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Global smooth solutions in a one-dimensional thermoviscoelastic model with temperature-dependent paramaters

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Abstract. This manuscript is concerned with the system

$$\begin{cases} u_{tt} = (\gamma(\Theta)u_{xt})_x + (a(x,t)u_x)_x + (f(\Theta))_x, \\ \Theta_t = D\Theta_{xx} + \gamma(\Theta)u_{xt}^2 + f(\Theta)u_{xt}, \end{cases}$$

which is used to describe thermoviscoelastic developments in one-dimensional Kelvin–Voigt materials.

It is assumed that a, γ and f are sufficiently smooth functions that satisfy

$$c_\gamma < \gamma(\zeta) < C_\gamma, \quad \gamma''(\zeta) \le 0,$$
$$f(0) = 0, \quad |f'(\zeta)| \le C_f \quad \text{and} \quad |f(\zeta)| \le C_f(1+\zeta)^\alpha \quad \text{for all } \zeta \ge 0$$

and some positive constants $c_\gamma, C_\gamma, C_f > 0$ and $\alpha \in (0, 5/6)$. Under these conditions, this study then establishes a result on the existence of global classical solutions for sufficiently smooth but arbitrarily large initial data.
Keywords: viscous wave equation, thermoviscoelasticity, Moser iteration.

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1 Introduction

We consider the one dimensional evolution system

$$\begin{cases} u_{tt} = (\gamma(\Theta)u_{xt})_x + (a(x,t)u_x)_x + (f(\Theta))_x, \\ \Theta_t = D\Theta_{xx} + \gamma(\Theta)u_{xt}^2 + f(\Theta)u_{xt}, \end{cases} \tag{1.1}$$

which can be derived from the more general systemclass for thermo visco elastic models

$$\begin{cases} u_{tt} = (\gamma(\Theta,u_x)u_{xt})_x + (a(\Theta,u_x)u_x)_x - (f(\Theta,u_x))_x, \\ \Theta_t = D\Theta_{xx} + \gamma(\Theta,u_x)u_{xt}^2 - f(\Theta,u_x)u_{xt}. \end{cases} \tag{1.2}$$

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Models of this type have two main physical use cases, to model the piezoelectric effect in Kelvin–Voigt type materials on the one handside and heat generation by acoustic waves in Kelvin–Voigt-type materials on the other (see e.g. [29] or [30]). In consideration of the first, piezoceramic are excited by external electrodes using electrical voltage, to set them into vibration. This process is key in many industrial applications, including nanopositioning, ultrasound generation, and medical technology. Not only is the heat generated in this process an indication of inefficiency and the thereby associated energy losses, it also poses a risk to the piezoelectric components used. These are known to be very susceptible to such damage caused by high temperatures, since the polarization of the piezo ceramics is essential for the piezoelectric effect, and this effect is lost when the Curie temperature is exceeded [31]. To model the heat generation caused by acoustic wave propagation, the mechanical strain S is coupled to the mechanical displacement u, the electric displacement field D and the strength of the electric field E along the corresponding piezoelectric coupling tensor e and the permittivity matrix ε by

$$D = \varepsilon E + eS, \quad \text{with} \quad \operatorname{div} D = 0, \quad \text{and} \quad S = \nabla^s u, \tag{1.3}$$

where $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ represent the symmetric gradient, respectively. When deployed in the underlying Kelvin–Voigt model of arbitrary spatial dimensions and in consideration of heat generation, in form of the temperature denoted as Θ, the resulting wave equation formulates to

$$\rho u_{tt} = \operatorname{div}(dS_t + CS - \Theta CB) - \operatorname{div}(e^T E), \tag{1.4}$$

in which ρ represents the material density, d the viscosity tensor, C the elastic parameter tensor and B the thermal dilation tensor. In the one-dimensional case, for the thermal equation, the conversion of mechanical work into thermal energy can be noted in view of mechanical strain S and thus displacement u by

$$Q = dS_t^2 = du_{xt}^2, \tag{1.5}$$

where the unit of the quantity $Q \in \mathbb{R}$ is that of the density of an energy [4]. Thus the thermal equation formulates to

$$\Theta_t = D\Theta_{xx} + du_{xt}^2 - \Theta CBu_{xt}. \tag{1.6}$$

If we assume that the permittivity, the piezoelectric coupling and the material density are constant, (1.2) results from choosing

$$\gamma = \frac{d}{\rho}, \qquad a = \frac{C}{\rho} + \frac{e^2}{\varepsilon\rho} \qquad \text{and} \qquad f = \frac{\Theta CB}{\rho},$$

while (1.1) occurs in cases where C may depend on space and time but not on the temperature Θ or the mechanical strain u_x.

Contribution.
Building on the foundational works in [9] and [10], a plethora of studies have emerged, investigating variations of the one-dimensional problem (1.2), where the parameters a and γ are independent of the temperature variable. These studies have yielded noteworthy results regarding various aspects of solvability. Despite our consideration of temperature-dependent γ and f, it is imperative to direct our attention to the solvability results within a broader context, particularly in regard to a more extensive range of systems.

In one spatial dimension, [26] demonstrated the existence of certain weak soultions to an associated initial-boundary value problem globally in time for constant γ and broad classes of functions $a = a(u_x)$ and $F = F(u_x)$ in $f = \Theta F$. Since global smooth solutions seem even for temperature independent γ hard to access in large data scenarios, the works [16], [32] and [5] rely on stronger but physically reasonable assumptions on the key ingredients. In [33] and [18] the existence of global solutions for small initial data is established.

Higher-dimensional variations of (1.2) have been analyzed in [3, 14, 19–21, 28–30] and viscosity-free thermoelastic models, which reduce to (1.2) with $\gamma \equiv 0$ in one spatial dimension, have been addressed in [17,23–25,34], and most recent in [1,6,27] and [2].

Recent experiments have shown a clear temperature dependence of the parameter functions [13], thus temperature dependent γ are not only of theoretical interest. In this case, the analysis of solvability in the more general context poses a significant challenge. As such, classical solutions have only been obtained for simplified versions or under very specific assumptions. In particular, the existence of global classical solutions has been demonstrated for sufficiently small and smooth initial data and for constant a, $f \equiv 0$ and a broad class of γ [8]. A similar result has been achieved for related systems, considering $a \equiv b \cdot \gamma + \mu$ for arbitrary $b > 0$, $\mu > 0$ and assuming F, f and γ as suitably smooth. With the help of a rather technical assumption regarding special inital data, the existence of global classical solutions has also been proven [12]. Another strong solvability result relies on the assumption that γ satisfies strict boundedness properties. In particular, $\gamma_0 \le \gamma < \gamma_0 + \delta$ is considered to hold for a certain $\delta > 0$ [37]. Other results rely on weaker solution concepts [8,36].

Main results.

In this work, we focus on functions $\gamma = \gamma(\Theta)$ which are bounded, sufficiently regular, and fulfill a kind of saturation condition for $\Theta \to \infty$ and, functions $a = a(x,t)$, which we assume as sufficiently smooth and positive. For f we restrict the standard assumption $f(\Theta) = \beta\Theta$ to sublinear growth by assuming $|f(\zeta)| \le C_f(1+\zeta)^\alpha$ with $0 < \alpha < 5/6$ and $|f'(\zeta)| \le C_f$. Under these assumptions, we are able to prove the existence of global classical solutions of

$$\begin{cases} u_{tt} = (\gamma(\Theta)u_{xt})_x + (a(x,t)u_x)_x + (f(\Theta))_x, & x \in \Omega,\ t > 0, \\ \Theta_t = D\Theta_{xx} + \gamma(\Theta)u_{xt}^2 + f(\Theta)u_{xt}, & x \in \Omega,\ t > 0, \\ u_x = \Theta_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_{0t}(x), \quad \Theta(x,0) = \Theta_0(x), & x \in \Omega. \end{cases} \tag{1.7}$$

Considering one spatial dimension is a crucial assumption to accomplish this, since Sobolev embeddings are particularly convenient for $n = 1$, a fact that will be utilised on two occasions. Firstly, we only require information about Θ in $W^{1,2}(\Omega)$ to utilize the extensibility criterion of the local solution theory used. Secondly, the 1-D Gagliardo–Nirenberg inequality will be applied in numerous critical instances.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}$ be an open bounded interval, let*

$$a \in C^2(\overline{\Omega} \times [0,\infty)) \quad \textit{be positive} \tag{1.8}$$

and suppose that

$$\gamma \in C^2([0,\infty)) \quad \textit{and} \quad f \in C^1([0,\infty)) \tag{1.9}$$

are such that

$$c_\gamma < \gamma(\zeta) < C_\gamma \quad \textit{and} \quad \gamma''(\zeta) \le 0 \quad \textit{for all } \zeta \ge 0, \tag{1.10}$$

and

$$f(0)=0, \quad |f'(\zeta)| \le C_f \quad \textit{and} \quad |f(\zeta)| \le C_f(\zeta+1)^\alpha \quad \textit{for all } \zeta \ge 0 \tag{1.11}$$

for some positive constants $c_\gamma < 1 < C_\gamma$, $C_f > 1$ *and*

$$0 < \alpha < \frac{5}{6}. \tag{1.12}$$

Then whenever

$$\begin{cases} u_0 \in W^{3,2}(\Omega) \textit{ is such that } u_{0x} = 0 \textit{ on } \partial\Omega, \\ u_{0t} \in W^{2,2}(\Omega) \textit{ is such that } u_{0tx} = 0 \textit{ on } \partial\Omega, \textit{ and} \\ \Theta_0 \in W^{2,2}(\Omega) \textit{ satisfies } \Theta_0 \ge 0 \textit{ in } \Omega \textit{ and } \Theta_{0x} = 0 \textit{ on } \partial\Omega, \end{cases} \tag{1.13}$$

there exists a pair of functions (u,Θ) *which solves* (1.7) *in the classical sense in* $\Omega \times (0,\infty)$*, while satisfying* $\Theta \ge 0$ *as well as*

$$\begin{cases} u \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,\infty)) \right) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \qquad \textit{and} \\ \Theta \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,\infty)) \right) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \end{cases} \tag{1.14}$$

and furthermore

$$u_t \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,\infty)) \right) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)). \tag{1.15}$$

2 Preliminaries

In the following chapter, we present the results of various studies and prepare them for our particular purposes. We refrain from providing detailed proofs, instead referring to the original works. We begin with a proposition on the existence of local solutions, which was examined in more detail in [12, Theorem 1.1]. The proof follows from minor adjustments, since the regularity of the parameter functions γ, f and a considered here is sufficient to the methods used there. Our proof of the existence of global solutions is based on the extensibility criterion contained therein.

Proposition 2.1. Let $\Omega \subset \mathbb{R}$ be an interval and suppose γ, a and f are such that (1.8)–(1.12) hold and (u_0, u_{0t}, Θ_0) to satisfy (1.13). Then there exists $T_{\max} \in (0,\infty]$ as well as a pair of functions (u,Θ) which solves (1.7) in the classical sense in $\Omega \times (0, T_{\max})$ while satisfying $\Theta \ge 0$ and

$$\begin{cases} u \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,T_{\max})) \right) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})) \qquad \text{and} \\ \Theta \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,T_{\max})) \right) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})) \end{cases} \tag{2.1}$$

and furthermore

$$u_t \in \left(\bigcup_{\mu\in(0,1)} C^{1+\mu,\frac{1+\mu}{2}}(\overline{\Omega}\times[0,T_{\max})) \right) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})), \tag{2.2}$$

and which have the additional property that

$$\text{if } T_{\max} < \infty, \quad \text{then} \quad \limsup_{t\nearrow T_{\max}} \|\Theta(\cdot,t)\|_{W^{1,2}(\Omega)} = \infty. \tag{2.3}$$

In preparation for applying a Moser iteration argument later on, we record the following result which can be found proven in [11, Lemma 2.1], for instance.

Lemma 2.2. Let $A \geq 0$ and $B \geq 1$, and suppose that $(M_k)_{k\geq 1} \subset (0,\infty)$ is such that

$$M_k \leq \max\left\{A^{2^k}, B^k M_{k-1}^2\right\} \quad \text{for all } k \geq 2. \tag{2.4}$$

Then

$$M_k^{\frac{1}{2^k}} \leq B^2 \cdot \max\{A, M_1\} \quad \text{for all } k \geq 1.$$

To cleanly prepare the utilization of a well-known result from the work of Porzio–Vespri in [22, Theorem 1.3], we note the following lemma, which represents a reduction to our homogeneous case of Theorem 1.3 therein. While the focus of the original work lies in establishing higher regularity properties for a given weak solution, we are interested in applying their results to already classical solutions as we want to make use of the fact that Porzio and Vespri carefully trace the dependency of the Hölder constants on other explicit bounds of said solutions. This will later allow us to establish a uniform-in-time modulus of continuity for our solution regarding their space variable and thus enable a crucial Ehrling-type inequality from the literature.

Lemma 2.3. Let $T > 0$, $q > 3$ and $w_0 \in W^{2,2}(\Omega)$, and suppose $w \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap L^\infty(\Omega \times (0,T))$ to be a classical solution of

$$\begin{cases} w_t = \left[g_1(x,t)w_x + g_2(x,t) + g_3(x,t)\right]_x & x \in \Omega, \quad t > 0, \\ w_x = 0 & x \in \partial\Omega, \quad t > 0, \\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

where $c_g \leq g_1 \leq C_g$, $g_2 \in L^\infty((0,T); L^2(\Omega))$ and $g_3 \in L^q(\Omega \times (0,T))$. Then there exist constants $\mu > 1$ and $\beta \in (0,1)$ such that

$$|w(x_1,t_1) - w(x_2,t_2)| \leq \mu\left(|x_1 - x_2|^\beta + |t_1 - t_2|^{\frac{\beta}{2}}\right)$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \overline{\Omega} \times (0,T)$. The constants $\mu > 1$ and $\beta \in (0,1)$ depend upon $\|w\|_{L^\infty(\Omega\times[0,T])}$ on c_g, C_g, $\|g_2\|_{L^\infty((0,T);L^2(\Omega))}$, $\|g_3\|_{L^q(\Omega\times(0,T))}$ and the norm $\|w_0\|_{C^0(\Omega)}$.

In line with the former, we introduce another lemma which derives a useful Ehrling-type inequality from the given Hölder continuity. For the proof, we refer to [35, Lemma A.1].

Lemma 2.4. Let $\Omega \subset \mathbb{R}$ be an interval, and let $\omega : (0,\infty) \to (0,\infty)$ be nondecreasing. Then for all $p \geq 1$ and each $\eta > 0$ there exists $C(p,\eta) > 0$ such that

$$\int_\Omega |\varphi_x|^{2p+2} \leq \eta \int_\Omega |\varphi_x|^{2p-2}\varphi_{xx}^2 + C(p,\eta)\|\varphi\|_{L^\infty(\Omega)}^{2p+2} \tag{2.6}$$

holds for all

$$\begin{aligned} \varphi \in S_\omega := \Bigg\{ \tilde{\varphi} \in C^2(\overline{\Omega}) \;\Bigg|\; \frac{\partial\tilde{\varphi}}{\partial\nu} = 0 \text{ on } \partial\Omega, \text{ and for all } \eta' > 0, \\ \text{we have } |\tilde{\varphi}(x) - \tilde{\varphi}(y)| < \eta' \\ \text{whenever } x, y \in \overline{\Omega} \text{ are such that } |x - y| < \omega(\eta') \Bigg\}. \end{aligned} \tag{2.7}$$

3 Global existence time. Proof of Theorem 1.1

In this work, whenever γ, a, f and (u_0, u_{0t}, Θ_0) are fixed such that (1.8)–(1.13) are met, we let $T_{\max}$, u and Θ be as found in Proposition 2.1.

With our toolbox full, we are now set out to prove Theorem 1.1, by deriving a bound for Θ in $W^{1,2}(\Omega)$. To do so, we will start with simple testing procedures to obtain information about the mechanical energy and the average temperature; in particular, we derive bounds for u_x and u_t in $L^2(\Omega)$ and for Θ in $L^1(\Omega)$.

Lemma 3.1. Let γ, a and f be such that (1.8)–(1.12) are satisfied and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). Then for every $T > 0$, there exists $C_1 = C_1(a, T) > 1$ such that

$$\int_\Omega u_t^2 \le C_1, \qquad \int_\Omega u_x^2 \le C_1, \quad \text{and} \quad \int_\Omega \Theta \le C_1 \tag{3.1}$$

for all $t \in (0, T) \cap (0, T_{\max})$.

Proof. Since $a \in C^2(\overline{\Omega} \times [0, \infty))$ is positive, there exists $c_1 = c_1(a, T) > 0$ such that $|a_t(x,t)| \le c_1$ and $\frac{1}{c_1} \le a(x,t) \le c_1$ on $\Omega \times [0, T]$. By simple testing procedures, with respect to $u_x = u_{tx} = \Theta_x = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a(x,t)u_x^2 + \frac{d}{dt}\int_\Omega \Theta \\
&\quad = -\int_\Omega \gamma(\Theta)u_{xt}^2 - \int_\Omega a(x,t)u_x u_{xt} - \int_\Omega f(\Theta)u_{xt} + \frac{1}{2}\int_\Omega a_t(x,t)u_x^2 \\
&\qquad + \int_\Omega a(x,t)u_x u_{xt} + \int_\Omega D\Theta_{xx} + \int_\Omega \gamma(\Theta)u_{xt}^2 + \int_\Omega f(\Theta)u_{xt} \\
&\quad = \frac{1}{2}\int_\Omega a_t(x,t)u_x^2 \\
&\quad \le c_1^2 \int_\Omega a(x,t)u_x^2 \qquad \text{for all } t \in (0, T) \cap (0, T_{\max}).
\end{aligned} \tag{3.2}$$

We define

$$y(t) := \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega a(x,t)u_x^2 + \int_\Omega \Theta \quad \text{for all } t \in (0, T) \cap (0, T_{\max}),$$

and hence from (3.2), we derive the differential inequality

$$y'(t) \le c_1^2 \cdot y(t) \quad \text{for all } t \in (0, T) \cap (0, T_{\max}),$$

and since

$$y(0) = \frac{1}{2}\int_\Omega u_{0t}^2 + \frac{1}{2}\int_\Omega a(\cdot, 0)u_{0x}^2 + \int_\Omega \Theta_0 < \infty,$$

due to (1.13), we infer by a comparison argument

$$y(t) \le y(0) \cdot e^{c_1^2 T} \quad \text{for all } t \in (0, T) \cap (0, T_{\max}),$$

which already establishes our claim (3.1) due to the nonnegativity of all the components of y. $\square$

Now, in view of our overall assumption that f satisfies (1.11) with $\alpha < 5/6 < 1$, we can control the ill-signed contributions to the evolution of functionals $\int_\Omega (\Theta + 1)^p$ in their concave range, where $p \in (0, 1)$.

Lemma 3.2. Let γ, a and f be such that (1.8)–(1.12) are satisfied and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). For all $p \in (0,1)$ and all $T_0 > 0$, there exists $C_2 = C_2(\gamma, a, f, \Omega, p, T_0) > 0$, such that

$$\int_0^T \int_\Omega (\Theta+1)^{p-2}\Theta_x^2 \le C_2 \quad \text{for all } T \in (0,T_0) \cap (0, T_{\max}) \tag{3.3}$$

Proof. We begin by fixing $p \in (0,1)$, $T_0 > 0$ and $T \in (0,T_0) \cap (0,T_{\max})$ and testing the second equation of (1.7) to obtain

$$\begin{aligned}
-\frac{1}{p}\frac{d}{dt}\left\{\int_\Omega (\Theta+1)^p\right\} &+ (1-p)D\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 \\
&= -\int_\Omega \gamma(\Theta)(\Theta+1)^{p-1}u_{xt}^2 - \int_\Omega (\Theta+1)^{p-1} f(\Theta) u_{xt} \\
&\le -\frac{1}{2}\int_\Omega \gamma(\Theta)(\Theta+1)^{p-1}u_{xt}^2 + \frac{C_f^2}{2c_\gamma}\int_\Omega (\Theta+1)^{p-1+2\alpha}
\end{aligned}$$

for all $t \in (0,T)$, where we used our assumed boundary condition $\Theta_x = 0$.

In preparation of a further estimation step, we first apply Young's inequality to conclude that since $\alpha \in (0,1)$, there exists $c_1 = c_1(\gamma, f, \Omega, p) > 0$ such that

$$\frac{C_f^2}{2c_\gamma}\int_\Omega (\Theta+1)^{p-1+2\alpha} \le c_1 + \int_\Omega (\Theta+1)^{p+1} \quad \text{for all } t \in (0,T),$$

and by an application of a Gagliardo–Nirenberg inequality, we infer that there exists $c_2 = c_2(\Omega, p) > 0$ such that

$$\|\varphi\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \le c_2 \|\varphi_x\|_{L^2(\Omega)}^{\frac{2(p+1)\lambda}{p}} \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{(1-\lambda)\frac{2(p+1)}{p}} + c_2 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}}$$

for every $\varphi \in C^1(\overline{\Omega})$ with $\lambda := \frac{p^2}{(p+1)^2} \in (0,1)$ and when applied to $\varphi = (\Theta+1)^{\frac{p}{2}}$ we infer

$$\begin{aligned}
\int_\Omega (\Theta+1)^{p+1} &= \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le c_2 \left\|\left((\Theta+1)^{\frac{p}{2}}\right)_x\right\|_{L^2(\Omega)}^{\frac{2(p+1)\lambda}{p}} \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{(1-\lambda)\frac{2(p+1)}{p}} + c_2 \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}}
\end{aligned} \tag{3.4}$$

for all $t \in (0,T)$. Moreover, since $(1-\lambda)(p+1) = \frac{2p+1}{p+1}$ and $\frac{(p+1)\lambda}{p} = \frac{p}{p+1} < 1$, we may apply Young's inequality to the powers $q = (p+1)/p$ and $\tilde{q} = p+1$, hence by Lemma 3.1, we are able to estimate the first term on the right side of (3.4) such that

$$\begin{aligned}
&c_2 \left\|\left((\Theta+1)^{\frac{p}{2}}\right)_x\right\|_{L^2(\Omega)}^{\frac{2(p+1)\lambda}{p}} \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{(1-\lambda)\frac{2(p+1)}{p}} \\
&= \left(\frac{(1-p)D}{p^2}\right)^{\frac{p}{p+1}} \left\|\left((\Theta+1)^{\frac{p}{2}}\right)_x\right\|_{L^2(\Omega)}^{\frac{2p}{p+1}} \cdot c_2 \left(\frac{p^2}{(1-p)D}\right)^{\frac{p}{p+1}} \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{(1-\lambda)\frac{2(p+1)}{p}} \\
&\le \frac{(1-p)D}{4^{\frac{p+1}{p}}}\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 + \frac{p^{2p}c_2^{p+1}}{(1-p)^p D^p}\left(\int_\Omega \Theta+1\right)^{2p+1} \\
&\le \frac{(1-p)D}{2}\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 + \frac{p^{2p}c_2^{p+1}}{(1-p)^p D^p}\left(C_1+|\Omega|\right)^{2p+1} \\
&\le \frac{(1-p)D}{2}\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 + c_3
\end{aligned}$$

for all $t \in (0,T)$ with $c_3 \equiv c_3(\gamma, a, f, \Omega, p) := \left(\frac{p^2}{(1-p)D}\right)^p c_2^{p+1} (C_1 + |\Omega|)^{2p+1} > 0$. Combining these two leads to the conclusion that

$$-\frac{1}{p}\frac{d}{dt}\int_\Omega (\Theta+1)^p + \frac{(1-p)D}{2}\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 \le c_1 + c_3 + c_2\,(C_1+|\Omega|)^{p+1}$$
$$= c_4$$

for all $t \in (0,T)$, where $c_4 \equiv c_4(\gamma, a, f, \Omega, p) := c_1 + c_3 + c_2\,(C_1 + |\Omega|)^{p+1} > 0$. Integrating in time over $(0,T)$ yields

$$\begin{aligned}\frac{(1-p)D}{2}\int_0^T\int_\Omega (\Theta+1)^{p-2}\Theta_x^2 &\le c_4 T + \frac{1}{p}\int_\Omega (\Theta(\cdot,T)+1)^p - \frac{1}{p}\int_\Omega (\Theta(\cdot,0)+1)^p \\ &\le c_4 T + \frac{C_1}{p} + \frac{2}{p}|\Omega|,\end{aligned}$$

which establishes (3.3). □

In the next step, we utilize our previously derived information in the context of another Gagliardo–Nirenberg inequality to obtain information about $\int_0^T \|\Theta\|_{L^q(\Omega)}^r$ for suitable $q, r > 1$. It is important to note that the argument used in this context is not applicable to higher spatial dimensions. The conditions on r with respect to q that are determined in this process will later clarify the restriction $\alpha < 5/6$ when applied to f.

Lemma 3.3. Let γ, a and f be such that (1.8)–(1.12) are satisfied and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). For any $T_0 > 0$, $q > 1$ and $r > 0$ which satisfy

$$r < \frac{2q}{q-1},$$

there exists $C_3 = C_3(q, r, \gamma, a, f, \Omega, T_0) > 0$ such that

$$\int_0^T \|\Theta+1\|_{L^q(\Omega)}^r \le C_3 \quad \text{for all } T \in (0,T_0)\cap(0,T_{\max}) \tag{3.5}$$

Proof. We fix $T_0 > 0$, $T \in (0,T_0)\cap(0,T_{\max})$, $q > 1$ and $p \in (0,1)$. For all $\varphi \in C^1(\overline{\Omega})$ and every $r \in (0,1]$ using Young's inequality, we conclude that

$$\|\varphi\|_{L^q(\Omega)}^r \le \|\varphi\|_{L^q(\Omega)}^{r'} + 1$$

for any $r' > 1$. It is therefore evident that the subsequent analyzes will exclusively address the case of $r > 1$. We fix $r > 1$ and infer by an application of a Gagliardo–Nirenberg inequality that there exists some $c_1 = c_1(r, q, p, \Omega) > 1$ such that

$$\|\varphi\|_{L^{\frac{2q}{p}}(\Omega)} \le c_1 \|\varphi_x\|_{L^2(\Omega)}^{\lambda} \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{(1-\lambda)} + c_1 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)} \quad \text{for all } \varphi \in C^1(\overline{\Omega})$$

with $\lambda := \frac{p(q-1)}{(p+1)q} \in (0,1)$. When applied to $\varphi = (\Theta+1)^{\frac{p}{2}}$, we obtain

$$\begin{aligned}\left\|\Theta+1\right\|_{L^q(\Omega)}^r &= \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2r}{p}} \\ &\le c_1 \left\|\left((\Theta+1)^{\frac{p}{2}}\right)_x\right\|_{L^2(\Omega)}^{\frac{2r}{p}\lambda} \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2r}{p}(1-\lambda)} + c_1 \left\|(\Theta+1)^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2r}{p}}\end{aligned}$$

for all $t \in (0,T)$. Since $r < \frac{2q}{q-1}$, another application of Young's inequality, when combined with Lemma 3.1 and Lemma 3.2, yields

$$
\begin{aligned}
&c_1 \int_0^T \left\| \left((\Theta+1)^{\frac{p}{2}}\right)_x \right\|_{L^2(\Omega)}^{\frac{2r(q-1)}{(p+1)q}} \left\| (\Theta+1)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2r}{p}(1-\lambda)} \\
&\leq \int_0^T \int_\Omega (\Theta+1)^{p-2}\Theta_x^2 + c_1^{\frac{(p+1)q}{(p+1)q-r(q-1)}} \int_0^T \left(\int_\Omega \Theta + 1 \right)^{\tilde{q}} \\
&\leq c_2 \equiv c_2(\gamma, a, f, \Omega, T_0) := C_2 + c_1^{\frac{(p+1)q}{(p+1)q-r(q-1)}} (C_1 + |\Omega|)^{\tilde{q}} T_0,
\end{aligned}
$$

where $\tilde{q} := \frac{rq+rp}{pq+q-rq+r}$. Since Lemma 3.1 also implies that

$$
c_1 \int_0^T \left\| (\Theta+1)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2r}{p}} \leq c_1 (C_1 + |\Omega|)^r T_0
$$

holds, we have already accomplished (3.5). $\square$

In order to apply Lemma 2.3, we require $L^\infty(\Omega)$ bounds, which are finite for $T_{\max} < \infty$. To derive these bounds, we use a limit process based on reducing higher-degree L^{p_k}-norms to $L^{p_{k-1}}$-norms, employing a Moser-type iteration and in particular Lemma 2.2.

Lemma 3.4. Let γ, a and f be such that (1.8)–(1.12) are satisfied for some $\alpha < 5/6$ and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). Then for all $T_0 > 0$, there exists $C_4 = C_4(\gamma, a, f, \Omega, T_0) > 0$ such that

$$
\|u_t(\cdot,t)\|_{L^\infty(\Omega)} \leq C_4 \quad \text{for all } t \in (0,T_0) \cap (0,T_{\max}). \tag{3.6}
$$

Proof. We fix $T \in (0,T_0) \cap (0,T_{\max})$ and write $I := (0,T)$. For integers $k \geq 1$, we let $p_k := 2^k$ and

$$
M_k(T) := \sup_{t \in I} \int_\Omega u_t^{p_k}(\cdot,t) < \infty. \tag{3.7}
$$

To prepare our later test procedure, we choose, with respect to $\alpha < 5/6$, some $q = q(\alpha) \in (2, \frac{1}{(3\alpha-2)_+})$ and define $p = p(\alpha) := \frac{2q}{q-2}$ as well as $\lambda = \lambda(q,f) := \frac{q+2}{3q}$. Hence, we note the observations

$$
\int_\Omega u_t^{p_k-2} f(\Theta)^2 \leq \int_\Omega u_t^{p_k} f(\Theta)^2 + \int_\Omega f(\Theta)^2 \quad \text{for all } t \in I \tag{3.8}
$$

and

$$
\int_\Omega u_t^{p_k} f(\Theta)^2 \leq \left\| u_t^{\frac{p_k}{2}} \right\|_{L^p(\Omega)}^2 \left\| f(\Theta) \right\|_{L^q(\Omega)}^2 \quad \text{for all } t \in I, \tag{3.9}
$$

by Young's and Hölder's inequalities. Moreover, our assumption on f (1.11) and our choice of λ guarantees $\frac{2\alpha}{1-\lambda} = \frac{3q\alpha}{q-1} < \frac{2q\alpha}{q\alpha-1}$ and thereby enables us to apply Lemma 3.3 for some $c_1 = c_1(\gamma, a, f, \Omega, T_0) > 0$ such that

$$
\int_0^T \left\| f(\Theta) \right\|_{L^q(\Omega)}^{\frac{2}{1-\lambda}} \leq C_f^{\frac{2}{1-\lambda}} \int_0^T \left\| \Theta + 1 \right\|_{L^{q\alpha}(\Omega)}^{\frac{2\alpha}{1-\lambda}} \leq c_1 \tag{3.10}
$$

and

$$
\int_0^T \left\| f(\Theta) \right\|_{L^2(\Omega)}^2 \leq C_f^2 \int_0^T \left\| \Theta + 1 \right\|_{L^{2\alpha}(\Omega)}^{2\alpha} \leq c_1. \tag{3.11}
$$

Due to a satisfying (1.9), we may utilize Lemma 3.1 to obtain $c_2 = c_2(a, T_0) > 1$ such that

$$\int_\Omega a^2 u_x^2 \leq c_2, \quad \text{for all } t \in I. \tag{3.12}$$

For the next preparation step, we apply a Gagliardo–Nirenberg inequality to infer that there exists $c_3 = c_3(\Omega) > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)}^2 \leq c_3 \|\varphi_x\|_{L^2(\Omega)}^{\frac{4}{3}} \|\varphi\|_{L^1(\Omega)}^{\frac{2}{3}} + c_3 \|\varphi\|_{L^1(\Omega)}^2 \tag{3.13}$$

for all $\varphi \in C^1(\overline{\Omega})$, and, in view of our choice of $p, \lambda \in (0,1)$, we may apply the inequality once more, to obtain for some $c_4 = c_4(\Omega) > 0$

$$\|\varphi\|_{L^p(\Omega)}^2 \leq c_4 \|\varphi_x\|_{L^2(\Omega)}^{2\lambda} \|\varphi\|_{L^1(\Omega)}^{2(1-\lambda)} + c_4 \|\varphi\|_{L^1(\Omega)}^2 \tag{3.14}$$

for all $\varphi \in C^1(\overline{\Omega})$. By testing the first equation and using (3.8) and (3.9), we obtain

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u_t^{p_k} &= p_k \int_\Omega u_t^{p_k - 1} \big[\gamma(\Theta) u_{tx} + a u_x + f(\Theta)\big]_x \\
&= -p_k (p_k - 1) \int_\Omega u_t^{p_k - 2} u_{tx} \big[\gamma(\Theta) u_{tx} + a u_x + f(\Theta)\big] \\
&\leq -\frac{p_k (p_k - 1) c_\gamma}{2} \int_\Omega u_t^{p_k - 2} u_{tx}^2 + \frac{p_k (p_k - 1)}{c_\gamma} \int_\Omega u_t^{p_k - 2} a^2 u_x^2 \\
&\quad + \frac{p_k (p_k - 1)}{c_\gamma} \int_\Omega u_t^{p_k - 2} f(\Theta)^2 \\
&\leq -\frac{p_k (p_k - 1) c_\gamma}{2} \int_\Omega u_t^{p_k - 2} u_{tx}^2 + \frac{p_k (p_k - 1)}{c_\gamma} \int_\Omega u_t^{p_k - 2} a^2 u_x^2 \\
&\quad + \frac{p_k (p_k - 1)}{c_\gamma} \int_\Omega f(\Theta)^2 + \frac{p_k (p_k - 1)}{c_\gamma} \left\| u_t^{\frac{p_k}{2}} \right\|_{L^p(\Omega)}^2 \left\| f(\Theta) \right\|_{L^q(\Omega)}^2
\end{aligned} \tag{3.15}$$

for all $t \in I$, where we have used the facts that $c_2 > 1$, $c_\gamma < 1$ and the vanishing boundary term, which is ensured by the boundary conditions $0 = u = u_t$ on $\partial\Omega$. To estimate the right hand side, we start by analyzing the second term, therefore we note $\|u_t^{p_k - 2}\|_{L^\infty(\Omega)} \leq \|u_t^{\frac{p_k}{2}}\|_{L^\infty(\Omega)}^2 + 1$ due to $p_k \geq 2$ and we conclude by applying (3.13) to $\varphi = u_t^{p_k/2}$ and Young's inequality

$$\begin{aligned}
\left\| u_t^{\frac{p_k}{2}} \right\|_{L^\infty(\Omega)}^2 &\leq c_3 \left\| \left(u_t^{\frac{p_k}{2}} \right)_x \right\|_{L^2(\Omega)}^{\frac{4}{3}} \left\| u_t^{\frac{p_k}{2}} \right\|_{L^1(\Omega)}^{\frac{2}{3}} + c_3 \left\| u_t^{\frac{p_k}{2}} \right\|_{L^1(\Omega)}^2 \\
&\leq \frac{c_\gamma^2}{4 c_2} \int_\Omega u_t^{p_k - 2} u_{tx}^2 + \left(16 \frac{c_2^2 c_3^3}{c_\gamma^4} + c_3 \right) \left(\int_\Omega u_t^{\frac{p_k}{2}} \right)^2 \\
&= \frac{c_\gamma^2}{4 c_2} \int_\Omega u_t^{p_k - 2} u_{tx}^2 + c_5 \left(\int_\Omega u_t^{\frac{p_k}{2}} \right)^2
\end{aligned} \tag{3.16}$$

for all $t \in I$, where $c_5 = c_5(\gamma, a, f, T_0) := 16 \frac{c_2^2 c_3^3}{c_\gamma^4} + c_3$. Combining (3.12) with (3.16) and the facts that $c_2 > 1 > c_\gamma > 0$ and $p_k/2 = p_{k-1}$, leads to

$$\begin{aligned}
&\frac{p_k(p_k-1)}{c_\gamma}\left(\int_\Omega u_t^{p_k-2}a^2u_x^2\right)\\
&\quad\le \frac{p_k(p_k-1)}{c_\gamma}\left(\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^\infty(\Omega)}+1\right)\int_\Omega a^2u_x^2\\
&\quad\le \frac{p_k(p_k-1)c_2}{c_\gamma}\left[1+\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^\infty(\Omega)}\right]\\
&\quad\le \frac{p_k(p_k-1)c_\gamma}{4}\int_\Omega u_t^{p_k-2}u_{tx}^2+\frac{p_k^2c_2c_5}{c_\gamma}\left(\int_\Omega u_t^{\frac{p_k}{2}}\right)^2+\frac{p_k^2c_2}{c_\gamma}
\end{aligned}\tag{3.17}$$

for all $t \in I$. For the last term on the right hand side of (3.15), we estimate by combining (3.14), Young's inequality and the fact that $A^2 \le A^{\frac{2}{1-\lambda}} + 1$ for all $A \ge 0$ and $\lambda \in (0,1)$

$$\begin{aligned}
\int_\Omega u_t^{p_k}f(\Theta)^2 &\le \left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^p(\Omega)}\left\|f(\Theta)\right\|^2_{L^q(\Omega)}\\
&\le \left(c_4\left\|\left(u_t^{\frac{p_k}{2}}\right)_x\right\|^{2\lambda}_{L^2(\Omega)}\left\|u_t^{\frac{p_k}{2}}\right\|^{2(1-\lambda)}_{L^1(\Omega)}+c_4\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^1(\Omega)}\right)\left\|f(\Theta)\right\|^2_{L^q(\Omega)}\\
&\le \frac{c_\gamma^2}{p_k^2}\left\|\left(u_t^{\frac{p_k}{2}}\right)_x\right\|^2_{L^2(\Omega)}+\left(\frac{p_k^2c_4}{c_\gamma^2}\right)^{\frac{\lambda}{1-\lambda}}c_4^{\frac{1}{1-\lambda}}\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^1(\Omega)}\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}\\
&\quad+c_4\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^1(\Omega)}\left\|f(\Theta)\right\|^2_{L^q(\Omega)}\\
&\le \frac{c_\gamma^2}{4}\int_\Omega u_t^{p_k-2}u_{tx}^2+p_k^{\frac{2\lambda}{1-\lambda}}\left(\frac{c_4}{c_\gamma^2}\right)^{\frac{\lambda}{1-\lambda}}\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^1(\Omega)}\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}\\
&\quad+c_4\left\|u_t^{\frac{p_k}{2}}\right\|^2_{L^1(\Omega)}\left(\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}+1\right)
\end{aligned}$$

for all $t \in I$. We now draw on our choice of $\lambda = \frac{q+2}{3q} \in (0,1)$ and $q > 2$ to infer $\frac{2\lambda}{1-\lambda} < 4$ and since $\|u_t^{\frac{p_k}{2}}\|^2_{L^1(\Omega)} = \|u_t^{p_{k-1}}\|^2_{L^1(\Omega)} \le M^2_{k-1}(T)$ for all $t \in I$, we obtain for $c_6 = c_6(\gamma, a, f, T_0, \Omega) := \left(\frac{c_4}{c_\gamma^2}\right)^{\frac{\lambda}{1-\lambda}} + c_4$ that

$$\begin{aligned}
&\frac{p_k(p_k-1)}{c_\gamma}\int_\Omega u_t^{p_k}f(\Theta)^2\\
&\quad\le \frac{p_k(p_k-1)c_\gamma}{4}\int_\Omega u_t^{p_k-2}u_{tx}^2+\frac{p_k^6c_6}{c_\gamma}M^2_{k-1}(T)\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}+\frac{p_k^2c_4}{c_\gamma}M^2_{k-1}(T)
\end{aligned}$$

for all $t \in I$. Inserting this together with (3.17) into (3.15), we may conclude that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_t^{p_k} &\le \frac{p_k^6c_6}{c_\gamma}M^2_{k-1}(T)\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}+\frac{p_k^2}{c_\gamma}\int_\Omega f(\Theta)^2+\frac{p_k^2(c_4+c_2c_5)}{c_\gamma}M^2_{k-1}(T)+\frac{p_k^2c_2}{c_\gamma}\\
&= \frac{p_k^6c_6}{c_\gamma}M^2_{k-1}(T)\left\|f(\Theta)\right\|^{\frac{2}{1-\lambda}}_{L^q(\Omega)}+\frac{p_k^2}{c_\gamma}\int_\Omega f(\Theta)^2+\frac{p_k^2c_7}{c_\gamma}M^2_{k-1}(T)+\frac{p_k^2c_2}{c_\gamma}
\end{aligned}\tag{3.18}$$

for all $t \in I$, where $c_7 = c_7(\gamma, a, f, T_0, \Omega) := c_4 + c_2 c_5$. Utilizing (3.10) in combination with the fact that $\int_\Omega u^{p_k}(\cdot, 0) \leq |\Omega| \cdot \|u_{0t}\|_{L^\infty(\Omega)}^{p_k}$, we infer by integrating (3.18) in time

$$
\begin{aligned}
&\int_\Omega u_t^{p_k}(\cdot,t) + \int_0^t \int_\Omega u_t^{p_k} \\
&\leq \frac{p_k^6 c_6}{c_\gamma} M_{k-1}^2(T) \int_0^T \left\| f(\Theta) \right\|_{L^q(\Omega)}^{\frac{2}{1-\lambda}} + \frac{p_k^2 c_7}{c_\gamma} T_0 M_{k-1}^2(T) \\
&\quad + \frac{p_k^2}{c_\gamma} \int_0^T \left\| f(\Theta) \right\|_{L^2(\Omega)}^2 + \frac{p_k^2 c_2}{c_\gamma} T_0 + \int_\Omega u_t^{p_k}(\cdot, 0) \\
&\leq p_k^6 \frac{c_1 c_6 + c_7 T_0}{c_\gamma} M_{k-1}^2(T) + p_k^2 \frac{c_1 + c_2 T_0}{c_\gamma} + |\Omega| \cdot \|u_{0t}\|_{L^\infty(\Omega)}^{p_k} \\
&\leq p_k^6 c_8 M_{k-1}^2(T) + p_k^2 c_9 + |\Omega| \cdot \|u_{0t}\|_{L^\infty(\Omega)}^{p_k}
\end{aligned}
$$

for all $t \in I$ with $c_8 = c_8(\gamma, a, f, \Omega, T_0) := \frac{c_1 c_6 + c_7 T_0}{c_\gamma}$ as well as $c_9 = c_9(\gamma, a, f, \Omega, T_0) := \frac{(c_1 + c_2 T_0)}{c_\gamma}$ and thus

$$M_k(T) \leq 2^{6k} c_8 M_{k-1}^2(T) + 2^{2k} c_9 + |\Omega| \cdot \|u_{0t}\|_{L^\infty(\Omega)}^{2^k}.$$

Since (1.13) ensures $\|u_{0t}\|_{L^\infty(\Omega)}$ being finite, we let $A := \max\left\{(1 + |\Omega|) \cdot \|u_{0t}\|_{L^\infty(\Omega)}, \, 2(1 + c_9)\right\}$ as well as $B := 2^6 \cdot \max\{c_8, 1\}$ and conclude from

$$M_k(T) \leq B^k M_{k-1}^2(T) + A^{2^k} \leq \max\{A^{2^k}, B^k M_{k-1}^2\}, \quad \text{for all } k \geq 2, \tag{3.19}$$

via induction that for $I_0 := (0, T_{\max,\varepsilon}) \cap (0, T_0)$ and for each $k \in \mathbb{N}$ we introduce by

$$\widehat{M}_k := \sup_{T \in I_0} M_k(T) \quad \text{for } k \geq 1$$

a finite number, which, in view of (3.19), satisfies

$$\widehat{M}_k \leq \max\{A^{2^k}, B^k \widehat{M}_{k-1}^2\}, \quad \text{for each } k \geq 2.$$

Thus, an application of Lemma 2.2 yields, due to the fact that $p_k := 2^k$,

$$\widehat{M}_k^{\frac{1}{2^k}} \leq B^2 \max\left\{A, \widehat{M}_1\right\}, \quad \text{for all } k \geq 1,$$

where the independence of $\widehat{M}_1 = \sup_{s \in I_0} \|u_t(\cdot, s)\|_{L^2(\Omega)}^2$ from $T_{\max,\varepsilon}$ and it's finiteness are ensured by Lemma 3.1. Taking the limits $k \to \infty$ establishes (3.6). □

As demonstrated in the preceding proof, it is evident that restricting $\alpha < 5/6$ is indispensable for our line of argument. Without it, we could not in general select $q \in (2, \frac{1}{(3\alpha-2)_+})$ as required to invoke Lemma 3.3 in (3.10).

We have now demonstrated all the prerequisites to derive the Hölder continuity of u_t from Lemma 2.3.

Lemma 3.5. Let γ, a and f be such that (1.8)–(1.12) are satisfied and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). Then for any $T_0 > 0$ there exist $\beta \in (0, 1)$ and $C_5 = C_5(\gamma, a, f, \Omega, \beta, T_0) > 0$ such that

$$|u_t(x_1, t_1) - u_t(x_2, t_2)| \leq C_5 \left(|x_1 - x_2|^\beta + |t_1 - t_2|^{\frac{\beta}{2}} \right) \tag{3.20}$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in \Omega \times \big((0, T_0) \cap (0, T_{\max})\big)$

Proof. We fix $T \in (0, T_0) \cap (0, T_{\max})$, $\alpha_0 \in (\alpha, 1)$ and $q = 3/\alpha_0$ and may infer by Lemma 3.1 and $a \in C^2(\overline{\Omega} \times [0, \infty))$ as well as (1.11), Lemma 3.3 and the fact that $3\alpha/\alpha_0 < 3$, we can infer that there exist constants $c_1 = c_1(a, T_0) > 0$ and $c_2 = c_2(\gamma, f, \Omega, T_0) > 0$ such that

$$\|a(x,t)u_x(x,t)\|_{L^2(\Omega)} \le c_1 \quad \text{for all } t \in (0,T) \text{ and } \int_0^T \|f(\Theta)\|^q_{L^q(\Omega)} \le C_f^{\frac{3\alpha}{\alpha_0}} \int_0^T \int_\Omega (\Theta+1)^{\frac{3\alpha}{\alpha_0}} \le c_2.$$

Furthermore, recalling $u_{0t} \in W^{2,2}(\Omega)$ and Lemma 3.4, yields $c_3 = c_3(\gamma, a, f, \Omega, T_0) > 0$ such that

$$\|u_t(\cdot,t)\|_{L^\infty(\Omega)} \le c_3$$

for all $t \in (0, T)$, we may apply Lemma 2.3 on $w = u_t$, $w_0 = u_{0t}$, $g_1(x,t) = \gamma(\Theta(x,t))$, $g_2(x,t) = a(x,t)u_x(x,t)$ and $g_3(x,t) = f(\Theta(x,t))$ to infer that there exist $\beta \in (0,1)$ and $\mu(\gamma, a, f, \Omega, \beta, T_0) > 0$ such that (3.20) holds. □

Having gathered all the information we need, we are now in a position to take the final step to prove Theorem 1.1.

Lemma 3.6. Let γ, a and f be such that (1.8)–(1.12) are satisfied and assume (u_0, u_{0t}, Θ_0) to fulfill (1.13). For every $T_0 > 0$ there exists $C_5 = C_5(\gamma, a, f, \Omega, T_0) > 0$ such that

$$\int_\Omega \Theta_x^2 \le C_6 \quad \text{for all } t \in (0, T_0) \cap (0, T_{\max}). \tag{3.21}$$

Proof. We start by fixing $T_0 > 0$. Since γ satisfies $\gamma''(\zeta) \le 0$ and $c_\gamma < \gamma(\zeta)$ on $[0, \infty)$ and since Θ is non-negative, there exists $c_1 = c_1(\gamma) > 0$ such that

$$|\gamma'(\zeta)| \le c_1 \quad \text{on } [0, \infty). \tag{3.22}$$

Furthermore, due to $a \in C^2(\overline{\Omega} \times [0, \infty))$ and a being positive, one can find $c_2 = c_2(a, T_0) > 0$ such that

$$\frac{1}{c_2} \le a(x,t), \qquad |a_t(x,t)| \le c_2 \quad \text{and} \quad |a_x(x,t)| \le c_2 \quad \text{on } \overline{\Omega} \times [0, T_0]. \tag{3.23}$$

Moreover, we note due to $0 = u_x = u_{tx}$ on $\partial\Omega$

$$\begin{aligned} -\int_\Omega u_{txx}\gamma'(\Theta)\Theta_x u_{tx} &+ \Big[u_{tx}^2 \gamma'(\Theta)\Theta_x\Big]_{\partial\Omega} \\ &= \int_\Omega u_{tx}\big(\gamma'(\Theta)\Theta_x u_{tx}\big)_x = \int_\Omega \gamma''(\Theta)\Theta_x^2 u_{tx}^2 + \int_\Omega \gamma'(\Theta)\Theta_{xx}u_{tx}^2 + \int_\Omega \gamma'(\Theta)\Theta_x u_{txx}u_{tx} \end{aligned} \tag{3.24}$$

for all $t \in (0, T_0) \cap (0, T_{\max})$, from which we conclude

$$-\int_\Omega \gamma'(\Theta)\Theta_x u_{txx}u_{tx} = \frac{1}{2}\int_\Omega \gamma''(\Theta)\Theta_x^2 u_{tx}^2 + \frac{1}{2}\int_\Omega \gamma'(\Theta)\Theta_{xx}u_{tx}^2 \tag{3.25}$$

for all $t \in (0, T_0) \cap (0, T_{\max})$. Now, we estimate by testing the first equation of (1.7) and an application of Young's inequality and the previously noted (3.22)–(3.25)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_{tx}^2 &= \int_\Omega u_{tx}u_{ttx}\\
&= -\int_\Omega u_{txx}\cdot\Big[\gamma(\Theta)u_{txx}+a(x,t)u_{xx}+a_x(x,t)u_x+f'(\Theta)\Theta_x\Big]\\
&\quad+\int_\Omega u_{tx}\Big(\gamma'(\Theta)\Theta_x u_{tx}\Big)_x\\
&\leq -\int_\Omega \gamma(\Theta)u_{txx}^2-\frac{1}{2}\frac{d}{dt}\int_\Omega a(x,t)u_{xx}^2+\frac{1}{2}\int_\Omega a_t(x,t)u_{xx}^2+\frac{c_2^2}{c_\gamma}\int_\Omega u_x^2+\frac{C_{f'}^2}{c_\gamma}\int_\Omega\Theta_x^2\\
&\quad+\frac{c_\gamma}{2}\int_\Omega u_{txx}^2+\frac{1}{2}\int_\Omega\gamma''(\Theta)\Theta_x^2u_{tx}^2+\frac{1}{2}\int_\Omega u_{tx}^2\gamma'(\Theta)\Theta_{xx}\\
&\leq -\frac{c_\gamma}{2}\int_\Omega u_{txx}^2-\frac{1}{2}\frac{d}{dt}\int_\Omega a(x,t)u_{xx}^2+\frac{c_2}{2}\int_\Omega u_{xx}^2+\frac{c_2^2}{c_\gamma}\int_\Omega u_x^2+\frac{C_{f'}^2}{c_\gamma}\int_\Omega\Theta_x^2\\
&\quad+\frac{c_1^2}{D}\int_\Omega u_{tx}^4+\frac{D}{4}\int_\Omega\Theta_{xx}^2
\end{aligned}\tag{3.26}$$

for all $t\in(0,T_0)\cap(0,T_{\max})$ and further for $c_3=c_3(\gamma,D):=\frac{C_\gamma^2D+1}{D^2}$ and $c_4=c_4(f):=C_f^4$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\Theta_x^2+\frac{D}{2}\int_\Omega\Theta_{xx}^2\leq c_3\int_\Omega u_{tx}^4+c_4\int_\Omega(\Theta+1)^{4\alpha}\tag{3.27}$$

for all $t\in(0,T_0)\cap(0,T_{\max})$. Moreover, Lemma 2.1 guarantees $u_t(\cdot,t)\in C^2(\overline{\Omega})$ for every $t\in(0,T_{\max,\varepsilon})$ and Lemma 3.5 provides constant $C_5=C_5(\gamma,a,f,\Omega,\beta,T_0)>0$ and $\beta\in(0,1)$ such that

$$|u_t(x_1,t)-u_t(x_2,t)|\leq C_5|x_1-x_2|^\beta\quad\text{for all } t\in(0,T_0)\cap(0,T_{\max,\varepsilon}).$$

We now let $\omega:(0,\infty)\to(0,\infty)$ being a non decreasing function defined by

$$\omega(\xi):=\left(\frac{\xi}{C_5}\right)^{1/\beta}\qquad\text{for }\xi>0$$

and S_ω defined as in (2.7)

$$\begin{aligned}
\varphi\in S_\omega:=\Big\{\tilde\varphi\in C^2(\overline{\Omega})\ \Big|\ &\tilde\varphi=0\text{ on }\partial\Omega,\text{ and for all }\eta'>0,\\
&\text{we have }|\tilde\varphi(x)-\tilde\varphi(y)|<\eta'\\
&\text{whenever }x,y\in\overline{\Omega}\text{ are such that }|x-y|<\omega(\eta')\Big\}.
\end{aligned}$$

Thus, $u_t(\cdot,t)\in S_\omega$ for every $t\in(0,T_0)\cap(0,T_{\max,\varepsilon})$ and we may apply Lemma 2.4 on $\eta=\frac{c_\gamma D}{4(c_1^2+c_3D)}>0$ and $\varphi=u_t(\cdot,t)$ for all $t\in(0,T_0)\cap(0,T_{\max,\varepsilon})$ to conclude for some $c_5=c_5(D,\gamma,a,f,\Omega,\beta,T_0)>0$

$$\left(\frac{c_1^2+c_3D}{D}\right)\int_\Omega u_{tx}^4\leq\frac{c_\gamma}{4}\int_\Omega u_{txx}^2+c_5\|u_t\|_{L^\infty(\Omega)}^4\quad\text{for all }t\in(0,T_0)\cap(0,T_{\max,\varepsilon}).\tag{3.28}$$

We note that since $\alpha \in (0, 5/6)$, we are always able to choose $\alpha_0 \in (5/6, 1)$ such that $\alpha < \alpha_0$ holds and thus $\int_\Omega (\Theta+1)^{4\alpha} \le |\Omega| + \int_\Omega (\Theta+1)^{4\alpha_0}$. We then infer by another application of a Gagliardo–Nirenberg inequality that there exists some $c_6 = c_6(\Omega, f) > 0$ such that

$$c_4 \|\varphi\|_{L^{4\alpha_0}(\Omega)}^{4\alpha_0} \le c_6 \|\varphi_x\|_{L^2(\Omega)}^{4\alpha_0\lambda} \|\varphi\|_{L^1(\Omega)}^{(1-\lambda)4\alpha_0} + c_6 \|\varphi\|_{L^1(\Omega)}^{4\alpha_0},$$

for all $\varphi \in C^1(\overline{\Omega})$, with $\lambda = \frac{4\alpha_0 - 1}{6\alpha_0}$, which, when applied to $\varphi = \Theta + 1$, leads to

$$\begin{aligned} c_4 \int_\Omega (\Theta+1)^{4\alpha_0} &= c_4 \|\Theta+1\|_{L^{4\alpha_0}(\Omega)}^{4\alpha_0} \\ &\le c_6 \|\Theta_x\|_{L^2(\Omega)}^{4\alpha_0\lambda} \|\Theta+1\|_{L^1(\Omega)}^{(1-\lambda)4\alpha_0} + c_6 \|\Theta+1\|_{L^1(\Omega)}^{4\alpha_0} \end{aligned} \tag{3.29}$$

for all $t \in (0, T_0) \cap (0, T_{\max,\varepsilon})$. Since $\lambda \cdot 2\alpha_0 = (4\alpha_0 - 1)/3 < 1$ we are able to employ Young's inequality once more to obtain

$$\begin{aligned} c_6 \|\Theta_x\|_{L^2(\Omega)}^{4\alpha_0\lambda} \|\Theta+1\|_{L^1(\Omega)}^{(1-\lambda)4\alpha_0} &\le c_6 \int_\Omega \Theta_x^2 + c_6 \left(\int_\Omega \Theta + 1 \right)^{\frac{2\alpha_0+1}{2-2\alpha_0}} \\ &\le c_6 \int_\Omega \Theta_x^2 + c_6 (C_1 + |\Omega|)^{12}, \end{aligned} \tag{3.30}$$

for all $t \in (0, T_0) \cap (0, T_{\max,\varepsilon})$, where we have used the fact that $C_1 > 1$. In view of Lemma 3.1 one can find $c_7 = c_7(a, M, T_0) > 0$ such that

$$c_6 \|\Theta+1\|_{L^1(\Omega)}^{4\alpha_0} \le c_7 \tag{3.31}$$

for all $t \in (0, T_0) \cap (0, T_{\max,\varepsilon})$. We compute by combining (3.22)–(3.31) with Lemma 3.1 and Lemma 3.4

$$\begin{aligned} &\frac{d}{dt} \frac{1}{2} \left\{ \int_\Omega u_{tx}^2 + \int_\Omega \Theta_x^2 + \int_\Omega a(x,t) u_{xx}^2 \right\} + \frac{c_\gamma}{4} \int_\Omega u_{txx}^2 \\ &\le \left(\frac{c_1^2}{D} + c_3 \right) \int_\Omega u_{tx}^4 + \frac{C_{f'}^2}{c_\gamma} \int_\Omega \Theta_x^2 + c_4 \int_\Omega (\Theta+1)^{4\alpha} + \frac{c_2}{2} \int_\Omega u_{xx}^2 + \frac{c_2^2}{c_\gamma} \int_\Omega u_x^2 \\ &\le \frac{c_\gamma}{4} \int_\Omega u_{txx}^2 + c_5 \|u_t\|_{L^\infty(\Omega)}^4 + c_8 \int_\Omega \Theta_x^2 + c_6 (|\Omega| + C_1)^{12} + c_7 + \frac{c_2^2}{2} \int_\Omega a(x,t) u_{xx}^2 + \frac{c_2^2}{c_\gamma} C_1 \\ &\le \frac{c_\gamma}{4} \int_\Omega u_{txx}^2 + c_8 \int_\Omega \Theta_x^2 + c_8 \int_\Omega a(x,t) u_{xx}^2 + c_9 \end{aligned} \tag{3.32}$$

for all $t \in (0, T_0) \cap (0, T_{\max,\varepsilon})$, where $c_8 = c_8(a, T_0, f, \gamma) := \frac{C_{f'}^2}{c_\gamma} + c_6 + \frac{c_2^2}{c_\gamma}$ and

$$c_9 = c_9(D, \gamma, a, f, T_0, \Omega) := c_5 C_4^4 + c_6 (|\Omega| + C_1)^{12} + \frac{c_2^2 C_1}{c_\gamma} + c_7.$$

To prepare a comparison argument, we define

$$y(t) := 1 + \frac{1}{2} \int_\Omega u_{tx}^2(\cdot, t) + \frac{1}{2} \int_\Omega \Theta_x^2(\cdot, t) + \frac{1}{2} \int_\Omega a(\cdot, t) u_{xx}^2(\cdot, t) \tag{3.33}$$

for all $t \in [0, T_{\max})$ and let

$$y(0) = 1 + \frac{1}{2} \int_\Omega u_{0tx}^2 + \frac{1}{2} \int_\Omega \Theta_{0x}^2 + \frac{1}{2} \int_\Omega a(\cdot, 0) u_{0xx}^2 =: c_{10},$$

where (1.13) and (3.22) ensure the finiteness of c_{10}. From (3.32), we may conclude that for $c_{11} = c_{11}(D, \gamma, f, a, T_0, \Omega) := \max\{2c_8, c_9\}$

$$y'(t) \le c_{11} y(t) \quad \text{for all } t \in (0, T_0) \cap (0, T_{\max})$$

and by a simple comparison argument

$$y(t) \le c_{10} \cdot e^{c_{11} t} \quad \text{for all } t \in [0, T_0) \cap [0, T_{\max}).$$

The claim (3.21) thereby readily follows from the nonnegativity of the terms in (3.33). □

With a bound for $\int_\Omega \Theta_x^2$ in hand, the proof of 1.1 is complete.

Proof of Theorem 1.1. In view of (2.3), the proof follows directly from Lemma 3.6. □

Data availability statement. Data sharing is not applicable to this article since no datasets were generated or analyzed during the current study.

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