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Mathematical study on the impact of word of mouth and social media advertisements on consumers

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Abstract. This paper proposes a marketing model based on an epidemic model that combines two essential strategies social media platform advertising and direct word-of-mouth advertising. This study investigates their vital impact on consumer behavior. The existence of equilibrium points is explored based on the basic reproduction number R_0 and the graphical representations of nullclines. The local stability of equilibrium points is calculated and the global stability of the inactive market equilibrium point is proven. According to our results, the model exhibits a transcritical bifurcation. A sensitivity analysis is also carried out to show how variations in parameters influence R_0. Therefore, the model provides actionable insights and useful information for creating successful marketing plans. These strategies encourage customer interaction and boost income.

Keywords: marketing, social media, stability, sensitivity analysis, bifurcation.

2020 Mathematics Subject Classification: 34C60, 37C75, 37N25.

1 Introduction

Marketing is a key driver of business growth and sustainability. Philip Kotler [25] defined marketing as the science and art of exploring, creating, and delivering value to satisfy the needs of a target market for profit. A product or service can be marketed in various ways. We focus on two of these methods: direct word-of-mouth marketing and social media advertisements, and the impact of these advertisements on interactions between people. Social media now plays a crucial role in our lives by facilitating online communication, content sharing, and information gathering. These platforms include social networking sites, bookmarking services, and forums. X, Facebook, Instagram, LinkedIn, Wikipedia, Pinterest, and Google Plus are some well-known examples [8,11,20].

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Word-of-mouth marketing (WOM) is the process of organic promotion through customer conversations, a strategy that is receiving increasing attention. The advantage of WOM is that customers see it as more trustworthy and credible than traditional advertising, since it originates from real user experiences. The WOM strategy has a significant impact on consumer purchase decisions. This impact includes cognitive (knowledge), affective (attitude), and conative (purchase intentions and actions) aspects [23]. According to statistical data, 87% of consumers base their purchase decisions on WOM [12,21]. WOM increases brand discovery by 49%, and 64% of marketing executives believe it to be the most successful kind of advertising [12].

A primary function of social media advertising is to move consumers from product unawareness to awareness, a crucial step for driving future sales. Social media are defined as a collection of web-based apps that build on the ideological and technical foundations of Web 2.0, and that enable the creation and distribution of user-generated [1]. Advertisements on social media are 55% more successful than traditional advertisements [2,24]. Social media is a vital brand tool, with over 91.9% of companies with more than 100 employees utilizing it for marketing purposes [2]. Social media platforms facilitate communication and engagement between consumers and brands. It gives marketers amazing opportunities to connect with customers and create enduring connections with them. Social media is regarded as a crucial medium for producing and disseminating brand-related information with a higher capacity to reach consumers than traditional media, such as radio, television, or print, because of its interactive features that enable participation, collaboration, and knowledge sharing. As a result, businesses will find it beneficial to include social media in their marketing mix [2].

Mathematical models use equations and algorithms to represent the real world [6,10,27]. They simplify complex systems, making them easier to study. We use models to organize data and make concrete predictions. The applications vary widely. In animal nutrition, for example, researchers use models to calculate feed quality and determine the specific nutrients animals need [13]. There are other examples, such as estimating the age of necrophagous fly larvae. It helps determine the time of death more effectively in criminal investigations [9]. Similarly, mathematical modeling can be used to model and predict climate change. Mathematical models have provided fresh insight into biological systems, physical systems, decision-making problems, space models, industrial, and economic problems [4].

Many fields of science utilize epidemic models to examine how various population classes interact with one another through a diffusion phenomenon. They are commonly predicated on the notion of contagion through human interaction. A well-known epidemic model SIR has three compartments of susceptible, infected, and recovered. The changes in the three compartments are illustrated by using differential equations. An epidemiological model is used to explore the interplay between sharing and advertising in viral videos [17]. The epidemic model is also used to study rumor spreading and to control its spread [18]. Researchers used an epidemiological approach to analyze and understand the phenomena of digital marketing [5].

The primary aim of this study is to investigate the impact of word-of-mouth and social media advertisements on consumer behavior using epidemic models. The model illustrates both inactive and active market equilibrium. We then study the role of the basic reproduction number R_0 in the model and its impact on marketing dynamics. Finally, the impact of various factors on marketing dynamics is determined through both theoretical analysis and simulation experiments. Based on these findings, some specific promotion strategies are recommended.

This paper has been organized in the following way: In Section 2 the model has been

proposed based on epidemic models. Section 3 provides a qualitative analysis of the model. Section 4 presents the transcritical bifurcation analysis of our model. Section 5 contains numerical simulations to deepen our understanding of the model. Finally, Section 6 presents the conclusion of this study.

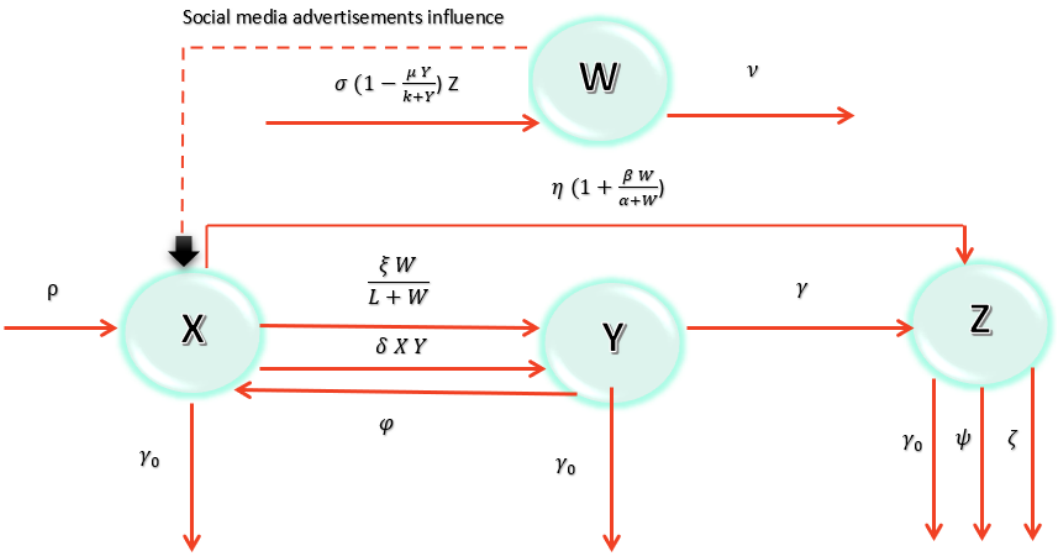


Figure 1.1: The scheme of the $XYZW$ model (2.1).

2 Mathematical model

Based on concepts derived from epidemic models, we propose a model to study the impact of direct word-of-mouth and social media advertising on marketing dynamics. This approach helps us understand how interpersonal communication and modern digital advertising work together to influence consumer behavior.

The entire population is categorized into three groups: $X(t)$ represents the number of individuals who are unaware of the product's existence, $Y(t)$ indicates the number of individuals who are aware of the product's existence, and $Z(t)$ refers to the number of actual purchasers of the product. At a specific time, the population size denoted by $M(t) = X(t) + Y(t) + Z(t)$ is the sum of individuals in three distinct groups.

We assume that individuals enter the market at a rate of ρ. The number of individuals who are unaware of the products becomes aware according to the interaction between them at the average of δXY. The rate ηXZ denotes the interaction between unaware individuals and those who have already purchased the product, leading the unaware individuals to become buyers. Aware individuals may return to the unaware class due to losing interest or forgetting the product, at a rate of φY. The natural death rate for every class is the average γ_0. Aware individuals turn into actual purchasers at a rate of γY due to their desire to try the product. Actual purchasers are decreasing at a rate of ψZ due to boredom with the product. Furthermore it is decreasing at a rate of ζZ due to better offers from other companies. Product marketing not only depends on word-of-mouth but also on internet ads in the current digital

Parameter	Description
ρ	The rate at which new individuals enter the unaware class.
δ	The contact rate between unaware individuals and aware individuals of the product.
η	The rate of transition from unaware individuals to actual purchasers is a result of interaction between them.
β	The efficacy rate of the advertising campaign.
α	Half-saturation point for the impact of social media advertisements on the interaction between unaware individuals to actual purchasers.
ξ	The transfer rate of unaware individuals to aware individuals through ads on social media about the product.
L	Half-saturation point for the impact of social media advertisements on unaware individuals.
φ	The transfer rate of aware individuals to unaware individuals is due to losing interest or forgetting the product.
γ_0	The rate of natural death for every class.
γ	The rate at which aware individuals turn into actual purchasers due to their desire to try the product.
ζ	The decreasing rate of actual purchasers because of the presence of other offers from competing companies.
ψ	The decreasing rate of actual purchasers due to boredom with the product.
σ	The growth rate of advertisements in social media.
μ	The rate of decline in advertisement effectiveness as awareness rises.
k	Half-saturation point for $\frac{\mu Y}{k+Y}$ as it reaches half of its maximum potential $\sigma\mu$ when aware individuals arrive at k.
ν	The decreasing rate of advertisements due to the gradual loss of their ability to persuade people to make a purchase.

Table 2.1: A description of the model's parameters

era. So, we use a compartment $W(t)$ that represents the number of advertisements on social media. Advertisements on social media platforms enhance the interaction between unaware and actual purchasers, at a rate of $\frac{\eta\beta W}{\alpha+W}XZ$. The contact rate η increases with increasing W, causing the number of actual purchasers to rise and eventually approach a maximum value of β as W becomes sufficiently large. The constant α represents the half-saturation point of this interaction. Furthermore advertisements on social media platforms convert unaware individuals into aware individuals at an average $\frac{\xi W}{L+W}X$, where ξ is the transfer rate of unaware individuals to aware individuals through ads on social media about the product. The parameter L represents the half-saturation point for the impact of social media advertisements on unaware individuals. Social media ads grow at a rate σZ as the number of actual purchasers increases. However, this rate decreases as the number of aware individuals Y increases due to the saturation effect. This decrease follows the rate $\frac{\mu Y}{k+Y}$, where $\mu \in (0,1)$ [20] represents the decline in advertisement effectiveness as awareness rises, and k is the half-saturation constant. Moreover, advertisements on social media decrease at a rate ν because they gradually lose

their ability to persuade people to buy. Based on these assumptions, the nonlinear system can be expressed as follows:

$$\begin{aligned}
\frac{dX}{dt} &= \rho - \delta XY - \eta\left(1+\frac{\beta W}{\alpha+W}\right)XZ - \left(\frac{\xi W}{L+W}+\gamma_0\right)X + \varphi Y,\\
\frac{dY}{dt} &= \delta XY + \frac{\xi W}{L+W}X - (\gamma_0+\varphi+\gamma)Y,\\
\frac{dZ}{dt} &= \eta\left(1+\frac{\beta W}{\alpha+W}\right)XZ - (\gamma_0+\psi+\zeta)Z + \gamma Y,\\
\frac{dW}{dt} &= \sigma\left(1-\frac{\mu Y}{k+Y}\right)Z - \nu W.
\end{aligned} \tag{2.1}$$

Table 2.1 describes the parameters of the model (2.1).

3 Qualitative analysis of model (2.1)

This section discusses the analysis of the model (2.1), focusing on the positivity and the boundedness of solutions, the equilibrium points, and their stability.

Theorem 3.1. *All solutions of the XYZW model* (2.1) *with non-negative initial conditions are positive for* $t \geq 0$.

Proof. It is obvious for nonnegative initial conditions that

$$\left.\frac{dX}{dt}\right|_{X=0} = \rho + \varphi Y \geq 0, \quad \left.\frac{dY}{dt}\right|_{Y=0} = \frac{\xi WX}{L+W} \geq 0, \quad \left.\frac{dZ}{dt}\right|_{Z=0} = \gamma Y \geq 0, \quad \text{and}$$
$$\left.\frac{dW}{dt}\right|_{W=0} = \sigma\left(1-\frac{\mu Y}{k+Y}\right)Z \geq 0, \quad \text{if} \quad \mu \leq 1.$$

Hence, for any nonnegative initial conditions in the interior of $\mathbb{R}^4_+$, solutions will remain in the closed positive orthant for all $t \geq 0$ [7]. This completes the proof. □

Theorem 3.2 (Boundedness)**.** *The system* (2.1) *has bounded solutions in* $\Gamma \subset \mathbb{R}^4_+$, *where*

$$\Gamma = \left\{(X,Y,Z,W) : 0 \leq M \leq \frac{\rho}{\gamma_0}, 0 \leq W \leq \frac{\sigma\rho}{\gamma_0\nu}\right\}.$$

Proof. For $M = X + Y + Z$, we find:

$$\begin{aligned}
\frac{dM}{dt} &= \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt}\\
&= \rho - \gamma_0 M - (\psi+\zeta)Z\\
&\leq \rho - \gamma_0 M,
\end{aligned}$$

utilizing the theorem of comparison [16], we get

$$M(t) \leq \frac{\rho}{\gamma_0} - \left(\frac{\rho}{\gamma_0} - M(0)\right)e^{-\gamma_0 t}. \tag{3.1}$$

For the final equation of model (2.1),

$$\begin{aligned}
\frac{dW}{dt} &= \sigma\left(1-\frac{\mu Y}{k+Y}\right)Z - \nu W\\
&\leq \frac{\sigma\rho}{\gamma_0} - \nu W,
\end{aligned}$$

similarly, by using the theorem of comparison, we get

$$W(t) \leq \frac{\sigma\rho}{\gamma_0\nu} - \left(\frac{\sigma\rho}{\gamma_0\nu} - W(0)\right) e^{-\nu t}. \tag{3.2}$$

From Eqs. (3.1) and (3.2), it is obvious that $M(t)$, $W(t)$ are bounded by $\frac{\rho}{\gamma_0}$ and $\frac{\sigma\rho}{\gamma_0\nu}$, respectively, as time progresses. Consequently, trajectories of the solution of the model, given the initial conditions, are bounded within $\Gamma \subset \mathbb{R}^4_+$. □

3.1 Existence of equilibrium points

The equilibrium points of model $XYZW$ (2.1) are determined by solving the following set of algebraic equations:

$$\begin{cases} \rho - \delta XY - \eta(1 + \frac{\beta W}{\alpha+W})XZ - \frac{\xi W}{L+W}X + \varphi Y - \gamma_0 X = 0, \\ \delta XY + \frac{\xi W}{L+W}X - \gamma_0 Y - \varphi Y - \gamma Y = 0, \\ \eta(1 + \frac{\beta W}{\alpha+W})XZ - \gamma_0 Z - \psi Z - \zeta Z + \gamma Y = 0, \\ \sigma(1 - \frac{\mu Y}{k+Y})Z - \nu W = 0. \end{cases} \tag{3.3}$$

Initially, if we put $Z = 0$, then the free equilibrium is $\varepsilon^0 = \left(\frac{\rho}{\gamma_0}, 0, 0, 0\right)$, which represents the case where no purchases occur. This equilibrium describes the state where all dynamic variables (Y, Z, W) are absent, and only the baseline variable X (representing the population unaware of the product) remains.

Next, we analyze the scenario where $Z \neq 0$. To find the equilibrium point $\varepsilon^* = (X^*, Y^*, Z^*, W^*)$ in this scenario, we have to find the intersection of the four equations in (3.3) within the positive quadrant. This will be carried out by analyzing the nullclines of the system. From the fourth equation of (3.3), we obtain:

$$W = \frac{\sigma(k + Y - \mu Y)Z}{\nu(k + Y)} = F(Y, Z), \tag{3.4}$$

substitute from Eq. (3.4) into the first equation of (3.3) and solve it for X, we have:

$$X = \frac{\varphi Y + \rho}{\gamma_0 + \delta Y + \eta Z + \frac{F(Y,Z)\xi}{F(Y,Z)+L} + \frac{\beta F(Y,Z)Z\eta}{F(Y,Z)+\alpha}}, \tag{3.5}$$

substitute from Eq. (3.4) and Eq. (3.5) into the second and third equations of system (3.3), followed by algebraic operations, we get:

$$G_1(Y, Z) = \frac{(\rho + \phi Y)}{\gamma_0 + \delta Y + \eta Z + \frac{\beta\eta F(Y,Z)Z}{F(Y,Z)+\alpha} + \frac{\xi F(Y,Z)}{L+F(Y,Z)}} \left(\delta Y + \frac{\xi F(Y, Z)}{L + F(Y, Z)}\right) - d_1 Y, \tag{3.6}$$

$$G_2(Y, Z) = d_1\eta YZ \left(1 + \frac{\beta F(Y, Z)}{F(Y, Z) + \alpha}\right) + (\gamma Y - d_2 Z) \left(\delta Y + \frac{\xi F(Y, Z)}{L + F(Y, Z)}\right), \tag{3.7}$$

where $d_1 = \gamma_0 + \varphi + \gamma$ and $d_2 = \gamma_0 + \psi + \zeta$. Therefore, the interior equilibrium points are the intersections of two nullclines (3.6) or (3.7).

Equation (3.6) shows that for $Z = 0$ then $Y_1^* = 0$ or $Y_2^* = \frac{-d_1\gamma_0+\delta\rho}{\delta(d_1-\varphi)}$, $(d_1 - \varphi) > 0$ and

$$\left.\frac{dZ}{dY}\right|_{(Y_1^*,0)} = -\frac{\frac{\partial G_1}{\partial Y}}{\frac{\partial G_1}{\partial Z}} = C,$$

where $C = \frac{L\gamma_0\nu}{\xi\rho\sigma}\left(d_1 - \frac{\delta\rho}{\gamma_0}\right) > 0$, if $d_1 > \frac{\delta\rho}{\gamma_0}$.
Also

$$\left.\frac{\partial G_1}{\partial Y}\right|_{(Y_2^*,0)} = A_1, \quad \text{and} \quad \left.\frac{\partial G_1}{\partial Z}\right|_{(Y_2^*,0)} = A_2,$$

where

$$A_1 = -\frac{(\gamma+\gamma_0)(-\delta\rho+\gamma_0 d_1)}{-\delta\rho+\gamma_0\varphi},$$

$$\begin{aligned} A_2 = &-\frac{\gamma+\gamma_0+\varphi}{L\delta\nu(-\delta\rho+\gamma_0\varphi)\left(\gamma_0^2+\gamma(\gamma_0-k\delta)-\delta\rho+\gamma_0(-k\delta+\varphi)\right)} \\ &\times\Bigg[-\gamma_0(\gamma+\gamma_0)\xi\sigma\Big(k\gamma\delta+k\gamma_0\delta+\gamma\gamma_0(\mu-1)+\gamma_0^2(\mu-1)-\delta(\mu-1)\rho+\gamma_0(\mu-1)\varphi\Big) \\ &+L\eta\nu\left(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi)\right)\left(\gamma_0^2+\gamma(\gamma_0-k\delta)-\delta\rho+\gamma_0(-k\delta+\varphi)\right)\Bigg]. \end{aligned}$$

If A_1 and A_2 have the same sign, then the curve enters the first quarter and intersects another nullcline.

Similarly, Eq. (3.7) shows that for $Z = 0$ then $Y_3^* = 0$ and

$$\left.\frac{dZ}{dY}\right|_{(Y_3^*,0)} = -\frac{\frac{\partial G_2}{\partial Y}}{\frac{\partial G_2}{\partial Z}} = \frac{\gamma_0}{d_2} > 0.$$

The mathematical analysis confirms that the nullclines (3.6) and (3.7) intersect at least once in the positive quadrant. The intersection points (Y^*, Z^*) of the two nullclines are determined first. Subsequently, the corresponding equilibrium components W^* and X^* are calculated using Eqs. (3.4) and (3.5), respectively. These components are positive provided that $\frac{\sigma(k+Y-\mu Y)Z}{\nu(k+Y)} > 0$.

3.2 Basic reproduction number

The basic reproduction number (R_0) is a concept borrowed from epidemiology. It represents the average number of secondary infections generated by a single infected individual. We apply this to model the spread and adoption of products in the market. In the $XYZW$ model (2.1), the reproduction number is a measure of how well social media advertising and the word-of-mouth process work together. When $R_0 > 1$, it indicates that a campaign is successful and sustained. The value of R_0 is determined using the next-generation matrix method [7,14, 26]. It is deduced from the following theorem.

Theorem 3.3. *The basic reproduction number, R_0, is given by*

$$R_0 = \frac{\rho}{d_2}\left(\frac{\gamma\xi\sigma}{L\nu(\gamma_0 d_1-\delta\rho)}+\frac{\eta}{\gamma_0}\right), \tag{3.8}$$

provided that $\gamma_0 d_1 - \delta\rho > 0$.

Proof. Following the standard next-generation matrix framework [26], we consider the active compartments of the model, denoted by the vector $G = (Y, Z, W)^T$. We decompose the active subsystem dynamics into the vector of rates of appearance of new purchasers and the vector of transition rates.

Evaluating the Jacobian matrices of these vectors at the inactive market equilibrium ε^0 yields the incidence matrix $\mathcal{U}$ and the transition matrix $\mathcal{S}$:

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 \\ \gamma & \frac{\eta\rho}{\gamma_0} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} d_1 - \frac{\delta\rho}{\gamma_0} & 0 & \frac{-\xi\rho}{L\gamma_0} \\ 0 & d_2 & 0 \\ 0 & -\sigma & \nu \end{bmatrix}.$$

For the transition matrix $\mathcal{S}$ to be a non-singular M-matrix, its diagonal entries must be strictly positive. This mathematically requires the parametric condition $\gamma_0 d_1 - \delta\rho > 0$. Under this condition, the inverse matrix $\mathcal{S}^{-1}$ is non-negative. Consequently, the next-generation matrix $\mathcal{U}\mathcal{S}^{-1}$ is a non-negative matrix. The basic reproduction number is determined by its dominant eigenvalue (spectral radius), which yields:

$$R_0 = \frac{\rho}{d_2}\left(\frac{\gamma\xi\sigma}{L\nu(\gamma_0 d_1 - \delta\rho)} + \frac{\eta}{\gamma_0}\right). \qquad \square$$

Furthermore, the number of feasible equilibrium points depends on the following conditions:

1. For $R_0 < 1$, the two nullclines intersect at one equilibrium point (trivial point).
2. For $R_0 > 1$, There are two cases as follows:
   (a) If $Y_2^* < 0$, then the nullcline G_1 enters the first quadrant from $(0, 0)$ and intersects the nullcline G_2 at two equilibrium points.
   (b) If $Y_2^* > 0$, then the nullcline G_1 enters the first quadrant from $(Y_2^*, 0)$ and intersects the nullcline G_2 at two equilibrium points.

The two nullclines intersect in the positive quadrant. Therefore, the following theorem can be stated.

Theorem 3.4. *For XYZW model (2.1), there exists*

1. *inactive market equilibrium point* $\varepsilon^0 = \left(\frac{\rho}{\gamma_0}, 0, 0, 0\right)$ *for* $R_0 < 1$.
2. *active market equilibrium point* $\varepsilon^* = (X^*, Y^*, Z^*, W^*)$ *when* $R_0 > 1$.

Now, we plot the two nullclines (3.6) and (3.7) to determine the possible number of intersection points. Figure 3.1(a) displays that two nullclines intersect at only one point $(0, 0)$, $\frac{dZ}{dY}|_{(Y_1^*,0)} = 7.22222 > 0$, and $\frac{dZ}{dY}|_{(Y_3^*,0)} = 0.664452 > 0$, where $R_0 = 0.829849 < 1$, indicating that no purchase occurs. Figure 3.1(b) shows that the two curves intersect at two points $(0, 0)$ and $(0.141258, 0.327738)$, $R_0 = 1.04731 > 1$, $Y_2^* = -0.793651$, $\frac{dZ}{dY}|_{(Y_1^*,0)} = 1.12613 > 0$, $\frac{\partial G_1}{\partial Y}|_{(Y_2^*,0)} = 0.0283784 > 0$, and $\frac{\partial G_1}{\partial Z}|_{(Y_2^*,0)} = 0.150631 > 0$. A positive equilibrium point indicates that the word-of-mouth and social media advertisements cause a positive impact on consumers. This leads to the product starting to spread gradually. In Fig. 3.1(c), we observe that $(0, 0)$ and $(0.955568, 4.22715)$ are equilibrium points, with $R_0 = 1.36955 > 0$, $Y_2^* = 5.39683 > 0$, $\frac{dZ}{dY}|_{(Y_1^*,0)} = -5.66667 < 0$, $\frac{\partial G_1}{\partial Y}|_{(Y_2^*,0)} = -0.0939474 < 0$, and $\frac{\partial G_1}{\partial Z}|_{(Y_2^*,0)} = -0.139062 < 0$. In

this case, word-of-mouth and social media advertisements work together to generate a self-perpetuating buying dynamic, with each reinforcing the other. This scenario reflects a mature and well-established market where continued sales growth depends more on social influence among consumers.

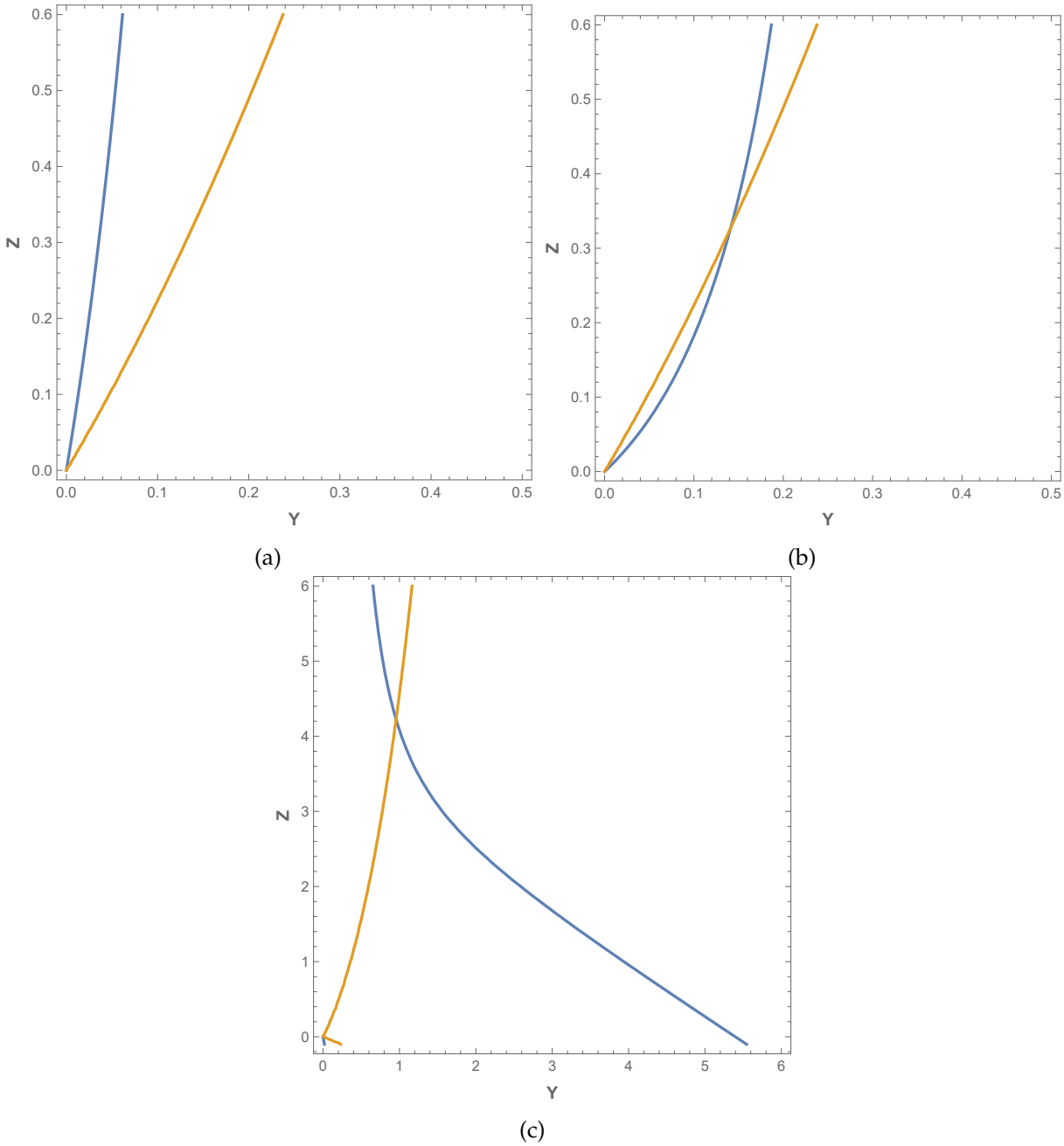


Figure 3.1: Nullclines and equilibrium points for model (2.1) for varying values of ρ: (a) $\rho = 3$, (b) $\rho = 3.7$, and (c) $\rho = 5$. All other parameters are fixed: $\eta = 0.01656$, $\xi = 0.02$, $\delta = 0.03$, $\varphi = 0.37$, $\gamma_0 = 0.2$, $\sigma = 0.15$, $\nu = 0.5$, $\mu = 0.8$, $\gamma = 0.01$, $\beta = 0.0986726$, $\zeta = 0.001$, $\psi = 0.1$, $\alpha = 3$, $L = 5$, and $k = 1$.

3.3 Local stability of equilibria

This subsection is devoted to analyzing the local stability of the equilibrium points.

Theorem 3.5. *The inactive equilibrium point* $\varepsilon^0 = \left(\frac{\rho}{\gamma_0}, 0, 0, 0\right)$ *is asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. The Jacobian matrix for the model (2.1) at ε^0 takes the form:

$$J|_{\varepsilon^0} = \begin{pmatrix} -\gamma_0 & \varphi - \frac{\delta\rho}{\gamma_0} & -\frac{\rho\eta}{\gamma_0} & -\frac{\xi\rho}{\gamma_0 L} \\ 0 & -d_1 + \frac{\delta\rho}{\gamma_0} & 0 & \frac{\xi\rho}{\gamma_0 L} \\ 0 & \gamma & -d_2 + \frac{\rho\eta}{\gamma_0} & 0 \\ 0 & 0 & \sigma & -\nu \end{pmatrix}. \tag{3.9}$$

Thus, the characteristic equation is:

$$(\lambda + \gamma_0)(\lambda + \nu)\left(\lambda + d_1 - \frac{\delta\rho}{\gamma_0}\right)(\lambda + d_2(1 - F_1(\lambda))) = 0, \tag{3.10}$$

where $F_1(\lambda) = \frac{1}{d_2}\left(\frac{\gamma\sigma\xi\rho}{L(\nu+\lambda)(\gamma_0(d_1+\lambda)-\delta\rho)} + \frac{\rho\eta}{\gamma_0}\right)$. Clearly, Eq. (3.10) has the three negative roots $-\gamma_0$,$-\nu$, and $\frac{\delta\rho}{\gamma_0} - d_1$. The following equation can be used to extract the remaining roots:

$$\lambda + d_2(1 - F_1(\lambda)) = 0, \tag{3.11}$$

if we assume that $\mathrm{Re}(\lambda) \geq 0$, then we have to prove that:

$$\mathrm{Re}(F_1(\lambda)) \geq 1. \tag{3.12}$$

Moreover, since we presume that $\mathrm{Re}(\lambda) \geq 0$, we can yield

$$\begin{aligned} |F_1(\lambda)| &= \left| \frac{\frac{\gamma\sigma\xi\rho}{L(\nu+\lambda)\left(\gamma_0(d_1+\lambda)-\delta\rho\right)} + \frac{\rho\eta}{\gamma_0}}{d_2} \right| \\ &\leq \left| \frac{\gamma\sigma\xi\rho}{Ld_2\nu\left(\gamma_0 d_1 - \delta\rho\right)} \right| + \left| \frac{\rho\eta}{\gamma_0 d_2} \right| \\ &= \frac{\gamma\sigma\xi\rho}{Ld_2\nu\left(\gamma_0 d_1 - \delta\rho\right)} + \frac{\rho\eta}{\gamma_0 d_2} = R_0 < 1. \end{aligned} \tag{3.13}$$

Hence, the inequality (3.13) demonstrates that if $R_0 < 1$ then $\mathrm{Re}(F_1(\lambda)) < 1$ which contradicts the condition stated in (3.12). Therefore, the assumption regarding λ is incorrect, and all the roots of Eq. (3.10) have negative real parts. If $R_0 < 1$, then the equilibrium point ε^0 is asymptotically stable else $R_0 > 1$ we find that ε^0 is unstable. □

Theorem 3.6. *The active equilibrium point* $\varepsilon^* = (X^*, Y^*, Z^*, W^*)$ *of system (2.1) is locally asymptotically stable if* $D_n > 0$ *for* $n = 0, 1, 2, 3$, *and* $D_3(D_2D_1 - D_0D_3) > D_1^2$.

Proof. At the equilibrium ε^*, the Jacobian matrix for model (2.1) is:

$$J|_{\varepsilon^*} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_5 & h_6 & 0 & h_7 \\ h_8 & \gamma & h_9 & h_{10} \\ 0 & h_{11} & h_{12} & -\nu \end{pmatrix}, \tag{3.14}$$

where the matrix entries and their signs are: $h_1 = -\gamma_0 - Y^*\delta - \frac{W^*\xi}{L+W^*} - \eta Z^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right) < 0$, $h_2 = \varphi - \delta X^*$, $h_3 = -\eta X^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right) < 0$, $h_4 = -\frac{LX^*\xi}{(L+W^*)^2} - \frac{\beta\eta\alpha X^*Z^*}{(\alpha+W^*)^2} < 0$, $h_5 = \delta Y^* + \frac{\xi W^*}{L+W^*} > 0$, $h_6 = -d_1 + \delta X^*$, $h_7 = \frac{L\xi X^*}{(L+W^*)^2} > 0$, $h_8 = \eta Z^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right) > 0$, $h_9 = -d_2 + \eta X^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right)$, $h_{10} = \frac{\alpha\beta\eta X^*Z^*}{(\alpha+W^*)^2} > 0$, $h_{11} = -\frac{\sigma\mu kZ^*}{(k+Y^*)^2} < 0$, $h_{12} = \sigma - \frac{\sigma\mu Y^*}{k+Y^*} > 0$.

The characteristic equation is $\det(J|_{\varepsilon^*} - \lambda I) = 0$, which gives:

$$\lambda^4 + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0 = 0, \tag{3.15}$$

where

$$\begin{aligned}
D_3 &= \nu - h_1 - h_6 - h_9, \\
D_2 &= h_1h_6 - h_2h_5 + h_1h_9 - h_3h_8 + h_6h_9 - h_7h_{11} - h_{10}h_{12} - (h_1 + h_6 + h_9)\nu, \\
D_1 &= -h_1h_6h_9 + h_2h_5h_9 - h_3h_5\gamma + h_3h_6h_8 + h_1h_7h_{11} - h_4h_5h_{11} + h_1h_{10}h_{12} - h_4h_8h_{12} \\
&\quad + h_6h_{10}h_{12} - h_7\gamma h_{12} + h_7h_9h_{11} + (h_1h_6 + h_1h_9 + h_6h_9 - h_2h_5 - h_3h_8)\nu, \\
D_0 &= h_{11}(h_4h_5h_9 - h_3h_5h_{10} - h_1h_7h_9 + h_3h_7h_8) \\
&\quad + h_{12}(h_1h_6h_{10} - h_1h_7\gamma - h_2h_5h_{10} + h_2h_7h_8 + h_4h_5\gamma - h_4h_6h_8) \\
&\quad - \nu(h_1h_6h_9 - h_2h_5h_9 + h_3h_5\gamma - h_3h_6h_8).
\end{aligned}$$

We use the equilibrium conditions (3.3) to determine the signs of the Jacobian entries. From the third equation in (3.3), we have:

$$-d_2 + \eta X^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right) = -(\gamma_0+\psi+\zeta) + \eta X^*\left(1+\frac{\beta W^*}{\alpha+W^*}\right) = -\gamma\frac{Y^*}{Z^*} < 0.$$

So, $h_9 < 0$. From the second equation in (3.3), we have:

$$-d_1 + \delta X^* = -(\gamma_0+\varphi+\gamma) + \delta X^* = -\frac{\xi W^*X^*}{(L+W^*)Y^*} < 0.$$

So, $h_6 < 0$. Also, $h_2 < 0$ when $X^* > \frac{\varphi}{\delta}$.

We now check the Routh–Hurwitz conditions analytically where possible. For D_3, since h_1, h_6, and h_9 are all negative:

$$D_3 = \nu + (-h_1) + (-h_6) + (-h_9) > 0.$$

Next, we evaluate the sign of D_2, which can be written as:

$$D_2 = H^+ - H^- - h_{10}h_{12},$$

where $H^+ = h_1h_6 + h_1h_9 + h_6h_9$ and $H^- = h_2h_5 + h_3h_8 + h_7h_{11} + \nu(h_1 + h_6 + h_9)$. Based on the established signs of the Jacobian entries, we have $H^+ > 0$. Furthermore, assuming the condition $X^* > \frac{\varphi}{\delta}$ holds (which ensures $h_2 < 0$), all terms within H^- are strictly negative, making $H^- < 0$. Because $h_{10}h_{12} > 0$, then, $D_2 > 0$ if $H^+ - H^- > h_{10}h_{12}$.

The expressions for D_1, D_0, and $D_3(D_2D_1 - D_0D_3) > D_1^2$ are too complex to sign analytically without specific parameter values. We verify these remaining conditions numerically in Section 5. □

Theorem 3.7. *The inactive market equilibrium point* $\varepsilon^0 = (X_0, 0, 0, 0)$, *where* $X_0 = \frac{\rho}{\gamma_0}$, *is globally asymptotically stable in* Γ *if* $R_0 + \Delta < 1$, *where* $\Delta = \frac{\eta\beta\rho}{\gamma_0 d_2}$.

Proof. Consider the linear Lyapunov function

$$V = c_1 Y + c_2 Z + c_3 W,$$

where $c_1, c_2, c_3 > 0$. Therefore, $V \geq 0$ in Γ, with $V = 0$ if and only if $Y = Z = W = 0$.

Differentiating V along the solutions of system (2.1), we obtain

$$\dot{V} = c_1 \dot{Y} + c_2 \dot{Z} + c_3 \dot{W}.$$

Using the bounds valid in the positively invariant region Γ, namely

$$X \leq X_0 \quad \text{and} \quad \frac{W}{L+W} \leq \frac{W}{L},$$

we have

$$\begin{aligned}
\dot{Y} &\leq (\delta X_0 - d_1) Y + \frac{\xi X_0}{L} W, \\
\dot{Z} &\leq \eta(1+\beta) X_0 Z - d_2 Z + \gamma Y, \\
\dot{W} &\leq \sigma Z - \nu W.
\end{aligned}$$

Consequently,

$$\dot{V} \leq c_1 \left[(\delta X_0 - d_1) Y + \frac{\xi X_0}{L} W \right] + c_2 \left[\eta(1+\beta) X_0 Z - d_2 Z + \gamma Y \right] + c_3 (\sigma Z - \nu W).$$

Collecting the coefficients of Y, Z, and W, we get

$$\dot{V} \leq \left[c_1(\delta X_0 - d_1) + c_2 \gamma \right] Y + \left[c_1 \frac{\xi X_0}{L} - c_3 \nu \right] W + \left[c_2 \big(\eta(1+\beta) X_0 - d_2 \big) + c_3 \sigma \right] Z.$$

By choosing

$$c_1 = \gamma L \nu, \quad c_2 = L\nu(d_1 - \delta X_0), \quad c_3 = \gamma \xi X_0,$$

the coefficients for Y and W vanish, yielding

$$\dot{V} \leq d_2 L \nu (d_1 - \delta X_0) \left(R_0 + \Delta - 1 \right) Z.$$

Since $d_1 - \delta X_0 > 0$ and $R_0 + \Delta < 1$, it follows that $\dot{V} \leq 0$ for all $(X, Y, Z, W) \in \Gamma$. Furthermore, $\dot{V} = 0$ if and only if $Z = 0$.

By LaSalle's Invariance Principle, the largest invariant set contained within $\{(X, Y, Z, W) \in \Gamma : \frac{d\dot{V}}{dt} = 0\}$ must satisfy $Z = 0$. If $Z = 0$, the system dynamics dictate that $\frac{dZ}{dt} = 0 \implies Y = 0$, and $\frac{dW}{dt} = 0 \implies W = 0$. Consequently, $X \to X_0$. Therefore, the largest invariant set is exactly the inactive market equilibrium point ε^0, proving it is globally asymptotically stable in Γ. □

Parameter	**value**	**Sensitivity index of R_0**
ρ	6	1.00016
γ	0.01	0.00136483
ξ	0.02	0.00139159
σ	0.15	0.00139159
η	0.01656	0.998608
L	5	-0.00139159
ν	0.5	-0.00139159
γ_0	0.2	-1.66515
φ	0.37	-0.000990173
δ	0.02	0.000160569
ζ	0.001	-0.00332226
ψ	0.1	-0.332226

Table 3.1: The sensitivity indices of R_0 to each parameters

3.4 Sensitivity analysis

A sensitivity analysis of the $XYZW$ model (2.1) is presented in this subsection. We focus on the parameters that affect the basic reproduction number R_0. The purpose of this analysis is to show how variations in these parameters influence R_0. Furthermore, it determines how best to generate product awareness and sales within the context of social media advertising and marketing dynamics. To find the sensitivity index, we utilize the method of differentiation of the reproduction number with respect to each parameter [15,19]. The sensitivity index is:

$$\vartheta_q = \frac{\partial R_0}{\partial q}\frac{q}{R_0}. \tag{3.16}$$

The sensitivity indices of the basic reproduction number to the parameters are summarized in Table 3.1. Fig. 4.1 displays that the difference in some parameters leads to a difference in the basic reproduction number. We observe that the most sensitive parameters are ρ, γ_0, η, ψ, and ξ, σ, L, ν have the same effect followed by γ, ζ, φ. Additionally, δ is the least sensitive to R_0.

4 Bifurcation analysis

This section analyzes the dynamical behavior of the $XYZW$ system (2.1) by varying a parameter around the inactive market equilibrium point ε^0. The Sotomayor theorem [3,22] is employed here to investigate the occurrence of a transcritical bifurcation.

We choose the transition from unaware individuals to actual purchasers η as the bifurcation parameter. At $R_0 = 1$, we find $\eta = \eta^{TB} = \frac{\gamma_0(\gamma_0+\zeta+\psi)}{\rho}\big(1 - \frac{\gamma\xi\rho\sigma}{L\nu(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))(\gamma_0+\zeta+\psi)}\big)$. The existence of a bifurcation is deduced from the following theorem.

Theorem 4.1. *The $XYZW$ model (2.1) undergoes a transcritical bifurcation at the equilibrium of the inactive market ε^0 when $\eta = \eta^{TB}$, provided that $B_2 \neq 0$, where B_2 is defined in the proof.*

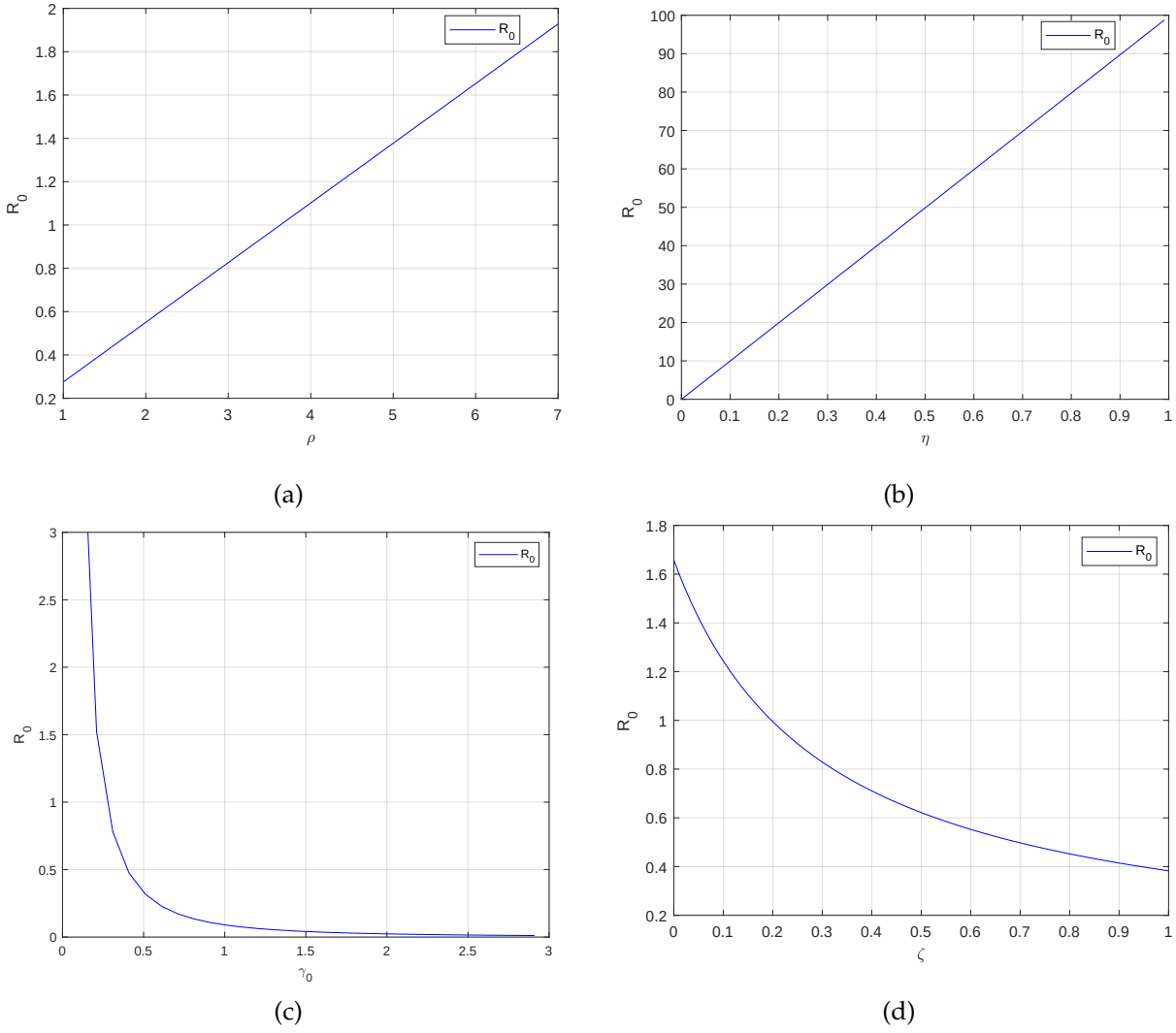


Figure 4.1: The variation of the basic reproduction number with respect to the critical parameters. The values of parameters are taken from Tabel 3.1. (a) ρ, (b) η, (c) γ_0, (d) ζ.

Proof. The Jacobian matrix at ε^0 when $\eta = \eta^{TB}$ is delivered by

$$J|_{(\varepsilon^0,\eta^{TB})} = \begin{bmatrix} -\gamma_0 & -\frac{\delta\rho}{\gamma_0}+\varphi & -d_2+\frac{\gamma\,\xi\,\rho\,\sigma}{L\nu(d_1\gamma_0-\delta\rho)} & -\frac{\xi\rho}{L\gamma_0} \\ 0 & -d_1+\frac{\delta\rho}{\gamma_0} & 0 & \frac{\xi\rho}{L\gamma_0} \\ 0 & \gamma & -\frac{\gamma\,\xi\,\rho\,\sigma}{L\nu(d_1\gamma_0-\delta\rho)} & 0 \\ 0 & 0 & \sigma & -\nu \end{bmatrix}. \tag{4.1}$$

It has a zero eigenvalue. The right and left eigenvectors corresponding to the zero eigenvalue satisfy, respectively, that is,

$$J|_{(\varepsilon^0,\eta^{TB})}\cdot v = 0\cdot v, \qquad J^T\Big|_{(\varepsilon^0,\eta^{TB})}\cdot u = 0\cdot u.$$

Then

$$v = \begin{pmatrix} -\frac{\xi\rho}{L(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))}-\frac{\nu(\gamma_0+\zeta+\psi)}{\gamma_0\sigma} \\ \frac{\xi\rho}{L(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))} \\ \frac{\nu}{\sigma} \\ 1 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 \\ \frac{L\gamma_0\nu}{\xi\rho} \\ \frac{L\nu(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))}{\gamma\xi\rho} \\ 1 \end{pmatrix}.$$

When $\eta = \eta^{TB}$, we find that

$$\begin{aligned} u^T\,\mathcal{M}_\eta(\varepsilon^0,\eta^{TB}) &= 0, \\ u^T\left[D\mathcal{M}_\eta(\varepsilon^0,\eta^{TB})\,v\right] &= B_1 \neq 0, \\ u^T\left[D^2\mathcal{M}(\varepsilon^0,\eta^{TB})\,(v,v)\right] &= B_2 \neq 0, \end{aligned}$$

where

$$\begin{aligned} B_1 &= \frac{L\nu^2(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))}{\gamma\gamma_0\xi\sigma}, \\ B_2 &= -\frac{\nu}{L\xi\rho(\delta\rho-\gamma_0(\gamma+\gamma_0+\varphi))^2}\Bigg(\frac{2\mu\xi^2\rho^2(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))}{k} \\ &\quad + 2\xi\Bigg[\rho\Bigg(\gamma^2\gamma_0^2+\gamma_0^4+\delta^2\rho^2-2\gamma_0\delta\rho\varphi+\gamma_0^3(\xi+2\varphi) \\ &\qquad + \gamma_0^2(-2\delta\rho+\varphi(\xi+\varphi))+\gamma\gamma_0(-2\delta\rho+\gamma_0(2\gamma_0+\xi+2\varphi))\Bigg) \\ &\qquad + \frac{L\gamma_0\nu}{\sigma}(\gamma+\gamma_0+\varphi)(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))(\gamma_0+\zeta+\psi)\Bigg] \\ &\quad + \frac{1}{\alpha\gamma\rho\sigma^2}(\delta\rho-\gamma_0(\gamma+\gamma_0+\varphi))\Big(\gamma\xi\rho\sigma+L\nu(\delta\rho-\gamma_0(\gamma+\gamma_0+\varphi))(\gamma_0+\zeta+\psi)\Big) \\ &\quad \times\Big(2\alpha\gamma_0\xi\rho\sigma+2L(-\delta\rho+\gamma_0(\gamma+\gamma_0+\varphi))(-\beta\rho\sigma+\alpha\nu(\gamma_0+\zeta+\psi))\Big)\Bigg). \end{aligned}$$

Since, $-\delta\rho + \gamma_0(\gamma + \gamma_0 + \varphi) \neq 0$ from Eq. (3.8) then $B_1 \neq 0$. All parameters are positive, and the vector of the first partial derivatives of the components of $\mathcal{M}$ (equations of our model) with respect to the critical parameter η is denoted by $\mathcal{M}_\eta$. Here, Sotomayor's theorem for a transcritical bifurcation at the inactive market equilibrium ε^0 when $\eta = \eta^{TB}$ is satisfied under the above conditions. □

5 Numerical simulations

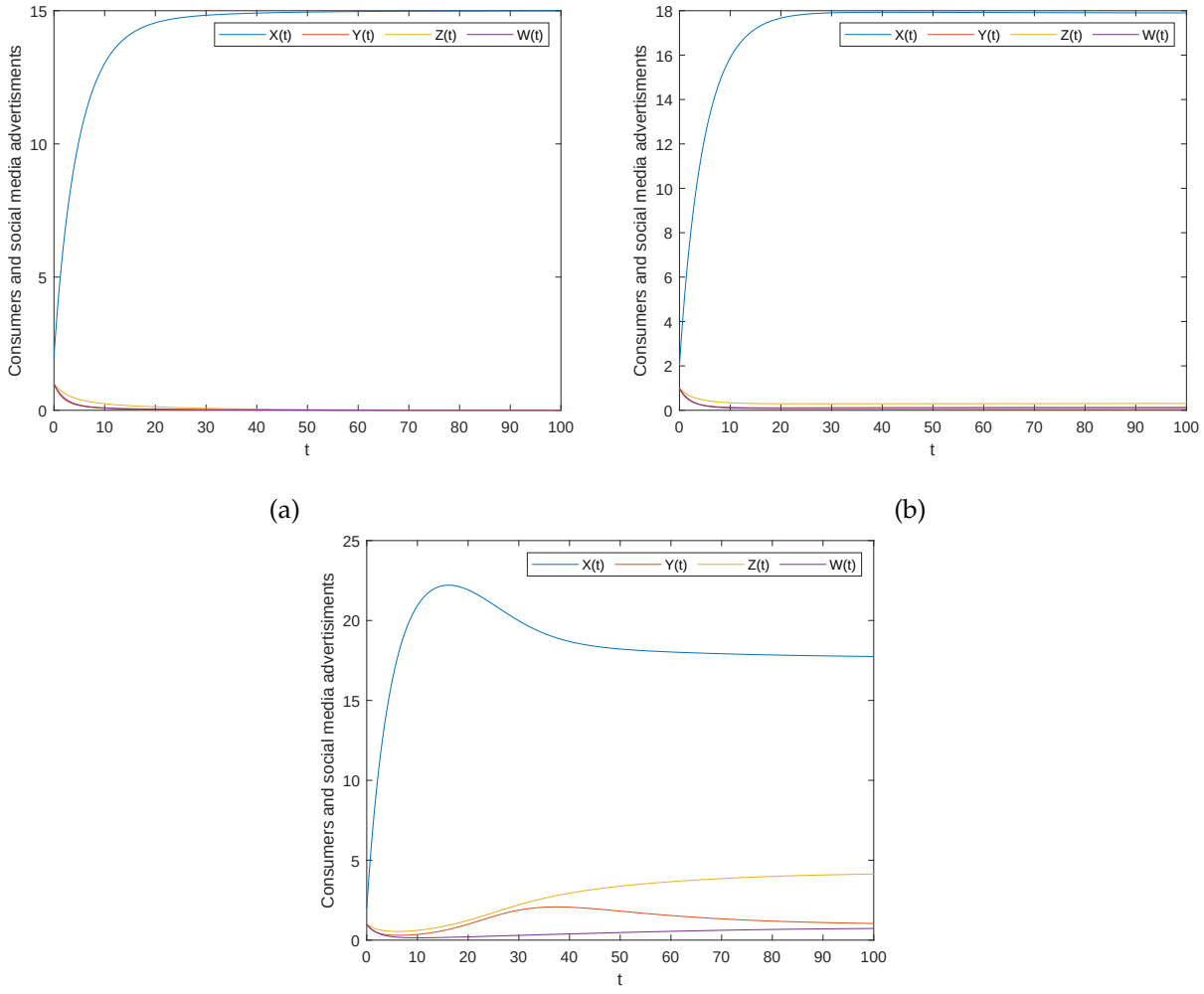


Figure 5.1: Time evolution of the $XYZW$ model for varying values of ρ: (a) $\rho = 3$, (b) $\rho = 3.7$, and (c) $\rho = 5$.

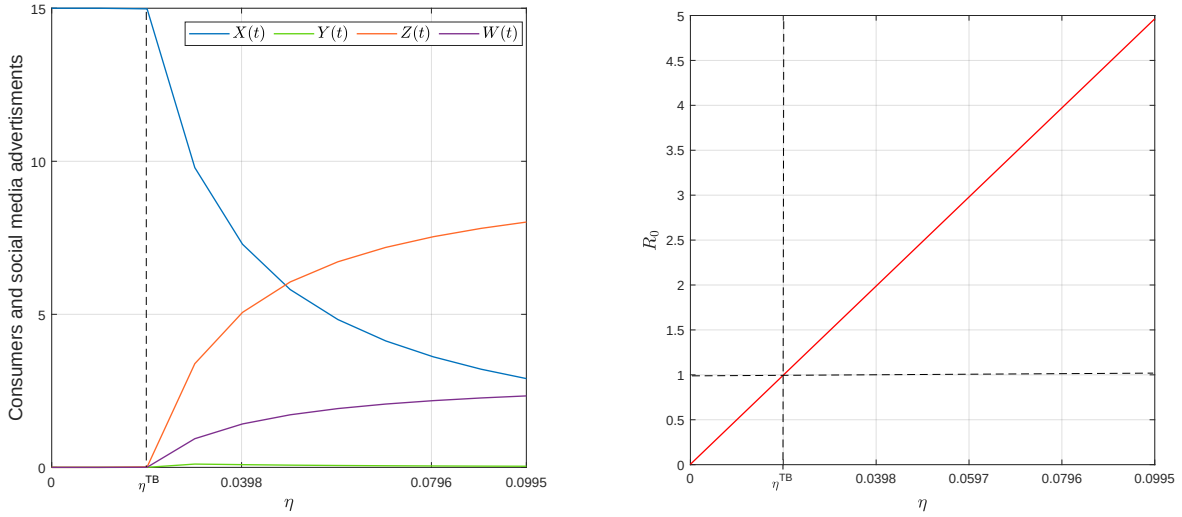
This section discusses the numerical simulations to verify the analytical results. By using hypothetical values for the parameters of nonlinear ordinary differential equations. In the numerical simulation, we select the parameter values as follows:

$$\rho = 3,\ \eta = 0.01656,\ \delta = 0.03,\ \xi = 0.02,\ \varphi = 0.37,\ \gamma_0 = 0.2,\ \zeta = 0.001,\ L = 5,\ \psi = 0.1,$$
$$\sigma = 0.15,\ \mu = 0.8,\ \gamma = 0.01,\ \beta = 0.0986726,\ \alpha = 3,\ k = 1,\ \nu = 0.5.$$

Using the parameters mentioned above, we yield that the $XYZW$ model (2.1) exhibits a single equilibrium point, $\varepsilon^0 = (15, 0, 0, 0)$, with $R_0 = 0.829849 < 1$ as in Theorem 3.4. This scenario leads to the inactive market equilibrium ε^0 being globally asymptotically stable. Fig. 5.1(a) displays that the density of unawareness of the product's existence, denoted by $X(t)$, settles at 15, while the densities of $Y(t)$, $Z(t)$, and $W(t)$ all decline to zero. In contrast, setting $\rho = 3.7$ yields a basic reproduction number of $R_0 = 1.04731 > 1$. Under this condition, the trivial equilibrium $\varepsilon^0 = (18.5, 0, 0, 0)$ remains but becomes unstable (as established in Theorem 3.5), and the active market equilibrium $\varepsilon^* = (17.8655, 0.141258, 0.327738, 0.0885857)$ emerges. Evaluating the Jacobian at ε^*, the characteristic equation defined in Eq. (3.15) becomes:

$$\lambda^4 + 0.758374\lambda^3 + 0.143583\lambda^2 + 0.00678061\lambda + 0.0000565585 = 0.$$

By substituting these numerical coefficients into the Routh–Hurwitz criteria, we verify that $D_n > 0$ for $n = 0, 1, 2, 3$ and $D_3(D_2D_1 - D_0D_3) > D_1^2$. Therefore, ε^* is locally asymptotically stable, which is visually confirmed by the trajectories in Fig. 5.1(b). Also, for $\rho = 5$, which results in $R_0 = 1.36955 > 1$, and our model exhibits two equilibrium points $(25, 0, 0, 0)$, $(17.6826, 0.955568, 4.22715, 0.772412)$ as in Theorem 3.4. The trivial equilibrium persists but becomes unstable, and the active market equilibrium emerges and is locally asymptotically stable, as shown in Fig. 5.1(c).



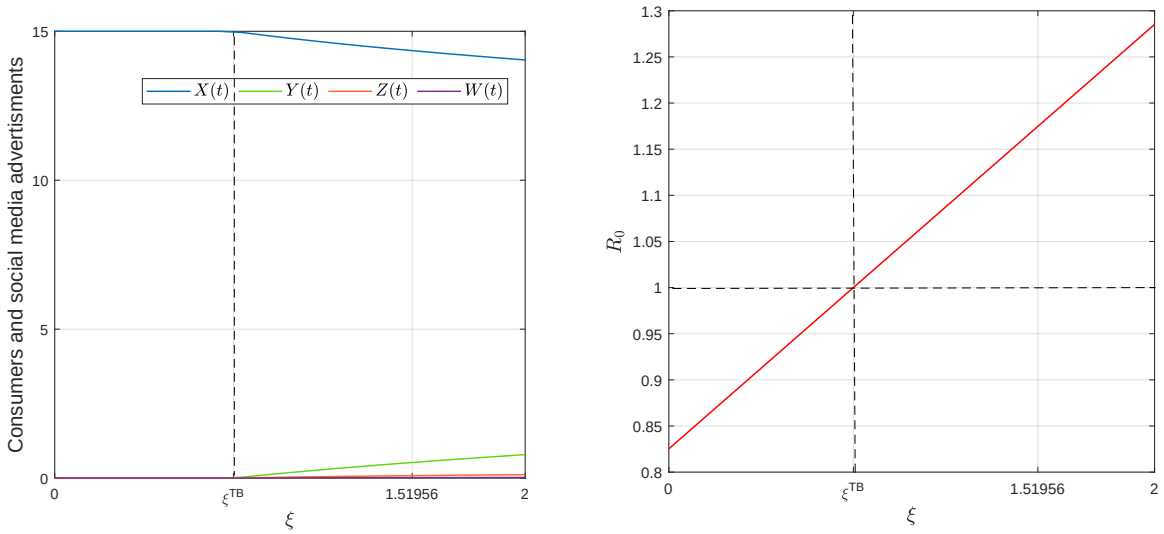
(a) (b)

Figure 5.2: The effect of the transition rate η on the system dynamics. (a) Bifurcation diagram of the state variables X, Y, Z, and W as η varies. (b) The corresponding variation in the basic reproduction number R_0. All other parameters remain fixed: $\rho = 3$, $\xi = 0.02$, $\delta = 0.03$, $\varphi = 0.37$, $\gamma_0 = 0.2$, $\sigma = 0.15$, $\nu = 0.5$, $\mu = 0.8$, $\gamma = 0.01$, $\beta = 0.0986726$, $\zeta = 0.001$, $\psi = 0.1$, $L = 5$, $\alpha = 3$, and $k = 1$.

The bifurcation diagram of the $XYZW$ model (2.1) is plotted to demonstrate the variation in fixed point stability with respect to η. This diagram captures the system's behavioral dynamics under increasing η, the direct transition rate from unaware individuals to purchasers, as depicted in Figs. 5.2 (a)–(b). Fig. 5.2 (a) displays the inactive market equilibrium point ε^0. However, when η reaches the threshold value $\eta^{TB} = 0.0199744$, a transcritical bifurcation (TB) occurs, leading to an active market equilibrium where $R_0 = 1$, which aligns with Theorem 4.1,

as seen in Fig. 5.2 (b). With increasing η, the density of unaware individuals $X(t)$ declines, whereas the density of purchasers $Z(t)$ rises. This occurs because a higher η accelerates the transition from unawareness directly to purchase. For all values of η, $Y(t)$ remains very small or nearly zero. This happens because individuals bypass the awareness stage and move directly to purchasing when η is high, thereby reducing the number of people who remain in the awareness stage. $W(t)$ increases gradually as η grows. This occurs since buyers generate advertisements as well as reviews. So, an increase in $Z(t)$ leads to more user-generated content and product-related posts on social media.

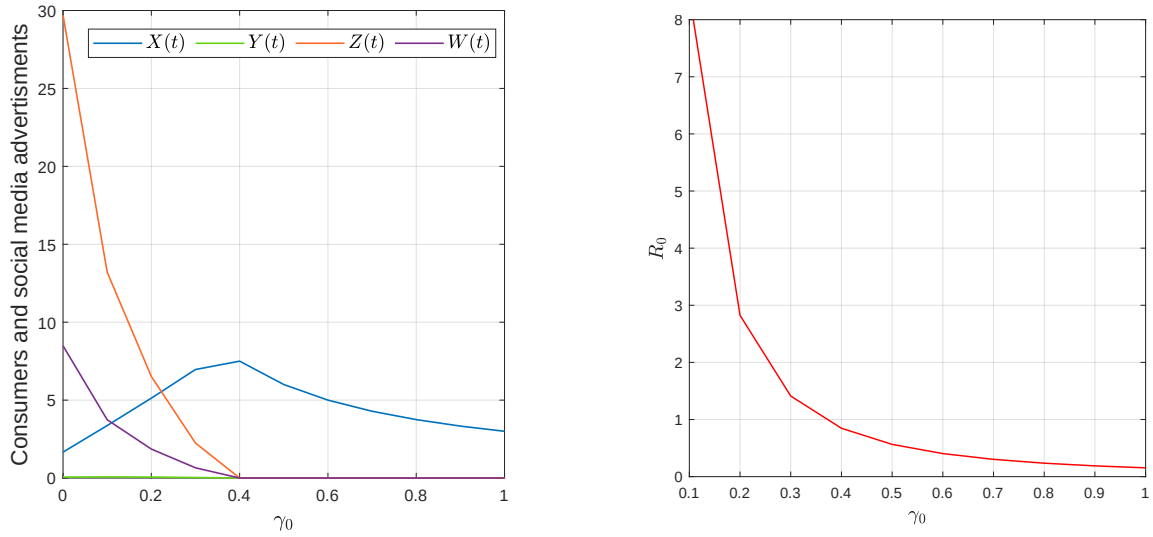
Figs. 5.3 (a) and 5.3 (b) show the effect of the parameter ξ, which represents the transfer rate of unaware individuals to aware individuals through ads on social media about the product, on the behavior of our model. The figures display the inactive market equilibrium point ε^0 until ξ reaches the critical value $\xi^{TB} = 0.759778$ leading to an active market equilibrium. Fig. 5.3 (a) demonstrates that as ξ increases, the density of unaware of the product's existence, $X(t)$, decreases, while the density of aware of the product's existence, $Y(t)$, increases. This occurs because social media advertisements are more effective in raising awareness about the product. The density of actual purchasers, $Z(t)$, remains almost unchanged and close to zero. This means that just raising awareness isn't enough to encourage people to buy anything; there need to be other variables that encourage the buying process.



(a) (b)

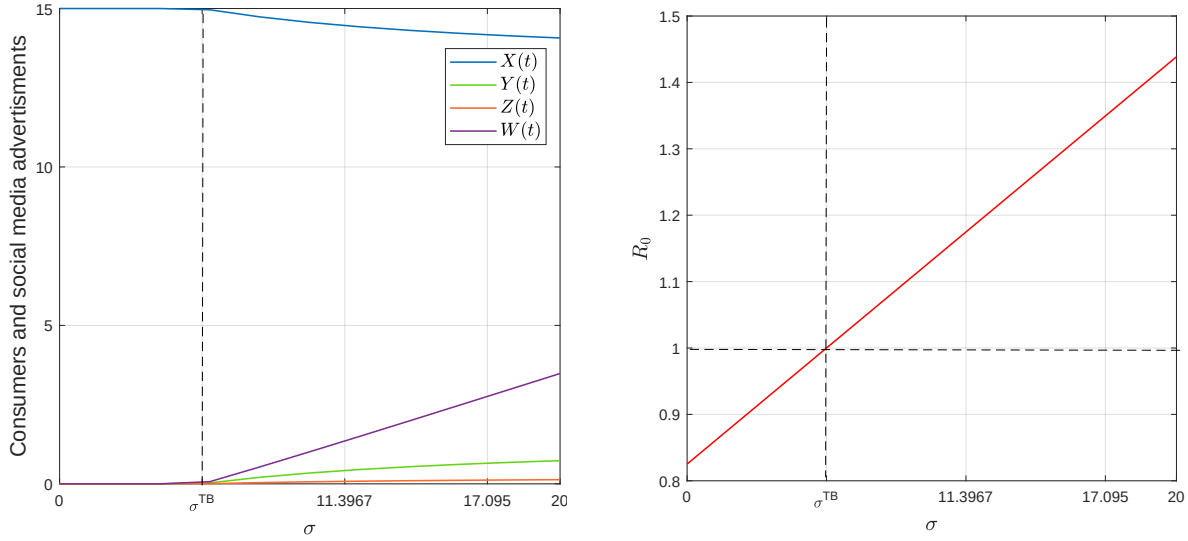
Figure 5.3: The effect of the transfer rate ξ (from unaware individuals to aware individuals through social media ads) on the system dynamics. (a) Bifurcation diagram of the state variables X, Y, Z, and W as ξ varies. (b) The corresponding variation in the basic reproduction number R_0. All other parameters remain fixed: $\rho = 3$, $\delta = 0.03$, $\varphi = 0.37$, $\gamma_0 = 0.2$, $\sigma = 0.15$, $\nu = 0.5$, $\mu = 0.8$, $\gamma = 0.01$, $\beta = 0.0986726$, $\zeta = 0.001$, $\psi = 0.1$, $L = 5$, $\alpha = 3$, and $k = 1$.

We present Figs. 5.4 (a) and 5.4 (b) to investigate the influence of the natural death rate of all classes, γ_0, on the system dynamics. For lower values of γ_0 and when $R_0 > 1$, the system exhibits an active market. However, as γ_0 increases, the equilibrium densities of $X(t)$, $Y(t)$, $Z(t)$, and $W(t)$ progressively decline, eventually leading the system toward an inactive market state.



(a) (b)

Figure 5.4: The effect of the parameter γ_0 (the rate of natural death for X, Y and Z) on the system dynamics. (a) Bifurcation diagram of the state variables X, Y, Z, and W as γ_0 varies. (b) The corresponding variation in the basic reproduction number R_0. All other parameters remain fixed: $\rho = 3$, $\xi = 0.02$, $\delta = 0.03$, $\varphi = 0.37$, $\sigma = 0.15$, $\nu = 0.5$, $\mu = 0.8$, $\gamma = 0.01$, $\beta = 0.0986726$, $\zeta = 0.001$, $\psi = 0.1$, $L = 5$, $\alpha = 3$, and $k = 1$.



(a) (b)

Figure 5.5: The effect of the parameter σ (the advertisement growth rate) on the system dynamics. (a) Bifurcation diagram of the state variables X, Y, Z, and W as σ varies. (b) The corresponding variation in the basic reproduction number R_0. All other parameters remain fixed: $\rho = 3$, $\xi = 0.02$, $\delta = 0.03$, $\varphi = 0.37$, $\gamma_0 = 0.2$, $\nu = 0.5$, $\mu = 0.8$, $\gamma = 0.01$, $\beta = 0.0986726$, $\zeta = 0.001$, $\psi = 0.1$, $L = 5$, $\alpha = 3$, and $k = 1$.

For parameter σ, Figs. 5.5 (a) and 5.5 (b) display the bifurcation diagrams of the model. When the growth rate σ of advertisements in social media is low, the market remains inactive. In contrast, when $\sigma = \sigma^{TB} = 5.69833$, the active market equilibrium point emerges. As σ increases, the density of unaware of the product's existence, $X(t)$, decreases, while the density of aware of the product's existence, $Y(t)$, increases. This occurs because an increase in growth rate leads to an increase in social media advertisements $W(t)$, which in turn increases awareness. The increase in actual purchasers $Z(t)$ may be slightly delayed, since the transition from $Y(t)$ to $Z(t)$ takes time due to the purchase decision process.

6 Conclusions

Social media platforms are central to our daily lives, as we use them for communication, learning, exchanging information, and commerce (buying and selling). Social media advertising and word-of-mouth marketing play a crucial role in influencing consumers and their purchasing decisions. Advertising also amplifies word-of-mouth among consumers. Based on epidemic models, we design our model to integrate the influence of both advertising on social media and word-of-mouth on customer purchasing decisions.

The population is divided into three distinct groups: individuals who are unaware of the product's existence (X), individuals who are aware of the product's existence (Y), and actual purchasers of the product (Z). Additionally, we model social media advertisements as a separate compartment W. Our model analyzes the combined impact of advertising on social media and word-of-mouth on consumers' behavior. A key threshold governing the system's equilibrium landscape is provided by the basic reproduction number R_0. The model has exhibited only one equilibrium point, the inactive market ε^0 when $R_0 < 1$. In contrast, when $R_0 > 1$, two equilibrium points emerge: the inactive equilibrium ε^0, and an additional active market equilibrium ε^*. The stability of the equilibrium points was examined. Additionally, we have explored the dynamics associated with transcritical bifurcation. The sensitivity analysis was conducted to determine the most sensitive parameters and their impact on R_0. It is found that the most sensitive parameters are ρ, γ_0, η, ψ, and ξ, σ, L, ν have the same effect, followed by γ, ζ, φ. Additionally, δ is the least sensitive to R_0.

Our model's numerical simulations highlighted the impact of both word-of-mouth and social media advertising on market dynamics and demonstrated how parameter changes have a major impact on consumer behavior. Increases in η (the interaction rate) accelerate the transition from unawareness of the product directly to purchase. Additionally, increases in ξ lead to higher levels of awareness. This indicates to more dynamic market and driving sales growth. On the other hand, an increase in γ_0 leads to a decline in all classes of our model. Finally, a higher growth rate of social media advertising investment also contributes to increased awareness.

Our results demonstrate the model's efficacy and highlight the critical role of parameters such as η, ξ, and σ in shaping social media and word-of-mouth marketing outcomes. By understanding these effects, businesses can optimize their advertising tactics, leveraging social media ads to raise customer awareness, thereby enhancing marketing performance and sales.

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Conflict of interest

With reference to this study, the authors claim that they do not have any conflict of interest.

Author contributions

All the authors have participated equally in all the aspects of this paper: conceptualization, methodology, investigation, formal analysis, writing-original draft preparation, writing-review and editing.

Availability of data and materials

The data used to support the findings of the study are available within the article.

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