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Existence and limiting behavior of normalized ground states for mass critical Kirchhoff equations with inhomogeneous interactions

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Abstract. We study the normalized ground state solutions for the following mass critical Kirchhoff equations with inhomogeneous interactions

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=\lambda u+\beta m(x)u^{p+1} & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=1, \end{cases}$$

where $a\geq 0$, $b>0$, $\beta>0$, $p=\frac{8}{N}$, $0<m(x)\leq 1$ and the function $V(x)$ is a trapping potential in $\mathbb{R}^N$ $(N=1,2,3)$. We prove that there exists $\beta^*>0$ such that the constraint minimizers exist if the interaction strength β satisfies $\beta\leq\beta^*$ when $a>0$ and if $\beta<\beta^*$ when $a=0$. Further, we investigate the limit behavior of minimizers as $a\to 0$ for $\beta=\beta^*$. We also prove that all the mass of the minimizers concentrate at a global minimum point x_0 of $V(x)$ as $a\to 0$, where x_0 is also a global maximum point of $m(x)$. Specially, we find that the limit behavior is related to the behavior of the functions $V(x)$ and $m(x)$ at their critical points.

Keywords: Kirchhoff equation, energy estimates, limit behavior, mass concentration.

2020 Mathematics Subject Classification: 35J60, 47J30.

1 Introduction

In this article, we devoted to the existence and limit behavior of normalized ground state solutions for the following mass critical Kirchhoff equation with inhomogeneous interactions

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=\lambda u+\beta m(x)u^{p+1} & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=1, \end{cases} \tag{1.1}$$

where $N=1,2,3$, $a\geq 0$, $b>0$, $\beta>0$, $p=\frac{8}{N}$ represent the mass critical exponent, $0<m(x)\leq 1$ represent the spatially inhomogeneous interaction and $\lambda\in\mathbb{R}$ appears as an Lagrange multiplier.

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The Kirchhoff problem arises from a natural and physically meaningful context, when investigating the steady state behavior of the hyperbolic equation for the free vibration of elastic strings,

$$u_{tt} - M\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u = 0, \quad \text{in } \mathbb{R}^N,$$

which was first proposed by Kirchhoff constitutes an extension of the classical d'Alembert wave equation, see [1,4–6,9] and the related works cited therein for more physical background details. From a mathematical standpoint, (1.1) is generally characterized as nonlocal, since the presence of the integral term $\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x$ shows the equation is no longer pointwise valid. This phenomenon poses mathematical challenges, making the study of Kirchhoff equations particularly valuable.

After Lions proposed an abstract functional analysis framework in [30] , extensive studies have focused on the existence and concentration phenomena of solutions to Kirchhoff equations. The role of the frequency λ provides us with two distinctly different solution strategies. In the first case, λ is regarded as a fixed and assigned parameter, where the research on the existence, nonexistence, and multiplicity of solutions to (1.1) can be referred to [2, 12, 15, 27, 31, 33, 34]. Notably, in this case, there is no information on the L^2-norm of solutions. In the second case, λ is treated as an unknown quantity in the problem when study the normalized solutions to (1.1). Thus, it has become a popular research topic in recent years. To the best of our knowledge, the pioneering research on normalized solutions was conducted by Jeanjean in [25].

In fact, we are concerned with the normalized solutions to (1.1) that possess minimal energy. In this paper, we focus on the following constrained minimization problem

$$e_\beta(a) = \inf_{u\in S} E_a^\beta(u). \tag{1.2}$$

It is well-know that a solution restricted to the mass constraint

$$S := \left\{u \in \mathcal{H} : \|u\|_2^2 := \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = 1\right\} \tag{1.3}$$

can be treated as a critical point of the associated energy functional

$$E_a^\beta(u) = \frac{1}{2}\int_{\mathbb{R}^N} [a|\nabla u|^2 + V(x)|u|^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u|^{p+2}\mathrm{d}x, \tag{1.4}$$

where $\mathcal{H}$ is defined as

$$\mathcal{H} = \left\{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < +\infty\right\}. \tag{1.5}$$

Note that $\mathcal{H}$ is a Hilbert space with the scalar product and norm given by

$$\langle u, v\rangle_{\mathcal{H}} := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\,\mathrm{d}x \quad \text{and} \quad \|u\|_{\mathcal{H}} := \langle u, u\rangle_{\mathcal{H}}^{1/2}. \tag{1.6}$$

As far as we know, studies on normalized solutions to (1.1) have been confined nearly solely to $V(x) \equiv 0$ or $m(x) \equiv 1$, see [7, 8, 24, 36–39]. We now review a few known results in this respect.

When $V(x) \equiv 0$, if $m(x) \equiv 1$, Ye [37] showed that $p = \frac{8}{N}$ is the mass critical exponent of Kirchhoff constraint minimization problem, and study the existence and nonexistence of minimizers which restricted to the following mass constraint

$$S_c := \left\{ u \in \mathcal{H} : \|u\|_2^2 := \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = c^2 \right\}.$$

For more related results regarding the existence and limit behavior of minimizers for (1.1), we refer the reader to [23,28,38,39] and their references. In recent work [8], Chen and Tang studied the existence of normalized solutions for (1.1) with Sobolev critical exponent and mixed nonlinearities. If $m(x) \not\equiv 1$, Chen, Rădulescu and Tang [7] verified the existence of normalized solutions by replacing the term $|u|^{p+1}$ in (1.1) with $f(u)$ that satisfies the general mass supercritical or mass subcritical conditions.

When $V(x) \not\equiv 0$ and $m(x) \equiv 1$, for $b = 0$ and with given $\|u\|_2$, the Bose–Einstein condensate (BEC) in a two-dimensional dilute trapped atomic gas is characterized by the Gross–Pitaevskii (GP) energy functional, the parameter β has to be interpreted as particle number times interaction strength, see [10,14,20] and citations therein for details. For $b \neq 0$, it is worth noting that Hu and Tang [24] investigated the mass critical constrained minimization problem for (1.2). They analyzed the existence and nonexistence of minimizers and verified the limit behavior of minimizers as $a \to 0$. In this case, we also refer the reader to [18,22,28] and their references for more related results of the existence and limit behavior of minimizers for (1.1).

Next, we naturally consider the case when $V(x) \not\equiv 0$ and $m(x) \not\equiv 1$, based on the result of [11,22], Guo, Liu and Zhao [19] studied the constrained minimization problem of the mass subcritical case for (1.2) when $N = p = 2$. They established the existence results and limit behavior of minimizers as $b \to 0$. In particular, they verified that all the mass of minimizers must concentrated at a global minimum point x_1 of $V(x)$ as $b \to 0$, where x_1 is also a global maximum point of $m(x)$. In recent work [13], Du, Gao and Tian investigated the existence of minimizers for (1.2) constrained to the set S_c, as well as the limit behavior of minimizers as $c \to \infty$ in the mass subcritical case.

Now, motivated by [13,19,24], a natural question arises: does (1.2) admit minimizers with normalized L^2-norm (i.e., L^2-norm equals 1) in the mass critical case where $V(x) \neq 0$ and $0 < m(x) \leq 1$? If such minimizers exist, what description can be given for limit behavior as $a \to 0$? The main purpose of this paper is to investigate the existence of minimizers for (1.2) when $a \geq 0$ and the existence of the threshold β^* described as $\beta^* = \frac{b}{2}\|Q\|_2^{p}$, where Q is the unique positive radial solution of (1.8), as well as the limit behavior of minimizers as $a \to 0$ with $\beta = \beta^*$.

Before stating the main results, since the existence of minimizers associated with the following Gagliardo–Nirenberg inequality [32]

$$\int_{\mathbb{R}^N} |u|^{p+2} \mathrm{d}x \leq \frac{p+2}{2\|Q\|_2^{p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \right)^2 \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\frac{p}{2}-1}, \tag{1.7}$$

for all $u \in H^1(\mathbb{R}^N)$, the equality is attained at $u(x) = Q(x)$. Here, $Q(x)$ denotes the unique positive radial solution of the semilinear equation

$$-2\Delta Q + \left(\frac{p}{2} - 1\right) Q = Q^{p+1} \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

In addition, for more details on $Q(x)$, refer to references [16,26].

We now proceed to the main results of this study. If u_a is a minimizer of $e_\beta(a)$, then $|u_a|$ is also a minimizer of $e_\beta(a)$ since $|\nabla|u|| = |\nabla u|$ holds a.e. in $\mathbb{R}^N$ (see [28]). Without loss of generality, we assume in the following sections that all minimizers of (1.2) are nonnegative functions.

We consider the trapping potential $V(x) : \mathbb{R}^N \to \mathbb{R}$ satisfies the following basic hypotheses

(V_1) $V(x) \in L^\infty_{loc}(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} V(x) = 0$ and $\lim_{|x|\to\infty} V(x) = \infty$.

Theorem 1.1. *Suppose that $V(x)$ satisfies (V_1) and $0 < m(x) \leq 1$, the following statements holds*

1. *When $a > 0$*

   (i) *If $0 < \beta \leq \beta^*$, then there exists at least one minimizer for (1.2).*

   (ii) *If $\beta > \beta^*$, then there is no minimizer for (1.2).*

2. *When $a = 0$*

   (iii) *If $0 < \beta < \beta^*$, then there exists at least one minimizer for (1.2).*

   (iv) *If $\beta \geq \beta^*$, then there is no minimizer for (1.2).*

Moreover, $e_{\beta^*}(0) = 0$ and $e_\beta(0) = -\infty$ for $\beta > \beta^*$.

Remark 1.2. Based on Theorem 1.1, it is evident that the minimizer u_a of $e_\beta(a)$ must converge to the minimizer u_0 of $e_\beta(0)$ as $a \to 0$ for $0 < \beta < \beta^*$. Since there are no minimizers for $e_\beta(0)$ when $\beta \geq \beta^*$, we next study the concentration property of the minimizers as $a \to 0$ for $\beta = \beta^*$.

Before stating our second result, we introduce the following assumptions about the trapping potential function $V(x)$ and the spatially inhomogeneous interaction function $m(x)$.

(V_2) There are numbers $q_j > 0$, $j = 0, 1, 2, \ldots, n$, and a function $h(x) \in C^1(\mathbb{R}^N)$ such that

$$V(x) = h(x)\prod_{j=0}^{n}|x - x_j|^{q_j}, \quad \text{with} \quad \frac{1}{C} \leq h(x) \leq C \quad \text{for all } x \in \mathbb{R}^N. \tag{1.9}$$

Define $Z_0 = \{x_0, x_1, \ldots, x_n\}$ as the zero set of $V(x)$.

(M_1) $0 < m(x) \leq 1$, x_0 is a unique maximum point of $m(x)$ and $m(x) = 1 - O(|x - x_0|^{s+2})$ as $x \to x_0$, where $s > 0$.

Denote

$$h = s - 2, \qquad l = \min\{q_0, h\} \tag{1.10}$$

$$\mu_0 = \lim_{x\to 0}\frac{V(x + x_0)}{|x|^{q_0}}\int_{\mathbb{R}^N}|x|^{q_0}Q^2(x)\mathrm{d}x \tag{1.11}$$

$$\kappa_0 = \lim_{x\to 0}\frac{1 - m(x + x_0)}{|x|^{s+2}}\int_{\mathbb{R}^N}|x|^{s+2}Q^{p+2}(x)\mathrm{d}x \tag{1.12}$$

Our results are as follows.

Theorem 1.3. *For any given $a > 0$, suppose $V(x)$ satisfies (V_1)–(V_2) and $m(x)$ satisfies (M_1). Let $\beta = \beta^*$ and u_a be a minimizer of $e_{\beta^*}(a)$, then each u_a has a unique global maximum point z_a satisfying*

$$\lim_{a\to 0} \bar{\epsilon}_a^{\frac{N}{2}} u_a(\bar{\epsilon}_a x + z_a) = \frac{Q(x)}{\|Q\|_2} \quad in\ H^1(\mathbb{R}^N) \tag{1.13}$$

and

$$z_a - x_0 = o(\bar{\epsilon}_a) \quad as\ a \to 0, \tag{1.14}$$

$$\lim_{a\to 0} \frac{e_{\beta^*}(a)}{a^{\frac{l}{l+4}}} = \frac{l+2}{2l}\left(\frac{l\theta}{2\|Q\|_2^2}\right)^{\frac{2}{l+2}}, \tag{1.15}$$

where

$$\bar{\epsilon}_a = \left(\frac{2a\|Q\|_2^2}{l\theta}\right)^{\frac{1}{l+2}} \quad and \quad \theta = \begin{cases} \mu_0, & q_0 < h, \\ \dfrac{(p+2)\mu_0 + b\kappa_0}{p+2}, & q_0 = h, \\ \dfrac{b\kappa_0}{p+2}, & q_0 > h. \end{cases} \tag{1.16}$$

The paper is organized as follows. In Section 2, we study the existence and nonexistence of minimizers, and give the proof of Theorem 1.1. In Section 3, we study the limit behavior of the minimizer u_a of $e_{\beta^*}(a)$ as $a \to 0$, and finish the proof of Theorem 1.3.

2 Existence and nonexistence of minimizers

In this section, we will prove the existence and nonexistence of minimizers for (1.2).

First of all, let us recall that $Q = Q(|x|)$ denotes the unique positive radially symmetric solutions of (1.8), and it follows from [16] satisfies the exponential decay

$$Q(|x|),\ |\nabla Q(|x|)| = O\left(|x|^{-\frac{N-1}{2}} e^{-|x|}\right), \quad \text{as } |x| \to \infty. \tag{2.1}$$

Moreover, see [32], using (1.7) which admits $Q(|x|)$ be a optimizer and satisfies

$$\int_{\mathbb{R}^N} |\nabla Q|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |Q|^2 \mathrm{d}x = \frac{2}{p+2}\int_{\mathbb{R}^N} |Q|^{p+2}\mathrm{d}x. \tag{2.2}$$

Before proving Theorem 1.1, we need to introduce the well-known compactness lemma proposed by Bartsch and Wang [3].

Lemma 2.1. *Let $V(x) \in L^\infty_{loc}(\mathbb{R}^N)$ satisfy (V_1). Then the embedding $\mathcal{H}$ into $L^p(\mathbb{R}^N)$ is compact for $p \in [2, 2^*), 2^* = \frac{2N}{N-2}$ if $N \geq 3, 2^* = +\infty$ if $N = 1, 2$.*

Proof. Here is the proof of the Theorem1.1, we divide our proof into two parts.

Case 1. When $a > 0$.

(i) We prove the existence of minimizers of (1.2) for $0 < \beta \leq \beta^*$. If $u \in \mathcal{H}$ and $\|u\|_2 = 1$, by (1.7) and $0 < m(x) \leq 1$, we obtain

$$\int_{\mathbb{R}^N} m(x)|u|^{p+2}\mathrm{d}x \leq \int_{\mathbb{R}^N} |u|^{p+2}\mathrm{d}x \leq \frac{p+2}{2\|Q\|_2^p}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2. \tag{2.3}$$

From (2.3) and the definition of $E_a^\beta(u)$, we have

$$E_a^\beta(u) \geq \frac{1}{2}\int_{\mathbb{R}^N} a|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2 \mathrm{d}x\right)^2 - \frac{\beta}{2\|Q\|_2^p}\left(\int_{\mathbb{R}^N}|\nabla u|^2 \mathrm{d}x\right)^2. \tag{2.4}$$

This gives $E_a^\beta(u)$ is bounded uniformly from below for $a > 0$. Take the minimizing sequence $\{u_n\} \subset \mathcal{H}$ of $e_\beta(a)$, that is, $\int_{\mathbb{R}^N}|u_n|^2 dx = 1$ and $\lim_{n\to\infty} E_a^\beta(u_n) = e_\beta(a)$. Using (1.7) again that for $n > 0$ large enough, we deduce from (2.4) that

$$e_\beta(a) + 1 \geq \frac{1}{2}\int_{\mathbb{R}^N} a|\nabla u_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x + \left(\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2 \mathrm{d}x\right)^2, \tag{2.5}$$

where we consider that $\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}$ is non-negative, i.e., when $0 < \beta \leq \frac{b}{2}\|Q\|_2^p = \beta^*$, it can be deduced that $\{u_n\}$ is uniformly bounded in $\mathcal{H}$. Combining with the compact embedding Lemma 2.1 and passing to a subsequence of $\{u_n\}$ if necessary, there exists $u_0 \in \mathcal{H}$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } \mathcal{H}, \qquad u_n \to u_0 \quad \text{strongly in } L^p(\mathbb{R}^N), \quad \text{for } p \in [2, 2^*). \tag{2.6}$$

We conclude that

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N} m(x)|u_n(x)|^{p+2}\mathrm{d}x &= \int_{\mathbb{R}^N} m(x)|u_0(x)|^{p+2}\mathrm{d}x, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x = 1 \quad &\text{and} \quad \int_{\mathbb{R}^N}|u_0|^2\mathrm{d}x = 1. \end{aligned} \tag{2.7}$$

Moreover, note that

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x &= \int_{\mathbb{R}^N}|\nabla u_n - \nabla u_0|^2\mathrm{d}x + 2\int_{\mathbb{R}^N}\nabla u_0\cdot\nabla(u_n - u_0)\mathrm{d}x \\ &\geq 2\int_{\mathbb{R}^N}\nabla u_0\cdot\nabla(u_n - u_0)\mathrm{d}x. \end{aligned} \tag{2.8}$$

Set the functional $T(u_n - u_0) = \int_{\mathbb{R}^N}\nabla u_0\cdot\nabla(u_n - u_0)\mathrm{d}x$. Taking $n \to \infty$ both sides of (2.8) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x \geq \lim_{n\to\infty} T(u_n - u_0) = 0, \tag{2.9}$$

together with the weak lower semicontinuity of the norm in $\mathcal{H}$, we have

$$\begin{aligned} 2E_a^\beta(u_0) &= \int_{\mathbb{R}^N}\left[a|\nabla u_0|^2 + V(x)u_0^2\right]\mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x\right)^2 - \frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_0|^{p+2}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} a|\nabla u_0|^2\mathrm{d}x + \left(\frac{b}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x - 1\right)\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x + \|u_0\|_{\mathcal{H}}^2 \\ &\quad - \frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_0|^{p+2}\mathrm{d}x \\ &\leq a\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x + \left(\frac{b}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x - 1\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x \\ &\quad + \lim_{n\to\infty}\|u_n\|_{\mathcal{H}}^2 - \lim_{n\to\infty}\frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_n|^{p+2}\mathrm{d}x \\ &= \lim_{n\to\infty} 2E_a^\beta(u_n) = 2e_\beta(a). \end{aligned} \tag{2.10}$$

Since $e_\beta(a) \le E_a^\beta(u_0)$, we have $e_\beta(a) = E_a^\beta(u_0)$. This implies that there exists at least a minimizer of (1.2) for $0 < \beta \le \beta^*$.

(ii) We show that there's no minimizers for (1.2) as soon as $\beta > \beta^*$. Choose a nonnegative cut-off function $\psi \in C_0^\infty(\mathbb{R}^N)$ with

$$|\nabla\psi(x)| \le 2, \qquad \psi(x) \equiv 1 \quad \text{if } |x| \le 1, \quad \psi(x) \equiv 0 \quad \text{if } |x| \ge 2, \quad \text{and} \\ 0 \le \psi(x) \le 1, \quad \text{if } 1 \le |x| \le 2. \tag{2.11}$$

For all $\tau > 0$, $R > 0$ and $x_0 \in \mathbb{R}^N$, we consider a trial function

$$K_{R,\tau}(x) = \frac{A_{R,\tau}\tau^{\frac{N}{2}}}{\|Q\|_2}\psi\left(\frac{|x-x_0|}{R}\right)Q\left(\tau|x-x_0|\right), \tag{2.12}$$

where $A_{R,\tau}$ is determined such that $\int_{\mathbb{R}^N}|K_{R,\tau}|^2dx = 1$. Due to the exponential decay of $Q(x)$, for $R\tau > 0$ large enough, we have

$$\begin{aligned}\frac{\|K_{R,\tau}\|_2^2}{A_{R,\tau}^2} = \frac{1}{A_{R,\tau}^2} &= \frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^N}\psi^2\left(\frac{x}{\tau R}\right)Q^2(x)\,\mathrm{d}x \\ &= 1 + O\left((R\tau)^{-\infty}\right), \quad \text{as } \tau \to +\infty.\end{aligned} \tag{2.13}$$

Here, $f(t) = O\left(t^{-\infty}\right)$ refers to a function f such that $\lim_{t\to\infty}|f(t)|t^s = 0$ for all $s > 0$. It follows that

$$\lim_{R\tau\to\infty} A_{R,\tau} = 1. \tag{2.14}$$

Choose $R = \tau^{-\frac{1}{2}}$, then we can obtain that $R\tau = \tau^{\frac{1}{2}} \to +\infty$ as $\tau \to +\infty$, using (2.1) and (2.2), we can obtain that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla K_{R,\tau}|^2\mathrm{d}x &= \frac{A_{R,\tau}^2}{\|Q\|_2^2}\int_{\mathbb{R}^N}\left[R^{-1}\nabla\psi\left(\frac{x}{\tau R}\right)Q(x) + \tau\psi\left(\frac{x}{\tau R}\right)\nabla Q(x)\right]^2\mathrm{d}x \\ &= \frac{A_{R,\tau}^2\tau^2}{\|Q\|_2^2}\left[\int_{\mathbb{R}^N}|\nabla Q(x)|^2\mathrm{d}x - \int_{\mathbb{R}^N\setminus B(0,\tau^{\frac{1}{2}})}|\nabla Q(x)|^2\mathrm{d}x\right. \\ &\quad \left. + \int_{\mathbb{R}^N\setminus B(0,\tau^{\frac{1}{2}})}\left|(R\tau)^{-1}\nabla\psi(x/R\tau)Q(x) + \psi(x/R\tau)\nabla Q(x)\right|^2\mathrm{d}x\right] \\ &= \tau^2 + O\left((R\tau)^{-\infty}\right), \quad \text{as } \tau \to +\infty\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}V(x)|K_{R,\tau}|^2\mathrm{d}x &= \frac{A_{R,\tau}^2}{\|Q\|_2^2}\int_{B(0,2\tau^{\frac{1}{2}})}V\left(\frac{x}{\tau}+x_0\right)\psi^2(x/R\tau)|Q(x)|^2\mathrm{d}x \\ &\to V(x_0), \quad \text{as } \tau \to +\infty.\end{aligned} \tag{2.16}$$

Similarly, we can derive

$$\begin{aligned}&\frac{\beta}{p+2}\int_{\mathbb{R}^N}m(x)|K_{R,\tau}|^{p+2}\mathrm{d}x \\ &= \frac{A_{R,\tau}^{p+2}\beta\tau^4}{(p+2)\|Q\|_2^{p+2}}\int_{\mathbb{R}^N}\left[1 - \left(1 - m\left(\frac{x}{\tau}+x_0\right)\right)\right]\psi^{p+2}\left(\frac{x}{R\tau}\right)Q^{p+2}(x)\mathrm{d}x \\ &= \frac{A_{R,\tau}^{p+2}\beta\tau^4}{2\|Q\|_2^p} - \frac{A_{R,\tau}^{p+2}\beta\tau^{2-s}}{(p+2)\|Q\|_2^{p+2}}\int_{B(0,\tau^{\frac{1}{2}})}\frac{1-m\left(\frac{x}{\tau}+x_0\right)}{\left|\frac{x}{\tau}\right|^{s+2}}|x|^{s+2}Q^{p+2}(x)\mathrm{d}x \\ &\quad + O\left((R\tau)^{-\infty}\right) \\ &= \frac{\beta\tau^4}{2\|Q\|_2^p} + O\left((R\tau)^{-\infty}\right), \quad \text{as } \tau \to +\infty.\end{aligned} \tag{2.17}$$

Hence, from (2.15)–(2.17), we have

$$
\begin{aligned}
e_\beta(a) &\leq \lim_{\tau\to+\infty} E_a^\beta(K_{R,\tau}) \\
&= \lim_{\tau\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N}\left[a|\nabla K_{R,\tau}|^2 + V(x)|K_{R,\tau}|^2\right]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla K_{R,\tau}|^2\mathrm{d}x\right)^2 \\
&\quad - \frac{\beta}{p+2}\int_{\mathbb{R}^N} m(x)|K_{R,\tau}|^{p+2}\mathrm{d}x \\
&= \lim_{\tau\to+\infty} \frac{a}{2}\tau^2 + \frac{1}{2}V(x_0) + \left(\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}\right)\tau^4,
\end{aligned}
\tag{2.18}
$$

where we consider that $\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}$ is negative, i.e., when $\beta > \frac{b}{2}\|Q\|_2^p = \beta^*$, it can be deduced that

$$
e_\beta(a) = \lim_{\tau\to+\infty} E_a^\beta(K_{R,\tau}) = -\infty. \tag{2.19}
$$

Consequently, this implies there is no minimizer for (1.2) as soon as $\beta > \beta^*$.

Case 2. When $a = 0$.

(iii) We prove the existence of minimizers of (1.2) for $0 < \beta < \beta^*$. From the definition of $E_0^\beta(u)$, we have

$$
E_0^\beta(u) = \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2 - \frac{\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u|^{p+2}\mathrm{d}x. \tag{2.20}
$$

Then let $\{u_n\} \subset \mathcal{H}$ be a minimizing sequence of $e_\beta(0)$ with $\|u_n\|_2 = 1$ and $\lim_{n\to\infty} E_0^\beta(u_n) = e_\beta(0)$. Now prove the boundedness of $\{u_n\}$ in $\mathcal{H}$. From (2.4), (2.20) and (1.7), for $n > 0$ large enough,

$$
\begin{aligned}
e_\beta(0) + 1 + \frac{\beta}{2\|Q\|_2^p}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2 &\geq E_0^\beta(u_n) + \frac{\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_n|^{p+2}\mathrm{d}x \\
&= \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_n|^2\mathrm{d}x,
\end{aligned}
\tag{2.21}
$$

which implies

$$
e_\beta(0) + 1 \geq \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_n|^2\mathrm{d}x + \left(\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2, \tag{2.22}
$$

where we consider that if and only if $\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}$ is positive, i.e, when $0 < \beta < \frac{b}{2}\|Q\|_2^p = \beta^*$, it can be deduced that $\{u_n\}$ is uniformly bounded in $\mathcal{H}$. From Lemma 2.1, passing to a subsequence if necessary, there exists $u_0 \in \mathcal{H}$ such that (2.6) and (2.7) holds.

Furthermore, applying the weak lower semicontinuity of the norm in $\mathcal{H}$ that

$$
\begin{aligned}
2E_0^{\beta}(u_0) &= \int_{\mathbb{R}^N} V(x)u_0^2 \mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^N} |\nabla u_0|^2 \mathrm{d}x\right)^2 - \frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_0|^{p+2}\mathrm{d}x \\
&= \left(\frac{b}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 \mathrm{d}x - 1\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 \mathrm{d}x + \|u_0\|_{\mathcal{H}}^2 - \frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_0|^{p+2}\mathrm{d}x \\
&\le \left(\frac{b}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x - 1\right)\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x \\
&\quad + \lim_{n\to\infty}\|u_n\|_{\mathcal{H}}^2 - \lim_{n\to\infty}\frac{2\beta}{p+2}\int_{\mathbb{R}^N} m(x)|u_n|^{p+2}\mathrm{d}x \\
&= \lim_{n\to\infty} 2E_0^{\beta}(u_n) = 2e_{\beta}(0).
\end{aligned}
\tag{2.23}
$$

Since $E_0^{\beta}(u_n) \ge e_{\beta}(0)$, we have $E_0^{\beta}(u_n) = e_{\beta}(0)$. This implies there exists at least a minimizer of (1.2) for $0 < \beta < \beta^*$.

(iv) Next, we show the nonexistence of minimizers for (1.2) for $\beta > \beta^*$. From (2.14)–(2.17)

$$
\begin{aligned}
e_{\beta}(0) &\le \lim_{\tau\to+\infty} E_0^{\beta}(K_{R,\tau}) \\
&= \lim_{\tau\to+\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} V(x)|K_{R,\tau}|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla K_{R,\tau}|^2 \mathrm{d}x\right)^2 \right.\\
&\quad \left. - \frac{\beta}{p+2}\int_{\mathbb{R}^N} m(x)|K_{R,\tau}|^{p+2}\mathrm{d}x\right] \\
&= \lim_{\tau\to+\infty}\left[\frac{1}{2}V(x_0) + \left(\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}\right)\tau^4\right],
\end{aligned}
\tag{2.24}
$$

where we consider that if and only if $\frac{b}{4} - \frac{\beta}{2\|Q\|_2^p}$ is negative, i.e, when $\beta > \frac{b}{2}\|Q\|_2^p = \beta^*$. It implies

$$
e_{\beta}(0) \le \lim_{\tau\to+\infty} E_0^{\beta}(K_{R,\tau}) = -\infty. \tag{2.25}
$$

So we prove the nonexistence of minimizers of (1.2) for $\beta > \beta^*$.

Finally, for $\beta = \beta^*$, we show that there's no minimizer for (1.2). Suppose that $e_{\beta^*}(0)$ is attained by a nonnegative minimizer $u \ne 0$, then u satisfies the Euler–Lagrange equation

$$
-\left(b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = \lambda u + \beta^* m(x)|u|^{p+1}, \tag{2.26}
$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier associate with minimizer u. From trail function (2.12), furthermore assume that $x_0 \in Z_0$, i.e. $V(x_0) = 0$, we have

$$
e_{\beta^*}(0) \le \lim_{\tau\to+\infty} E_0^{\beta^*}(K_{R,\tau}) = \lim_{\tau\to+\infty}\left(\frac{b}{4} - \frac{\beta^*}{2\|Q\|_2^p}\right)\tau^4 + \frac{1}{2}V(x_0) = 0. \tag{2.27}
$$

In addition, for any $u \in \mathcal{H}$ with $\|u\|_2 = 1$, using (1.7) to obtain that

$$
\begin{aligned}
E_0^{\beta^*}(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \left(\frac{\beta^*}{2\|Q\|_2^p}\right)\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 \\
&\quad + \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} [1 - m(x)]|u|^{p+2}\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x + \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} [1 - m(x)]|u|^{p+2}\mathrm{d}x \ge 0,
\end{aligned}
\tag{2.28}
$$

which implies $e_{\beta^*}(0) \geq 0$. From (2.27), we have $e_{\beta^*}(0) = 0$ and u would have to have compact support for $\int_{\mathbb{R}^N} V(x)|u|^2 dx = 0$. Then

$$0 = e_{\beta^*}(0) = E_0^{\beta^*}(u) = \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} m(x)|u|^{p+2}\mathrm{d}x. \tag{2.29}$$

Multiplying both sides of (2.26) by u and integrating, then

$$\lambda = \int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \beta^* \int_{\mathbb{R}^N} m(x)|u|^{p+2}\mathrm{d}x. \tag{2.30}$$

Combining (2.29) and (2.30) with $p = \frac{8}{N} > 2$, we obtain

$$\lambda = -\frac{p}{2}\int V(x)u^2 \mathrm{d}x - \frac{(p-2)}{4} b\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 < 0. \tag{2.31}$$

Since u has compact support, denote it by Ω such that $Z_0 \subset \Omega$. Hence, $\lambda < 0$ and u satisfies the equation

$$-\left(b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u - \lambda u = \beta^* m(x)|u|^{p+1} \quad \text{in } \Omega \subset\subset \mathbb{R}^N, \quad u = 0 \text{ on } \partial\Omega. \tag{2.32}$$

Next, applying the standard elliptic regularity theory and the strong maximum principle [3], we conclude that $u = 0$ in $\mathbb{R}^N$, which is a contradiction.

Consequently, the proof of Theorem 1.1 is complete. □

3 Limiting behavior of minimizer

Fixed $\beta = \beta^*$. In this section, we will prove the limit behavior of the minimizer u_a for (1.1) as $a \to 0$, and give the proof of Theorem 1.3. To better understand the proof of our results, we first show the following several lemmas.

Lemma 3.1. *For any $a > 0$, suppose that $V(x)$ satisfies (V_1)–(V_2) and $m(x)$ satisfies (M_1), let u_a be a nonnegative minimizer of $e_{\beta^*}(a)$, then the following statements hold.*

(i) *$e_{\beta^*}(a) > 0$ satisfies*

$$e_{\beta^*}(a) \to e_{\beta^*}(0) = 0 \quad \text{as } a \to 0, \tag{3.1}$$

$$\int_{\mathbb{R}^N} a|\nabla u_a|^2 \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)|u_a|^2 \mathrm{d}x \to 0 \quad \text{as } a \to 0. \tag{3.2}$$

(ii) *Define*

$$\epsilon_a := \left(\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)^{-\frac{1}{2}}, \quad \text{where } \epsilon_a > 0 \tag{3.3}$$

satisfies $\epsilon_a \to 0$ as $a \to 0$.

Proof. (i) Since $u_a \in \mathcal{H}$ with $\|u_a\|_2 = 1$, we get from the definition of $E_a^{\beta^*}(u)$ and using (2.3), the following equation holds

$$\begin{aligned} e_{\beta^*}(a) = E_a^{\beta^*}(u_a) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[a|\nabla u_a|^2 + V(x)u_a^2\right] \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)^2 \\ &\quad - \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} \left[a|\nabla u_a|^2 + V(x)u_a^2\right] \mathrm{d}x > 0, \end{aligned} \tag{3.4}$$

hence $e_{\beta^*}(a) > 0$, define the same cut-off function $\psi(x)$ and trail function $K_{R,\tau}$ with $\|K_{R,\tau}\|_2 = 1$ as (2.11) and (2.12). Taking $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, using (2.14)–(2.17) to get that

$$\begin{aligned} 0 < e_{\beta^*}(a) \le E_a^{\beta^*}(K_{R,\tau}) &= \frac{a}{2}\tau^2 + \frac{1}{2}V(x_0) + \frac{b}{4}\tau^4 - \frac{\beta^*\tau^4}{2\|Q\|_2^2} + o(e^{-R\tau}) + o(1) \\ &= \frac{a}{2}\tau^2 + o(e^{-R\tau}) + o(1), \end{aligned} \tag{3.5}$$

where $o(1)$ denotes an infinitely small value as $\tau \to +\infty$. Letting $a \to 0$ first, then $\tau \to +\infty$, we have

$$0 < e_{\beta^*}(a) \to 0 \quad \text{as } a \to 0. \tag{3.6}$$

So (3.1) holds, combining with (3.5) and (3.6) we deduce that

$$\int_{\mathbb{R}^N} a|\nabla u_a|^2 \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)|u_a|^2 \mathrm{d}x \to 0 \quad \text{as } a \to 0 \tag{3.7}$$

and (3.2) holds, the proof of (i) has been completed.

(ii) Next, we claim that

$$\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x \to +\infty \quad \text{as } a \to 0. \tag{3.8}$$

Otherwise, if (3.8) is false, then there exists a sequence of $\{a_k\}$ that satisfies $a_k \to 0$ as $k \to +\infty$ such that $\{u_{a_k}\}$ is a bounded sequence in $\mathcal{H}$. According to Lemma 2.1, up to a subsequence if necessary, there exists $u_0 \in \mathcal{H}$ such that

$$u_{a_k} \rightharpoonup u_0 \text{ weakly in } \mathcal{H} \quad \text{and} \quad u_{a_k} \to u_0 \text{ strongly in } L^p(\mathbb{R}^N), \text{ as } k \to \infty, \text{ for } p \in [2, 2^*). \tag{3.9}$$

Furthermore, due to $e_{\beta^*}(0) = 0$ and (3.1), we conclude that

$$0 = e_{\beta^*}(0) \le E_0^{\beta^*}(u_0) \le \liminf_{k\to\infty} E_{a_k}^{\beta^*}(u_{a_k}) = \lim_{k\to\infty} e_{\beta^*}(a_k) = 0 = e_{\beta^*}(0), \tag{3.10}$$

which shows that u_0 is a minimizer of $e_{\beta^*}(0)$, this contradicts the non-existence of minimizers for $e_{\beta^*}(0)$ (i.e. Theorem 1.1-iv). Therefore, we have $\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x \to +\infty, \text{as } a \to 0$, so (3.8) holds. Hence, $\epsilon_a \to 0$ as $a \to 0$ holds by (3.3) and (3.8), and we completes the proof of (ii). □

Lemma 3.2. *For any $a > 0$, suppose that $V(x)$ satisfies (V_1)–(V_2) and $m(x)$ satisfies (M_1), let u_a be a nonnegative minimizer of $e_{\beta^*}(a)$ and z_a be a global maximum point of u_a. Set the normalized function*

$$W_a(x) = \epsilon_a^{\frac{N}{2}} u_a(\epsilon_a x + z_a), \quad \textit{where } \epsilon_a \textit{ is defined by (3.3)}, \tag{3.11}$$

then the following statements hold.

(i)

$$\int_{\mathbb{R}^N} |\nabla W_a|^2 \mathrm{d}x = 1, \qquad \int_{\mathbb{R}^N} |W_a|^2 \mathrm{d}x = 1, \tag{3.12}$$

$$\eta_a := \frac{b}{4} - \frac{\beta^*}{p+2} \int_{\mathbb{R}^N} m(\epsilon_a x + z_a)|W_a|^{p+2} \mathrm{d}x \to 0, \quad \textit{as } a \to 0. \tag{3.13}$$

(ii) There exists positive constants $M > 0$ such that

$$\liminf_{a\to 0} \int_{B_2(0)} |W_a|^2 \mathrm{d}x \ge M > 0. \tag{3.14}$$

(*iii*) *Moreover, for any sequence* $\{a_k\}$ *satisfies* $a_k \to 0$ *as* $k \to +\infty$*, there exists a subsequence still denoted by* $\{a_k\}$*, and exists* $z_0 \in \mathbb{R}^N$ *such that*

$$\lim_{a\to 0} z_a = z_0, \quad \textit{satisfies } V(z_0) = 0. \tag{3.15}$$

Proof. (i) Since u_a is the minimizer of $e_{\beta^*}(a)$ with $\|u_a\|_2 = 1$, according to (1.7), we have

$$\begin{aligned} 0 &\le \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)^2 - \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \\ &= \frac{b}{4}\epsilon_a^{-4} - \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \\ &\le E_a^{\beta^*}(u_a) = e_{\beta^*}(a) \to 0 \quad \text{as } a \to 0, \end{aligned} \tag{3.16}$$

which leads to

$$\epsilon_a^4 \int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \to \frac{b(p+2)}{4\beta^*} \quad \text{as } a \to 0. \tag{3.17}$$

We can also know from the definition of $W_a(x)$ that

$$\epsilon_a^4 \int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x = \int_{\mathbb{R}^N} m(\epsilon_a x + z_a)|W_a|^{p+2}\mathrm{d}x \to \frac{b(p+2)}{4\beta^*} \quad \text{as } a \to 0, \tag{3.18}$$

which implies

$$\frac{b}{4} - \frac{\beta^*}{p+2}\int_{\mathbb{R}^N} m(\epsilon_a x + z_a)|W_a|^{p+2}\mathrm{d}x \to 0, \quad \text{as } a \to 0, \tag{3.19}$$

so (3.13) holds. Using the definitions of ϵ_a and $W_a(x)$ (i.e., from (3.3) and (3.11)), we have

$$\begin{cases} \int_{\mathbb{R}^N} |W_a|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |u_a|^2 \mathrm{d}x = 1 \\ \int_{\mathbb{R}^N} |\nabla W_a|^2 \mathrm{d}x = \epsilon_a^2 \int_{\mathbb{R}^N} |\nabla, u_a|^2 \mathrm{d}x = 1 \quad \text{as } a \to 0. \end{cases} \tag{3.20}$$

Thus (3.12) holds and the prove of (i) is complete.

(ii) Let u_a be a minimizer of $e_{\beta^*}(a)$, then it satisfies the following Euler–Lagrange equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)\Delta u_a + V(x)u_a = \lambda_a u_a + \beta^* m(x) u_a^{p+1} \quad \text{in } \mathbb{R}^N, \tag{3.21}$$

where λ_a is the associated Lagrange multiplier and

$$\lambda_a = \int_{\mathbb{R}^N} \left[a|\nabla u_a|^2 + V(x)u_a^2\right]\mathrm{d}x + b\left(\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)^2 - \beta^* \int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x, \tag{3.22}$$

hence, we can obtain that

$$\begin{aligned} \epsilon_a^4 \lambda_a &= \epsilon_a^4\left(\int_{\mathbb{R}^N} \left[a|\nabla u_a|^2 + V(x)u_a^2\right]\mathrm{d}x\right) + \epsilon_a^4 b\left(\int_{\mathbb{R}^N} |\nabla u_a|^2 \mathrm{d}x\right)^2 - \epsilon_a^4\beta^* \int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \\ &= \epsilon_a^4\left(\int_{\mathbb{R}^N} \left[a|\nabla u_a|^2 + V(x)u_a^2\right]\mathrm{d}x\right) + b - \epsilon_a^4\beta^* \int_{\mathbb{R}^N} m(x)|u_a|^{p+2}\mathrm{d}x \\ &\to \frac{b(2-p)}{4} < 0 \quad \text{where } p > 2, \quad \text{as } a \to 0. \end{aligned} \tag{3.23}$$

Following (3.21), W_a satisfies the following equation

$$-(\epsilon_a^2 a+b)\Delta W_a+\epsilon_a^4 V(\epsilon_a x+z_a)W_a=\lambda_a\epsilon_a^4 W_a+\beta^* m(\epsilon_a x+z_a)W_a^{p+1} \quad x\in\mathbb{R}^N. \tag{3.24}$$

Since z_a is a global maximum point of u_a, then a global maximum of W_a is attained at $x=0$, we have $-\Delta W_a(0)\geq 0$. Then, according to $V(x)\geq 0$ that leads the left side of (3.24) is positive, using (3.23) we have

$$W_a^p(0)\geq\frac{-\lambda_a\epsilon_a^4}{\beta^*}\geq\frac{b(p-2)}{4\beta^*}>0, \quad \text{as } a\to 0, \tag{3.25}$$

we claim that there exists some $\delta=\left[\frac{b(p-2)}{4\beta^*}\right]^{\frac{1}{p}}>0$ such that

$$W_a(0)\geq\delta. \tag{3.26}$$

As a small enough, it follows from (3.24) that

$$-(\epsilon_a^2 a+b)\Delta W_a\leq\beta^* W_a^{p+1} \quad \text{in } \mathbb{R}^N. \tag{3.27}$$

Applying De Giorgi–Nash–Moser theory [17], we deduce that for any $\xi\in\mathbb{R}^N$,

$$A\left(\int_{B_2(\xi)}|W_a|^2\mathrm{d}x\right)^{\frac{1}{2}}\geq\max_{B_1(\xi)}W_a(x)\geq W_a(\xi) \tag{3.28}$$

holds, where A is a constant depending only on the bound of $\|W_a\|_{L^{p/2}(B_2(\xi))}$ and ξ is an arbitrary point in $\mathbb{R}^N$. If (3.26) is false, then for any $r>0$, passing to a subsequence if necessary, we have

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,r)}|W_a|^2\mathrm{d}x\to 0 \quad \text{as } a\to 0. \tag{3.29}$$

Then, since $0<m(x)\leq 1$, the vanishing lemma in [35] shows that $\int_{\mathbb{R}^N}|W_a|^{p+2}\mathrm{d}x\to 0$, and

$$0\leq\int_{\mathbb{R}^N}m(\epsilon_a x+z_a)|W_a|^{p+2}\mathrm{d}x\leq\int_{\mathbb{R}^N}|W_a|^{p+2}\mathrm{d}x\to 0 \quad \text{as } a\to 0. \tag{3.30}$$

This contradicts with (3.20), hence, (3.26) holds, and choosing $\xi=0$ in (3.28), we have

$$\liminf_{a\to 0}\int_{B_2(0)}|W_a|^2\mathrm{d}x\geq M>0 \quad \text{for some } M>0, \tag{3.31}$$

which implies (3.14) holds.

(iii) We claim $\{z_a\}\subset\mathbb{R}^N$ is bounded uniformly as $a\to 0$. For any $\{z_{a_k}\}$ satisfies $a_k\to 0$, there is a subsequence, still denoted by $\{z_{a_k}\}$, so that $z_{a_k}\to z_0$ as $a_k\to 0$. Otherwise, from $V(x)\to\infty$ as $|x|\to\infty$, we assume that $V(z_0)\geq\omega>0$, then we can infer from Fatou's lemma and (3.14) that

$$\begin{aligned}0<\widehat{\omega}\leq\int_{B_2(0)}\liminf_{a_k\to 0}V(\epsilon_{a_k}x+z_{a_k})|W_{a_k}|^2\mathrm{d}x&\leq\int_{\mathbb{R}^N}\liminf_{a_k\to 0}V(\epsilon_{a_k}x+z_{a_k})|W_{a_k}|^2\mathrm{d}x\\&\leq\liminf_{a_k\to 0}\int_{\mathbb{R}^N}V(\epsilon_{a_k}x+z_{a_k})|W_{a_k}|^2\mathrm{d}x,\end{aligned} \tag{3.32}$$

which contradicts with Lemma 3.1 (i) since it shows

$$\int_{\mathbb{R}^N}V(x)u_{a_k}^2\mathrm{d}x=\int_{\mathbb{R}^N}V(\epsilon_{a_k}x+z_{a_k})|W_{a_k}|^2\mathrm{d}x\to 0 \quad \text{as } a_k\to 0, \tag{3.33}$$

hence there exists $z_0\in Z_0$ such that (3.15) holds.

This completes the proof. □

Lemma 3.3. *Suppose that $V(x)$ satisfies (V_1)–(V_2) and $m(x)$ satisfies (M_1), let u_a be a nonnegative minimizer of $e_{\beta^*}(a)$ and z_a be a global maximum point of u_a. Then*

$$\lim_{a\to 0} \epsilon_a^{\frac{N}{2}} u_a(\epsilon_a x + z_a) \to \frac{Q(x)}{\|Q\|_2} \quad \text{in } H^1(\mathbb{R}^N). \tag{3.34}$$

Moreover, $z_a \in \mathbb{R}^N$ is the unique maximum point of u_a, and

$$\lim_{a\to 0} z_a = x_0, \tag{3.35}$$

where x_0 satisfies $V(x_0) = 0$ and $m(x_0) = 1$.

Proof. Let u_a be a minimizer of $e_{\beta^*}(a)$. It follows from (3.11) and (3.12) that $\{W_a\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, and passing to subsequence, there exists a nonnegative function $W_0 \in H^1(\mathbb{R}^N)$ such that

$$W_a \rightharpoonup W_0 \quad \text{in } H^1(\mathbb{R}^N) \quad \text{as } a \to 0. \tag{3.36}$$

Since W_a satisfies (3.24), by passing to the weak limit, we can check from (3.23) that W_0 satisfies the following equation

$$-b\Delta W_0 + \frac{b(p-2)}{4} W_0 = \beta^* m(z_0) W_0^{p+1}, \quad x \in \mathbb{R}^N. \tag{3.37}$$

It follows from (3.14) that $W_0 \neq 0$, and $W_0 > 0$ by the strong maximum principle. Note that (3.37) is equivalent to

$$-2\Delta W_0 + \frac{4-N}{N} W_0 = \|Q\|_2^p m(z_0) W_0^{p+1}. \tag{3.38}$$

Due to $Q(x)$ is the unique positive solution of (1.8), comparing (1.8) and (3.38), then using a simple rescaling, we know that

$$W_0(x) = \frac{Q(x-x_0)}{\|Q\|_2 \, m(z_0)^{\frac{1}{p}}} \quad \text{for some } x_0 \in \mathbb{R}^N, \tag{3.39}$$

which implies that

$$\frac{1}{m(z_0)^{\frac{2}{p}}} = \|\nabla W_0\|_2^2 = \|W_0\|_2^2. \tag{3.40}$$

Moreover, by (2.2), (3.36) and (3.39), we conclude that

$$\frac{1}{m(z_0)^{\frac{2}{p}}} = \|W_0\|_2^2 \leq \lim_{a\to 0} \|W_a\|_2^2 = 1. \tag{3.41}$$

Since $0 < \frac{2}{p} < 1$ and $0 < m(x) \leq 1$, which implies $m(z_0) = 1$. Together with (3.15), we have $V(z_0) = 0$ and $m(z_0) = 1$, and $z_0 = x_0$ through the conditions (V_2) and (M_1). Then, (3.40) implies

$$\|\nabla W_0\|_2^2 = \|W_0\|_2^2 = 1. \tag{3.42}$$

Combining with (3.36) and (3.42), we have

$$W_a \to W_0 = \frac{Q(x-x_0)}{\|Q\|_2} \quad \text{in } H^1(\mathbb{R}^N) \text{ as } a \to 0. \tag{3.43}$$

Since $V(x) \in C^{\alpha}_{loc}(\mathbb{R}^N, \mathbb{R}^+)$ for some $\alpha \in (0,1)$. From (3.24) and (3.43), we have

$$W_a \to W_0 \quad \text{in } C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N) \quad \text{as } a \to 0 \quad \text{for some } \alpha \in (0,1). \tag{3.44}$$

By the definition of W_a, we have $x = 0$ is a critical point of $W_a(x)$ for all $a > 0$, then it is also a critical point of W_0 by (3.44). Since $Q(x)$ is radially symmetric about the origin and strictly monotonous about $|x|$ [26], W_0 has a unique global maximum point at $x = 0$ and $x_0 = 0$. Hence,

$$W_a \to W_0 = \frac{Q(x)}{\|Q\|_2} \quad \text{in } H^1(\mathbb{R}^N) \text{ as } a \to 0, \tag{3.45}$$

and (3.34) holds by (3.45). Moreover, by (3.44), the argument analogous to the proof of [21], we conclude that z_a is the unique maximum point of u_a and z_a satisfies (3.35). □

Proof of Theorem 1.3. Lemma 3.3 ensures that for each u_a, there exists a unique global maximum point z_a such that $z_a \to x_0$ as $a \to 0$, and $V(x_0) = 0$, $m(x_0) = 1$. Given the function $K_{R,\tau}$ defined by (2.12) and $R = \tau^{-\frac{1}{2}}$, then from (2.16) and (2.17) we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|K_{R,\tau}|^2 \mathrm{d}x &= \frac{A^2_{R,\tau}}{\|Q\|_2^2} \int_{\mathbb{R}^N} V\left(\frac{x}{\tau} + x_0\right) \psi^2(x/R\tau)|Q(x)|^2 \mathrm{d}x \\ &= \frac{A^2_{R,\tau}}{\|Q\|_2^2 \tau^{q_0}} \int_{\mathbb{R}^N} \frac{V\left(\frac{x}{\tau} + x_0\right)}{|\frac{x}{\tau}|^{q_0}} \psi^2(x/R\tau)|Q(x)|^2 |x|^{q_0} \mathrm{d}x \\ &= \frac{\mu_0}{\|Q\|_2^2 \tau^{q_0}} (1 + o(1)) \quad \text{as } \tau \to +\infty,\end{aligned} \tag{3.46}$$

$$\begin{aligned}&\frac{\beta^*}{p+2} \int_{\mathbb{R}^N} m(x)|K_{R,\tau}|^{p+2} \mathrm{d}x \\ &= \frac{A^{p+2}_{R,\tau} \beta^* \tau^4}{(p+2)\|Q\|_2^{p+2}} \int_{\mathbb{R}^N} \left[1 - \left(1 - m\left(\frac{x}{\tau} + x_0\right)\right)\right] \psi^{p+2}\left(\frac{x}{R\tau}\right) Q(x)^{p+2} \mathrm{d}x \\ &= \frac{A^{p+2}_{R,\tau} \beta^* \tau^4}{2\|Q\|_2^p} - \frac{A^{p+2}_{R,\tau} \beta^* \tau^{2-s}}{(p+2)\|Q\|_2^{p+2}} \int_{B(0,\tau^{\frac{1}{2}})} \frac{1 - m\left(\frac{x}{\tau} + x_0\right)}{|\frac{x}{\tau}|^{s+2}} |x|^{s+2} Q(x)^{p+2} \mathrm{d}x + O\left((R\tau)^{-\infty}\right) \\ &= \frac{b}{4}\tau^4 - \frac{\tau^{2-s}\beta^* A^{p+2}_{R,\tau} \kappa_0}{(p+2)\|Q\|_2^{p+2}} (1 + o(1)) \\ &= \frac{b}{4}\tau^4 - \frac{\tau^{2-s} b \kappa_0}{2(p+2)\|Q\|_2^2} (1 + o(1)) \quad \text{as } \tau \to +\infty.\end{aligned} \tag{3.47}$$

Let u_a be a minimizer of $e_{\beta^*}(a)$, with $\|K_{R,\tau}\|_2^2 = 1$, we conclude that

$$\begin{aligned}e_{\beta^*}(a) &\le E_a^{\beta^*}(K_{R,\tau}) \\ &= \frac{a}{2}\tau^2 + \frac{\mu_0}{2\|Q\|_2^2 \tau^{q_0}} + \frac{b}{4}\tau^4 - \left(\frac{b}{4}\tau^4 - \frac{\tau^{2-s} b \kappa_0}{2(p+2)\|Q\|_2^2}\right)(1 + o(1)) \\ &= \frac{a}{2}\tau^2 + \frac{\mu_0}{2\|Q\|_2^2 \tau^{q_0}} (1 + o(1)) + \frac{b \kappa_0}{2(p+2)\|Q\|_2^2 \tau^{s-2}} (1 + o(1)) \quad \text{as } \tau \to +\infty.\end{aligned} \tag{3.48}$$

Let $h = s - 2$, we next continue the analysis by discussing separately three different cases.

Case 1. When $0 < q_0 < h$, take

$$\tau = \left(\frac{q_0 \mu_0}{2a\|Q\|_2^2}\right)^{\frac{1}{q_0+2}}, \quad \text{then } \tau \to \infty \quad \text{as } a \to 0. \tag{3.49}$$

Following from (3.48) we can obtain that

$$\begin{aligned} e_{\beta^*}(a) \le E_a^{\beta^*}(K_{R,\tau}) &= \frac{a}{2}\left(\frac{q_0\mu_0}{2a\|Q\|_2^2}\right)^{\frac{2}{q_0+2}} + \frac{\mu_0}{2\|Q\|_2^2}\left(\frac{q_0\mu_0}{2a\|Q\|_2^2}\right)^{\frac{-q_0}{q_0+2}}(1+o(1)) \\ &= \frac{a^{\frac{q_0}{q_0+2}}}{2}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}} + \frac{a^{\frac{q_0}{q_0+2}}}{q_0}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}}(1+o(1)) \quad \text{as } a \to 0, \end{aligned} \tag{3.50}$$

which implies

$$\begin{aligned} \limsup_{a\to 0} \frac{e_{\beta^*}(a)}{a^{\frac{q_0}{q_0+2}}} &\le \frac{1}{2}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}} + \frac{1}{q_0}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}} \\ &= \frac{q_0+2}{2q_0}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}}. \end{aligned} \tag{3.51}$$

By (3.51), we know $\limsup_{a\to 0} \frac{e_{\beta^*}(a)}{a^{\frac{q_0}{q_0+2}}}$ has an upper estimates. Therefore, we need only to show that the limit has the same lower bound. We first claim that

$$\left\{\frac{z_a - x_0}{\epsilon_a}\right\} \text{ is bounded.} \tag{3.52}$$

Assume to the contrary that there is a subsequence denoted by $\{a_k\}$ with $a_k \to 0$, such that $\left\{\frac{z_a - x_0}{\epsilon_a}\right\} \to +\infty$, where $z_{a_k} \triangleq z_a$. Hence, for M large enough,

$$\begin{aligned} \liminf_{a\to 0} \frac{1}{\epsilon_a^{q_0}} \int_{\mathbb{R}^N} V(x)|u_a|^2 dx &= \liminf_{a\to 0} \frac{1}{\epsilon_a^{q_0}} \int_{\mathbb{R}^N} V(\epsilon_a x + z_a)|W_a|^2 dx \\ &= \liminf_{a\to 0} \int_{\mathbb{R}^N} \frac{V(\epsilon_a x + z_a)}{|\epsilon_a x + z_a - x_0|^{q_0}} \left| x + \frac{z_a - x_0}{\epsilon_a}\right|^{q_0} |W_a|^2 dx \ge M. \end{aligned} \tag{3.53}$$

Following from (1.7) and Young's inequality, we see that

$$\begin{aligned} e_{\beta^*}(a) = E_a^{\beta^*}(u_a) &\ge \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_a|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u_a^2 dx \\ &\ge \frac{a}{2}\epsilon_a^{-2} + \frac{M}{2}{\epsilon_a}^{q_0} \\ &\ge C a^{\frac{q_0}{q_0+2}} M^{\frac{2}{q_0+2}}, \end{aligned} \tag{3.54}$$

which contradicts with (3.51) due to M is arbitrarily large and (3.52) holds. Therefore, passing to a subsequence, we may assume that there exists $y_0 \in \mathbb{R}^N$ such that

$$\frac{z_a - x_0}{\epsilon_a} \to y_0 \quad \text{as } a \to 0. \tag{3.55}$$

Since $\frac{V(\epsilon_a x + z_a)}{|\epsilon_a x + z_a - x_0|^{q_0}}$ converges uniformly to a constant as $a \to 0$ and the integrand satisfies the dominated convergence condition, then it follows from (3.34) that

$$\begin{aligned}
\liminf_{a\to 0}\frac{1}{\epsilon_a^{q_0}}\int_{\mathbb{R}^N}V(x)|u_a|^2\mathrm{d}x &= \liminf_{a\to 0}\frac{1}{\epsilon_a^{q_0}}\int_{\mathbb{R}^N}V(\epsilon_a x+z_a)|W_a|^2\mathrm{d}x\\
&= \liminf_{a\to 0}\int_{\mathbb{R}^N}\frac{V(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{q_0}}\left|x+\frac{z_a-x_0}{\epsilon_a}\right|^{q_0}|W_a|^2\mathrm{d}x\\
&= \liminf_{a\to 0}\frac{V(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{q_0}}\frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^N}|x+y_0|^{q_0}Q^2(x)\,\mathrm{d}x \qquad (3.56)\\
&\geq \liminf_{a\to 0}\frac{V(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{q_0}}\frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^N}|x|^{q_0}Q^2(x)\mathrm{d}x\\
&= \frac{\mu_0}{\|Q\|_2^2}.
\end{aligned}$$

Therefore, using Young's inequality again we can obtain that

$$\begin{aligned}
e_{\beta^*}(a) = E_a^{\beta^*}(u_a) &\geq \frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_a|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_a^2\mathrm{d}x\\
&\geq \frac{a}{2}\epsilon_a^{-2}+\frac{\mu_0}{2\|Q\|_2^2}\epsilon_a{}^{q_0}(1+o(1)) \qquad (3.57)\\
&\geq \frac{a^{\frac{q_0}{q_0+2}}(q_0+2)\,(1+o(1))}{2q_0}\left(\frac{\mu_0 q_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}} \quad \text{as } a\to 0.
\end{aligned}$$

Hence,

$$\liminf_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{q_0}{q_0+2}}}\geq\frac{q_0+2}{2q_0}\left(\frac{q_0\mu_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}}, \tag{3.58}$$

where the equality holds if and only if $y_0=0$ and

$$\lim_{a\to 0}\frac{\epsilon_a}{\left(\frac{2a\|Q\|_2^2}{q_0\mu_0}\right)^{\frac{1}{q_0+2}}}=1. \tag{3.59}$$

Moreover, it follows from (3.51) and (3.58) that

$$\lim_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{q_0}{q_0+2}}}=\frac{q_0+2}{2q_0}\left(\frac{\mu_0 q_0}{2\|Q\|_2^2}\right)^{\frac{2}{q_0+2}}. \tag{3.60}$$

Case 2. When $0<h<q_0$, choosing

$$\tau=\left(\frac{hb\kappa_0}{2a(p+2)\|Q\|_2^2}\right)^{\frac{1}{h+2}}, \quad \text{then } \tau\to\infty \quad \text{as } a\to 0. \tag{3.61}$$

From (3.48), which implies

$$\begin{aligned}
e_{\beta^*}(a) &\leq E_a^{\beta^*}(K_{R,\tau})\\
&= \frac{a}{2}\left(\frac{hb\kappa_0}{2a(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}+\frac{b\kappa_0}{2(p+2)\|Q\|_2^2}\left(\frac{hb\kappa_0}{2a(p+2)\|Q\|_2^2}\right)^{-\frac{h}{h+2}}(1+o(1)) \qquad (3.62)\\
&= \frac{a^{\frac{h}{h+2}}}{2}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}+\frac{a^{\frac{h}{h+2}}}{h}\left(\frac{hb\kappa}{2(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}(1+o(1)) \quad \text{as } a\to 0,
\end{aligned}$$

which implies

$$\begin{aligned}\limsup_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{h}{h+2}}} &\leq \frac{1}{2}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}+\frac{1}{h}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}\\ &= \frac{h+2}{2h}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^2}\right)^{\frac{2}{h+2}}.\end{aligned}\tag{3.63}$$

We claim that

$$\left\{\frac{z_a-x_0}{\epsilon_a}\right\} \text{ is bounded.}\tag{3.64}$$

By contradiction, there exists a subsequence, denoted by $\{a_k\}$, of $\{a_k\}$ such that $\left\{\frac{z_a-x_0}{\epsilon_a}\right\}\to +\infty$, where $z_{a_k}\triangleq z_a$. Hence, for any large constant M,

$$\begin{aligned}&\liminf_{a\to 0}\frac{1}{\epsilon_a^h}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x\\ &=\lim_{a\to 0}\frac{1}{\epsilon_a^{h+4}}\int_{\mathbb{R}^N}[1-m(\epsilon_a x+z_a)]|W_a|^{p+2}\mathrm{d}x\\ &=\lim_{a\to 0}\int_{\mathbb{R}^N}\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}\left|x+\frac{z_a-x_0}{\epsilon_a}\right|^{h+4}|W_a|^{p+2}\mathrm{d}x\geq M.\end{aligned}\tag{3.65}$$

Similar to (3.54), we have

$$\begin{aligned}e_{\beta^*}(a)=E_a^{\beta^*}(u_a) &= \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla u_a|^2+V(x)|u_a|^2]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_a|^2\mathrm{d}x\right)^2\\ &\quad -\frac{\beta^*}{p+2}\int_{\mathbb{R}^N}|u_a|^{p+2}\mathrm{d}x+\frac{\beta^*}{p+2}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x\\ &\geq \frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_a|^2\mathrm{d}x+\frac{\beta^*}{p+2}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x\\ &\geq \frac{a}{2}{\epsilon_a}^{-2}+\frac{\beta^*}{p+2}M{\epsilon_a}^h\\ &\geq Ca^{\frac{h}{h+2}}M^{\frac{2}{h+2}},\end{aligned}\tag{3.66}$$

which leads to a contradiction with (3.63) since M is arbitrary large and (3.64) holds. Therefore, passing to a subsequence, we may assume that there exists $y_0\in\mathbb{R}^N$ such that

$$\frac{z_a-z_0}{\epsilon_a}\to y_0 \quad \text{as } a\to 0.\tag{3.67}$$

Since $\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}$ converges uniformly to a constant as $a\to 0$ and the integrand satisfies the dominated convergence condition, then it follows from (3.34) that

$$
\begin{aligned}
&\liminf_{a\to 0}\frac{1}{\epsilon_a^h}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x\\
&=\liminf_{a\to 0}\frac{1}{\epsilon_a^{h+4}}\int_{\mathbb{R}^N}[1-m(\epsilon_a x+z_a)]|W_a|^{p+2}\mathrm{d}x\\
&=\liminf_{a\to 0}\int_{\mathbb{R}^N}\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}\left|x+\frac{z_a-x_0}{\epsilon_a}\right|^{h+4}|W_a|^{p+2}\mathrm{d}x\\
&=\liminf_{a\to 0}\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}\frac{1}{\|Q\|_2^{p+2}}\int_{\mathbb{R}^N}|x+y_0|^{h+4}Q^{p+2}(x)\mathrm{d}x\\
&\geq\liminf_{a\to 0}\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}\frac{1}{\|Q\|_2^{p+2}}\int_{\mathbb{R}^N}|x|^{h+4}Q^{p+2}(x)\mathrm{d}x\\
&=\frac{\kappa_0}{\|Q\|_2^{p+2}}.
\end{aligned}
\tag{3.68}
$$

Therefore, using Young's inequality again we can obtain that

$$
\begin{aligned}
e_{\beta^*}(a)=E_a^{\beta^*}(u_a)&\geq\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_a|^2\mathrm{d}x+\frac{\beta^*}{p+2}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x\\
&\geq\frac{a}{2}{\epsilon_a}^{-2}+\frac{\beta^*\kappa_0}{(p+2)\|Q\|_2^{p+2}}{\epsilon_a}^{h}(1+o(1))\\
&\geq\frac{a^{\frac{h}{h+2}}(h+2)\,(1+o(1))}{2h}\left(\frac{h\beta^*\kappa_0}{(p+2)\|Q\|_2^{p+2}}\right)^{\frac{2}{h+2}}\quad\text{as } a\to 0.
\end{aligned}
\tag{3.69}
$$

Hence,

$$
\begin{aligned}
\liminf_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{h}{h+2}}}&\geq\frac{h+2}{2h}\left(\frac{h\beta^*\kappa_0}{(p+2)\|Q\|_2^{p+2}}\right)^{\frac{2}{h+2}}\\
&=\frac{h+2}{2h}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^{2}}\right)^{\frac{2}{h+2}},
\end{aligned}
\tag{3.70}
$$

where the equality holds if and only if $y_0=0$ and

$$
\lim_{a\to 0}\frac{\epsilon_a}{\left(\frac{2a(p+2)\|Q\|_2^2}{hb\kappa_0}\right)^{\frac{1}{h+2}}}=1.
\tag{3.71}
$$

Moreover, it follows from (3.63) and (3.70) that

$$
\lim_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{h}{h+2}}}=\frac{h+2}{2h}\left(\frac{hb\kappa_0}{2(p+2)\|Q\|_2^{2}}\right)^{\frac{2}{h+2}}.
\tag{3.72}
$$

Case 3. When $q_0=h$, choosing

$$
\tau=\left(\frac{(p+2)h\mu_0+hb\kappa_0}{2a(p+2)\|Q\|_2^2}\right)^{\frac{1}{h+2}}=\left\{\frac{h}{2a(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right]\right\}^{\frac{1}{h+2}}.
\tag{3.73}
$$

Following from (3.48), we have

$$\begin{aligned} e_{\beta^*}(a) \le E_a^{\beta^*}(K_R,\tau) &\le \frac{a}{2}\tau^2 + \frac{(p+2)\mu_0 + b\kappa_0}{2(p+2)\|Q\|_2^2}\tau^{-h} \\ &= \frac{(h+2)a^{\frac{h}{h+2}}}{2h}\left\{\frac{h}{2(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right]\right\}^{\frac{2}{h+2}}(1+o(1)) \quad \text{as } a\to 0, \end{aligned} \tag{3.74}$$

which implies

$$\limsup_{a\to 0}\frac{e_\beta^*(a)}{a^{\frac{h}{h+2}}} \le \frac{h+2}{2h}\left\{\frac{h}{2(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right]\right\}^{\frac{2}{h+2}}. \tag{3.75}$$

Similar to (3.52)–(3.54) and (3.64)–(3.66), we claim that there exists $y_0\in\mathbb{R}^N$ such that

$$\frac{z_a - x_0}{\epsilon_a} \to y_0 \quad \text{as } a\to 0. \tag{3.76}$$

From (3.56) and (3.68), we conclude that

$$\begin{aligned} \liminf_{a\to 0}\frac{1}{\epsilon_a^h}\int_{\mathbb{R}^N}V(x)|u_a|^2\mathrm{d}x &= \liminf_{a\to 0}\frac{1}{\epsilon_a^h}\int_{\mathbb{R}^N}V(\epsilon_a x+z_a)|W_a|^2\mathrm{d}x \\ &\ge \liminf_{a\to 0}\frac{V(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^h}\frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^N}|x|^hQ^2(x)\mathrm{d}x \\ &= \frac{\mu_0}{\|Q\|_2^2} \end{aligned} \tag{3.77}$$

and

$$\begin{aligned} &\liminf_{a\to 0}\frac{1}{\epsilon_a^h}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x \\ &= \liminf_{a\to 0}\frac{1}{\epsilon_a^{h+4}}\int_{\mathbb{R}^N}[1-m(\epsilon_a x+z_a)]|W_a|^{p+2}\mathrm{d}x \\ &\ge \liminf_{a\to 0}\frac{1-m(\epsilon_a x+z_a)}{|\epsilon_a x+z_a-x_0|^{h+4}}\frac{1}{\|Q\|_2^{p+2}}\int_{\mathbb{R}^N}|x|^{h+4}Q^{p+2}(x)\mathrm{d}x \\ &= \frac{\kappa_0}{\|Q\|_2^{p+2}}. \end{aligned} \tag{3.78}$$

Using (1.7) and Young's inequality again, which implies

$$\begin{aligned} e_{\beta^*}(a) = E_a^{\beta^*}(u_a) &\ge \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla u_a|^2+V(x)|u_a|^2]\mathrm{d}x + \frac{\beta^*}{p+2}\int_{\mathbb{R}^N}[1-m(x)]|u_a|^{p+2}\mathrm{d}x \\ &\ge \frac{a}{2}{\epsilon_a}^{-2} + \left[\frac{\mu_0}{2\|Q\|_2^2}{\epsilon_a}^h + \frac{\beta^*\kappa_0}{(p+2)\|Q\|_2^{p+2}}{\epsilon_a}^h\right](1+o(1)) \\ &= \frac{a}{2}{\epsilon_a}^{-2} + \frac{{\epsilon_a}^h}{2(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right](1+o(1)) \\ &\ge \frac{a^{\frac{h}{h+2}}(h+2)(1+o(1))}{2h}\left\{\frac{h}{2(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right]\right\}^{\frac{2}{h+2}} \quad \text{as } a\to 0. \end{aligned} \tag{3.79}$$

Hence,

$$\liminf_{a\to 0}\frac{e_{\beta^*}(a)}{a^{\frac{h}{h+2}}} \ge \frac{h+2}{2h}\left\{\frac{h}{2(p+2)\|Q\|_2^2}\left[(p+2)\mu_0+b\kappa_0\right]\right\}^{\frac{2}{h+2}}, \tag{3.80}$$

where the equality holds if and only if $y_0 = 0$ and

$$\lim_{a\to 0} \frac{\epsilon_a}{\left\{\frac{2a(p+2)\|Q\|_2^2}{h[(p+2)\mu_0+b\kappa_0]}\right\}^{\frac{1}{h+2}}} = 1. \tag{3.81}$$

Moreover, it follows from (3.75) and (3.80) that

$$\lim_{a\to 0} \frac{e_{\beta^*}(a)}{a^{\frac{h}{h+2}}} = \frac{h+2}{2h} \left\{\frac{h}{2(p+2)\|Q\|_2^2} [(p+2)\mu_0 + b\kappa_0]\right\}^{\frac{2}{h+2}}. \tag{3.82}$$

It is worth noting that equations (3.60), (3.72) and (3.82) illustrate that inequalities (3.58), (3.70) and (3.80) all become equalities, and $y_0 = 0$. Then, using (1.16) and $l = \min\{q_0, h\}$, from (3.59), (3.71) and (3.81) we conclude that

$$\lim_{a\to 0} \frac{\epsilon_a}{\bar{\epsilon}_a} = 1. \tag{3.83}$$

Therefore, (1.13) holds by (3.34) and (3.83). Also, (1.14) holds by (3.55), (3.67) and (3.76). Moreover, (1.15) follows from (3.60), (3.72) and (3.82). □

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References

[1] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348**(1996), No. 1, 305–330. https://doi.org/10.1090/s0002-9947-96-01532-2; MR1333386; Zbl 0858.35083

[2] C. O. Alves, F. Corrêa, T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.* **49**(2005), No. 1, 85–93. https://doi.org/10.1016/j.camwa.2005.01.008; MR2123187; Zbl 1130.35045

[3] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$: Existence and multiplicity results, *Comm. Partial Differential Equations* **20**(1995), No. 9–10, 1725–1741. https://doi.org/10.1080/03605309508821149; MR1349229; Zbl 0837.35043

[4] G. F. Carrier, On the non-linear vibration problem of the elastic string, *Quart. Appl. Math.* **3**(1945), No. 2, 157–165. https://doi.org/10.1090/qam/12351; MR0012351; Zbl 0063.00715

[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation, *Adv. Differential Equations* **6**(2001), No. 6, 701–730. https://doi.org/10.57262/ade/1357140586; MR1829093;Zbl 1007.35049

[6] G. F. Carrier, A note on the vibrating string, *Quart. Appl. Math.* **7**(1949), No. 1, 97–101. https://doi.org/10.1090/qam/28511; MR0028511; Zbl 0033.03003

[7] S. Chen, V. D. Rădulescu, X. Tang, Normalized solutions of nonautonomous Kirchhoff equations: sub- and super-critical cases, *Appl. Math. Optim.* **84**(2021), No. 1, 773–806. https://doi.org/10.1007/s00245-020-09661-8; MR4802999; Zbl 1473.35149

[8] S. Chen, X. Tang, Normalized solutions for Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities, *Math. Ann.* **391**(2025), No. 2, 2783–2836. https://doi.org/10.1007/s00208-024-02982-x; MR4853031; Zbl 1556.35137

[9] P. D'ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **108**(1992), No. 2, 247–262. https://doi.org/10.1007/BF02100605; MR1161092; Zbl 0785.35067

[10] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, S. Stringari, Theory of Bose–Einstein condensation in trapped gases, *Rev. Modern Phys.* **71**(1999), 463. https://doi.org/10.1103/revmodphys.71.463

[11] Y. Deng, Y. Guo, L. Lu, On the collapse and concentration of Bose–Einstein condensates with inhomogeneous attractive interactions, *Calc. Var. Partial Differential Equations* **54**(2015), No. 1, 99–118. https://doi.org/10.1007/s00526-014-0779-9; MR3385154; Zbl 1328.35209

[12] Y. Deng, S. Peng, W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$, *J. Funct. Anal.* **269**(2015), No. 11, 3500–3527. https://doi.org/10.1016/j.jfa.2015.09.012; MR3406860; Zbl 1343.35081

[13] M. Du, X. Gao, L. Tian, Existence and limit behavior of normalized ground state solutions for a class of non-autonomous Kirchhoff equations, *Z. Angew. Math. Phys.* **75**(2024), No. 5, Paper No. 190, 25 pp. https://doi.org/10.1007/s00033-024-02328-2; MR4803508; Zbl 1550.35226

[14] A. L. Fetter, Rotating trapped Bose–Einstein condensates, *Rev. Modern Phys.* **81**(2009), No. 2, 647–691. https://doi.org/10.1103/revmodphys.81.647

[15] G. M. Figueiredo, N. Ikoma, J. R. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, *Arch. Ration. Mech. Anal.* **213**(2014), No. 3, 931–979. https://doi.org/10.1007/s00205-014-0747-8; MR3218834; Zbl 1302.35356

[16] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$, Mathematical analysis and applications, New York–London, 1981. MR0634248; Zbl 0469.35052

[17] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin–New York, 1977. MR0473443; Zbl 0361.35003

[18] H. Guo, H. Liu, L. Zhao, Concentration behavior and local uniqueness of normalized solutions for Kirchhoff type equation, *Z. Angew. Math. Phys.* **75**(2024), No. 3, Paper No. 89, 19 pp. https://doi.org/10.1007/s00033-024-02231-w; MR4735980; Zbl 1539.35093

[19] H. Guo, H. Liu, L. Zhao, Properties of the constrained minimizers for Kirchhoff type energy functionals with inhomogeneous interactions, *Math. Methods Appl. Sci.* **48**(2025), No. 14, 13344–13354. https://doi.org/10.1002/mma.11106; MR4946437; Zbl 08095309

[20] Y. Guo, R. Seiringer, On the mass concentration for Bose–Einstein condensates with attractive interactions, *Lett. Math. Phys.* **104**(2014), No. 2, 141–156. https://doi.org/10.1007/s11005-013-0667-9; MR3152151; Zbl 1311.35241

[21] Y. Guo, X. Zeng, H. S. Zhou, Concentration behavior of standing waves for almost mass critical nonlinear Schrödinger equations, *J. Differential Equations* **256**(2014), No. 7, 2079–2100. https://doi.org/10.1016/j.jde.2013.12.012; MR3160437; Zbl 1285.35109

[22] H. Guo, H. S. Zhou, Properties of the minimizers for a constrained minimization problem arising in Kirchhoff equation, *Discrete Contin. Dyn. Syst.* **41**(2021), No. 3, 1023–1050. https://doi.org/10.3934/dcds.2020308; MR4201831; Zbl 1459.35157

[23] T. Hu, L. Lu, Concentration and local uniqueness of minimizers for mass critical degenerate Kirchhoff energy functional, *J. Differential Equations* **363**(2023), 275–306. https://doi.org/10.1016/j.jde.2023.03.023; MR4564934; Zbl 1514.35138

[24] T. Hu, C. L. Tang, Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations, *Calc. Var. Partial Differential Equations* **60**(2021), No. 6, Paper No. 210, 26 pp. https://doi.org/10.1007/s00526-021-02018-1; MR4305432; Zbl 1473.35266

[25] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.* **28**(1997), No. 10, 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1; MR1430506; Zbl 0877.35091

[26] K. M. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.* **105**(1989), No. 3, 243–266. https://doi.org/10.1007/BF00251502; MR0969899; Zbl 0676.35032

[27] G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differential Equations* **257**(2014), No. 2, 566–600. https://doi.org/10.1016/j.jde.2014.04.011; MR3200382; Zbl 1290.35051

[28] G. Li, H. Ye, On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials, *J. Differential Equations* **266**(2019), No. 11, 7101–7123. https://doi.org/10.1016/j.jde.2018.11.024; MR3926094; Zbl 1423.35106

[29] E. H. Lieb, M. Loss, *Analysis*, American Mathematical Society, 2001. https://doi.org/10.1353/plo.2020.0074; MR1817225; Zbl 0873.26002

[30] J. L. Lions, On some questions in boundary value problems of mathematical physics. Contemporary developments in: *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, North-Holland Math. Stud., Vol. 30, North-Holland, Amsterdam–New York, 1978, pp. 284–346. MR0519648; Zbl 0404.35002

[31] T. F. Ma, J. E. M. Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, *Appl. Math. Lett.* **16**(2003), No. 2, 243–248. https://doi.org/10.1016/S0893-9659(03)80038-1; MR1962322; Zbl 1135.35330

[32] I. Michael, Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* **87**(1982), No. 4, 567–576. https://doi.org/10.1007/BF01208265; MR0691044; Zbl 0527.35023

[33] K. Perera, Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations* **221**(2006), No. 1, 246–255. https://doi.org/10.1016/j.jde.2005.03.006; MR2193850; Zbl 1357.35131

[34] J. Wang, L. Tian, J. Xu, F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differential Equations* **253**(2012), No. 7, 2314–2351. https://doi.org/10.1016/j.jde.2012.05.023; MR2946975; Zbl 1402.35119

[35] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhäuser Boston Inc, Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[36] W. Xie, H. Chen, Existence and multiplicity of normalized solutions for the nonlinear Kirchhoff type problems, *Comput. Math. Appl.* **76**(2018), No. 3, 579–591. https://doi.org/10.1016/j.camwa.2018.04.038; MR3812371; Zbl 1422.35057

[37] H. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, *Math. Methods Appl. Sci.* **38**(2015), No. 13, 2663–2679. https://doi.org/10.1002/mma.3247; MR3382697; Zbl 1331.35134

[38] H. Ye, The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.* **66**(2015), No. 4, 1483–1497. https://doi.org/10.1007/s00033-014-0474-x; MR3377698; Zbl 1322.35032

[39] H. Ye, The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.* **67**(2016), No. 2, 29. https://doi.org/10.1007/s00033-016-0624-4; MR3483878; Zbl 1341.35059