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Concentrated solutions to fractional Kirchhoff equations with inhomogeneous perturbation

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Abstract. This paper is concerned with normalized solutions for a class of fractional Kirchhoff equations with an inhomogeneous perturbation in $\mathbb{R}^2$. We study the constrained minimization problem associated with the corresponding nonlocal energy functional under a prescribed L^2-mass. The interaction between the Kirchhoff term, the fractional Laplacian and the spatially dependent coefficient $f(x)$ leads to several compactness and asymptotic difficulties. For the zero-potential case, we establish existence and nonexistence results for constrained minimizers. When a trapping potential is present, we prove the existence of minimizers in the subcritical case and characterize the threshold in the critical case $p = 4s$. In the mass-critical case $p = 2s$, we analyze the concentration behavior of non-negative minimizers as the Kirchhoff parameter tends to zero. We prove that the minimizers concentrate at global minimum points of the potential and, after rescaling, converge to the unique positive radial solution of the limiting fractional scalar field equation. A sharper description of the concentration location and the energy asymptotics is obtained when the potential is almost homogeneous near its minimum points.

Keywords: limiting behavior, energy estimate, blow-up analysis, normalized solutions.

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1 Introduction

Fractional and nonlocal equations have attracted considerable attention in recent decades. They appear in several models where the state of the system is influenced not only by local variations but also by long-range interactions, such as anomalous diffusion, quantum mechanics, finance, phase transitions, obstacle problems and conformal geometry. From the variational point of view, these problems are naturally formulated in fractional Sobolev spaces. In this setting, the lack of locality changes both the compactness mechanism and the scaling properties of the associated energy functionals, and it often leads to concentration phenomena which have no direct analogue in the classical local case. For general background and related applications, we refer to [5,7,9,19,20,26,30].

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We also mention some recent works in fractional analysis which are related to the broader development of the field. Commutator estimates for fractional integral operators on vanishing Morrey spaces were studied in [29]. Fractional derivatives associated with the Riemann zeta function and their relation with prime numbers were considered in [14], while hybrid fractional differential equations involving proportional derivatives were investigated in [1]. Connections among fractional calculus, zeta functions and Shannon entropy were discussed in [15]. Nonlinear fractional differential inclusions with a non-singular Mittag-Leffler kernel were treated in [2]. These contributions illustrate different directions in current fractional analysis and provide additional motivation for studying nonlocal variational problems such as the one considered here.

In this paper we consider the following fractional Kirchhoff equation with an inhomogeneous perturbation:

$$\left(1+b\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)(-\Delta)^s u+V(x)u=\beta f(x)|u|^p u+\lambda u \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $s\in\left[\frac{1}{2},1\right), b,\beta>0, \lambda\in\mathbb{R}\setminus\{0\}$, and $p\in(0,2_s^*-2)$ with $2_s^*=\frac{2}{1-s}$. Throughout the paper,the potential $V(x)$ and the coefficient $f(x)$ satisfy the following conditions:

($\mathbf{V_1}$) $0\le V(x)\in C^1(\mathbb{R}^2)$, $\lim_{|x|\to\infty}V(x)=\infty$, and $\inf_{x\in\mathbb{R}^2}V(x)=0$ is attained.

($\mathbf{F_1}$) $0<f(x)\le 1$, and there exists $\bar{x}\in\mathbb{R}^2$ such that $f(\bar{x})=\sup_{x\in\mathbb{R}^2}f(x)=1$.

($\mathbf{F_2}$) $0<f(x)\le 1$, x_0 is a unique maximum point of $f(x)$, and

$$1-f(x)=O(|x-x_0|^{q+2s}) \quad \text{as } x\to x_0,$$

where $q\in(0,4s^2+4s+2)$.

($\mathbf{F_\infty}$) $0<f(x)\le 1$, and $\lim_{|x|\to\infty}f(x)<1$.

For $s\in(0,1)$, the fractional Laplacian is defined by

$$(-\Delta)^s v(x)=C_s P.V.\int_{\mathbb{R}^2}\frac{v(x)-v(y)}{|x-y|^{2+2s}}dy=C_s\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus B_\varepsilon(x)}\frac{v(x)-v(y)}{|x-y|^{2+2s}}dy,$$

for $v\in\mathcal{S}(\mathbb{R}^2)$, where P.V. denotes a Cauchy principal value, $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space of rapidly decaying C^∞ function, $B_\varepsilon(x)$ is the ball centered at x with radius ε, and

$$C_s=\left(\int_{\mathbb{R}^2}\frac{1-\cos(\xi_1)}{|\xi|^{2+2s}}\right)^{-1}.$$

For more details on the fractional Laplacian and related Sobolev spaces, see [8,9,34] and the references therein.

We are interested in normalized solutions of (1.1), namely solutions obtained under the prescribed mass constraint

$$\int_{\mathbb{R}^N}|u|^2dx=\|u\|_2^2=1.$$

Such a constraint arises in the mathematical description of nonlinear optics and Bose-Einstein condensates, where the L^2-norm of the wave function represents the optical power or the total number of particles, respectively. Under this constraint, the parameter λ is not prescribed in advance but appears as the Lagrange multiplier associated with the mass constraint. Consequently, the study of normalized solutions is closely connected with constrained variational problems and provides useful information on the existence, orbital stability and concentration behavior of ground states.

When $s = 1$, equation (1.1) is related to Kirchhoff's model for the transversal vibration of an elastic string,

$$u_{tt} - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = h(x,u), \quad \text{in } \mathbb{R}^N,$$

where u is the displacement, h is an external force, b is the initial tension, and a depends on intrinsic properties of the string. Kirchhoff [18] introduced this model to describe the effect of the change of the string length during oscillations. The presence of the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$ leads to the loss of locality in the equation. Since the pioneering work of Lions [21], Kirchhoff-type problems have been extensively studied; see, for instance, [6,11,16,17,24,28,31,32,35] and the references therein.

Fractional Kirchhoff equations combine Kirchhoff-type nonlocality with the nonlocal diffusion induced by the fractional Laplacian. Consequently, both the compactness properties of the associated functional and the scaling behavior of solutions become more delicate, so that their analysis requires variational techniques that differ from those employed for classical local Schrödinger equations and standard fractional Schrödinger equations.

If the frequency λ in (1.1) is fixed, a classical approach is to search for critical points of the functional $J : \mathcal{H} \to \mathbb{R}$ given by

$$\begin{aligned} J(u) = &\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + (V(x)-\lambda)|u|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx\right)^2 \\ &- \frac{\beta}{p+2}\int_{\mathbb{R}^2} f(x)|u|^{p+2}dx, \end{aligned}$$

where

$$\mathcal{H} := \left\{u \in H^s(\mathbb{R}^2) : \int_{\mathbb{R}^2}(V(x)-\lambda)u^2 dx < \infty\right\}.$$

Here $H^s(\mathbb{R}^2)$ denotes the fractional Sobolev space endowed with the norm

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + u^2\right)dx,$$

and

$$D^{s,2}(\mathbb{R}^2) := \left\{u \in L^{2_s^*}(\mathbb{R}^2) : (-\Delta)^{\frac{s}{2}}u \in L^2(\mathbb{R}^2)\right\}$$

denotes the homogeneous fractional Sobolev space with the norm

$$\|u\|_{D^{s,2}}^2 := \int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx.$$

For $s \in (0,1)$, there exists a best constant $R_s > 0$ such that

$$R_s = \inf_{u \in D^{s,2}\setminus\{0\}} \frac{\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx}{\left(\int_{\mathbb{R}^2}|u|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}.$$

Related fractional Kirchhoff problems have been studied in [3,4,13,17,23] and the references therein.

In the present paper we take a different point of view and study (1.1) through the following constrained minimization problem:

$$d^f_\beta(b,p) := \inf_{u\in M} E^f_{\beta,b,p}(u), \tag{1.2}$$

where

$$\begin{aligned} E^f_{\beta,b,p}(u) = &\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + V(x)|u|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 \\ &- \frac{\beta}{p+2}\int_{\mathbb{R}^2} f(x)|u|^{p+2}dx, \quad u\in E. \end{aligned} \tag{1.3}$$

Here

$$E := \left\{u\in H^s(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u|^2dx < \infty\right\}, \tag{1.4}$$

with norm

$$\|u\|^2 = \int_{\mathbb{R}^2}\left[\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + (1+V(x))u^2\right]dx$$

and

$$M := \left\{u\in E : \int_{\mathbb{R}^2}|u|^2dx = 1\right\}. \tag{1.5}$$

The minimization problem is driven by the competition between the positive quadratic and Kirchhoff terms and the negative inhomogeneous nonlinear term. By a scaling argument, one can show that $d^f_\beta(b,p) = -\infty$ for $p > 4s$ and $b > 0$, which motivates us to restrict our attention to the range $0 < p \le 4s$.

We first consider the case of zero potential. It follows from Theorem 1.1 and Remark 1.1 in [10] (see also Lemma 2.1 below) that the equation

$$(-\Delta)^s u + \frac{p(s-1)+2s}{p}u - \frac{2s}{p}u^{p+1} = 0, \quad p\in(0,2^*_s-2) \quad\text{and}\quad u\in H^s(\mathbb{R}^2), \tag{1.6}$$

admits, up to translations, a unique positive radial solution. We denote this solution by $Q_p = Q_p(|x|)$. Our first result gives the exact existence threshold when $V\equiv 0$.

Theorem 1.1. *If $V(x)=0$, $p\in(0,4s]$, $s\in\left[\frac{1}{2},1\right)$, $b>0$, $f(x)$ satisfies (F_1), (F_2) and (F_∞), define*

$$\hat\beta_p = \begin{cases} 0, & p\in(0,2s), \\ \|Q_p\|_2^{2s}, & p=2s, \\ 2\|Q_p\|_2^p\left(\frac{s}{4s-p}\right)^{\frac{4s-p}{2s}}\left(\frac{sb}{2p-4s}\right)^{\frac{p-2s}{2s}}, & p\in(2s,4s), \end{cases} \tag{1.7}$$

and

$$\hat\beta_{4s} = \begin{cases} \frac{b}{2}\|Q_{4s}\|_2^{4s}, & p=4s \text{ and } s\in\left(\frac{1}{2},1\right), \\ bR_s^2, & p=4s \text{ and } s=\frac{1}{2}. \end{cases} \tag{1.8}$$

Then,

(i) Problem (1.2) admits a minimizer if and only if $\beta > \hat\beta_p$ for $p\in(0,2s]$, or $\beta\ge\hat\beta_p$ for $p\in(2s,4s)$.

(ii) For $p=4s$, problem (1.2) has no minimizers for any $\beta>0$.

For fractional Schrödinger equations with L^2-constraints, Du, Wang, Tian and Zhang [10] studied

$$e(a) := \inf_{u \in M} E_a(u), \tag{1.9}$$

where the energy functional $E_a(u)$ is defined by

$$E_a(u) = \int_{\mathbb{R}^2} \left(\left| (-\Delta)^{\frac{s}{2}} u \right|^2 + V(x)|u|^2 \right) dx - \frac{a}{1+s} \int_{\mathbb{R}^N} |u|^{2+2s} dx, \quad u \in E.$$

They proved existence for $a \in [0, \beta^*)$ and nonexistence for $a \in [\beta^*, +\infty)$, where $\beta^* = \|Q\|_2^{2s}$ and Q is the unique positive radial solution of

$$(-\Delta)^s u + su - |u|^{2s} u = 0, \quad \text{in } \mathbb{R}^2. \tag{1.10}$$

Moreover, under suitable assumptions on the local behavior of V near its zero set, they also derived the concentration behavior of non-negative minimizers as $a \nearrow \beta^*$. In the present setting, the Kirchhoff term changes the scaling of the energy functional, while the spatially dependent coefficient $f(x)$ introduces an additional inhomogeneous effect. These features make the variational analysis substantially more involved.

To state our remaining results, define

$$\beta_p := \frac{b}{2} \|Q_p\|_2^p. \tag{1.11}$$

We next consider the case where a trapping potential is present. The following two theorems describe the existence and nonexistence of minimizers in this setting.

Theorem 1.2. *Let $p \in (0, 4s)$, $s \in \left[\frac{1}{2}, 1\right)$, and $b > 0$. Suppose that $V(x)$ satisfies (V_1), $f(x)$ satisfies (F_1) and (F_2). Then, for any fixed $\beta > 0$, problem (1.2) has at least one minimizer.*

Theorem 1.3. *Let $p = 4s$ with $s \in \left(\frac{1}{2}, 1\right)$ and $b > 0$. Assume that $V(x)$ satisfies (V_1), $f(x)$ satisfies (F_1) and (F_2). Then the following statements hold:*

(i) If $0 < \beta \leq \beta_{4s} := \frac{b}{2} \|Q_{4s}\|_2^{4s}$, then $d_\beta^f(b, 4s) > 0$ and problem (1.2) has at least one minimizer.

(ii) If $\beta > \beta_{4s}$, then $d_\beta^f(b, 4s) = -\infty$ and problem (1.2) has no minimizers.

We next consider the mass-critical case $p = 2s$ and investigate the limiting behavior as the Kirchhoff parameter tends to zero. This regime is more delicate, since the associated fractional Gagliardo–Nirenberg inequality is attained at the critical threshold.

Theorem 1.4. *Let $p = 2s$, $s \in \left(\frac{1}{2}, 1\right)$ and suppose that $V(x)$ satisfies (V_1), $f(x)$ satisfies (F_1) and (F_2). For any given $\beta \geq \beta^*$, let u_k be a non-negative minimizer of $d_\beta^f(b_k, 2s)$ with $b_k \to 0^+$ as $k \to \infty$. Then, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that u_k concentrates at a global minimum point x_0 of $V(x)$ in the following sense: for each large k, u_k has a unique global maximum point $\tilde{z}_k \in \mathbb{R}^2$ and satisfies*

$$\epsilon_k u_k(\epsilon_k x + \tilde{z}_k) \xrightarrow{k} \frac{Q_{2s}(x)}{(\beta^*)^{\frac{1}{2s}}} \quad \text{in } H^s(\mathbb{R}^2), \tag{1.12}$$

where $Q_{2s}(x) = Q(x)$ *is the unique positive solution of* (1.10), $\tilde{z}_k \xrightarrow{k} x_0$ *, and* $\epsilon_k \xrightarrow{k} 0^+$ *is given by*

$$\epsilon_k = \begin{cases} \left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx\right)^{-\frac{1}{2s}}, & \beta = \beta^*, \\ \left(\frac{b_k \beta^*}{\beta-\beta^*}\right)^{\frac{1}{2s}}, & \beta > \beta^*. \end{cases} \tag{1.13}$$

Moreover, if $\beta > \beta^*$*,then*

$$d^f_\beta(b_k, 2s) = -\frac{1}{4b_k}\left(\frac{\beta-\beta^*}{\beta^*}\right)^2 (1+o(1)) \quad \text{as } k \to \infty. \tag{1.14}$$

To obtain a sharper description of the concentration point, we impose additional assumptions on the local structure of the potential $V(x)$ near its zero set. In particular, we assume that $V(x)$ possesses exactly m global minimum points,

$$Y := \left\{x \in \mathbb{R}^2 : V(x) = 0\right\} = \{x_1, x_2, \dots, x_m\}, \quad \text{where } m \geq 1. \tag{1.15}$$

Assume further that V is almost homogeneous of degree $\eta_i > 0$ near each x_i, namely, there exists $V_i \in C^2_{\text{loc}}(\mathbb{R}^2)$ with $\lim_{|x|\to\infty} V_i(x) = \infty$, homogeneous of degree η_i, such that

$$\lim_{x\to 0} \frac{V(x+x_i)}{V_i(x)} = 1, \quad i = 1, 2, \dots, m. \tag{1.16}$$

For $i = 1, 2, \dots, m$, define

$$M_i(y) := \int_{\mathbb{R}^2} V_i(x+y) Q^2_{2s}(x) dx. \tag{1.17}$$

Set

$$\eta := \max_{1\leq i\leq m} \eta_i, \quad \bar{Y} := \{x_i \in Y : \eta_i = \eta\} \subset Y \tag{1.18}$$

and

$$\theta_0 := \min_{i\in\kappa} \theta_i, \quad \text{where } \theta_i = \min_{y\in\mathbb{R}^2} M_i(y) \quad \text{and} \quad \kappa := \left\{i : x_i \in \bar{Y}\right\}. \tag{1.19}$$

Denote

$$Y_0 := \left\{x_i \in \bar{Y} : \theta_i = \theta_0\right\}. \tag{1.20}$$

Under the above assumptions, the concentration location and the leading-order energy can be identified more precisely.

Theorem 1.5. *Under the assumptions of Theorem 1.4, suppose further that* $V(x)$ *satisfies the additional condition* (1.16)*. Let* u_k *and* $\tilde{z}_k$ *be obtained in Theorem 1.4, then we have*

$$\tilde{\epsilon}_k u_k(\tilde{\epsilon}_k x + \tilde{z}_k) \xrightarrow{k} \frac{Q_{2s}(x)}{(\beta^*)^{\frac{1}{2s}}} \quad \text{in } H^s(\mathbb{R}^2), \tag{1.21}$$

where $\tilde{\epsilon}_k$ *is given by*

$$\tilde{\epsilon}_k = \begin{cases} \left[\frac{2sb_k(\beta^*)^{\frac{1}{s}}}{\eta\theta_0}\right]^{\frac{1}{\eta+4s}}, & \beta = \beta^*, \\ \left(\frac{b_k\beta^*}{\beta-\beta^*}\right)^{\frac{1}{2s}}, & \beta > \beta^*, \end{cases} \tag{1.22}$$

and x_0 *satisfies*

$$\frac{\tilde{z}_k - x_0}{\tilde{\epsilon}_k} \xrightarrow{k} y_0, \tag{1.23}$$

with $x_0 = x_{i_0} \in Y_0$ *for some* $1 \le i_0 \le m$*, and* $y_0 \in \mathbb{R}^2$ *satisfying* $M_{i_0}(y_0) = \theta_0$*. Moreover, if* $\beta = \beta^*$*, we get*

$$\frac{d^f_{\beta^*}(b_k, 2s)}{b_k^{\frac{\eta}{\eta+4s}}} \xrightarrow{k} \left(\frac{1}{s}\right)^{\frac{4s}{\eta+4s}} \frac{\eta + 4s}{4\eta} \left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}} \left(\frac{1}{\beta^*}\right)^{\frac{4}{\eta+4s}}. \tag{1.24}$$

To provide a visual illustration of the concentration behavior established in Theorems 1.4 and 1.5, we present a schematic one-dimensional profile of the normalized minimizers in Figure 1.1. As $b \to 0^+$, the profiles become increasingly localized near the concentration point.

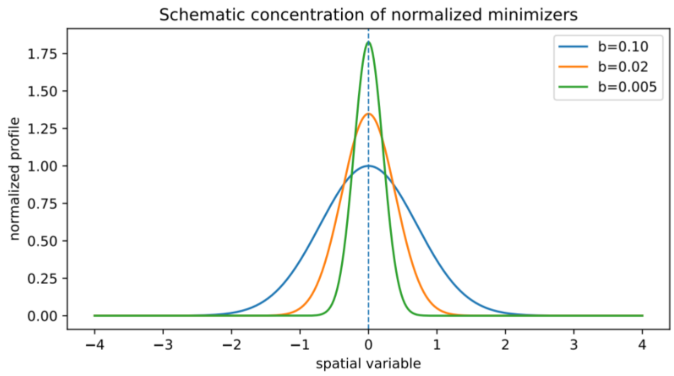

Figure 1.1: Schematic concentration profiles of normalized minimizers as $b \to 0^+$

Throughout this paper, we use the following notation:

- For $\rho > 0$ and $z \in \mathbb{R}^2$, $B_\rho(z)$ denotes the circle of radius ρ centered at z .
- The symbol $\rightharpoonup$ denotes weak convergence, and the symbol $\to$ denotes strong convergence.
- $L^q(\mathbb{R}^2)$ denotes the usual Lebesgue space with norm $\|u\|_q := \left(\int_{\mathbb{R}^2} |u|^q dx\right)^{\frac{1}{q}}$, $1 \le q \le \infty$.
- C, C_i $(i = 1, 2, 3, \ldots)$ denotes various positive constants, which may vary from one line to another and which is not important for the analysis of the problem.

The remainder of the paper is organized as follows. In Section 2, we collect some preliminary results. Section 3 is devoted to the zero-potential case and contains the proof of Theorem 1.1. In Section 4, we prove Theorems 1.2 and 1.3. Section 5 is devoted to energy estimates and blow-up analysis, which are used in the proof of Theorem 1.4. Finally, Section 6 contains the proof of Theorem 1.5.

2 Preliminaries

This section is devoted to several preliminary results that will be used throughout the sequel. We first recall the fractional Gagliardo–Nirenberg–Sobolev inequality established in [10].

Lemma 2.1. *For $u \in H^s(\mathbb{R}^2)$ and $0 < p < 2_s^* - 2$, the fractional Gagliardo–Nirenberg–Sobolev inequality*

$$\int_{\mathbb{R}^2} |u|^{p+2} dx \leq C_{opt} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{p}{2s}} \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^{\frac{sp+2s-p}{2s}}, \tag{2.1}$$

is attained at a function $Q_p(x)$ with the following properties:

(i) $C_{opt} = \frac{p+2}{2\|Q_p\|_2^p}$.

(ii) $Q_p(x)$ *is radial, positive, and strictly decreasing in* $|x|$.

(iii) $Q_p(x)$ *is the unique solution of the fractional Schrödinger equation (1.6).*

(iv) $Q_p(x)$ *belongs to* $H^{2s+1}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ *and satisfies*

$$\frac{C_1}{1+|x|^{2+2s}} \leq Q_p(x) \leq \frac{C_2}{1+|x|^{2+2s}}, \quad x \in \mathbb{R}^2,$$

where C_1, C_2 are positive constants.

We next recall the fractional Pohožaev identity established in Lemma 2.1 of [25].

Lemma 2.2. *Let $u \in H^s(\mathbb{R}^N)$, $N \geq 2$, satisfy the equation*

$$(-\Delta)^s u = g(u),$$

then it holds that

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx = N \int_{\mathbb{R}^N} G(u) dx,$$

where $G(s) = \int_0^s g(t)dt$.

Let Q_p be the unique (up to translations) radially symmetric positive solution of (1.6). It follows from Lemma 2.1 and Lemma 2.2 that a direct computation yields

$$\int_{\mathbb{R}^2} \left| (-\Delta)^{\frac{s}{2}} Q_p \right|^2 dx = \int_{\mathbb{R}^2} |Q_p|^2 dx = \frac{2}{p+2} \int_{\mathbb{R}^2} |Q_p|^{p+2} dx. \tag{2.2}$$

Lemma 2.3. *Let $s \in (0,1)$. For any $u \in H^s(\mathbb{R}^2)$, the following inequality holds:*

$$\int_{\mathbb{R}^4} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{2+2s}} dxdy \leq \int_{\mathbb{R}^4} \frac{(u(x) - u(y))^2}{|x-y|^{2+2s}} dxdy,$$

where u^* denotes the symmetric radial decreasing rearrangement of u.

From [10, 22], we obtain the following two compactness results.

Lemma 2.4. *Suppose that (V_1) holds. Then, the embedding $E \hookrightarrow L^q(\mathbb{R}^2)$ is compact for all $q \in [2, 2_s^*)$.*

Lemma 2.5. *Let $N \geq 2$, then $H_r^s(\mathbb{R}^N)$ is compactly embedding into $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$.*

The following vanishing lemma for fractional Sobolev spaces is taken from [10].

Lemma 2.6. *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_\rho(y)} |u_n|^2 dx = 0,$$

where $\rho > 0$. Then, $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.

Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq 2$. Define

$$Q_\tau(x) = \eta(x/\tau) Q_p(x), \quad x \in \mathbb{R}^2, \tag{2.3}$$

for any $\tau > 0$, where $Q_p(x)$ is given by Lemma 2.1. The following result follows from Lemma 3.2 in [10].

Lemma 2.7. *Let $s \in (0,1)$. Then, the following estimate holds true:*

$$\int_{\mathbb{R}^4} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{2+2s}} dxdy \leq \int_{\mathbb{R}^4} \frac{|Q_p(x) - Q_p(y)|^2}{|x-y|^{2+2s}} dxdy + O(\tau^{-4s}) \quad \text{as } \tau \to \infty.$$

3 Existence and nonexistence for (1.2) with $V(x) \equiv 0$

For simplicity, we rewrite (1.2) as follows:

$$\tilde{d}^f_\beta(b,p) := \inf_{u \in \tilde{M}} \tilde{E}^f_{\beta,b,p}(u), \tag{3.1}$$

where

$$\tilde{E}^f_{\beta,b,p}(u) := \frac{1}{2} \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 - \frac{\beta}{p+2} \int_{\mathbb{R}^2} f(x)|u|^{p+2} dx \tag{3.2}$$

and

$$\tilde{M} = \left\{ u \in H^s(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = 1 \right\}.$$

We first show that (3.1) is well defined for $p \in (0, 4s)$ and

$$\tilde{d}^f_\beta(b,p) \leq 0 \quad \text{for all } \beta > 0 \quad \text{and} \quad \tilde{d}^f_\beta(b,p) = 0 \quad \text{for all } \beta \leq 0. \tag{3.3}$$

Indeed, for any $\beta > 0$ and $u \in \tilde{M}$, by Lemma 2.1 and (F_1) , we deduce that

$$\begin{aligned} \tilde{E}^f_{\beta,b,p}(u) &\geq \frac{1}{2} \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - \frac{\beta}{p+2} \int_{\mathbb{R}^2} |u|^{p+2} dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - \frac{\beta}{2\|Q_p\|_2^p} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{p}{2s}}, \end{aligned} \tag{3.4}$$

which implies that $\tilde{E}^f_{\beta,b,p}$ is well defined since $p \in (0, 4s)$ and $\frac{p}{2s} \in (0,2)$. Moreover, it follows from (3.2) that $\tilde{d}^f_\beta(b,p) \geq 0$ for all $\beta \leq 0$. For $u \in \tilde{M}$, we set $u_t(x) = tu(tx)$ for $t > 0$. Then $u_t \in \tilde{M}$ and

$$\begin{aligned} \tilde{d}^f_\beta(b,p) \leq \tilde{E}^f_{\beta,b,p}(u_t) = \frac{t^{2s}}{2} \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{bt^{4s}}{4} \left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 \\ - \frac{\beta t^p}{p+2} \int_{\mathbb{R}^2} f\left(\frac{x}{t}\right) |u|^{p+2} dx \to 0 \quad \text{as } t \to 0, \end{aligned} \tag{3.5}$$

which implies that (3.3) holds.

Lemma 3.1. *Let $p \in (0, 4s)$. Then, $\tilde{d}^f_\beta(b, p) < 0$ if and only if $\beta > \hat{\beta}_p$, where $\hat{\beta}_p$ is defined by (1.7). $\tilde{d}^f_\beta(b, p) = 0$ for all $\beta \le \hat{\beta}_p$.*

Proof. Case 1: $p \in (0, 2s)$, It follows from (1.7) that $\hat{\beta}_p = 0$. Hence, (3.5) yields that $\tilde{d}^f_\beta(b, p) < 0$ for all $\beta > \hat{\beta}_p$, while (3.3) implies that $\tilde{d}^f_\beta(b, p) = 0$ for all $\beta \le \hat{\beta}_p$.

Case 2: $p = 2s$, If $\beta \le \hat{\beta}_p = \|Q_p\|_2^{2s}$. Let $u \in \tilde{M}$, it follows from Lemma 2.1 and (3.13) and (F_1) that

$$\tilde{E}^f_{\beta,b,p}(u) \ge \frac{\hat{\beta}_p - \beta}{2\hat{\beta}_p} \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 > 0,$$

which implies that $\tilde{d}^f_\beta(b, p) = 0$ since (3.3). If $\beta > \hat{\beta}_p = \|Q_p\|_2^{2s}$. Define

$$u_t = \frac{tQ_p(t(x - \bar{x}))}{\|Q_p\|_2} \quad \text{for } t > 0. \tag{3.6}$$

We have that $u_t \in \tilde{M}$ and $Q_p \in L^{p+2}(\mathbb{R}^2)$. It follows from (2.2), Lemma 2.1 and assumption (F_1) that

$$\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_t\right|^2 dx = t^{2s} \tag{3.7}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} f(x)|u_t|^{p+2} dx &= \frac{t^p}{\|Q_p\|_2^{p+2}} \int_{\mathbb{R}^2} f(\bar{x} + \frac{y}{t})|Q_p(y)|^{p+2} dy \\ &= \frac{t^p}{\|Q_p\|_2^{p+2}} \int_{\mathbb{R}^2} |Q_p(y)|^{p+2} dy + o(t^p) \\ &= \frac{(p+2)t^p}{2\|Q_p\|_2^p} + o(t^p) = \frac{(p+2)t^p}{2\hat{\beta}_p} + o(t^p). \end{aligned} \tag{3.8}$$

Then, we have

$$\tilde{E}^f_{\beta,b,p}(u_t) = \left(\frac{1}{2} - \frac{\beta}{2\hat{\beta}_p} \right) t^{2s} + \frac{t^{4s}}{4} + o(t^{2s}),$$

which shows that

$$\tilde{d}^f_\beta(b, p) \le \inf_{t>0} \tilde{E}^f_{\beta,b,p}(u_t) = -\frac{1}{4b} \left(\frac{\beta}{\hat{\beta}_p} - 1 \right)^2 < 0 \quad \text{as } \beta > \hat{\beta}_p.$$

Case 3: $p \in (2s, 4s)$, Let u_t be given by (3.6). Using (3.7) and (3.8), we obtain

$$\tilde{E}^f_{\beta,b,p}(u_t) = \frac{1}{2} t^{2s} + \frac{b}{4} t^{4s} - \frac{\beta}{2\|Q\|_2^p} t^p + o(t^p). \tag{3.9}$$

By the Young's inequality, we have

$$\frac{1}{2} t^{2s} + \frac{b}{4} t^{4s} \ge \left(\frac{mt^{2s}}{2} \right)^{\frac{1}{m}} \left(\frac{bnt^{4s}}{4} \right)^{\frac{1}{n}} = t^{2s + \frac{2s}{n}} \left(\frac{m}{2} \right)^{\frac{1}{m}} \left(\frac{bn}{4} \right)^{\frac{1}{n}}, \tag{3.10}$$

where $\frac{1}{m} + \frac{1}{n} = 1$, and equality holds if and only if

$$\frac{mt^{2s}}{2} = \frac{bnt^{4s}}{4}.$$

Let $\frac{2}{n} = \frac{p}{s} - 2$, so that $\frac{2}{m} = 4 - \frac{p}{s}$. Then (3.10) yields that

$$\frac{1}{2}t^{2s} + \frac{b}{4}t^{4s} \geq t^p \left(\frac{s}{4s-p}\right)^{\frac{4s-p}{2s}} \left(\frac{sb}{2p-4s}\right)^{\frac{p-2s}{2s}}. \tag{3.11}$$

Now, we define

$$\hat{\beta}_p = 2\|Q_p\|_2^p \left(\frac{s}{4s-p}\right)^{\frac{4s-p}{2s}} \left(\frac{sb}{2p-4s}\right)^{\frac{p-2s}{2s}}. \tag{3.12}$$

If $\beta > \hat{\beta}_p$, we take t_0 such that $\frac{mt_0^{2s}}{2} = \frac{bnt_0^{4s}}{4}$, then

$$\begin{aligned}\tilde{d}_\beta^f(b,p) \leq \tilde{E}_{\beta,b,p}^f(u_{t_0}) &= \frac{1}{2}t_0^{2s} + \frac{b}{4}t_0^{4s} - \frac{\beta}{2\|Q_p\|_2^p}t_0^p + o(t_0^p)\\ &= \frac{\hat{\beta}_p - \beta}{2\|Q_p\|_2^p}t_0^p + o(t_0^p) < 0.\end{aligned}$$

If $\beta \leq \tilde{\beta}_p$, it follows from Lemma 2.1, (F_1), and (3.6)–(3.8) that, for any $u \in \tilde{M}$,

$$\tilde{E}_{\beta,b,p}^f(u) \geq \frac{\hat{\beta}_p}{2\|Q_p\|_2^p}\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^{\frac{p}{2s}} - \frac{\beta}{2\|Q_p\|_2^p}\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^{\frac{p}{2s}} \geq 0,$$

which means that $\tilde{d}_\beta^f(b,p) \geq 0$. Hence, $\tilde{d}_\beta^f(b,p) = 0$ since (3.3). □

Lemma 3.2. *If $p = p^* = 4s$ with $s \in \left[\frac{1}{2}, 1\right)$, and suppose that $f(x)$ satisfies (F_1) and (F_2), then $\tilde{d}_\beta^f(b,4s) = 0$ for all $\beta \leq \hat{\beta}_{4s}$ and $\tilde{d}_\beta^f(b,4s) = -\infty$ for all $\beta > \hat{\beta}_{4s}$, where $\hat{\beta}_{4s}$ is defined by (1.8).*

Proof. Case 1: $s \in (\frac{1}{2}, 1)$, it follows from $p = 4s$, Lemma 2.1, and (F_1) that, for $u \in \tilde{M}$, we have

$$\begin{aligned}\tilde{E}_{\beta,b,p}^f(u) &\geq \frac{1}{2}\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2 - \frac{\beta}{2\|Q_{4s}\|_2^{4s}}\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2\\ &= \frac{1}{2}\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{1}{4}\left(b - \frac{2\beta}{\|Q_{4s}\|_2^{4s}}\right)\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2,\end{aligned}$$

by (1.8) and $\beta \leq \hat{\beta}_{4s}$, we can deduce that

$$b - \frac{2\beta}{\|Q_{4s}\|_2^{4s}} \geq b - \frac{2\hat{\beta}_{4s}}{\|Q_{4s}\|_2^{4s}} = 0,$$

which implies that $\tilde{d}_\beta^f(b,4s) = 0$ for all $\beta \leq \tilde{\beta}_{4s}$.

Case 2: $s = \frac{1}{2}$, then $p = p^* = 4s = 2$. Equation (1.6) still admits a solution. However, it does not belong to $L^2(\mathbb{R}^2)$, and thus we redefine $\hat{\beta}_{4s} = bR_s^2$. Then, for any $\beta \in (0, \hat{\beta}_{4s})$ and $p = 2$, it follows from the definition of R_s that

$$\begin{aligned}\tilde{E}_{\beta,b,p}^f(u) \geq \frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\\ - \frac{\beta}{2\|Q_p\|_2^p}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^{\frac{p}{2s}},\end{aligned} \tag{3.13}$$

which implies that $\tilde{E}^f_{\beta,b,p}$ is well defined since $p \in (0, 4s)$ and $\frac{p}{2s} \in (0, 2)$. Moreover, (3.13) shows that $\tilde{d}^f_\beta(b, p) \geq 0$ for all $\beta \leq \hat{\beta}_{4s}$. For $u \in \tilde{M}$, letting $u_t(x) = tu(tx)$ $(t > 0)$, then $u_t \in \tilde{M}$ and

$$\begin{aligned}\tilde{E}^f_{\beta,b,p}(u_t) = &\frac{t^{2s}}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{bt^{4s}}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\\ &-\frac{\beta t^{4s}}{4s+2}\int_{\mathbb{R}^2}f\left(\frac{x}{t}\right)|u|^{4s+2}dx \to 0 \quad \text{as } t \to 0,\end{aligned}$$

which implies that $\tilde{d}^f_\beta(b, 4s) \leq 0$. Then, we know that $\tilde{d}^f_\beta(b, 4s) = 0$ for all $\beta \leq \hat{\beta}_{4s}$.

Case 3: $s \in (\frac{1}{2}, 1)$, by $p = 4s$ Equation (1.6) has solution Q_{4s}, Define

$$u_t(x) = \frac{tQ_{4s}(t(x - \bar{x}))}{\|Q_{4s}\|_2}, \quad t > 0.$$

From (3.6)–(3.8), we know that $u_t \in \tilde{M}$ and

$$\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_t\right|^2dx = t^{2s}$$

and

$$\int_{\mathbb{R}^2}f(x)|u_t|^{4s+2}dx = \frac{t^{4s}}{\|Q_{4s}\|_2^{4s+2}}\int_{\mathbb{R}^2}f\left(\bar{x} + \frac{y}{t}\right)|Q_{4s}(y)|^{4s+2}dy,$$

when $t \to \infty$, $f\left(\bar{x} + \frac{x}{t}\right) \to f(\bar{x}) = 1$ and $\|Q_{4s}(x)\|^{4s+2} \in L^1(\mathbb{R}^2)$. By dominated convergence theorem and Lemma 2.1, we have

$$\int_{\mathbb{R}^2}f(x)|u_t|^{4s+2}dx = \frac{(4s+2)t^{4s}}{2\|Q_{4s}\|_2^{4s}} + o(t^{4s}), \quad t \to \infty$$

and

$$\tilde{E}^f_{\beta,b,4s}(u_t) = \frac{1}{2}t^{2s} + \frac{b}{4}t^{4s} - \frac{\beta}{4s+2}\frac{4s+2}{2\|Q_{4s}\|_2^{4s}}t^{4s} + o(t^{4s}) = \frac{1}{2}t^{2s} + \frac{b}{4}\left(1 - \frac{\beta}{\hat{\beta}_{4s}}\right) + o(t^{4s}),$$

Since $1 - \frac{\beta}{\hat{\beta}_{4s}} < 0$, we deduce that

$$\tilde{d}^f_\beta(b, 4s) \leq \tilde{E}^f_{\beta,b,4s}(u_t) \to -\infty,$$

which implies $\tilde{d}^f_\beta(b, 4s) = -\infty$.

For $\beta > \tilde{\beta}_{4s}$, by [8] and $s = \frac{1}{2}$, we know that R_s is attained in $\mathbb{R}^2$ by

$$\bar{u}(x) = J(2, s)\left(\frac{1}{1 + |x|^2}\right)^{\frac{1}{2}},$$

with $J(2, s)$ chosen so that

$$\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}\bar{u}|^2dx = \int_{\mathbb{R}^2}|\bar{u}|^{2^*_s}dx = \int_{\mathbb{R}^2}|\bar{u}|^4dx = R_s^2. \tag{3.14}$$

Set

$$u^*(x) = \frac{J(2, s)}{(\varepsilon^2 + |x - x_0|^2)^{\frac{1}{2}}}, \quad V_\varepsilon(x) = \varepsilon^{\frac{1}{2}}u^*(x), \quad x \in \mathbb{R}^2, \varepsilon > 0.$$

Let $\eta \in C_0^\infty(\mathbb{R}^2)$ be such that $\eta = 1$ on $B_R(0)$, $\eta = 0$ on $\mathbb{R}^2 \setminus B_{2R}(0)$, $0 \le \eta \le 1$, and $|\nabla \eta| \le C$. There exists $R_0 \ge R$, R_0 satisfies $bR_s^2 - \beta(1-\delta_{R_0}) < 0$ and define $\delta_{R_0} := C(2R_0)^{q+2s}$. Define $u_\varepsilon(x) = \eta(x-x_0)V_\varepsilon(x)$ for any $\varepsilon > 0$ and $x \in \mathbb{R}^2$. Now, we define

$$w_\varepsilon(x) := A_\varepsilon \varepsilon^{-\frac{1}{2}} \eta(x-x_0)\varepsilon^{\frac{1}{2}} \frac{J(2,s)}{(\varepsilon^2 + |x-x_0|^2)^{\frac{1}{2}}} = A_\varepsilon \varepsilon^{-\frac{1}{2}} u_\varepsilon(x),$$

where A_ε is chosen so that $\|w_\varepsilon(x)\|_2^2 = 1$ and $\operatorname{supp} w_\varepsilon(x) \subset B_{2R}(x_0)$. Hence, we can get that

$$\begin{aligned} 1 &= \int_{\mathbb{R}^2} |w_\varepsilon|^2 dx = J^2(2,s) A_\varepsilon^2 \varepsilon^{-2} \int_{\mathbb{R}^2} \eta^2 (x-x_0) \frac{1}{1 + |\frac{x}{\varepsilon} - \frac{x_0}{\varepsilon}|^2} dx \\ &\ge J^2(2,s) A_\varepsilon^2 \varepsilon^{-2} \int_{|x-x_0| \le R} \frac{1}{1 + |\frac{x}{\varepsilon} - \frac{x_0}{\varepsilon}|^2} dx \\ &= J^2(2,s) A_\varepsilon^2 \varepsilon^{-2} \int_0^{\frac{R}{\varepsilon}} \frac{\rho}{1+\rho^2} d\rho \to +\infty \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} 1 &= \int_{\mathbb{R}^2} |w_\varepsilon|^2 dx = J^2(2,s) A_\varepsilon^2 \varepsilon^{-2} \int_{\mathbb{R}^2} \eta^2 (x-x_0) \frac{1}{1 + |\frac{x}{\varepsilon} - \frac{x_0}{\varepsilon}|^2} dx \\ &\le J^2(2,s) A_\varepsilon^2 \varepsilon^{-2} \int_{|x-x_0| \le 2R} \frac{1}{1 + |\frac{x}{\varepsilon} - \frac{x_0}{\varepsilon}|^2} dx \\ &= J^2(2,s) A_\varepsilon^2 \int_0^{\frac{2R}{\varepsilon}} \frac{\rho}{1+\rho^2} d\rho \le C A_\varepsilon^2 \ln \frac{2R}{\varepsilon} \quad \text{as } \varepsilon \text{ small enough.} \end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that $A_\varepsilon^2 \to 0$ and $A_\varepsilon^4 \varepsilon^{-2} \to +\infty$ as $\varepsilon \to 0$. Moreover, (3.14) yields that

$$\begin{aligned} \int_{\mathbb{R}^2} |w_\varepsilon|^4 dx &= A_\varepsilon^4 \varepsilon^{-2} \int_{\mathbb{R}^2} \varepsilon^2 \eta^4 (x-x_0) \frac{J^4(2,s)}{(\varepsilon^2 + |x-x_0|^2)^2} dx \\ &= A_\varepsilon^4 \varepsilon^{-2} \left(\int_{\mathbb{R}^2} \frac{\varepsilon^2 J^4(2,s)}{(\varepsilon^2 + |x-x_0|^2)^2} dx + \int_{\mathbb{R}^2} \left(\eta^4 (x-x_0) - 1 \right) \varepsilon^2 \frac{J^4(2,s)}{(\varepsilon^2 + |x-x_0|^2)^2} dx \right) \\ &= A_\varepsilon^4 \varepsilon^{-2} \left(\int_{\mathbb{R}^2} \frac{\varepsilon^2 J^4(2,s)}{(\varepsilon^2 + |x-x_0|^2)^2} dx + \int_{B_R^c(x_0)} \left(\eta^4 (x-x_0) - 1 \right) \varepsilon^2 \frac{J^4(2,s)}{(\varepsilon^2 + |x-x_0|^2)^2} dx \right) \\ &= A_\varepsilon^4 \varepsilon^{-2} \left(\int_{\mathbb{R}^2} |\bar{u}|^4 dx + \int_{B^c_{\frac{R}{\varepsilon}}(0)} \left(\eta^4(\varepsilon x) - 1 \right) \frac{J^4(2,s)}{(1+|x|^2)^2} dx \right) \\ &\approx A_\varepsilon^4 \varepsilon^{-2} \left(\int_{\mathbb{R}^2} |\bar{u}|^4 dx + \int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{J^4(2,s)}{|x|^4} dx \right) \\ &= A_\varepsilon^4 \varepsilon^{-2} \left(R_s^2 + O(\varepsilon^2) \right). \end{aligned} \tag{3.17}$$

Now, we show that the following assertions hold true:

(1) for any $x \in \mathbb{R}^2$ and $y \in B_R^c(x_0)$ with $|x-y| \le \frac{R}{2}$,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon^{\frac{1}{2}} |x-y|; \tag{3.18}$$

(2) for any $x, y \in B_R^c(x_0)$,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\varepsilon^{\frac{1}{2}} \min\{1, |x-y|\}, \tag{3.19}$$

for any $\varepsilon > 0$. In fact, for $x \in B^c_{\frac{R}{2}}(x_0)$, we have

$$\begin{aligned} |\nabla u_\varepsilon(x)| &\leq C\varepsilon^{\frac{1}{2}} \left| \frac{\nabla(\eta(x-x_0))}{(\varepsilon^2 + |x-x_0|^2)^{\frac{1}{2}}} + \frac{\eta(x-x_0)|x-x_0|}{(\varepsilon^2+|x-x_0|^2)^{\frac{3}{2}}} \right| \\ &\leq C\varepsilon^{\frac{1}{2}} \left[\frac{1}{(\varepsilon^2+R^2)^{\frac{1}{2}}} + \frac{|x-x_0|^2}{R(\varepsilon^2+|x-x_0|^2)^{\frac{3}{2}}} \right] \leq C\varepsilon^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Let $x \in \mathbb{R}^2$ and $y \in B_R^c(x_0)$ with $|x-y| \leq \frac{R}{2}$. Let ϑ lie on the segment joining x and y, so that $\vartheta = tx + (1-t)y$ for some $t \in [0,1]$. Hence, we obtain

$$|\vartheta - x_0| \geq |y - x_0| - t|x-y| \geq R - \frac{tR}{2} \geq \frac{R}{2}. \tag{3.21}$$

By (3.20) and (3.21) and a first-order Taylor expansion, we deduce that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\varepsilon^{\frac{1}{2}}|x-y|,$$

which shows that (3.18) holds. Let $x, y \in B_R^c(x_0)$, if $|x-y| \leq 1$, then (3.19) follows from (3.18). If $|x-y| > 1$, then

$$|u_\varepsilon(x)| \leq |V_\varepsilon(x)| \leq \frac{C\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{1}{2}}} \leq \frac{C\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+R^2)^{\frac{1}{2}}} \leq C\varepsilon^{\frac{1}{2}}, \tag{3.22}$$

which implies that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq |u_\varepsilon(x)| + |u_\varepsilon(y)| \leq C\varepsilon^{\frac{1}{2}}.$$

This completes the proof of (3.19). Define

$$\mathbf{Z_1} := \left\{ (x,y) \in \mathbb{R}^4 : x \in B_R(x_0),\ y \in B_R^c(x_0) \text{ and } |x-y| > R/2 \right\},$$

and

$$\mathbf{Z_2} := \left\{ (x,y) \in \mathbb{R}^4 : x \in B_R(x_0),\ y \in B_R^c(x_0) \text{ and } |x-y| \leq R/2 \right\}.$$

According to the definition of u_ε, we have

$$\begin{aligned} \int_{\mathbb{R}^4} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy = & \int_{B_R(x_0)\times B_R(x_0)} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy \\ & + 2\int_{\mathbf{Z_1}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy + 2\int_{\mathbf{Z_2}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy \\ & + \int_{B_R^c(x_0)\times B_R^c(x_0)} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy. \end{aligned} \tag{3.23}$$

From (3.19), it follows that

$$\int_{B_R^c(x_0)\times B_R^c(x_0)} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}} dxdy \leq C\varepsilon \int_{B_{2R}^c(x_0)\times\mathbb{R}^2} \frac{\min\{1, |x-y|^2\}}{|x-y|^{2+2s}} dxdy = O(\varepsilon). \tag{3.24}$$

From (3.18), we also have

$$
\begin{aligned}
\int_{\mathbf{Z_2}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy &\le C\varepsilon \int_{x\in B_R(x_0),\ y\in B_R^c(x_0), |x-y|\le R/2} \frac{|x-y|^2}{|x-y|^{2+2s}}dxdy \\
&\le C\varepsilon \int_{B_R(x_0)} dx \int_0^{\frac{R}{2}} d\rho = O(\varepsilon).
\end{aligned} \tag{3.25}
$$

For $(x,y) \in \mathbf{Z_1}$, we know that

$$
\begin{aligned}
|u_\varepsilon(x)-u_\varepsilon(y)|^2 = |V_\varepsilon(x)-u_\varepsilon(y)|^2 &= |V_\varepsilon(x)-V_\varepsilon(y)+V_\varepsilon(y)-u_\varepsilon(y)|^2 \\
&\le |V_\varepsilon(x)-V_\varepsilon(y)|^2 + |V_\varepsilon(y)-u_\varepsilon(y)|^2 \\
&\quad + 2|V_\varepsilon(x)-V_\varepsilon(y)||V_\varepsilon(y)-u_\varepsilon(y)|.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\int_{\mathbf{Z_1}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy \le& \int_{\mathbf{Z_1}} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy + \int_{\mathbf{Z_1}} \frac{|V_\varepsilon(y)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy \\
&+ 2\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(x)-V_\varepsilon(y)||V_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{2+2s}}dxdy.
\end{aligned} \tag{3.26}
$$

From (3.22), we can obtain that

$$
\begin{aligned}
\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(y)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy &\le 4\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy \\
&\le C\varepsilon \int_{\mathbf{Z_1}} \frac{1}{|x-y|^{2+2s}}dxdy \le C\varepsilon \int_{B_R(x_0)} dx \int_{\frac{R}{2}}^{+\infty} \rho^{-2} d\rho = O(\varepsilon).
\end{aligned} \tag{3.27}
$$

Setting $v = x-y$, we obtain that

$$
\begin{aligned}
\int_{\mathbf{A}} \frac{|V_\varepsilon(x)||V_\varepsilon(y)|}{|x-y|^{2+2s}}dxdy &\le C\varepsilon \int_{\mathbf{A}} \frac{1}{(\varepsilon^2+|x-x_0|^2)^{\frac{1}{2}}\,|x-y|^{2+2s}}dxdy \\
&= C\varepsilon \int_{x\in B_{R/\varepsilon}(0),\ y\in B_{R/\varepsilon}^c(0), |x-y|>R/2\varepsilon} \frac{1}{(1+|x|^2)^{\frac{1}{2}}\,|x-y|^{2+2s}}dxdy \\
&\le C\varepsilon \int_{B_{R/\varepsilon}(0)} \frac{1}{|x|} \int_{|v|>R/2\varepsilon} \frac{1}{|v|^3}dv = O(\varepsilon).
\end{aligned} \tag{3.28}
$$

By the definition of u_ε and (3.28), we get that

$$
\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(x)||V_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{2+2s}}dxdy \le 2\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(x)||V_\varepsilon(y)|}{|x-y|^{2+2s}}dxdy \le O(\varepsilon) \tag{3.29}
$$

and

$$
\begin{aligned}
\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(y)||V_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{2+2s}}dxdy &\le 2\int_{\mathbf{Z_1}} \frac{|V_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy \\
&\le C\varepsilon \int_{B_R(x_0)} dx \int_{R/2}^{+\infty} \rho^{-2} d\rho = O(\varepsilon).
\end{aligned} \tag{3.30}
$$

From (3.14), (3.23)–(3.27), (3.29) and (3.30), it follows that

$$\begin{aligned}\int_{\mathbb{R}^4}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy &\le \int_{\mathbb{R}^4}\frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy+O(\varepsilon)\\ &=\varepsilon\int_{\mathbb{R}^4}\frac{\left|\frac{J(2,s)}{(\varepsilon^2+|x-x_0|^2)^{\frac12}}-\frac{J(2,s)}{(\varepsilon^2+|y-x_0|^2)^{\frac12}}\right|^2}{|x-y|^{2+2s}}dxdy+O(\varepsilon)\\ &=\varepsilon^{-1}\int_{\mathbb{R}^4}\frac{\varepsilon^4\left|\frac{J(2,s)}{(1+|x|^2)^{\frac12}}-\frac{J(2,s)}{(1+|y|^2)^{\frac12}}\right|}{|x-y|^{2+2s}\varepsilon^3}dxdy+O(\varepsilon)\\ &=\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}\bar{u}\right|^2dx+O(\varepsilon)=R_s^2+O(\varepsilon).\end{aligned}\tag{3.31}$$

Since $f(x)$ satisfies (F_2), there exists $R_0>0$ and $C>0$, for all x satisfies $|x-x_0|\le 2R_0$, we can get that

$$1-f(x)\le C|x-x_0|^{q+2s}\le C(2R_0)^{q+2s},$$
$$\operatorname{supp} w_\varepsilon\subset B_{2R}(x_0)\subset B_{2R_0}(x_0)$$

and

$$\delta_{R_0}:=C(2R_0)^{q+2s},\quad \delta_{R_0}\to 0\quad \text{as } R_0\to 0.\tag{3.32}$$

For all $x\in\operatorname{supp} w_\varepsilon$ and $\varepsilon>0$, we have

$$f(x)\ge 1-C(2R_0)^{q+2s}=1-\delta_{R_0},$$

then,

$$\int_{\mathbb{R}^2}f(x)|w_\varepsilon|^4dx\ge(1-\delta_{R_0})\int_{\mathbb{R}^2}|w_\varepsilon|^4dx=(1-\delta_{R_0})A_\varepsilon^4\varepsilon^{-2}(R_s^2+O(\varepsilon^2)).\tag{3.33}$$

By (3.17) , (3.31) and (3.33), we deduce that

$$\begin{aligned}\tilde{d}_\beta^f(b,4s)\le \tilde{E}^f_{\beta,b,4s}(w_\varepsilon)&=\frac12A_\varepsilon^2\varepsilon^{-1}\int_{\mathbb{R}^4}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy\\ &\quad+\frac{b}{4}A_\varepsilon^4\varepsilon^{-2}\left(\int_{\mathbb{R}^4}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{2+2s}}dxdy\right)^2-\frac{\beta}{4}\int_{\mathbb{R}^2}f(x)|w_\varepsilon|^4dx\\ &\le\frac12A_\varepsilon^2\varepsilon^{-1}\left(R_s^2+O(\varepsilon)\right)+\frac{b}{4}A_\varepsilon^4\varepsilon^{-2}\left(R_s^2+O(\varepsilon)\right)^2-\frac{\beta}{4}((1-\delta_{R_0})A_\varepsilon^4\varepsilon^{-2}\left(R_s^2+O(\varepsilon^2)\right)\\ &=A_\varepsilon^4\varepsilon^{-2}\left(\frac12A_\varepsilon^{-2}\varepsilon\left(R_s^2+O(\varepsilon)\right)\right)+\frac{b}{4}\left(R_s^2+O(\varepsilon)\right)^2-\frac{\beta}{4}\left((1-\delta_{R_0})\left(R_s^2+O(\varepsilon^2)\right)\right)\end{aligned}\tag{3.34}$$

and

$$\frac{b}{4}R_s^4-\frac{\beta}{4}(1-\delta_{R_0})R_s^2=\frac{R_s^2}{4}\left(bR_s^2-\beta(1-\delta_{R_0})\right),$$

for ε small enough, we know that there exist $C_0>0$,independent of ε,we have

$$\tilde{E}^f_{\beta,b,4s}(w_\varepsilon)\le -C_0A_\varepsilon^4\varepsilon^{-2}+o(1),$$

letting $\varepsilon\to 0$, we can get that $\tilde{d}_\beta^f(b,4s)=-\infty$. □

Lemma 3.3. *If $p\in(0,4s)$, $s\in\left[\frac12,1\right)$, $f(x)$ satisfies $(F_1),(F_\infty)$ and $\tilde{d}_\beta^f(b,p)<0$, then (3.1) has at least a non-negative minimizer.*

Proof. Let $\{u_n\} \subset \tilde{M}$ be a minimizing sequence of $\tilde{d}^{f}_{\beta}(b,p)$. By (3.13), $\tilde{d}^{f}_{\beta}(b,p) < 0$ and $p \in (0,4s)$, we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^2)$. Then, Lemma 2.3 shows that there exists $\{u_n^*\} \subset H^s_r(\mathbb{R}^2)$, which is also a minimizing sequence for $\tilde{d}^{f}_{\beta}(b,p)$. Thus, Lemma 2.3 and Lemma 2.5 show that there exists a subsequence, still denoted by $\{u_n^*\}$, and some $u_0 \in H^s_r(\mathbb{R}^2)$ such that

$$\begin{cases} u_n^* \rightharpoonup u_0, & \text{in } H^s_r(\mathbb{R}^2), \\ u_n^* \to u_0, & \text{in } L^q(\mathbb{R}^2) \text{ for } q \in (2, 2_s^*). \end{cases} \tag{3.35}$$

Since $\tilde{E}^{f}_{\beta,b,p}(u_n) \to \tilde{d}^{f}_{\beta}(b,p) < 0$, for n large enough, we can get that $\tilde{E}^{f}_{\beta,b,p}(u_n) \le -C_0 < 0$ and

$$\begin{aligned} \frac{\beta}{p+2}\int f(x)|u_n|^{p+2} &= \frac{1}{2}\int_{\mathbb{R}^2} |(-\Delta)^{\frac{1}{2}} u_n|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{1}{2}} u_n|^2 dx\right)^2 - \tilde{E}^{f}_{\beta,b,p}(u_n) \\ &\ge -\tilde{E}^{f}_{\beta,b,p}(u_n) \ge C_0, \end{aligned}$$

there exist $C_1 > 0$, such that

$$\int_{\mathbb{R}^2} f(x)|u_n|^{p+2}\, dx \ge C_1, \tag{3.36}$$

by assumption (F_∞), there exists $\delta \in (0,1)$ and $R > 0$ such that

$$f(x) \le 1-\delta \quad \text{for } |x| \ge R,$$

then

$$\int_{\mathbb{R}^2} f(x)|u_n|^{p+2}\, dx \le \int_{B_R} |u_n|^{p+2}\, dx + (1-\delta)\int_{B_R^c} |u_n|^{p+2}\, dx,$$

Combining (3.35) and (3.36), there exists $\eta_0 > 0$ such that

$$\int_{B_R} |u_n|^{p+2}\, dx \ge \eta_0 \quad \text{for } n \text{ large},$$

by Hölder's inequality, we know that

$$\int_{B_R} |u_n|^2\, dx \ge C_2 > 0. \tag{3.37}$$

From (3.35) and (3.37), we can get that

$$\int_{B_R} |u_0|^2 = \lim_{n\to\infty}\int_{B_R} |u_n|^2 \ge C_2 > 0,$$

which means that $u_0 \not\equiv 0$.

By the compactness of $\{u_n\}$ and Lemma 2.6, we deduce that

$$u_n \to u_0 \quad \text{in } L^2(\mathbb{R}^2),$$

which implies that $\|u_0\|_2^2 = 1$ and $u_0 \in \tilde{M}$.

By (3.35) and (F_1), there exists $R_1 > 0$, we have

$$\int f|u_n|^{p+2} = \int_{B_{R_1}} f|u_n|^{p+2} + \int_{B_{R_1}^c} f|u_n|^{p+2},$$

$$\int_{B_{R_1}} f|u_n|^{p+2} \to \int_{B_{R_1}} f|u_0|^{p+2},$$

$$\sup_n \int_{B_{R_1}^c} f|u_n|^{p+2} \le \sup_n \int_{B_{R_1}^c} |u_n|^{p+2} \to 0 \quad \text{as } R_1 \to \infty.$$

Then, we can deduce that

$$\int_{\mathbb{R}^2} f|u_n|^{p+2} \to \int_{\mathbb{R}^2} f|u_0|^{p+2}.$$

By weak lower semicontinuity,

$$\tilde{E}^f_{\beta,b,p}(u_0) \le \liminf_{n\to\infty} \tilde{E}^f_{\beta,b,p}(u_n) = \tilde{d}^f_\beta(b,p),$$

Since $u_0 \in \tilde{M}$ and $\tilde{E}^f_{\beta,b,p}(u_0) \ge \tilde{d}^f_\beta(b,p)$, we conclude that u_0 is a minimizer of $\tilde{d}^f_\beta(b,p)$. Moreover, by the definition of $\tilde{d}^f_\beta(b,p)$, it follows that $|u_0|$ is also a minimizer. □

Proof of Theorem 1.1. We first show that $\tilde{d}^f_\beta(b,p)$ has no minimizer for $\beta < \hat{\beta}_p$ and $p \in (0,4s)$. Indeed, suppose by contradiction that there exists $\beta < \hat{\beta}_p$ such that $\tilde{d}^f_\beta(b,p)$ admits a minimizer $u \in \tilde{M}$. Then, by Lemma 3.1, we have that

$$0 = \tilde{d}^f_{\hat{\beta}_p}(b,p) \le \tilde{E}^f_{\hat{\beta}_p,b,p}(u) < \tilde{E}^f_{\beta,b,p}(u) = 0,$$

which gives a contradiction.

Then, we prove that $\tilde{d}^f_\beta(b,p)$ has no minimizer for $\beta = \hat{\beta}_p$ and $p \in (0,2s]$.

For $p \in (0,2s)$, then $\beta = \hat{\beta}_p = 0$. Assume that there exists a minimizer u_0 for $\tilde{d}^f_\beta(b,p)$. By using (3.13) and Lemma 3.1, we have $u_0 \equiv 0$, which leads to a contradiction since $u_0 \in \tilde{M}$.

For $p = 2s$, then $\hat{\beta}_p > 0$. Lemma 3.1 means that $\tilde{d}^f_\beta(b,p) = 0$. If there exists $u_0 \in \tilde{M}$ such that $\tilde{E}^f_{\beta,b,p}(u_0) = \tilde{d}^f_\beta(b,p) = 0$, by applying Lemma 2.1 and (F_1), we can deduce that

$$\frac{1}{2}\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_0\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_0\right|^2 dx\right)^2 \le \frac{\beta}{2+2s}\int_{\mathbb{R}^2} |u_0|^{2+2s} dx \le \frac{1}{2}\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_0\right|^2 dx,$$

which implies that $u_0 \equiv 0$. This is impossible since $u_0 \in \tilde{M}$.

Next, we show the existence of minimizer.

By Lemma 3.1 and Lemma 3.3, we deduce that $\tilde{d}^f_\beta(b,p)$ admits a non-negative minimizer for $\beta > \hat{\beta}_p$ and $p \in (0,4s)$. We now prove that it also admits a minimizer for $\beta = \hat{\beta}_p$ and $p \in (2s,4s)$.

Let $\beta_n = \hat{\beta}_p + \frac{1}{n}$, $n \in \mathbb{N}$. Then, for each β_n, Lemma 3.1 and Lemma 3.3 ensure the existence of $u_n \in \tilde{M}$ such that

$$\tilde{d}^f_{\beta_n}(b,p) = \tilde{E}^f_{\beta_n,b,p}(u_n) < 0, \tag{3.38}$$

and $\{u_n\}$ is bounded in $H^s(\mathbb{R}^2)$. Then, Lemma 2.3 and Lemma 2.5 show that there exists $u_n^* \in H^s_r(\mathbb{R}^2) \cap \tilde{M}$, which is also a minimizer of $\tilde{d}^f_{\beta_n}(b,p)$ and $u_n^* \in H^s_r(\mathbb{R}^2)$ is bounded. Thus, there exists $u_0 \in H^s_r(\mathbb{R}^2)$ such that

$$\begin{cases} u_n^* \rightharpoonup u_0, & \text{in } H^s_r(\mathbb{R}^2), \\ u_n^* \to u_0, & \text{in } L^q(\mathbb{R}^2) \text{ for } q \in (2, 2^*_s). \end{cases}$$

From the definition of $\tilde{E}^f_{\beta_n,b,p}(u_n^*)$ and (3.38), it follows that

$$\tilde{d}^f_{\hat{\beta}_p}(b,p) - \frac{1}{n(p+2)}\int_{\mathbb{R}^2} f(x)|u_n^*|^{p+2}dx \le \tilde{E}^f_{\beta_n,b,p}(u_n^*) = \tilde{d}^f_{\beta_n}(b,p) < 0,$$

which implies that $\tilde{d}^f_{\beta_n}(b,p) \to 0$ as $n \to \infty$ due to Lemma 3.1. Similar to the argument of Lemma 3.3, we can obtain that $u_0 \not\equiv 0$ and u_0 is a minimizer of $\tilde{d}^f_{\hat{\beta}_p}(b,p)$.

We finally show that $\tilde{d}^f_\beta(b,4s)$ admits no minimizer for any $\beta > 0$.

Indeed, Lemma 3.2 shows that $\tilde{d}^f_\beta(b,4s)$ has no minimizer for any $\beta > \hat{\beta}_{4s}$. Suppose that there exists some $\beta \le \hat{\beta}_{4s}$ such that $\tilde{d}^f_\beta(b,4s)$ has a minimizer $u_0 \in \tilde{M}$. By Lemma 2.1 and Lemma 3.2 and (F_1), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_0\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_0\right|^2dx\right)^2\\
&= \frac{\beta}{2+4s}\int_{\mathbb{R}^2} f(x)|u_0|^{2+4s}dx \le \frac{\beta}{2+4s}\int_{\mathbb{R}^2}|u_0|^{2+4s}dx\\
&\le \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_0\right|^2dx\right)^2,
\end{aligned}$$

which implies that $u_0 \equiv 0$. This is impossible due to $u_0 \in \tilde{M}$. Hence, $\tilde{d}^f_\beta(b,4s)$ has no minimizer for any $\beta > 0$. □

4 Existence and nonexistence for (1.2) with $V(x) \not\equiv 0$

In this section, we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. For any fixed $\beta > 0$, $p \in (0,4s)$, and $u \in M$, by using (1.11) and Lemma 2.1 and (F_1), we obtain that

$$\begin{aligned}
E^f_{\beta,b,p}(u) &\ge \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + V(x)|u|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 - \frac{\beta}{p+2}\int_{\mathbb{R}^2}|u|^{p+2}dx\\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2 + V(x)|u|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\\
&\quad - \frac{\beta}{2\|Q_p\|^2}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^{\frac{p}{2s}},
\end{aligned}$$

which implies that $d^f_\beta(b,p) > -\infty$, since $\frac{p}{2s} \in (0,2)$. Let $\{u_n\}$ be a minimizing sequence of $d^f_\beta(b,p)$. It follows that $\{u_n\}$ is bounded in E. Up to a subsequence, by Lemma 2.4, we may assume that there exists $u \in M$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } E,\\ u_n \to u & \text{in } L^q(\mathbb{R}^2) \text{ for } q \in [2,2_s^*). \end{cases}$$

Then, we can deduce that

$$\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx,$$

$$\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\leq\liminf_{n\to\infty}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^2$$

and

$$\int_{\mathbb{R}^2}V(x)|u|^2dx\leq\liminf_{n\to\infty}\int_{\mathbb{R}^2}V(x)|u_n|^2dx.$$

Since $f(x)\in L^\infty$, we have

$$\left|\int_{\mathbb{R}^2}f(x)(|u_n|^{p+2}-|u|^{p+2})dx\right|\leq\|f\|_\infty\||u_n|^{p+2}-|u|^{p+2}\|_{L^1},$$

by $u_n\to u$ in L^{p+2}, we deduce that

$$\||u_n|^{p+2}-|u|^{p+2}\|_{L^1}\to 0\quad\text{as }n\to\infty,$$

thus, we obtain that

$$\int_{\mathbb{R}^2}f(x)|u_n|^{p+2}dx\to\int_{\mathbb{R}^2}f(x)|u|^{p+2}dx\quad\text{as }n\to\infty.$$

Hence, we get

$$d_\beta^f(b,p)=\liminf_{n\to\infty}E^f_{\beta,b,p}(u_n)\geq E^f_{\beta,b,p}(u)\geq d^f_\beta(b,p),$$

which means that u is a minimizer of $d^f_\beta(b,p)$ for $\beta>0$. □

Proof of Theorem 1.3. (i) For $u\in M, p=4s$, and $s\in\left(\frac{1}{2},1\right)$, by Lemma 2.1 and Lemma 2.4, we obtain that there exists a constant $\varsigma>0$ such that

$$\begin{aligned}E^f_{\beta,b,4s}(u)&\geq\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2+V(x)|u|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\\&\quad-\frac{\beta}{4s+2}\int_{\mathbb{R}^2}|u|^{4s+2}dx\\&\geq\frac{1}{2}\int_{\mathbb{R}^2}\left(|(-\Delta)^{\frac{s}{2}}u|^2+V(x)|u|^2\right)dx+\frac{\beta_{4s}-\beta}{2\|Q_{4s}\|_2^{4s}}\left(\int_{\mathbb{R}^2}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2\\&\geq\varsigma\|u\|_2^2=\varsigma>0,\end{aligned}\tag{4.1}$$

which means that $d^f_\beta(b,4s)>0$. Let $\{u_n\}$ be a minimizing sequence of $d^f_\beta(b,4s)$, then (4.1) shows that $\{u_n\}$ is bounded in E. Similar to the proof of Theorem 1.2, we can conclude that $d^f_\beta(b,4s)$ has at least one minimizer if $0<\beta\leq\beta_{4s}:=\frac{b}{2}\|Q_{4s}\|_2^{4s}$.

(ii) Choose a non-negative cut-off function $\eta\in C_0^\infty(\mathbb{R}^2)$ be such that $\eta=1$ on $B_1(0),\eta=0$ on $\mathbb{R}^2\setminus B_2(0), 0\leq\eta\leq 1$, and $|\nabla\eta|\leq 2$. For $x_0\in\mathbb{R}^2,\tau>0$, let

$$u_\tau(x):=\frac{A_\tau\tau}{\|Q_{4s}\|_2}\eta\left(\frac{x-x_0}{\tau}\right)Q_{4s}\left(\tau(x-x_0)\right),$$

where $A_\tau>0$ is chosen so that $\|u_\tau\|_2^2=1$, since the polynomial decay of Q_{4s}, such that

$$A_\tau^2=\frac{\|Q_{4s}\|_2^2}{\int_{\mathbb{R}^2}\eta^2\left(\frac{x}{\tau^2}\right)Q^2_{4s}(x)dx}=1+O(\tau^{-4-8s})\quad\text{as }\tau\to\infty,\tag{4.2}$$

which implies that

$$\lim_{\tau\to\infty} A_\tau^2 = 1. \tag{4.3}$$

From (2.2), (4.3), and Lemma 2.1 and Lemma 2.7, suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1), it follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u_\tau|^2 dx &= \frac{A_\tau^2 \tau^{2s}}{\|Q_{4s}\|_2^2} \int_{\mathbb{R}^4} \frac{|\eta(x/\tau^2)Q_{4s}(x) - \eta(y/\tau^2)Q_{4s}(y)|^2}{|x-y|^{2+2s}} dxdy \\ &\leq \frac{A_\tau^2 \tau^{2s}}{\|Q_{4s}\|_2^2} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} Q_{4s}|^2 dx + O(\tau^{-8s}) \right) \\ &= A_\tau^2 \tau^{2s} + O(\tau^{-6s}) \quad \text{as } \tau \to \infty,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)|u_\tau|^{2+4s} &= \frac{A_\tau^{2+4s}\tau^{4s}}{\|Q_{4s}\|_2^{2+4s}} \int_{\mathbb{R}^2} \eta^{2+4s}\left(\frac{x}{\tau^2}\right) Q_{4s}^{2+4s}(x) f\left(\frac{x}{\tau} + x_0\right) dx \\ &= \frac{A_\tau^{2+4s}\tau^{4s}}{\|Q_{4s}\|_2^{2+4s}} \left(\int_{\mathbb{R}^2} \eta^{2+4s}\left(\frac{x}{\tau^2}\right) Q_{4s}^{2+4s}(x) dx \right. \\ &\quad \left. - \int_{\mathbb{R}^2} \eta^{2+4s}\left(\frac{x}{\tau^2}\right) Q_{4s}^{2+4s}(x) \left(1 - f\left(\frac{x}{\tau} + x_0\right)\right) dx \right) \\ &\geq \frac{bA_\tau^{2+4s}\tau^{4s}(2+4s)}{4\beta_{4s}} - \frac{A_\tau^{2+4s}\tau^{4s}}{\|Q_{4s}\|_2^{2+4s}} \int_{\mathbb{R}^2} Q_{4s}^{2+4s}(x) \left(1 - f\left(\frac{x}{\tau} + x_0\right)\right) dx \\ &\quad + O(\tau^{-4-20s-16s^2}) \quad \text{as } \tau \to \infty,\end{aligned} \tag{4.5}$$

and

$$\int_{\mathbb{R}^2} V(x) u_\tau^2 dx = \frac{A_\tau^2}{\|Q_{4s}\|_2^2} \int_{\mathbb{R}^2} V\left(\frac{x}{\tau} + x_0\right) \eta^2\left(\frac{x}{\tau^2}\right) Q_{4s}^2(x) dx = o(1) \quad \text{as } \tau \to \infty. \tag{4.6}$$

By $\beta > \beta_{4s}$ and (4.3)–(4.6), suppose that $f(x)$ satisfies (F_1)–(F_2), we obtain that

$$\begin{aligned} d_\beta^f(b, 4s) \leq E_{\beta,b,4s}^f(u_\tau) &\leq \frac{1}{2} A_\tau^2 \tau^{2s} + \frac{b}{4} \tau^{4s} \left(A_\tau^4 - \frac{\beta}{\beta_{4s}} A_\tau^{2+4s} \right) \\ &\quad + \frac{A_\tau^{2+4s}\tau^{4s}}{\|Q_{4s}\|_2^{2+4s}} \int_{\mathbb{R}^2} Q_{4s}^{2+4s}(x) \left(1 - f\left(\frac{x}{\tau} + x_0\right)\right) dx + O(\tau^{-4s}) + o(1) \\ &\to -\infty \quad \text{as } \tau \to \infty,\end{aligned}$$

which implies that $d_\beta^f(b, 4s)$ has no minimizer if $\beta > \beta_{4s}$. □

5 Concentration behavior under general trapping potential

In this section, we investigate whether new phenomena arise when $p = 2s$ and $b \to 0^+$. Theorem 1.1 shows that $\tilde{d}_\beta^f(b, 2s)$ admits a minimizer if and only if $\beta > \beta^* := \|Q_{2s}\|_2^{2s}$. Setting $p = 2s$ in (2.2), it follows that

$$(\beta^*)^{\frac{1}{s}} = \int_{\mathbb{R}^2} \left| (-\Delta)^{\frac{s}{2}} Q_{2s} \right|^2 dx = \int_{\mathbb{R}^2} |Q_{2s}|^2 dx = \frac{1}{s+1} \int_{\mathbb{R}^2} |Q_{2s}|^{2s+2} dx. \tag{5.1}$$

We establish several energy estimates, which play a key role in the proof of the concentration behavior.

Lemma 5.1. *For any $\beta > \beta^*$ and $p = 2s$, suppose that we have $f(x)$ satisfies (F_1), we have $\tilde{d}^f_\beta(b,2s) \geq -\frac{1}{4b}(\frac{\beta-\beta^*}{\beta^*})^2$, if $f(x)$ satisfies (F_2), there exists a set of trial functions, $u_t \in H^s(\mathbb{R}^2)$, such that*

$$\tilde{d}^f_\beta(b,2s) \leq -\frac{1}{4b}\left(\frac{\beta-\beta^*}{\beta^*}\right)^2 + O(t^{-q}).$$

In particular, we have

$$\tilde{d}^f_\beta(b,2s) = -\frac{1}{4b}\left(\frac{\beta-\beta^*}{\beta^*}\right)^2, \qquad t \to \infty. \tag{5.2}$$

Define

$$x_b := \frac{\beta-\beta^*}{b\beta^*}.$$

Proof. For any $u \in H^s(\mathbb{R}^2)$ with $\|u\|_2^2 = 1$, by using Lemma 2.1 and (F_1), we have that

$$\begin{aligned}
\tilde{E}^f_{\beta,b,2s}(u) &= \frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 - \frac{\beta}{2s+2}\int_{\mathbb{R}^2}f(x)|u|^{2s+2}dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 - \frac{\beta}{2s+2}\int_{\mathbb{R}^2}|u|^{2s+2}dx \\
&\geq \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 - \frac{\beta-\beta^*}{2\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx.
\end{aligned}$$

Let

$$I(x) := \frac{b}{4}x^2 - \frac{\beta-\beta^*}{2\beta^*}x, \qquad x \in [0,+\infty). \tag{5.3}$$

It is easy to see that $I(x)$ attains its global minimum at $x_b = \frac{\beta-\beta^*}{b\beta^*}$. This implies that

$$\tilde{d}^f_\beta(b,2s) \geq -\frac{1}{4b}\left(\frac{\beta-\beta^*}{\beta^*}\right)^2. \tag{5.4}$$

Now, we set

$$u_t(x) = \frac{t}{\beta^*}Q_{2s}(tx), \quad t > 0,$$

then we have $\|u_t\|_2^2 = 1$. By (5.1), it follows that

$$\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_t\right|^2dx = \frac{t^{2s}}{(\beta^*)^{\frac{1}{s}}}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}Q_{2s}\right|^2dx = t^{2s} \tag{5.5}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2}f(x)|u_t|^{2+2s}dx &= \frac{t^{2s}}{(\beta^*)^{1+\frac{1}{s}}}\int_{\mathbb{R}^2}f\left(x_0+\frac{x}{t}\right)|Q_{2s}|^{2+2s}dx \\
&= \frac{t^{2s}}{(\beta^*)^{1+\frac{1}{s}}}\int_{\mathbb{R}^2}|Q_{2s}|^{2+2s}dx - \frac{t^{2s}}{(\beta^*)^{1+\frac{1}{s}}}\int_{\mathbb{R}^2}\left(1-f\left(x_0+\frac{x}{t}\right)\right)|Q_{2s}|^{2+2s}dx.
\end{aligned}$$

By Lemma 2.1 and $q \in (0, 4s^2+4s+2)$, suppose that $f(x)$ satisfies (F_2) , we deduce that

$$1-f\left(x_0+\frac{x}{t}\right) = O\left(\left|\frac{x}{t}\right|^{q+2s}\right) = O\left(|x|^{q+2s}\right)t^{-(q+2s)}$$

and

$$\int_{\mathbb{R}^2} |Q_{2s}|^{q+2s} O(|x|^{q+2s}) dx < \infty,$$

which means that

$$\int_{\mathbb{R}^2} f(x)|u_t|^{2+2s} dx = \frac{t^{2s}}{(\beta^*)^{1+\frac{1}{s}}} \int_{\mathbb{R}^2} |Q_{2s}|^{2+2s} dx + O(t^{-q}) = \frac{(1+s)t^{2s}}{\beta^*} + O(t^{-q}). \tag{5.6}$$

Then,

$$\tilde{d}^f_\beta(b,2s) \le \tilde{E}^f_{\beta,b,2s}(u_t) = \frac{b}{4} t^{4s} - \frac{\beta-\beta^*}{2\beta^*} t^{2s} + O(t^{-q}) = I(t^{2s}) + O(t^{-q}).$$

Let $t = (x_b)^{\frac{1}{2s}}$, it follows that

$$\tilde{d}^f_\beta(b,2s) \le I(x_b) + O(t^{-q}) = -\frac{1}{4b}\left(\frac{\beta-\beta^*}{\beta^*}\right)^2 + O(t^{-q}). \tag{5.7}$$

Combining (5.4) and (5.7), we obtain that (5.2) holds. □

Lemma 5.2. *For any given $\beta > \beta^*$, suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1), let u_b is a non-negative minimizer of $d^f_\beta(b,2s)$. Then, we have*

$$\begin{cases} 0 \le d^f_\beta(b,2s) - \tilde{d}_\beta(b,2s) \to 0, & \text{as } b \to 0^+, \\ \int_{\mathbb{R}^2} V(x)|u_b|^2\, dx \to 0, & \text{as } b \to 0^+, \\ \int_{\mathbb{R}^2} (1-f(x))|u_b|^{2s+2}\, dx \to 0, & \text{as } b \to 0^+. \end{cases}$$

Denote

$$\tilde{d}_\beta(b,2s) := \inf\left\{ \tilde{E}_{\beta,b,2s}(u) : u \in H^s(\mathbb{R}^2), \int_{\mathbb{R}^2} |u|^2 = 1 \right\}$$

$$\tilde{E}_{\beta,b,2s}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx\right)^2 - \frac{\beta}{2s+2}\int_{\mathbb{R}^2} |u|^{2s+2} dx.$$

Proof. Since (F_1) and (V_1) holds, for all u satisfies $\|u\|_2^2 = 1$, we have

$$-\frac{\beta}{2s+2}\int_{\mathbb{R}^2} f(x)|u|^{2s+2} dx = -\frac{\beta}{2s+2}\int_{\mathbb{R}^2} |u|^{2s+2} dx + \frac{\beta}{2s+2}\int_{\mathbb{R}^2} (1-f(x))\,|u|^{2s+2} dx$$

and

$$E^f_{\beta,b,2s}(u) = \tilde{E}_{\beta,b,2s}(u) + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u|^2 dx + \frac{\beta}{2s+2}\int_{\mathbb{R}^2} (1-f(x))|u|^{2s+2} dx \ge \tilde{E}_{\beta,b,2s}(u),$$

which means that

$$d^f_\beta(b,2s) \ge \tilde{d}_\beta(b,2s). \tag{5.8}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be the cut-off function given in Lemma 5.1 of [10], satisfying $\varphi = 1$ on $B_1(0)$, $\varphi = 0$ on $\mathbb{R}^2 \setminus B_2(0)$, $0 \le \varphi \le 1$, and $|\nabla\varphi| \le C$. For a fixed $x_0 \in \mathbb{R}^2$, we set

$$\hat{u}_b(x) = A_b \varphi\left(x_b^{-\frac{2}{s}}(x-x_0)\right) \bar{u}_b(x-x_0) = A_b \left(\frac{x_b}{\beta^*}\right)^{\frac{1}{2s}} \varphi\left(x_b^{-\frac{2}{s}}(x-x_0)\right) Q_{2s}\left(x_b^{\frac{1}{2s}}(x-x_0)\right),$$

where $\bar{u}_b(x)$ is the unique non-negative minimizer of $\tilde{d}_\beta(b,2s)$ and $A_b > 0$ is chosen so that $\|\hat{u}_b\|_2^2 = 1$. Then, we deduce that

$$0 \le A_b^2 - 1 \le C x_b^{-\frac{5}{s}-10} \quad \text{as } b \to 0^+ \tag{5.9}$$

A direct computation shows that

$$\begin{aligned}
0 &\le d_\beta^f(b,2s) - \tilde{d}_\beta(b,2s) \le E^f_{\beta,b,2s}(\hat{u}_b) - \tilde{E}_{\beta,b,2s}(\bar{u}_b) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\hat{u}_b\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\bar{u}_b\right|^2 dx \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\hat{u}_b\right|^2 dx\right)^2 - \frac{b}{4}\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\bar{u}_b\right|^2 dx\right)^2 \\
&\quad + \frac{\beta}{2s+2}\int_{\mathbb{R}^2} |\bar{u}_b|^{2s+2} dx - \frac{\beta}{2s+2}\int_{\mathbb{R}^2} f(x)|\hat{u}_b|^{2s+2} dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|\hat{u}_b|^2 dx.
\end{aligned}$$

From Lemma 5.2 in [10], we know that

$$\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\hat{u}_b\right|^2 dx - \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\bar{u}_b\right|^2 dx \le C x_b^{-\frac{1}{4}} \quad \text{as } b \to 0^+, \tag{5.10}$$

$$\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\hat{u}_b\right|^2 dx\right)^2 - \left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}\bar{u}_b\right|^2 dx\right)^2 \le C x_b^{-\frac{1}{4}} \quad \text{as } b \to 0^+, \tag{5.11}$$

$$\int_{\mathbb{R}^2} |\hat{u}_b|^{2s+2} dx = \int_{\mathbb{R}^2} |\bar{u}_b|^{2s+2} dx - C x_b^{-\frac{5}{s}-19-10s} \quad \text{as } b \to 0^+, \tag{5.12}$$

$$\int_{\mathbb{R}^2} V(x)\,|\hat{u}_b|^2\, dx = V(x_0) + o(1) = o(1) \quad \text{as } b \to 0^+. \tag{5.13}$$

We now estimate

$$\int_{\mathbb{R}^2} |\bar{u}_b|^{2s+2} dx - \int_{\mathbb{R}^2} f(x)|\hat{u}_b|^{2s+2} dx.$$

By (5.12), it suffices to estimate

$$\int_{\mathbb{R}^2} f(x)|\hat{u}_b|^{2s+2} dx - \int_{\mathbb{R}^2} |\hat{u}_b|^{2s+2} dx.$$

Since $f(x)$ satisfies (F_2), there exist constants $C > 0$ and $\rho > 0$ such that, for $|x - x_0| < \rho$, we have

$$0 \le 1 - f(x) \le C|x - x_0|^{q+2s},$$

as b small enough, we can deduce that

$$\operatorname{supp} \hat{u}_b \subset B_\rho(x_0),$$

$$0 \le \int_{\mathbb{R}^2} (1 - f(x))\,|\hat{u}_b|^{2s+2}\, dx \le C\int_{\mathbb{R}^2} |x - x_0|^{q+2s}\,|\hat{u}_b|^{2s+2}\, dx$$

and

$$
\begin{aligned}
& C \int_{\mathbb{R}^2} |x-x_0|^{q+2s} |\hat{u}_b|^{2s+2}\, dx \\
&\quad = C \int_{\mathbb{R}^2} |x-x_0|^{q+2s} A_b^{2s+2} \left(\frac{x_b}{\beta^*}\right)^{\frac{2s+2}{2s}} \varphi^{2s+2}\left(x_b^{-\frac{2}{s}}(x-x_0)\right) Q_{2s}^{2s+2}\left(x_b^{\frac{1}{2s}}(x-x_0)\right) dx \\
&\quad \leq C \int_{\mathbb{R}^2} |x-x_0|^{q+2s} x_b^{\frac{2s+2}{2s}} Q_{2s}^{2s+2}\left(x_b^{\frac{1}{2s}}(x-x_0)\right) dx \\
&\quad = C \int_{\mathbb{R}^2} x_b^{-\frac{q}{2s}} x^{q+2s} Q_{2s}^{2s+2}(x) dx.
\end{aligned}
$$

By the proof of Lemma 5.1, we have

$$
\int_{\mathbb{R}^2} x^{q+2s} Q_{2s}^{2s+2}(x) dx < \infty
$$

and

$$
0 \leq \int_{\mathbb{R}^2} (1-f(x)) |\hat{u}_b|^{2s+2}\, dx \leq O\left(x_b^{-\frac{q}{2s}}\right),
$$

for $\beta > \beta^*$ and $b \to 0^+$, such that $x_b \to \infty$ and

$$
\int_{\mathbb{R}^2} f(x) |\hat{u}_b|^{2s+2}\, dx = \int_{\mathbb{R}^2} |\hat{u}_b|^{2s+2}\, dx + o(1) \quad \text{as } b \to 0^+. \tag{5.14}
$$

From (5.9)–(5.14), we can deduce that

$$
0 \leq d_\beta^f(b,2s) - \tilde{d}_\beta(b,2s) \to 0 \quad \text{as } b \to 0^+. \tag{5.15}
$$

Letting u_b is a non-negative minimizer of $d_\beta^f(b,2s)$, such that

$$
d_\beta^f(b,2s) = E^f_{\beta,b,2s}(u_b) = \tilde{E}_{\beta,b,2s}(u_b) + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u_b|^2 dx + \frac{\beta}{2s+2}\int_{\mathbb{R}^2} (1-f(x)) |u_b|^{2s+2} dx,
$$

with $\tilde{E}_{\beta,b,2s}(u_b) \geq \tilde{d}_\beta(b,2s)$, we have

$$
d_\beta^f(b,2s) - \tilde{d}_\beta(b,2s) \geq \frac{1}{2} V(x)|u_b|^2 dx + \frac{\beta}{2s+2}\int_{\mathbb{R}^2} (1-f(x)) |u_b|^{2s+2} dx
$$

by (5.15), it follows that

$$
\int_{\mathbb{R}^2} V(x)|u_b|^2 dx \to 0 \quad \text{as } b \to 0^+
$$

and

$$
\int_{\mathbb{R}^2} (1-f(x)) |u_b|^{2s+2} dx \to 0 \quad \text{as } b \to 0^+.
$$

which means Lemma 5.2 holds. □

Lemma 5.3. *For any given $\beta > \beta^*$, suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1), and let u_b is a non-negative minimizer of $d_\beta^f(b,2s)$. Then, we have*

$$
\frac{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_b\right|^2 dx}{x_b} \to 1 \quad \text{and} \quad \frac{\int_{\mathbb{R}^2} f(x)|u_b|^{2+2s} dx}{x_b} \to \frac{1+s}{\beta^*} \quad \text{as } b \to 0^+,
$$

where x_b is defined in Lemma 5.1.

Proof. For any u be such that $\|u\|_2^2 = 1$, it follows from (F_1) and Lemma 2.1 that

$$\int_{\mathbb{R}^2} f(x)|u|^{2s+2}dx \le \int_{\mathbb{R}^2} |u|^{2s+2}dx \le \frac{s+1}{\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx,$$

then

$$\begin{aligned}
E^f_{\beta,b,2s}(u) \ge{}& \frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u|^2dx \\
&- \frac{\beta}{2s+2}\frac{s+1}{\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx \\
={}& \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx\right)^2 - \frac{\beta-\beta^*}{2\beta}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u|^2dx \\
={}& I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx\right) + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u|^2dx,
\end{aligned}$$

where $I(x)$ is defined in (5.3) and $I_{min} = I(x_b)$. Then, let $u = u_b$, we obtain that

$$\begin{aligned}
d^f_\beta(b,2s) = E^f_{\beta,b,2s}(u_b) &\ge I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx\right) + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u_b|^2dx \\
&\ge I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx\right).
\end{aligned} \tag{5.16}$$

From Lemma 5.1 and Lemma 5.2 in [10], it follows that

$$d^f_\beta(b,2s) = \tilde{d}_\beta(b,2s) + o(1) \le I(x_b) + o(1) \quad \text{as } b \to 0^+. \tag{5.17}$$

Suppose, by contradiction, that there exists $\gamma \ge 0$, $\gamma \ne 1$, such that

$$\frac{\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx}{x_b} \to \gamma \quad \text{as } b \to 0^+.$$

If $\gamma \in [0,1)$, then there exists a subsequence $\{b_k\}$, $b_k \to 0$ as $k \to \infty$. Let $\delta \in (\gamma, 1)$, for k large enough, we have

$$\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_{b_k}\right|^2 dx \le \delta x_{b_k}$$

and

$$I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_{b_k}\right|^2 dx\right) \ge h(\delta x_{b_k}) > I(x_{b_k}), \tag{5.18}$$

with (5.16) and (5.17), we can get that

$$I(\delta x_{b_k}) \le I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_{b_k}\right|^2 dx\right) \le d^f_\beta(b_k,2s) \le I(x_{b_k}) + o(1)$$

which gives a contradiction by (5.18). Similarly, we can also get a contradiction when $\gamma > 1$. Hence,

$$\frac{\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx}{x_b} \to 1 \quad \text{as } b \to 0^+. \tag{5.19}$$

Since

$$\begin{aligned}
d^f_\beta(b,2s) ={}& \frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)|u_b|^2dx \\
&- \frac{\beta}{2s+2}\int_{\mathbb{R}^2} f(x)|u_b|^{2s+2}dx,
\end{aligned}$$

then, we can deduce that

$$d_\beta^f(b,2s) - I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right) = \frac{\beta}{2\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)|u_b|^2dx \\ - \frac{\beta}{2s+2}\int_{\mathbb{R}^2}f(x)|u_b|^{2s+2}dx,$$

by Lemma 5.2, we have

$$d_\beta^f(b,2s) - I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right) \to 0 \quad \text{as } b \to 0^+.$$

Then,

$$\begin{aligned}\frac{\beta}{2s+2}\int_{\mathbb{R}^2}f(x)|u_b|^{2s+2}dx &= \frac{\beta}{2\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)|u_b|^2dx \\ &\quad - \left(d_\beta^f(b,2s) - I\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right)\right) \\ &= \frac{\beta}{2\beta^*}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx + o(1) \quad \text{as } b \to 0^+.\end{aligned}$$

Since (5.19) holds, we obtain that

$$\frac{\int_{\mathbb{R}^2}f(x)|u_b|^{2+2s}dx}{x_b} \to \frac{1+s}{\beta^*} \quad \text{as } b \to 0^+, \tag{5.20}$$

which means Lemma 5.3 holds. □

Lemma 5.4. *For $\beta = \beta^*$, suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1), and let u_b is a non-negative minimizer of $d_{\beta^*}^f(b,2s)$. Then, we have*

$$\begin{cases} d_{\beta^*}^f(b,2s) \to d_{\beta^*}^f(0,2s) = 0 & \text{as } b \to 0^+, \\ \int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx \to +\infty & \text{as } b \to 0^+, \\ b\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right)^2 \to 0 & \text{as } b \to 0^+, \\ \int_{\mathbb{R}^2}V(x)|u_b|^2dx \to 0 & \text{as } b \to 0^+. \end{cases}$$

Proof. From Theorem 1.1 in [12], we know that $d_{\beta^*}^f(0,2s) = 0$. By Lemma 2.1, we have

$$\begin{aligned} d_{\beta^*}^f(b,2s) &= E_{\beta^*,b,2s}^f(u_b) \\ &\geq \frac{b}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2}V(x)|u_b|^2dx > d_{\beta^*}^f(0,2s) = 0. \end{aligned} \tag{5.21}$$

For fixed $x_0 \in \mathbb{R}^2$ and $\tau > 0$, define

$$u_\tau(x) = A_\tau\frac{\tau}{\|Q_{2s}\|_2}\varphi(x-x_0)Q_{2s}(\tau(x-x_0)), \tag{5.22}$$

where φ is given in Lemma 5.2 and $A_\tau > 0$ is chosen so that $\|u_2\|_2^2 = 1$. By Lemma 2.1, we deduce that

$$A_\tau^2 = \frac{\|Q_{2s}\|_2^2}{\int_{\mathbb{R}^2}\varphi^2\left(\frac{x}{\tau}\right)Q_{2s}^2(x)dx} = 1 + O(\tau^{-2-4s}) \quad \text{as } \tau \to \infty. \tag{5.23}$$

From (5.1), (5.23), and Lemma 2.7, it follows that

$$\begin{aligned}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_\tau\right|^2dx&=\frac{A_\tau^2\tau^{2s}}{\|Q_{2s}\|_2^2}\int_{\mathbb{R}^4}\frac{\left|\varphi\left(\frac{x}{\tau}\right)Q_{2s}(x)-\varphi\left(\frac{y}{\tau}\right)Q_{2s}(y)\right|^2}{|x-y|^{2+2s}}dxdy\\&\leq\frac{\tau^{2s}}{\|Q_{2s}\|_2^2}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}Q_{2s}\right|^2dx+O(\tau^{-4s})\right)\\&\leq C(\tau^{-2s}+\tau^{2s})\quad\text{as }\tau\to\infty,\end{aligned}\tag{5.24}$$

from Lemma 2.1 and Lemma 2.2 and (F_2), we can deduce that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_\tau\right|^2dx-\frac{\beta^*}{2s+2}\int_{\mathbb{R}^2}f(x)|u_\tau|^{2+2s}dx\\&\leq\frac{\tau^{2s}}{2\|Q_{2s}\|_2^2}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}Q_{2s}\right|^2dx-\frac{1}{1+s}\int_{\mathbb{R}^2}f\left(x_0+\frac{x}{\tau}\right)|Q_{2s}|^{2+2s}dx+O(\tau^{-4s})\right)\\&=\frac{\tau^{2s}}{2\|Q_{2s}\|_2^2}\left(\frac{1}{1+s}\int_{\mathbb{R}^2}|Q_{2s}|^{2+2s}dx-\frac{1}{1+s}\int_{\mathbb{R}^2}f\left(x_0+\frac{x}{\tau}\right)|Q_{2s}|^{2+2s}dx+O(\tau^{-4s})\right)\\&=\frac{\tau^{2s}}{2(1+s)\|Q_{2s}\|_2^2}\left(\int_{\mathbb{R}^2}\left(1-f\left(x_0+\frac{x}{\tau}\right)\right)|Q_{2s}|^{2+2s}dx+O(\tau^{-4s})\right)\\&\leq C\tau^{-(q+2s)}+C\tau^{-2s}\quad\text{as }\tau\to\infty.\end{aligned}\tag{5.25}$$

Moreover, by a direct computation, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2}V(x)|u_\tau|^2dx&=\frac{A_\tau^2}{(\beta^*)^{\frac{1}{s}}}\int_{\mathbb{R}^2}V\left(\frac{x}{\tau}+x_0\right)\varphi^2\left(\frac{x}{\tau}\right)Q_{2s}^2(x)dx\\&=V(x_0)+o(1)=o(1)\quad\text{as }\tau\to\infty,\end{aligned}\tag{5.26}$$

where $o(1)\to 0$ as $\tau\to\infty$. From (5.24)–(5.26), it follows that

$$0<d_{\beta^*}^f(b,2s)\leq E_{\beta^*,b,2s}^f(u_\tau)\leq Cb(\tau^{-4s}+\tau^{4s})+Cb+C\tau^{-2s}+C\tau^{-(q+2s)}+o(1).\tag{5.27}$$

Letting $\tau=b^{-\frac{1}{8s}}$ and $b\to 0^+$, then (5.27) shows that

$$0<d_{\beta^*}^f(b,2s)\to 0\quad\text{as }b\to 0^+.$$

Hence, (5.21) implies that

$$b\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\right)^2\to 0\quad\text{and}\quad\int_{\mathbb{R}^2}V(x)|u_b|^2dx\to 0\quad\text{as }b\to 0^+.\tag{5.28}$$

We next show that

$$\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_b\right|^2dx\to+\infty\quad\text{as }b\to 0^+.\tag{5.29}$$

If (5.29) is false, then (5.28) shows that there exists a sequence of $\{b_k\}$ with $b_k\xrightarrow{k}0^+$ such that the sequence $\{u_k\}$ is bounded in E, where $u_k:=u_{b_k}$. By Lemma 2.4, passing to a subsequence, there exists $u_0\in E$ such that

$$\begin{cases}u_k\rightharpoonup u_0&\text{in }E,\\u_k\to u_0&\text{in }L^q(\mathbb{R}^2)\text{ for }q\in[2,2_s^*).\end{cases}$$

Thus, it follows that

$$d^f_{\beta^*}(0,2s) \leq E^f_{\beta^*,0,2s}(u_0) \leq \liminf_{k\to\infty} E^f_{\beta^*,b_k,2s}(u_k) = \lim_{k\to\infty} d^f_{\beta^*}(b_k,2s) = 0 = d^f_{\beta^*}(0,2s),$$

which indicates that u_0 is a minimizer of $d^f_{\beta^*}(0,2s)$. This is impossible since Theorem 1.1 in [12], which shows that (5.29) holds. □

Lemma 5.5. *Let $s \in \left(\frac{1}{2},1\right)$, and suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1). For any given $\beta \geq \beta^*$, let u_k be a non-negative minimizer of $d^f_\beta(b_k,2s)$ with $b_k \xrightarrow{k} 0^+$. Then, there exists a sequence $\{y_{\epsilon_k}\}$, $R_0 > 0$, and $\alpha > 0$ such that the function*

$$w_k(x) := \epsilon_k u_k(\epsilon_k x + \epsilon_k y_{\epsilon_k}), \quad \text{where } \epsilon_k \text{ is defined by (1.13)}, \tag{5.30}$$

satisfies

$$\liminf_{k\to\infty} \int_{B_{R_0}(0)} |w_k|^2 dx \geq \alpha > 0. \tag{5.31}$$

Moreover, for any sequence $\{b_k\}$ with $b_k \xrightarrow{k} 0^+$, there exists a subsequence, still denoted by $\{b_k\}$, such that $z_k := \epsilon_k y_{\epsilon_k} \xrightarrow{k} x_0$ and $x_0 \in \mathbb{R}^2$ is a global minimum point of $V(x)$, i.e., $V(x_0) = 0$. In addition, for any $\rho > 0$ small enough, we have

$$u_k(x) = \frac{1}{\epsilon_k} w_k\left(\frac{x - z_k}{\epsilon_k}\right) \xrightarrow{k} 0 \quad \text{for any } x \in B^c_\rho(x_0). \tag{5.32}$$

Proof. We divide the proof into six steps.

Step 1. Define $\tilde{w}(x) = \epsilon_k u_k(\epsilon_k x)$, where ϵ_k is given by (1.13). For $\beta = \beta^*$, it follows from Lemma 5.4 and assumption (F_1) that

$$\begin{aligned}\frac{\int_{\mathbb{R}^2} f(x)|u_k|^{2+2s}dx}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx} &= \frac{2s+2}{\beta^*}\left[\frac{1}{2} + \frac{1}{2}\frac{\int_{\mathbb{R}^2} V(x)|u_k|^2dx}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx}\right.\\ &\quad \left. + \frac{b}{4}\frac{\left(\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx\right)^2}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx} - \frac{d^f_{\beta^*}(b_k,2s)}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx}\right]\\ &\to \frac{s+1}{\beta^*} \quad \text{as } k \to \infty\end{aligned}$$

and

$$\int_{\mathbb{R}^2} |\tilde{w}|^{2+2s}dx = \frac{\int_{\mathbb{R}^2} |u_k|^{2+2s}dx}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx} \geq \frac{\int_{\mathbb{R}^2} f(x)|u_k|^{2+2s}dx}{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}}u_k\right|^2 dx},$$

then,

$$\liminf_{k\to\infty} \int_{\mathbb{R}^2} |\tilde{w}|^{2+2s}dx \geq \frac{s+1}{\beta^*} > 0. \tag{5.33}$$

For $\beta > \beta^*$, then Lemma 5.3 and (F_1) shows that

$$\frac{\int_{\mathbb{R}^2} f(x)|u_k|^{2+2s}dx}{x_b} \to \frac{1+s}{\beta^*} \quad \text{as } k \to \infty$$

and

$$\int_{\mathbb{R}^2} |\tilde{w}|^{2+2s}dx = \frac{\int_{\mathbb{R}^2} |u_k|^{2+2s}dx}{x_b} \geq \frac{\int_{\mathbb{R}^2} f(x)|u_k|^{2+2s}dx}{x_b},$$

then, we obtain that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^2}|\tilde{w}|^{2+2s}dx \geq \frac{s+1}{\beta^*}>0. \tag{5.34}$$

Next, we show that there exists a sequence $\{y_{\epsilon_k}\}\subset\mathbb{R}^2$ and $R_0,\alpha>0$ such that

$$\liminf_{\epsilon_k\to 0^+}\int_{B_{R_0}(y_{\epsilon_k})}|\tilde{w}_k|^2dx \geq \alpha>0. \tag{5.35}$$

Suppose by contradiction that for any $R>0$, there exists a subsequence of $\{\tilde{w}_k\}$ such that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y)}|\tilde{w}_k|^2dx=0.$$

From Lemma 2.6, we get that $\tilde{w}_k \overset{k}{\to} 0$ in $L^r(\mathbb{R}^2)$ for $2<r<2_s^*$. Hence, $\tilde{w}_k \overset{k}{\to} 0$ in $L^{2+2s}(\mathbb{R}^2)$, which contradicts (5.33) and (5.34). Therefore, (5.35) shows that (5.31) holds.

Step 2. By Lemma 5.2 and Lemma 5.4, it follows that

$$\int_{\mathbb{R}^2}V(x)|u_k|^2dx=\int_{\mathbb{R}^2}V(\epsilon_k x+\epsilon_k y_{\epsilon_k})|w_k|^2dx\to 0 \quad \text{as } k\to\infty,$$

which implies that

$$0=\liminf_{k\to\infty}\int_{\mathbb{R}^2}V(\epsilon_k x+\epsilon_k y_{\epsilon_k})|w_k|^2dx \geq \liminf_{k\to\infty}\int_{B_{R_0}(0)}V(\epsilon_k x+\epsilon_k y_{\epsilon_k})|w_k|^2dx.$$

Since $V(x)\to\infty$ as $|x|\to\infty$, (5.31) implies that $\{\epsilon_k y_{\epsilon_k}\}$ is bounded in $\mathbb{R}^2$, and passing to a subsequence if necessary, there exists a $x_0\in\mathbb{R}^2$ such that $z_k:=\epsilon_k y_{\epsilon_k}\overset{k}{\to} x_0$. By (5.31) and Fatou's lemma, it follows that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^2}V(\epsilon_k x+\epsilon_k y_{\epsilon_k})|w_k|^2dx \geq V(x_0)\int_{B_{R_0}(0)}\lim_{k\to\infty}|w_k|^2dx\geq V(x_0)\alpha,$$

which implies that $V(x_0)=0$.

Step 3. Since u_k is a non-negative minimizer for $d_\beta^f(b_k,2s)$, we have

$$\left(1+b_k\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2dx\right)(-\Delta)^s u_k+V(x)u_k=\beta f(x)|u_k|^{2s+1}+\lambda_k u_k \quad \text{in } \mathbb{R}^2, \tag{5.36}$$

where $\lambda_k\in\mathbb{R}$ is a Lagrange multiplier. Moreover, we also have

$$\begin{aligned} d_\beta^f(b_k,2s)=&\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2+V(x)|u_k|^2\right)dx+\frac{b_k}{4}\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2dx\right)^2\\ &-\frac{\beta}{2s+2}\int_{\mathbb{R}^2}f(x)|u_k|^{2s+2}dx \end{aligned} \tag{5.37}$$

and

$$\lambda_k=\int_{\mathbb{R}^2}\left(\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2+V(x)|u_k|^2\right)dx+b_k\left(\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2dx\right)^2-\beta\int_{\mathbb{R}^2}f(x)|u_k|^{2s+2}dx. \tag{5.38}$$

From (1.13), (5.37) and Lemma 5.4, we deduce that for $\beta = \beta^*$,

$$\begin{cases} \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} w_k\right|^2 dx = \int_{\mathbb{R}^2} |w_k|^2 dx = 1, \\ \epsilon_k^{2s} \int_{\mathbb{R}^2} f(x)|u_k|^{2s+2} dx \to \frac{s+1}{\beta^*} \quad \text{as } k \to \infty. \end{cases} \tag{5.39}$$

From (1.13), (5.30), and Lemma 5.3, we obtain that for $\beta > \beta^*$,

$$\begin{cases} \int_{\mathbb{R}^2} |w_k|^2 dx = 1, \\ \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} w_k\right|^2 dx = \epsilon_k^{2s} \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx = \frac{\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx}{x_{b_k}} \to 1 \quad \text{as } k \to \infty, \\ \epsilon_k^{2s} \int_{\mathbb{R}^2} f(x)|u_k|^{2s+2} dx = \frac{\int_{\mathbb{R}^2} f(x)|u_k|^{2s+2} dx}{x_{b_k}} \to \frac{s+1}{\beta^*} \quad \text{as } k \to \infty. \end{cases} \tag{5.40}$$

Hence, by (5.38)–(5.40), it follows that for any $\beta \geq \beta^*$,

$$\lambda_k \epsilon_k^{2s} \to -\frac{s\beta}{\beta^*} \quad \text{as } k \to \infty. \tag{5.41}$$

Combining (5.35) with (5.41), we have

$$\begin{aligned} &\left(1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx\right) (-\Delta)^s w_k + \epsilon_k^{2s} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) w_k \\ &\quad = \lambda_k \epsilon_k^{2s} w_k + \beta f(\epsilon_k x + \epsilon_k y_{\epsilon_k}) w_k^{2s+1} \quad \text{in } \mathbb{R}^2. \end{aligned} \tag{5.42}$$

Since $z_k \to x_0$, $\epsilon_k \to 0$, and f is continuous, we obtain that for any fixed $R > 0$, it holds that

$$\sup_{|x| \leq R} |f(\epsilon_k x + \epsilon_k y_{\epsilon_k}) - f(x_0)| \to 0,$$

which implies that

$$f(\epsilon_k x + \epsilon_k y_{\epsilon_k}) \xrightarrow{k} f(x_0) = 1.$$

From (5.39) and (5.40), we deduce that $\{w_k\}$ is bounded in $H^s(\mathbb{R}^2)$. Hence, up to a subsequence, there exists $w_0 \in H^s(\mathbb{R}^2)$ such that

$$w_k \rightharpoonup w_0 \geq 0 \quad \text{in } H^s(\mathbb{R}^2) \text{ as } k \to \infty. \tag{5.43}$$

By using (5.41)–(5.43) and passing to the weak limit, we can obtain that $w_0(x)$ satisfies, in the weak sense,

$$(-\Delta)^s w_0 + s w_0 = \beta^* w_0^{2s+1} \quad \text{in } \mathbb{R}^2. \tag{5.44}$$

It follows from (5.31) that $w_0 \not\equiv 0$. Arguing as in the proof of Proposition 4.4 in [33], we deduce that $w_0 \in C^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$. Then, by Lemma 3.2 in [9], we have

$$(-\Delta)^s w_0(x) = -\frac{1}{2} C_s \int_{\mathbb{R}^2} \frac{w_0(x+y) + w_0(x-y) - 2w_0(x)}{|x-y|^{2+2s}} dy \quad \forall x \in \mathbb{R}^2.$$

We next show that $w_0 > 0$. Suppose, by contradiction, that there exists $x_0 \in \mathbb{R}^2$ such that $w_0(x_0) = 0$. Then it follows that

$$(-\Delta)^s w_0(x_0) = -s w_0(x_0) + \beta^* w_0^{2s+1}(x_0) = 0,$$

Since $w_0 \geq 0$ and $w_0 \not\equiv 0$, we can deduce that

$$(-\Delta)^s w_0(x_0) = -\frac{1}{2}C_s \int_{\mathbb{R}^2} \frac{w_0(x_0+y)+w_0(x_0-y)}{|x_0-y|^{2+2s}}dy < 0$$

which gives a contradiction. Hence, $w_0 > 0$ for all $x \in \mathbb{R}^2$. Now, by (5.44) and the fact that $Q_{2s}(x)$ is the unique positive radial solution of (1.10), it follows that

$$w_0(x) = \left(\frac{1}{\beta^*}\right)^{\frac{1}{2s}} Q_{2s}(|x-y_0|) \quad \text{for some } y_0 \in \mathbb{R}^2. \tag{5.45}$$

A direct computation shows that

$$\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} w_0\right|^2 dx = \int_{\mathbb{R}^2} |w_0|^2 dx = 1.$$

Combining (5.39), (5.40) and (5.43), we obtain

$$w_k \to w_0 = \frac{Q_{2s}(|x-y_0|)}{(\beta^*)^{\frac{1}{2s}}} \quad \text{in } H^s(\mathbb{R}^2) \text{ as } k \to \infty. \tag{5.46}$$

Step 4. For k large enough, and $f(x)$ satisfies (F_1), it follows from (5.41) that

$$(-\Delta)^s w_k \leq \beta f(x) w_k^{2s+1} \leq \beta w_k^{2s+1}. \tag{5.47}$$

Next, we prove that $\|w_k\|_\infty \leq C$ uniformly for large k. Indeed, for $\alpha \geq 1$ and $T > 0$, let

$$\nu(t) = \begin{cases} 0, & t \leq 0, \\ t^p, & 0 < t < T, \\ pT^{p-1}(t-T)+T^p, & t \geq T. \end{cases}$$

Since ν is convex and Lipschitz continuous, we get

$$(-\Delta)^s \nu(w_k) \leq \nu'(w_k)(-\Delta)^s w_k, \tag{5.48}$$

in the weak sense. Using the Sobolev inequality, (5.47)–(5.48), and the estimates $\nu'(w_k)\nu(w_k) \leq p w_k^{2p-1}$ and $w_k \nu'(w_k) \leq p\nu(w_k)$, together with integration by parts, it follows that

$$\begin{aligned}
\|\nu(w_k)\|_{2_s^*}^2 &\leq C\int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} \nu(w_k)\right|^2 dx = C\int_{\mathbb{R}^2} \nu(w_k)(-\Delta)^s \nu(w_k) dx \\
&\leq C\int_{\mathbb{R}^2} \nu(w_k)\nu'(w_k)(-\Delta)^s w_k dx \\
&\leq C\int_{\mathbb{R}^2} \nu(w_k)\nu'(w_k)\beta w_k^{2s+1} dx \\
&\leq C\int_{\mathbb{R}^2} \nu(w_k)\nu'(w_k)\left(1+w_k^{2_s^*-1}\right) dx \\
&= C\left(\int_{\mathbb{R}^2} \nu(w_k)\nu'(w_k) dx + \int_{\mathbb{R}^2} \nu(w_k)\nu'(w_k) w_k^{2_s^*-1} dx\right) \\
&\leq Cp\left(\int_{\mathbb{R}^2} w_k^{2\alpha-1} dx + \int_{\mathbb{R}^2} (\nu(w_k))^2 w_k^{2_s^*-2} dx\right).
\end{aligned} \tag{5.49}$$

Since $p \geq 1$ and that $\nu(w_k)$ is linear when $w_k \geq T$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} (\nu(w_k))^2 w_k^{2_s^*-2} dx &= \int_{\{w_k \leq T\}} (\nu(w_k))^2 w_k^{2_s^*-2} dx + \int_{\{w_k > T\}} (\nu(w_k))^2 w_k^{2_s^*-2} dx \\ &\leq T^{2p-2} \int_{\mathbb{R}^2} w_k^{2_s^*} dx + C \int_{\mathbb{R}^2} w_k^{2_s^*} dx < +\infty,\end{aligned}$$

which implies that $\int_{\mathbb{R}^2} (\nu(w_k))^2 w_k^{2_s^*-2} dx$ is well defined for every T.

Now, we choose p in (5.49) such that $2p - 1 = 2_s^*$ and define $p_1 = \frac{2_s^*+1}{2}$. Let $R > 0$ be fixed later, then by Hölder's inequality, it follows that

$$\begin{aligned}\int_{\mathbb{R}^2} (\nu(w_k))^2 w_k^{2_s^*-2} dx &= \int_{\{w_k \leq R\}} (\nu(w_k))^2 w_k^{2_s^*-2} dx + \int_{\{w_k > R\}} (\nu(w_k))^2 w_k^{2_s^*-2} dx \\ &\leq R^{2_s^*-1} \int_{\{w_k \leq R\}} \frac{(\nu(w_k))^2}{w_k} dx \\ &\quad + \left(\int_{\{w_k > R\}} w_k^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^2} (\nu(w_k))^{2_s^*} dx \right)^{\frac{2}{2_s^*}}.\end{aligned} \tag{5.50}$$

From (5.46), it follows that $\{w_k\}$ converges strongly in $H^s(\mathbb{R}^2)$, and hence in $L^{2_s^*}(\mathbb{R}^2)$. Consequently, there exists $R > 0$ sufficiently large such that

$$\left(\int_{\{w_k > R\}} w_k^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2Cp_1}. \tag{5.51}$$

By (5.49)–(5.51), we obtain that

$$\left(\int_{\mathbb{R}^2} (\nu(w_k))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq 2Cp_1 \left(\int_{\mathbb{R}^2} w_k^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^2} \frac{(\nu(w_k))^2}{w_k} dx \right). \tag{5.52}$$

According to the fact $\nu(w_k) \leq w_k^{p_1}$ and letting $T \to \infty$, we know that

$$\left(\int_{\mathbb{R}^2} w_k^{2_s^* p_1} dx \right)^{\frac{2}{2_s^*}} \leq 2Cp_1 \left(\int_{\mathbb{R}^2} w_k^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^2} w_k^{2_s^*} dx \right) < \infty,$$

which shows that

$$w_k \in L^{2_s^* p_1}(\mathbb{R}^2). \tag{5.53}$$

Suppose that $p > p_1$. Letting $T \to \infty$ in (5.49), we can see that

$$\left(\int_{\mathbb{R}^2} w_k^{2_s^* p} dx \right)^{\frac{2}{2_s^*}} \leq Cp \left(\int_{\mathbb{R}^2} w_k^{2p-1} dx + \int_{\mathbb{R}^2} w_k^{2p+2_s^*-2} dx \right). \tag{5.54}$$

Set

$$w_k^{2p-1} = w_k^a w_k^b,$$

where $a = \frac{2_s^*(2_s^*-1)}{2(p-1)}$ and $b = 2p - 1 - a$. Moreover, $p > p_1$ implies that $0 < a, b < 2_s^*$, and by using Young's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^2} w_k^{2p-1} dx &\leq \frac{a}{2_s^*} \int_{\mathbb{R}^2} w_k^{2_s^*} dx + \frac{2_s^* - a}{2_s^*} \int_{\mathbb{R}^2} w_k^{\frac{2_s^* b}{2_s^* - a}} dx \\ &\leq \int_{\mathbb{R}^2} w_k^{2_s^*} dx + \int_{\mathbb{R}^2} w_k^{2p+2_s^*-2} dx \\ &\leq C \left(1 + \int_{\mathbb{R}^2} w_k^{2p+2_s^*-2} dx \right).\end{aligned} \tag{5.55}$$

It then follows from (5.54) and (5.55) that

$$\left(\int_{\mathbb{R}^2} w_k^{2_s^* p} dx\right)^{\frac{2}{2_s^*}} \leq Cp\left(1+\int_{\mathbb{R}^2} w_k^{2p+2_s^*-2} dx\right), \tag{5.56}$$

which implies that

$$\left(1+\int_{\mathbb{R}^2} w_k^{2_s^* p} dx\right)^{\frac{1}{2_s^*(p-1)}} \leq (Cp)^{\frac{1}{2(p-1)}}\left(1+\int_{\mathbb{R}^2} w_k^{2p+2_s^*-2} dx\right)^{\frac{1}{2(p-1)}}.$$

Iterating this argument, we deduce that

$$\left(1+\int_{\mathbb{R}^2} w_k^{2_s^* p_{i+1}} dx\right)^{\frac{1}{2_s^*(p_{i+1}-1)}} \leq (Cp_{i+1})^{\frac{1}{2(p_{i+1}-1)}}\left(1+\int_{\mathbb{R}^2} w_k^{2_s^* p_i} dx\right)^{\frac{1}{2(p_i-1)}},$$

where

$$2p_{i+1}+2_s^*-2 = 2_s^* p_i \quad \text{and} \quad p_{i+1} = \left(\frac{2_s^*}{2}\right)^i (p_i-1)+1.$$

Let $C_{i+1} = Cp_{i+1}$ and

$$U_i = \left(1+\int_{\mathbb{R}^2} w_k^{2_s^* p_i} dx\right)^{\frac{1}{2_s^*(p_i-1)}}.$$

Hence, there exists a constant $C > 0$ independent of i such that

$$U_{i+1} \leq \prod_{i=2}^{i+1} C_i^{\frac{1}{2(p_i-1)}} U_1 \leq CU_1,$$

which shows that

$$\|w_k\|_\infty \leq C \quad \text{uniformly for large } k.$$

Step 5. We prove that $w_k(x) \to 0$ as $|x| \to \infty$ uniformly for large enough k.

We rewrite problem (5.42) as follows:

$$(-\Delta)^s w_k + w_k = h_k(x) \quad \text{in } \mathbb{R}^2,$$

where

$$h_k(x) = w_k + \frac{-\epsilon_k^{2s} V(\epsilon_k x+\epsilon_k y_{\epsilon_k}) w_k + \lambda_k \epsilon_k^{2s} w_k + \beta f(\epsilon_k x+\epsilon_k y_{\epsilon_k}) w_k^{2s+1}}{1+b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx}.$$

By Step 4, it follows that $h_k \in L^\infty(\mathbb{R}^2)$, and $\{h_k\}$ is uniformly bounded for sufficiently large k. Using the interpolation inequality and the strong convergence of $\{w_k\}$ in $H^s(\mathbb{R}^2)$, we deduce that $h_k \to h$ in $L^q(\mathbb{R}^2)$ for any $q \in [2, +\infty)$, where $h \in L^q(\mathbb{R}^2)$. Hence, by [12], it follows that

$$w_k(x) = \int_{\mathbb{R}^2} \mathcal{K}(x-y) h_k(y) dy,$$

where $\mathcal{K}$ is a Bessel potential and it satisfies the following:

$(\mathcal{K}_1)$ $\mathcal{K}$ is positive, radially symmetric, and smooth in $\mathbb{R}^2 \setminus \{0\}$.

$(\mathcal{K}_2)$ There exists $C > 0$ such that $\mathcal{K}(x) \leq \frac{C}{|x|^{2+2s}}$ for $x \in \mathbb{R}^2 \setminus \{0\}$.

$(\mathcal{K}_3)$ $\mathcal{K} \in L^r(\mathbb{R}^2)$ for $r \in [1, \frac{1}{1-s})$.

For any $\xi > 0$, we have

$$
\begin{aligned}
0 \le w_k(x) &\le \int_{\mathbb{R}^2} \mathcal{K}(x-y)|h_k(y)|dy \\
&= \int_{\{|x-y|\ge \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k(y)|dy + \int_{\{|x-y|< \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k(y)|dy.
\end{aligned}
$$

By $(\mathcal{K}_2)$, we have

$$
\begin{aligned}
\int_{\{|x-y|\ge \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k(y)|dy &\le C\xi^s \|h_k\|_\infty \int_{\{|x-y|\ge \frac{1}{\xi}\}} \frac{1}{|x-y|^{2+s}} dy \\
&\le C\xi^s \int_{\{|x-y|\ge \frac{1}{\xi}\}} \frac{1}{|x-y|^{2+s}} dy = C\xi^{2s}.
\end{aligned} \tag{5.57}
$$

On the other hand, by using Hölder's inequality and $(\mathcal{K}_3)$, we deduce that

$$
\begin{aligned}
\int_{\{|x-y|< \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k(y)|dy &\le \int_{\{|x-y|< \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k-h|dy + \int_{\{|x-y|< \frac{1}{\xi}\}} \mathcal{K}(x-y)|h|dy \\
&\le \left(\int_{\mathbb{R}^2} |\mathcal{K}|^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |h_k-h|^2 dy\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\mathbb{R}^2} |\mathcal{K}|^2 dy\right)^{\frac{1}{2}} \left(\int_{\{|x-y|< \frac{1}{\xi}\}} |h|^2 dy\right)^{\frac{1}{2}},
\end{aligned}
$$

which implies that there exists $R_0 > 0$ independent of $\xi > 0$ such that

$$
\int_{\{|x-y|< \frac{1}{\xi}\}} \mathcal{K}(x-y)|h_k(y)|dy \le \xi \quad \text{uniformly for large } k \text{ and } |x| \ge R_0, \tag{5.58}
$$

where we have used the fact $s > \frac{1}{2}$ so that $2 < \frac{1}{1-s}$ and $\left(\int_{\{|x-y|< \frac{1}{\xi}\}} |h|^2 dy\right)^{\frac{1}{2}} \to 0$ as $|x| \to \infty$. It then follows from (5.57) and (5.58) that

$$
0 \le w_k(x) \le C(\xi^{2s} + \xi) \quad \text{as } |x| \ge R_0 \text{ uniformly for large } k,
$$

which implies

$$
w_k(x) \to 0 \quad \text{as } |x| \to \infty \text{ uniformly for large } k.
$$

Step 6. We prove that (5.32) holds.

In fact, by Lemma 5.3 and Lemma 5.4, we obtain that there exists $C_1 > 0$ such that

$$
1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx \le C_1 \quad \text{uniformly for large } k. \tag{5.59}
$$

From Lemma 4.3 in [12], there exists a function χ such that

$$
0 < \chi \le \frac{C}{1+|x|^{2+2s}}
$$

and

$$
(-\Delta)^s \chi + \frac{s\beta}{2C_1\beta^*} \chi = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R_1}(0),
$$

for some suitable constant $R_1 > 0$. Moreover, by Step 5, (5.41) and (5.59), we know that there exists $R_2 > 0$ large enough such that for large enough k,

$$\begin{aligned}(-\Delta)^s w_k + \frac{s\beta}{2C_1\beta^*} w_k &\le (-\Delta)^s w_k + \frac{s\beta}{2\beta^*\left(1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx\right)} w_k \\ &= \frac{-\epsilon_k^{2s} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) w_k + \lambda_k \epsilon_k^{2s} w_k + \beta f(\epsilon_k x + \epsilon_k y_{\epsilon_k}) w_k^{2s+1} + \frac{s\beta}{2\beta^*} w_k}{1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx} \le 0,\end{aligned}$$

for $|x| \ge R_2$. Then, similar to the Proof of Lemma 5.6 in [33], it follows that

$$w_k(x) \le \frac{C}{1 + |x|^{2+2s}} \quad \text{uniformly for large } k. \tag{5.60}$$

For any $x \in B_\rho^c(x_0)$, we have

$$\frac{|x - z_k|}{\epsilon_k} \ge \frac{|x - x_0|}{2\epsilon_k} \ge \frac{\rho}{2\epsilon_k} \xrightarrow{k} +\infty. \tag{5.61}$$

From (5.60) and (5.61), it follows that

$$\begin{aligned}u_k(x) = \frac{1}{\epsilon_k} w_k\left(\frac{x - z_k}{\epsilon_k}\right) &\le \frac{1}{\epsilon_k} \frac{C}{1 + \left|\frac{x - z_k}{\epsilon_k}\right|^{2+2s}} \\ &\le \frac{1}{\epsilon_k} \frac{C}{1 + \left|\frac{\rho}{2\epsilon_k}\right|^{2+2s}} \xrightarrow{k} 0 \quad \forall x \in B_\rho^c(x_0),\end{aligned}$$

which means Lemma 5.5 holds. □

Proof of Theorem 1.4. Set $\bar{z}_k$ a global maximum point of u_k. Then, (5.36) and (5.41) show that

$$u_k(\bar{z}_k) \ge \left(\frac{-\lambda_k}{\beta}\right)^{\frac{1}{2s}} \ge C\epsilon_k^{-1}.$$

Thus, Lemma 5.5 implies that $\bar{z}_k \to x_0$ as $k \to \infty$. Let

$$\tilde{w}_k := \epsilon_k u_k(\epsilon_k x + \bar{z}_k). \tag{5.62}$$

From (5.36), we have

$$\begin{aligned}&\left(1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx\right) (-\Delta)^s \tilde{w}_k + \epsilon_k^{2s} V(\epsilon_k x + \bar{z}_k) \tilde{w}_k \\ &\quad = \lambda_k \epsilon_k^{2s} \tilde{w}_k + \beta f(\epsilon_k x + \bar{z}_k) \tilde{w}_k^{2s+1} \quad \text{in } \mathbb{R}^2.\end{aligned} \tag{5.63}$$

Similar to the proof of Step 3 in Lemma 5.5, there exists a subsequence of $\{\tilde{w}_k\}$, still denoted by $\{\tilde{w}_k\}$, such that $\tilde{w}_k \xrightarrow{k} \tilde{w}_0 > 0$, where $\tilde{w}_0$ satisfies (5.44) for some $\beta > 0$. Now, we claim that

$$\tilde{w}_k \to \tilde{w}_0 \quad \text{in } C_{\mathrm{loc}}^{2,\delta}(\mathbb{R}^2) \text{ as } k \to \infty \text{ for some } \delta \in (0,1). \tag{5.64}$$

In fact, we rewrite problem (5.63) as follows:

$$(-\Delta)^s \tilde{w}_k = \frac{-\epsilon_k^{2s} V(\epsilon_k x + \bar{z}_k) \tilde{w}_k + \lambda_k \epsilon_k^{2s} \tilde{w}_k + \beta f(\epsilon_k x + \bar{z}_k) \tilde{w}_k^{2s+1}}{1 + b_k \int_{\mathbb{R}^2} \left|(-\Delta)^{\frac{s}{2}} u_k\right|^2 dx} =: P(\tilde{w}_k).$$

For large k, similar to the proof of Step 4 in Lemma 5.5, we know that $\|\tilde{w}_k\|_\infty \le C$. Then, $P(\tilde{w}_k) \in L^\infty(\mathbb{R}^2)$ for large k. Applying Proposition 2.9 in [30], we have that for large k,

$$\|\tilde{w}_k\|_{C^{1,\alpha}(\mathbb{R}^2)} \le C\left(\|\tilde{w}_k\|_\infty + \|P(\tilde{w}_k)\|_\infty\right) \le C \quad \text{for some } \alpha \in (0,1). \tag{5.65}$$

Moreover, by (V_1), $\|\tilde{w}_k\|_\infty \le C$ and (5.65), we deduce that

$$\begin{aligned}\left\|V(\epsilon_k x + \bar{z}_k)\tilde{w}_k\right\|_{C^{0,\alpha}(\mathbb{R}^2)} &\le C + \sup_{x\neq y}\frac{\left|V(\epsilon_k x+\bar{z}_k)\tilde{w}_k(x) - V(\epsilon_k y+\bar{z}_k)\tilde{w}_k(y)\right|}{|x-y|^\alpha}\\ &\le C + \sup_{x\neq y}\frac{\left|V(\epsilon_k x+\bar{z}_k) - V(\epsilon_k y+\bar{z}_k)\right|\left|\tilde{w}_k(x)\right|}{|x-y|^\alpha}\\ &\quad + \sup_{x\neq y}\frac{\left|V(\epsilon_k y+\bar{z}_k)\right|\left|\tilde{w}_k(x)-\tilde{w}_k(y)\right|}{|x-y|^\alpha} \quad \text{for large } k.\end{aligned} \tag{5.66}$$

For any fixed $R > 0$, then we have

$$\begin{cases}\dfrac{|\tilde{w}_k(x)-\tilde{w}_k(y)|}{|x-y|^\alpha} \le CR^{1-\alpha} \le C & \text{if } |x-y| \le R,\\ \dfrac{|\tilde{w}_k(x)-\tilde{w}_k(y)|}{|x-y|^\alpha} \le CR^{-\alpha} \le C & \text{if } |x-y| > R.\end{cases} \tag{5.67}$$

From (5.66) and (5.67),

$$\left\|V(\epsilon_k x+\bar{z}_k)\tilde{w}_k\right\|_{C^{0,\alpha}(\mathbb{R}^2)} \le C \quad \text{for large } k. \tag{5.68}$$

Similarly, by (F_1), we also have

$$\|\tilde{w}_k\|_{C^{0,\alpha}(\mathbb{R}^2)} \le C \quad \text{and} \quad \|f(\epsilon_k x+\bar{z}_k)\tilde{w}_k^{2s+1}\|_{C^{0,\alpha}(\mathbb{R}^2)} \le C \quad \text{for large } k. \tag{5.69}$$

Then, (5.68) and (5.69) show that $\|P(\tilde{w}_k)\|_{C^{0,\alpha}(\mathbb{R}^2)} \le C$ for large k. Thus, by Proposition 2.8 in [19], we have that

$$\|\tilde{w}_k\|_{C^{1,\varpi}(\mathbb{R}^2)} \le C\left(\|\tilde{w}_k\|_\infty + \|P(\tilde{w}_k)\|_{C^{0,\alpha}(\mathbb{R}^2)}\right) \le C \quad \text{for large } k, \tag{5.70}$$

where $\varpi = \alpha + 2s - 1$. We now differentiate the equation to obtain

$$(-\Delta)^s(\tilde{w}_k)_{x_i} = P'(\tilde{w}_k)(\tilde{w}_k)_{x_i}$$

for $i = 1,2$ with $P'(\tilde{w}_k) = \frac{-\epsilon_k^{2s}V(\epsilon_k x+\bar{z}_k)+\lambda_k\epsilon_k^{2s}+\beta f(\epsilon_k x+\bar{z}_k)\tilde{w}_k^{2s}}{1+b_k\int_{\mathbb{R}^2}\left|(-\Delta)^{\frac{s}{2}}u_k\right|^2dx}$. Clearly, (5.70) implies that

$$\|(\tilde{w}_k)_{x_i}\|_{C^{0,\varpi}(\mathbb{R}^2)} \le C \quad \text{and} \quad \|P'(\tilde{w}_k)\|_{C^{0,\varpi}(\mathbb{R}^2)} \le C \quad \text{for large } k. \tag{5.71}$$

From (5.70), (5.71), and (V_1), we deduce that

$$\begin{aligned}\left\|V(\epsilon_k x+\bar{z}_k)(\tilde{w}_k)_{x_i}\right\|_{C^{0,\alpha}(\mathbb{R}^2)} &= \sup\left|V(\epsilon_k x+\bar{z}_k)(\tilde{w}_k)_{x_i}\right|\\ &\quad + \sup_{x\neq y}\frac{\left|V(\epsilon_k x+\bar{z}_k)D\tilde{w}_k(x) - V(\epsilon_k y+\bar{z}_k)D\tilde{w}_k(y)\right|}{|x-y|^\alpha}\\ &\le C + \sup_{x\neq y}\frac{\left|V(\epsilon_k x+\bar{z}_k) - V(\epsilon_k y+\bar{z}_k)\right|\left|D\tilde{w}_k(x)\right|}{|x-y|^\alpha}\\ &\quad + \sup_{x\neq y}\frac{\left|V(\epsilon_k y+\bar{z}_k)\right|\left|D\tilde{w}_k(x)-D\tilde{w}_k(y)\right|}{|x-y|^\alpha} \le C \quad \text{for large } k.\end{aligned} \tag{5.72}$$

Similarly, by (F_1), we also can obtain that

$$\|f(\epsilon_k x+\bar{z}_k)\tilde{w}_k^{2s}(\tilde{w}_k)_{x_i}\|_{C^{0,\alpha}(\mathbb{R}^2)}\le C\quad\text{for large } k. \tag{5.73}$$

Thus, by (5.72)–(5.73) and Proposition 2.8 in [30], it follows that for some $\delta\in(0,1)$,

$$\|(\tilde{w}_k)_{x_i}\|_{C^{1,\delta}(\mathbb{R}^2)}\le C\left(\|(\tilde{w}_k)_{x_i}\|_\infty+\|P'(\tilde{w}_k)(\tilde{w}_k)_{x_i}\|_\infty\right)\le C\quad\text{for large } k,$$

which implies that $\|\tilde{w}_k\|_{C^{2,\delta}(\mathbb{R}^2)}\le C$ for large k. Therefore, (5.64) holds.

Since $x=0$ is a critical point of $\tilde{w}_k$ for all k, it follows from (5.64) that $x=0$ is also a critical point of $\tilde{w}_0$. By Lemma 2.1, $Q_{2s}(x)$ is radially symmetric about the origin and strictly monotone with respect to $|x|$. Hence, $\tilde{w}_0$ attains its unique global maximum at $x=0$ and

$$\tilde{w}_0(x)=\frac{Q_{2s}(|x|)}{(\beta^*)^{\frac{1}{2s}}}.$$

Combining $\tilde{w}_k\ge\left(\frac{s}{2\beta^*}\right)^{\frac{1}{2s}}$ at each global maximum point and $\tilde{w}_k$ decays to zero uniformly in k as $|x|\to\infty$, we get that each global maximum point of $\tilde{w}_k$ stays in a finite ball in $\mathbb{R}^2$. From (5.64) and that the origin is the only critical point of $\tilde{w}_0$, then all global maximum points of $\{\tilde{w}_k\}$ must approach to the origin and, hence, stay in a small ball $B_\rho(0)$ as $k\to\infty$. Let $\rho>0$ small enough such that $\tilde{w}_0''(\tau)<0$ for $0<\tau<\rho$. By Lemma 4.2 in [27], we know that $\{\tilde{w}_k\}$ has no critical points other than the origin. Hence, we can deduce that $\{u_k\}$ has a unique global maximum point $\bar{z}_k\in\mathbb{R}^2$ and $\bar{z}_k$ concentrates at a unique global minimum point of potential $V(x)$ as $k\to\infty$. □

6 Concentration behavior under homogeneous potential

We first derive a detailed estimate of the energy $d^f_{\beta^*}(b,2s)$ as $b\to 0^+$.

Lemma 6.1. *Suppose that $f(x)$ satisfies (F_1)–(F_2) and $V(x)$ satisfies (V_1) and (1.15) and (1.16) hold, then we have*

$$\limsup_{b\to 0^+}\frac{d^f_{\beta^*}(b,2s)}{b^{\frac{\eta}{\eta+4s}}}\le\frac{\eta+4s}{4\eta}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}}\frac{1}{(\beta^*)^{\frac{4}{\eta+4s}}}\frac{1}{s^{\frac{4s}{\eta+4s}}}, \tag{6.1}$$

where η and θ_0 are given by (1.18) and (1.19), respectively.

Proof. Take $x_0\in Y_0$ and y_0 satisfying $M_{i_0}(y_0)=\theta_0$. Let u_τ be given by (5.22) with $x_0=x_{i_0}+\frac{y_0}{\tau}$. By direct computation, we deduce that

$$\int_{\mathbb{R}^2}V(x)|u_\tau|^2dx=\frac{\theta_0}{(\beta^*)^{\frac{1}{s}}\tau^\eta}(1+o(1))\quad\text{as } \tau\to\infty. \tag{6.2}$$

By (6.2) and (5.24)–(5.25), we conclude that

$$d^f_{\beta^*}(b,2s)\le E^f_{\beta^*,b,2s}(u_\tau)\le\frac{b}{4}\tau^{4s}+Cb\tau^{-4s}+Cb+C\tau^{-2s}+C\tau^{-(q+2s)}+\frac{\theta_0}{2(\beta^*)^{\frac{1}{s}}\tau^\eta}(1+o(1)). \tag{6.3}$$

Take $\tau = [\frac{\eta\theta_0}{2sb(\beta^*)^{\frac{1}{s}}}]^{\frac{1}{\eta+4s}}$, and then, $\tau \to \infty$ as $b \to 0^+$. Thus, it follows from (6.3) that

$$d^f_{\beta^*}(b, 2s) \le \frac{1}{4}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}}\left(\frac{1}{\beta^*}\right)^{\frac{4}{\eta+4s}} s^{\frac{-4s}{\eta+4s}} b^{\frac{\eta}{\eta+4s}} + \frac{1}{\eta}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}}\left(\frac{1}{\beta^*}\right)^{\frac{4}{\eta+4s}} s^{\frac{\eta}{\eta+4s}} b^{\frac{\eta}{\eta+4s}} + Cb^{\frac{2s}{\eta+4s}} \quad \text{as } b \to 0^+,$$

which implies that (6.1) holds. □

Proof of Theorem 1.5. For $\beta = \beta^*$, let u_k be a non-negative minimizer for $d^f_{\beta^*}(b_k, 2s)$ and $\tilde{w}_k$ be defined by (5.62), where $b_k \to 0^+$. From Theorem 1.4, we know that u_k has a unique global maximum point $\tilde{z}_k \in \mathbb{R}^2$ such that $\tilde{z}_k \to x_0$ as $k \to \infty$ with $V(x_0) = 0$. Assume that $x_0 = x_{i_0}$ for some $1 \le i_0 \le m$. Next, we prove that

$$\eta_{i_0} = \eta \quad \text{and} \quad \left\{\frac{\tilde{z}_k - x_{i_0}}{\epsilon_k}\right\} \quad \text{is bounded,} \tag{6.4}$$

where ϵ_k is given by (1.13). In fact, if (6.4) is false, then we have $\eta_{i_0} < \eta$ or $\lim_{k\to\infty}|\frac{\tilde{z}_k - x_{i_0}}{\epsilon_k}| = +\infty$. By using (1.12) and (1.16), we can deduce that for any $T > 0$ large enough,

$$\begin{aligned}&\liminf_{k\to\infty} \frac{1}{\epsilon_k^\eta}\int_{\mathbb{R}^2} V(x)|u_k|^2 dx \\ &= \liminf_{k\to\infty} \frac{1}{\epsilon_k^{\eta-\eta_{i_0}}}\int_{\mathbb{R}^2} \frac{V(\epsilon_k x + \tilde{z}_k)}{V_{i_0}(\epsilon_k x + \tilde{z}_k - x_{i_0})} V_{i_0}\left(x + \frac{\tilde{z}_k - x_{i_0}}{\epsilon_k}\right)|\tilde{w}_k|^2 dx \ge T.\end{aligned}$$

By Lemma 2.1 and Young's inequality, we have

$$d^f_{\beta^*}(b_k, 2s) = E^f_{\beta^*, b_k, 2s}(u_k) \ge \frac{b_k}{4\epsilon_k^{4s}} + \frac{T\epsilon_k^\eta}{2} \ge CT^{\frac{4s}{\eta+4s}} b_k^{\frac{\eta}{\eta+4s}},$$

which gives a contradiction by Lemma 6.1. Hence, (6.4) holds. Passing to a subsequence, we may assume that there exists $y_0 \in \mathbb{R}^2$ such that

$$\frac{\tilde{z}_k - x_{i_0}}{\epsilon_k} \to y_0 \quad \text{as } k \to \infty. \tag{6.5}$$

From (1.12), (1.16), and (6.5), it follows that

$$\begin{aligned}\liminf_{k\to\infty} \frac{1}{\epsilon_k^\eta}\int_{\mathbb{R}^2} V(x)|u_k|^2 dx &= \liminf_{k\to\infty} \int_{\mathbb{R}^2} \frac{V(\epsilon_k x + \tilde{z}_k)}{V_{i_0}(\epsilon_k x + \tilde{z}_k - x_{i_0})} V_{i_0}\left(x + \frac{\tilde{z}_k - x_{i_0}}{\epsilon_k}\right)|\tilde{w}_k|^2 dx \\ &= \frac{1}{(\beta^*)^{\frac{1}{s}}}\int_{\mathbb{R}^2} V_{i_0}(x + y_0) Q_{2s}^2 dx \ge \frac{\theta_{i_0}}{(\beta^*)^{\frac{1}{s}}} \ge \frac{\theta_0}{(\beta^*)^{\frac{1}{s}}}.\end{aligned} \tag{6.6}$$

By using (6.6) and Young's inequality, we deduce that

$$\begin{aligned}d^f_{\beta^*}(b_k, 2s) = E^f_{\beta^*, b_k, 2s}(u_k) &\ge \frac{b_k}{4\epsilon_k^{4s}} + \frac{\theta_0\epsilon_k^\eta}{2(\beta^*)^{\frac{1}{s}}}(1 + o(1)) \\ &\ge b_k^{\frac{\eta}{\eta+4s}} \frac{\eta + 4s}{4\eta}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}} \frac{1}{(\beta^*)^{\frac{4}{\eta+4s}}} \frac{1}{s^{\frac{4s}{\eta+4s}}}(1 + o(1)),\end{aligned}$$

which implies that

$$\limsup_{k\to\infty}\frac{d^f_{\beta^*}(b_k,2s)}{b_k^{\frac{\eta}{\eta+4s}}}\geq\frac{\eta+4s}{4\eta}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}}\frac{1}{(\beta^*)^{\frac{4}{\eta+4s}}}\frac{1}{s^{\frac{4s}{\eta+4s}}}. \tag{6.7}$$

where the equality holds if and only if $\theta_{i_0}=\theta_0$, $M_{i_0}(y_0)=\theta_0$, and

$$\lim_{k\to\infty}\frac{\epsilon_k}{\tilde{\epsilon}_k}=1,\quad \text{where } \tilde{\epsilon}_k \text{ is defined in (1.22).} \tag{6.8}$$

By Lemma 6.1, we have that

$$\lim_{k\to\infty}\frac{d^f_{\beta^*}(b_k,2s)}{b_k^{\frac{\eta}{\eta+4s}}}=\frac{\eta+4s}{4\eta}\left(\frac{\eta\theta_0}{2}\right)^{\frac{4s}{\eta+4s}}\frac{1}{(\beta^*)^{\frac{4}{\eta+4s}}}\frac{1}{s^{\frac{4s}{\eta+4s}}}, \tag{6.9}$$

which implies that all inequalities in (6.6) and (6.7) become equalities and $M_{i_0}(y_0)=\theta_0=\theta_{i_0}$. Therefore, (1.12) and (6.8) yield (1.21), while (6.5) implies (1.23). For $\beta>\beta^*$, we take $x_{i_0}\in Y_0$ and y_0 satisfying $M_{i_0}(y_0)=\theta_0$. Set

$$\hat{u}_k(x)=A_k\varphi\left(x_{b_k}^{-\frac{2}{s}}(x-x_{i_0}-\tilde{\epsilon}_k y_0)\right)\bar{u}_k(x-x_{i_0}-\tilde{\epsilon}_k y_0),$$

where $\bar{u}_k:=\bar{u}_{b_k}$ and φ are given in Lemma 5.2, respectively, and $A_k>0$ is chosen so that $\|\hat{u}_k\|_2^2=1$. By (1.22) and definition of x_b, we get that $x_{b_k}=\tilde{\epsilon}_k^{-2s}$. Similar to (5.10)–(5.11) and (5.13), it then follows from (5.16) and the dominated convergence theorem that

$$\begin{aligned}
d^f_\beta(b_k,2s)-\tilde{d}_\beta(b_k,2s)&\leq E^f_{\beta,b_k,2s}(\hat{u}_k)-\tilde{E}_{\beta,b_k,2s}(\bar{u}_k)\leq\frac{1}{2}\int_{\mathbb{R}^2}V(x)|\hat{u}_b|^2dx+o(1)\\
&=\frac{\tilde{\epsilon}_k^\eta}{2(\beta^*)^{\frac{1}{s}}}\int_{\mathbb{R}^2}\frac{V(\tilde{\epsilon}_k x+x_{i_0}+\tilde{\epsilon}_k y_0)}{V_{i_0}(\tilde{\epsilon}_k x+\tilde{\epsilon}_k y_0)}V_{i_0}(x+y_0)\varphi^2(\tilde{\epsilon}_k^5 x)Q_{2s}^2(x)dx+o(1)\\
&=\frac{\tilde{\epsilon}_k^\eta\theta_0}{2(\beta^*)^{\frac{1}{s}}}\big(1+o(1)\big)\quad\text{as } k\to\infty,
\end{aligned}$$

which shows that

$$\limsup_{k\to\infty}\frac{d^f_\beta(b_k,2s)-\tilde{d}_\beta(b_k,2s)}{\tilde{\epsilon}_k^\eta}\leq\frac{\theta_0}{2(\beta^*)^{\frac{1}{s}}}. \tag{6.10}$$

Denote u_k be a non-negative minimizer for $d^f_\beta(b_k,2s)$. From Theorem 1.4, we know that u_k has a unique global maximum point $\tilde{z}_k\in\mathbb{R}^2$ such that $\tilde{z}_k\to x_0$ as $k\to\infty$ with $V(x_0)=0$. Assume that $x_0=x_{i_0}$ for some $1\leq i_0\leq m$. From (1.12) and (1.16), it follows that

$$\begin{aligned}
\liminf_{k\to\infty}\frac{d^f_\beta(b_k,2s)-\tilde{d}_\beta(b_k,2s)}{\tilde{\epsilon}_k^\eta}&\geq\liminf_{k\to\infty}\frac{1}{2\tilde{\epsilon}_k^\eta}\int_{\mathbb{R}^2}V(x)|u_k|^2dx\\
&=\liminf_{k\to\infty}\frac{1}{2\tilde{\epsilon}_k^{\eta-\eta_{i_0}}}\int_{\mathbb{R}^2}\frac{V(\tilde{\epsilon}_k x+\tilde{z}_k-x_{i_0}+x_{i_0})}{V_{i_0}(\tilde{\epsilon}_k x+\tilde{z}_k-x_{i_0})}V_{i_0}\left(x+\frac{\tilde{z}_k-x_{i_0}}{\tilde{\epsilon}_k}\right)|\tilde{w}_k|^2dx.
\end{aligned}$$

Thus, similar to (6.4), we can also obtain that $\eta_{i_0}=\eta$ and $\{\frac{\tilde{z}_k-x_{i_0}}{\tilde{\epsilon}_k}\}$ is bounded sequence. By passing to a subsequence, we may assume that there exists $y_0\in\mathbb{R}^2$ such that

$$\frac{\tilde{z}_k-x_{i_0}}{\tilde{\epsilon}_k}\to y_0\quad\text{as } k\to\infty. \tag{6.11}$$

By (1.12), (1.16) and (6.11), we deduce that

$$\liminf_{k\to\infty} \frac{d^f_\beta(b_k,2s) - \tilde{d}_\beta(b_k,2s)}{\tilde{\epsilon}_k^\eta} \geq \frac{1}{2(\beta^*)^{\frac{1}{s}}} \int_{\mathbb{R}^2} V_{i_0}(x+y_0) Q_{2s}^2 dx \geq \frac{\theta_{i_0}}{2(\beta^*)^{\frac{1}{s}}} \geq \frac{\theta_0}{2(\beta^*)^{\frac{1}{s}}}, \tag{6.12}$$

where (6.12) becomes equalities if and only if $\theta_{i_0} = \theta_0$ and $M_{i_0}(y_0) = \theta_0$. Clearly, (6.10) and (6.12) show that

$$\lim_{k\to\infty} \frac{d^f_\beta(b_k,2s) - \tilde{d}_\beta(b_k,2s)}{\tilde{\epsilon}_k^\eta} = \frac{\theta_0}{2(\beta^*)^{\frac{1}{s}}},$$

which implies that (6.12) becomes equalities. Therefore, (6.11) shows that (1.23) holds. □

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Conflict of interest:

The authors declare that they have no known competing financial interests or personal.

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