thus

$$\{|f| > t\} \subset \{|f_{2,N_t}| > t/2\}.$$

By Chebyshev's inequality and Parseval's identity,

$$\big|\{|f_{2,N_t}| > t/2\}\big| \leq \frac{4}{t^2} \|f_{2,N_t}\|^2_{L^2_{2\pi}} = \frac{4}{t^2} \sum_{|k|\geq N_t} |\widehat{f}(k)|^2.$$

Hence,

$$2\pi \|f\|^p_{L^p_{2\pi}} \leq 4p \sum_{k\in\mathbb{Z}^*} |\widehat{f}(k)|^2 \int_0^{\alpha_k} t^{p-3} dt \leq 4 \frac{p}{p-2} \sum_{k\in\mathbb{Z}^*} \alpha_k^{p-2} |\widehat{f}(k)|^2,$$

where

$$\alpha_k = \frac{4\|f\|_{H^s_{2\pi}}}{(1-2s)^{1/2}} 2^{1/p} |k|^{1/p}.$$

Finally, using $1 - 2s = 2/p$, we obtain

$$\|f\|^p_{L^p_{2\pi}} \leq \tilde{R}^p_p \|f\|^p_{H^s_{2\pi}},$$

with

$$\tilde{R}_p = \left(3(4/\pi)^{p-1} p^{(p-2)/2} \right)^{1/p}.$$

Second case: Let $f \in C^\infty_{2\pi}$ be arbitrary.

Set $g = f - \widehat{f}(0)$. By the first case,

$$\|g\|_{L^p_{2\pi}} \leq \tilde{R}_p \|g\|_{H^s_{2\pi}}.$$

Then,

$$\|f\|_{L^p_{2\pi}} \leq \|g\|_{L^p_{2\pi}} + |\widehat{f}(0)| \leq \tilde{R}_p \|g\|_{H^s_{2\pi}} + |\widehat{f}(0)| \leq 2\tilde{R}_p \|f\|_{H^s_{2\pi}}.$$

Third case: Let $f \in H^s_{2\pi}$.

Since $C^\infty_{2\pi}$ is dense in $H^s_{2\pi}$ and the estimate above is continuous, we conclude

$$\|f\|_{L^p_{2\pi}} \leq 2\tilde{R}_p \|f\|_{H^s_{2\pi}}, \qquad \forall f \in H^s_{2\pi}.$$

This completes the proof. □

Conflicts of interest

The author declare no conflicts of interest.

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