Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **27**, 1–13; https://doi.org/10.14232/ejqtde.2026.1.27 www.math.u-szeged.hu/ejqtde/

Existence of solutions for a class of nonlocal $p(b(u))\&q(b(u))$ elliptic problems

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Received 5 February 2026, appeared 16 July 2026
Communicated by Gennaro Infante

Abstract. In this work, we consider the following nonlocal quasilinear elliptic problem

$$-M(\mathcal{A}(u))\,\mathrm{div}\left(a(|\nabla u|^{p(b(u))})|\nabla u|^{p(b(u))-2}\nabla u\right) = f(x,u),$$

with a homogeneous Dirichlet boundary condition. Under suitable conditions on the functions M, a, p, b and f, employing an approximation method combined with fixed point arguments, we obtain the existence of a weak solution for the above problem.

Keywords: $p(b(u))\&q(b(u))$ elliptic problems, weak solutions, fixed point theorem.

2020 Mathematics Subject Classification: 35J60, 35J105, 35A15.

1 Introduction

In this work, we study the solvability of the following quasilinear elliptic problem

$$\begin{aligned} -M(\mathcal{A}(u))\,\mathrm{div}\left(a(|\nabla u|^{p(b(u))})|\nabla u|^{p(b(u))-2}\nabla u\right) &= f(x,u) \quad \text{in } \Omega,\\ u &= 0 \qquad\quad\ \ \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 3$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-function, and M, p, b, f are functions satisfying conditions to be stated later.

In 1883, Kirchhoff proposed the following model, extending the classical D'Alembert's wave equation for free vibrations of elastic strings by accounting for the change in length of the string during vibration:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.2}$$

The corresponding generalized stationary version, involving the p-Laplacian operator, is the problem

$$\begin{aligned} -M\left(\int_\Omega |\nabla u|^p\,dx\right)\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= f(x,u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

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where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. This is commonly known as the p-Kirchhoff-type equation; it is a nonlocal problem due to the presence of the term $M\big(\int_\Omega |\nabla u|^p\,dx\big)$. Partial differential equations involving the p-Laplacian arise in many branches of pure mathematics, including the theory of quasiregular and quasiconformal mappings [4], as well as in mathematical physics, biophysics, and the study of chemical reactions. More precisely, the prototype for these models can be written in the form

$$u_t = \operatorname{div}(D(u)\nabla u) + C(x,u), \tag{1.4}$$

where $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$ and a_p, b_q are positive constants. In this framework, the function u generally stands for a concentration, the term $\operatorname{div}(D(u)\nabla u)$ corresponds to the diffusion with diffusion coefficient $D(u)$, and $f(x,u)$ is the reaction term related to source and loss processes, see [8,20].

Nonlocal elliptic problems and their applications to biological and physical systems have been studied extensively. Applications include thermal runaway in Ohmic heating [6,24], collections of particles in thermodynamic equilibrium interacting through gravitational (Coulombic) potentials [19, 23, 28], and the population density of organisms (e.g., bacteria) inside a container [9]. In particular, the study of Kirchhoff-type equations has been extended to the case involving the p-Laplacian [7, 11, 13, 14] and the $p(x)$-Laplacian [3, 12, 17, 25, 27], among others.

The quasilinear equation of type (1.1) was studied in [21, 22] under suitable conditions on the function a and the exponent $p(x,u,b(u)) = p(x)$. In [10], Chipot and de Oliveira studied the case $p(x,u,b(u)) = p(u)$, $M = 1$, $a = 1$. The cases $p(x,u,b(u)) = p(u)$, $M = 1$, $a(t) = 1 + t^{\frac{q-p}{p}}$ and $p(x,u,b(u)) = p(x,|\nabla u|)$, $M = 1$, $a(t) = 1 + t^{\frac{q-p}{p}}$ were studied by Bahrouni et al. [1] and Fadil et al. [15], respectively. In this work, we extend and complement the results of Chipot [10] and Hurtado [21] to the case where the operator is of $p\&q$-Kirchhoff type, with the exponent p depending on a nonlocal quantity $b(u)$ and the external source f depending on u.

The paper is organized as follows. Section 2 collects the necessary background and several technical results. Section 3 is devoted to the proof of the main result.

2 Preliminaries and technical results

Let $1 < s < \infty$ and p' defined by $\frac{1}{s} + \frac{1}{s'} = 1$. The Lebesgue space $L^s(\Omega)$ is endowed with the standard norm

$$\|u\|_s = \left(\int |u(x)|^s\,dx\right)^{\frac{1}{s}}, \quad \text{for all } u \in L^s(\Omega).$$

The Sobolev space $W_0^{1,s}(\Omega)$ is equipped with the usual norm

$$\|u\|_{1,s} = \left(\int |\nabla u(x)|^s\,dx\right)^{\frac{1}{s}}, \quad \text{for all } u \in W_0^{1,s}(\Omega).$$

Let $p, q : \mathbb{R} \to [1, +\infty[$ be the nonlinear exponent functions such that

$$\begin{gathered} p, q \text{ are Lipschitz-continuous functions, } 1 < \alpha < p(\lambda) \le \beta < \infty, \\ N > q(\lambda) > p(\lambda) \ge 2 \text{ for each } \lambda \in \mathbb{R} \text{ and for some constants } \alpha, \beta. \end{gathered} \tag{2.1}$$

We consider a mapping $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$ which is continuous and bounded.

Remark. It is important to observe that $p(b(u))$ and $q(b(u))$ are real numbers (since $b(u) \in \mathbb{R}$ and $p, q : \mathbb{R} \to [1, +\infty[$), and are not variable exponent functions. Consequently, all Sobolev spaces involved in this work are the classical Sobolev spaces with constant exponents. Also, we set

$\gamma(\lambda) = (1 - H(k_3))p(\lambda) + H(k_3)q(\lambda)$, where $H(k) = 1$ if $k > 0$ and $H(k) = 0$ if $k = 0$. Here, H plays the role of a switch: when $k_3 = 0$ (i.e., the operator reduces to the p-Laplacian case), $\gamma = p(\lambda)$; when $k_3 > 0$ (the full $p\&q$ case), $\gamma = q(\lambda)$.

We introduce the functional spaces

$$W_0^{1,p(b(u))}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(b(u))}\, dx < +\infty \right\},$$

and $X = W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,\gamma(b(u))}(\Omega)$ endowed with the norm

$$\begin{aligned} \|u\| &:= \|u\|_{1,p(b(u))} + H(k_3)\|u\|_{1,q(b(u))} \\ &= \|\nabla u\|_{p(b(u))} + H(k_3)\|\nabla u\|_{q(b(u))}. \end{aligned}$$

Observe that, because $q(\lambda) > p(\lambda)$ and $H = 0$ or $H = 1$ we have $\gamma(\lambda) \geq p(\lambda)$ and so

$$X = W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,\gamma(b(u))}(\Omega) = W_0^{1,\gamma(b(u))}(\Omega).$$

Next, we consider the following assumptions.

(M_0) $M : [0, +\infty[\to]m_0, m_1[$ is continuous and increasing function with $m_0, m_1 > 0$.

(a_0) The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^1 and there exist positive constants $k_0, k_1 > 0$, $k_2, k_3 \geq 0$, such that

$$k_0 + H(k_3)k_2 t^{(q-p)/p} \leq a(t) \leq k_1 + k_3 t^{(q-p)/p} \quad \text{for all } \ t \geq 0.$$

The operator $\mathcal{A} : X \to \mathbb{R}$ is given by

$$\mathcal{A}(u) = \frac{1}{p(b(u))} \int_\Omega A\left(|\nabla u|^{p(b(u))}\right) dx, \quad \text{where} \quad A(\xi) = \int_0^\xi a(s)ds.$$

(a_1) The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ given by $g(t) = a(t^p)t^{p-2}$ is increasing.

(F_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x, u)| \leq c_1 + c_2|u|^{\alpha-1} \quad \text{a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}, \ c_1, c_2 > 0,$$

with $1 \leq \alpha < p^*$ where $p^* = Np/(N - p)$, if $N > p$.

Proposition 2.1. *Let us define the functional* $\Phi : X \to \mathbb{R}$ *by*

$$\Phi(u) = \mathcal{M}\left(\mathcal{A}(u)\right)$$

where $\mathcal{M}(t) = \int_0^t M(s)\, ds, \ t \geq 0$. *Then*

i) Φ *is well defined and of the class* C^1, *with Fréchet derivative*

$$\langle \Phi'(u), v \rangle = M\left(\mathcal{A}(u)\right) \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v\, dx, \quad \forall u, v \in X.$$

ii) The mapping $\Phi' : X \to X'$ is bounded, continuous, strictly monotone and it is of type (S_+). Namely $u_\nu \rightharpoonup u$ and $\lim_{\nu\to+\infty}\sup \langle \Phi'(u_\nu), u_\nu - u\rangle \le 0$ then $u_\nu \to u$ in X.

iii) Φ' is a homeomorphism.

Proof. The proof of statement i) is immediate.

ii) Define the mapping $L_a : X \to X'$ by

$$\langle L_a u, v\rangle = \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v\, dx.$$

Then $\Phi'(u) = M(\mathcal{A}(u)) L_a u$. Since $M(\mathcal{A}(.))$ and L_a are continuous and bounded, so is Φ'. We know, from [21, Lemma 6.1], that $L_a = \mathcal{A}'$ is strictly monotone. So $\mathcal{A}$ is strictly convex. Also, because M is increasing, $\mathcal{M}$ is convex in $[0,+\infty[$. Then

$$\mathcal{M}(\mathcal{A}(\gamma u + \delta v)) < \mathcal{M}(\gamma\mathcal{A}(u) + \delta\mathcal{A}(v)) \le \gamma\mathcal{M}(\mathcal{A}(u)) + \delta\mathcal{M}(\mathcal{A}(v))$$

for every $u, v \in X$ with $u \neq v$, and $\gamma, \delta \in]0,1[$ with $\gamma + \delta = 1$. Thus Φ is strictly convex. Therefore, Φ' is strictly monotone.

To prove that Φ' is of type (S_+), first we note that $M(\mathcal{A}(u)) > 0$ for all $u \in X \setminus \{0\}$. Now, let $(u_\nu)_\nu \subset X$ any bounded sequence for which $M(\mathcal{A}(u_\nu)) \to 0$. Thus

$$\|u_\nu\|_X \le \frac{1}{m_0} M(\mathcal{A}(u_\nu))\|u_\nu\|_X \le C_0 M(\mathcal{A}(u_\nu)) \to 0.$$

So, $u_\nu \to 0$ in X. Since L_a is of type (S_+) ([21, Lemma 6.2]), the assertion follows by virtue of [16, Proposition 2.1].

iii) We note that

$$\|\Phi'(u)\|_{X'} = \sup_{v\in X\setminus\{0\}} \frac{|\langle \Phi'(u), v\rangle|}{\|v\|} \ge \frac{|\langle \Phi'(u), u\rangle|}{\|u\|} \ge m_0 a_0 \|u\|^{p-1} \to \infty \quad \text{as } \|u\| \to \infty.$$

Hence by ii), Minty–Browder's Theorem [5, Theorem 5.16] and [2, Lemma 5.2], we conclude that Φ' is a homeomorphism. □

Lemma 2.2. *Let $1 < \alpha < p < q < N$, $p, q \in \mathbb{R}$. Assume that conditions $(M_0), (a_0), (a_1)$ hold. If f is a Carathéodory function satisfying*

$(F_1)'$ $|f(x,s)| \le \hat{a} + \hat{b}|s|^{\alpha-1}$,

for some $\hat{a}, \hat{b} > 0$, then there exists

$$u \in \widehat{X} = W_0^{1,p}(\Omega) \cap W_0^{1,\gamma}(\Omega), \quad \gamma = (1 - H(h_3))\, p + H(h_3)\, q,$$

such that

$$M(\mathcal{A}(u)) \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega f(x,u) v\, dx, \quad \forall v \in \widehat{X}.$$

Proof. The proof is standard, and several approaches could be used; here, we will use a topological method, which we will briefly outline for completeness. We define the operator $T, S : \hat{X} \to \hat{X}'$ by

$$\langle Tu, v\rangle = M(\mathcal{A}(u)) \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx, \tag{2.2}$$

$$\langle Su, v\rangle := \langle N_f(u), v\rangle = \int_\Omega f(x,u)v\,dx. \tag{2.3}$$

Thus, $u \in \widehat{X}$ is a weak solution of (2.2) if and only if $G := (T-S)(u) = 0$ in $\hat{X}'$. It is obvious that G is continuous and bounded. From proposition 2.1ii), T is of type (S+). By using the compact embedding $X \hookrightarrow L^\alpha(\Omega)$ we deduce that the Nemytskii operator $S = N_f$ is compact. Noting that the sum of an (S_+) type mapping and a compact mapping is of type (S_+), it follows that the mapping $G = T - S$ is of type (S+). Also, for $\|u\|$ large enough, we have

$$\langle Gu, u\rangle \geq m_0 a_0 c_\epsilon \|u\|^p - \widehat{c}_1\|u\| - \widehat{c}_2 > \widehat{c}_3\|u\|^p > 0.$$

Then, by the degree theory for (S_+) mappings [26], for $R > 0$ large enough, we have

$$\deg(G, B(0,R), 0) = 1.$$

Therefore, the equation $G(u) = 0$ has at least one solution $u \in B(0,R)$. □

Lemma 2.3. *Assume the conditions of Lemma 2.2 hold. For a given $u \in \widehat{X}$, the following statements are equivalent:*
(i) The function u is a weak solution to the Kirchhoff equation:

$$M(\mathcal{A}(u))\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\cdot\nabla v dx = \int_\Omega f(x,u)v dx, \quad \forall v \in \widehat{X}$$

(ii) The function u satisfies the Minty-type variational inequality:

$$M(\mathcal{A}(v))\int_\Omega a(|\nabla v|^p)|\nabla v|^{p-2}\nabla v\cdot\nabla(v-u)\,dx \geq \int_\Omega f(x,u)(v-u)\,dx, \quad \forall v \in \widehat{X}$$

Proof. We already have the operators $T, S : \widehat{X} \to \widehat{X}'$ given in (2.2), (2.3) respectively.

(i) $\Longrightarrow$ (ii): Assume $u \in \widehat{X}$ satisfies $\langle Tu, v\rangle = \langle Su, v\rangle$ for all $v \in \widehat{X}$. By the monotonicity of T, we have:

$$\langle Tv - Tu, v - u\rangle \geq 0, \quad \forall v \in \widehat{X},$$

which implies

$$\langle Tv, v-u\rangle \geq \langle Tu, v-u\rangle = \langle Su, v-u\rangle = \int_\Omega f(x,u)(v-u)\,dx.$$

Thus, we obtain ii).

(ii) $\Longrightarrow$ (i): Suppose that (ii) holds. Let $w \in \widehat{X}$ be an arbitrary test function and let $\epsilon > 0$. By setting $v = u + \epsilon w$ in the variational inequality, we have

$$\langle T(u+\epsilon w), (u+\epsilon w) - u\rangle \geq \langle Su, (u+\epsilon w) - u\rangle,$$

then

$$\epsilon\langle T(u+\epsilon w), w\rangle \geq \epsilon\langle Su, w\rangle.$$

Dividing by $\epsilon > 0$ and applying the limit as $\epsilon \to 0^+$, we exploit the hemicontinuity of T to get

$$\lim_{\epsilon\to 0^+}\langle T(u+\epsilon w), w\rangle = \langle Tu, w\rangle \geq \langle Su, w\rangle.$$

Since $w \in \widehat{X}$ is arbitrary, we may replace w with $-w$, yielding

$$\langle Tu, -w\rangle \geq \langle Su, -w\rangle \implies \langle Tu, w\rangle \leq \langle Su, w\rangle.$$

Consequently, $\langle Tu, w\rangle = \langle Su, w\rangle$ for all $w \in \widehat{X}$. □

Now, we give a compactness-type result inspired by the work of Zhikov [29], which was later proven elementally by Chipot [10].

Lemma 2.4. *Assume that* $1 < \alpha \le h_\nu(x), g_\nu(x) \le \beta, \forall \nu \in \mathbb{N}$, *a.e.* $x \in \Omega$ *for some constants* α *and* β, *with* $h_\nu, g_\nu(x) \in C_+(\bar{\Omega}) = \{h : h \in C(\bar{\Omega})$ *such that* $h(x) > 1, \forall x \in \bar{\Omega}\}$, $h_\nu \to h$, $g_\nu \to g$ *a.e. in* Ω *as* $\nu \to \infty$

$$\nabla u_\nu \to \nabla u \text{ in } \left(L^1(\Omega)\right)^N \quad \text{as } \nu \to \infty,$$

$$\left\| |\nabla u_\nu|^{h_\nu(x)} \right\|_{L^1(\Omega)}, \left\| |\nabla u_\nu|^{g_\nu(x)} \right\|_{L^1(\Omega)} \le c, \text{ for some constant } c > 0.$$

Then $|\nabla u|^{h(x)}, |\nabla u|^{g(x)} \in L^1(\Omega)$ *and*

$$\liminf_{\nu\to\infty} \int_\Omega |\nabla u_\nu|^{h_\nu(x)}\, dx \ge \int_\Omega |\nabla u|^{h(x)} dx$$

and

$$\liminf_{\nu\to\infty} \int_\Omega |\nabla u_\nu|^{g_\nu(x)}\, dx \ge \int_\Omega |\nabla u|^{g(x)} dx.$$

The following lemma is the key insight to prove our main result.

Lemma 2.5. *For* $\nu \in \mathbb{N}$, *let* u_ν *be the solution to the problem*

$$\begin{aligned} &u_\nu \in X_\nu = W_0^{1,p_\nu}(\Omega) \cap W_0^{1,\gamma_\nu}(\Omega), \\ &M\left(\mathcal{A}(u_\nu)\right) \int_\Omega a\left(|\nabla v_\nu|^{p_\nu}\right) |\nabla u_\nu|^{p_\nu - 2} \nabla u_\nu \nabla v\, dx = \int_\Omega f(x, u_\nu) v\, dx, \end{aligned} \tag{2.4}$$

$\forall v \in X_\nu$.
Suppose that

$$p_\nu \to p \quad \text{and} \quad q_\nu \to q \quad \text{as } \nu \to \infty, \quad \text{where} \quad 1 < \alpha < p < q < N. \tag{2.5}$$

Then

$$u_\nu \to u \quad \text{in } W_0^{1,\alpha}(\Omega) \quad \text{as } \nu \to \infty, \tag{2.6}$$

where u *is the solution to the problem*

$$\begin{aligned} &u \in \widehat{X}, \\ &M\left(\mathcal{A}(u)\right) \int_\Omega a\left(|\nabla u|^p\right) |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_\Omega f(x, u) v\, dx, \end{aligned} \tag{2.7}$$

$\forall v \in \widehat{X}$.

Proof. The proof is done in two parts.

Part 1. Weak convergence: Since $p_\nu \to p$ and $q_\nu \to q$ as $\nu \to \infty$ and $1 < \alpha < p < q < N$, we may assume that

$$\alpha < p_\nu < q_\nu < q + 1. \tag{2.8}$$

First, we shall prove that (u_ν) is bounded in X_ν.

Taking $v = u_\nu$ as test function in (2.4), we get

$$M\left(\mathcal{A}(u_\nu)\right) \int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right) |\nabla u_\nu|^{p_\nu}\, dx = \int_\Omega f(x, u_\nu) u_\nu\, dx. \tag{2.9}$$

By (a_0), (F_1) and Holder's inequality, we derive

$$\begin{aligned} m_0k_0\int_\Omega |\nabla u_\nu|^{p_\nu}\,dx + m_0k_2H(k_3)\int_\Omega |\nabla u_\nu|^{q_\nu}\,dx &\leq \|f(.,u_\nu)\|_{\alpha'}\|u_\nu\|_\alpha \\ &\leq C_4\left(\|u_\nu\|_\alpha + \|u_\nu\|_\alpha^\alpha\right)\right) \leq C_5\left(\|\nabla u_\nu\|_\alpha + \|\nabla u_\nu\|_\alpha^\alpha\right)\right). \end{aligned} \tag{2.10}$$

We now proceed by contradiction. Suppose that, up to a subsequence, $\|u_\nu\|_{X_\nu} \to +\infty$. From (2.8), we have

$$\|\nabla u_\nu\|_\alpha \leq C_6\|u_\nu\|_{X_\nu}. \tag{2.11}$$

If $k_3 = 0$, by (2.10) and (2.11), we get

$$C_7\|u_\nu\|_{X_\nu}^{p_\nu} = C_7\|\nabla u_\nu\|_{p_\nu}^{p_\nu} \leq C_8\left(\|u_\nu\|_{X_\nu} + \|u_\nu\|_{X_\nu}^\alpha\right),$$

which contradicts (2.8).

If $k_3 > 0$, we consider the following cases:

(i) $\|\nabla u_\nu\|_{p_\nu} \to +\infty$ and $\|\nabla u_\nu\|_{q_\nu} \to +\infty$ as $\nu \to +\infty$,

ii) $\|\nabla u_\nu\|_{p_\nu} \to +\infty$ and $(\|\nabla u_\nu\|_{q_\nu})_\nu$ is bounded.

iii) $(\|\nabla u_\nu\|_{p_\nu})_\nu$ is bounded and $\|\nabla u_\nu\|_{q_\nu} \to +\infty$.

In the case i), for ν large enough, $\|\nabla u_\nu\|_{q_\nu}^{q_\nu - p_\nu} \geq 1$, $\|\nabla u_\nu\|_{p_\nu} \geq 1$. Then $\|\nabla u_\nu\|_{q_\nu}^{q_\nu} \geq \|\nabla u_\nu\|_{q_\nu}^{p_\nu}$ and $\|\nabla u_\nu\|_{p_\nu}^\alpha \geq \|\nabla u_\nu\|_{p_\nu}$. Hence, by (2.10) and (2.11), we have

$$\begin{aligned} \left(\|\nabla u_\nu\|_{p_\nu} + \|\nabla u_\nu\|_{q_\nu}\right)^{p_\nu} &\leq C_9\left(\|\nabla u_\nu\|_{p_\nu} + \|\nabla u_\nu\|_{q_\nu}^\alpha\right) \\ &\leq C_{10}\left(\|\nabla u_\nu\|_{p_\nu} + \|\nabla u_\nu\|_{q_\nu}\right)^\alpha, \end{aligned}$$

which is a contradiction.

In the case ii), using (2.10) and (2.11), we obtain

$$m_0k_0\|\nabla u_\nu\|_{p_\nu}^{p_\nu} \leq C_9\left(\|\nabla u_\nu\|_{p_\nu} + \|\nabla u_\nu\|_{q_\nu}^\alpha\right),$$

it yields

$$m_0k_0 \leq \left(\frac{1}{\|\nabla u_\nu\|_{p_\nu}^{p_\nu - 1}} + \frac{\|\nabla u_\nu\|_{q_\nu}^\alpha}{\|\nabla u_\nu\|_{p_\nu}^{p_\nu}}\right).$$

Hence, letting $\nu \to +\infty$, we obtain a contradiction.

Case iii) is identical to case ii), mutatis mutandis, and it will be omitted.

Therefore $(u_\nu)_\nu$ is bounded in X_ν. Then, by (2.11), $(u_\nu)_\nu$ is bounded in $W_0^{1,\alpha}(\Omega)$. Thus, along a relabeled subsequence we have

$$\nabla u_\nu \rightharpoonup \nabla u, \quad \text{in } L^\alpha(\Omega). \tag{2.12}$$

Due to the convergences (2.5) and (2.8), the boundedness of $(u_\nu)_\nu$ in X_ν and (2.12), we deduce from Lemma 2.4, that

$$\liminf_{\nu\to\infty}\int_\Omega |\nabla u_\nu|^{p_\nu}\,dx \geq \int_\Omega |\nabla u|^p dx$$

and

$$\liminf_{\nu\to\infty}\int_\Omega |\nabla u_\nu|^{q_\nu}\,dx \geq \int_\Omega |\nabla u|^q dx.$$

Consequently

$$u \in \widehat{X}. \tag{2.13}$$

Thanks to Lemma 2.3, equation (2.4) is equivalent to

$$M\left(\mathcal{A}(v)\right)\int_\Omega a(|\nabla v|)^{p_\nu}|\nabla v|^{p_\nu-2}\nabla v\cdot\nabla(v-u_\nu)\,dx \geq \int_\Omega f(x,u_\nu)(v-u_\nu)\,dx, \quad \forall v\in\widehat{X}_\nu. \tag{2.14}$$

Choosing $v \in C_0^\infty(\Omega)$, using the convergences (2.5) and (2.12), observing that, from the continuity of the Nemytskii operator we have

$$f(x,u_\nu)\to f(x,u) \quad \text{in } L^{\alpha'}(\Omega), \tag{2.15}$$

and applying the Dominated Convergence Theorem, we can take the limit as $\nu \to +\infty$ in inequality (2.14), to obtain

$$M\left(\mathcal{A}(v)\right)\int_\Omega a(|\nabla v|)^{p}|\nabla v|^{p-2}\nabla v\cdot\nabla(v-u)\,dx \geq \int_\Omega f(x,u)(v-u)\,dx, \quad \forall v\in C_0^\infty(\Omega). \tag{2.16}$$

By density, we can take $v \in \widehat{X}$ in(2.16). Next, we choose $v = u \pm t\varphi$, where $\varphi \in \widehat{X}$, $t > 0$, and letting $t \to 0^+$, after simplifying the resulting inequality, we find

$$M\left(\mathcal{A}(u)\right)\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx = \int_\Omega f(x,u)\varphi\,dx \quad \forall\varphi\in\widehat{X}.$$

Therefore, we conclude that $u \in \widehat{X}$ is the solution to the problem (2.7).

Part 2: Strong convergence: In this part, we will show that the convergence (2.12) is strong. To get this, we choose $v = u_\nu$ in (2.7), taking into account (2.12) and (2.15), then letting $\nu \to \infty$ we find

$$\begin{aligned} M\left(\mathcal{A}(u_\nu)\right)\int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu}\,dx &= \int_\Omega f(x,u_\nu)u_\nu\,dx \\ &\to \int_\Omega f(x,u)u\,dx = M\left(\mathcal{A}(u)\right)\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p\,dx. \end{aligned} \tag{2.17}$$

Now, we consider the following cases:

i) Case $p_\nu \geq p$ $\forall \nu \in \mathbb{N}$: As (u_ν) is bounded in $X_\nu \hookrightarrow W_0^{1,p}(\Omega)$, then $(\mathcal{A}(u_\nu))_\nu$ is bounded in $\mathbb{R}$. So, passing to a subsequence, if necessary, we may assume that $\mathcal{A}(u_\nu) \to t_0$ as $\nu \to \infty$, and, as M is continuous

$$\lim_{\nu\to\infty} M(\mathcal{A}(u_\nu)) \to M(t_0) \geq m_0. \tag{2.18}$$

From (2.12), the boundedness of (u_ν) in $L^\alpha(\Omega)$ and (2.15) we get

$$\begin{aligned} &M\left(\mathcal{A}(u_\nu)\right)\int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx \\ &\quad= \underbrace{\int_\Omega \left(f(x,u_\nu)-f(x,u)\right)(u_\nu-u)\,dx}_{\searrow 0} + \underbrace{M\left(\mathcal{A}(u)\right)\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u(\nabla u_\nu-\nabla u)\,dx}_{\searrow 0}. \end{aligned} \tag{2.19}$$

Therefore, from (2.18) and (2.19), we deduce that

$$\lim_{\nu\to\infty}\int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx = 0. \tag{2.20}$$

Invoking the inequality given in [18, Lemma 2.4], (a_0), (a_1), (2.20), using the fact that

$$|a\left(|\nabla u|^{p_\nu}\right)|\nabla u|^{p_\nu-2}\nabla u| \leq (k_1+k_3)\left(\max\{1,|\nabla u|\}\right)^q \in L^{p'}(\Omega), \tag{2.21}$$

we obtain

$$\begin{aligned} 0 &\leq \frac{a_0}{4}\int_\Omega |\nabla u_\nu - \nabla u|^{p_\nu}\,dx \\ &\leq \int_\Omega \left[a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu-2}\nabla u_\nu - a\left(|\nabla u|^{p_\nu}\right)|\nabla u|^{p_\nu-2}\nabla u\right](\nabla u_\nu - \nabla u)\,dx \qquad (2.22)\\ &\leq \underbrace{\left|\int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx\right|}_{\searrow 0} + \underbrace{\left|\int_\Omega a\left(|\nabla u|^{p_\nu}\right)|\nabla u|^{p_\nu-2}\nabla u(\nabla u_\nu-\nabla u)\,dx\right|}_{\searrow 0}. \end{aligned}$$

Hence

$$\lim_{\nu\to\infty}\int_\Omega |\nabla u_\nu - \nabla u|^{p_\nu}\,dx = 0. \tag{2.23}$$

Relation (2.23) and the Hölder's inequality imply that

$$\lim_{\nu\to\infty}\int_\Omega |\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu - \nabla u)\,dx = 0. \tag{2.24}$$

If $k_3 = 0$, from (2.23) we get $u_\nu \to u$ in X_ν as $\nu \to \infty$. On the other hand, if $k_3 > 0$, we find

$$\begin{aligned} &\int_\Omega a\left(|\nabla u_\nu|^{p_\nu}\right)|\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx \\ &\geq C\left(\underbrace{\int_\Omega |\nabla u_\nu|^{p_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx}_{I_1} + \int_\Omega |\nabla u_\nu|^{q_\nu-2}\nabla u_\nu(\nabla u_\nu-\nabla u)\,dx\right) \\ &\geq C\left(I_1 + \int_\Omega \left(|\nabla u_\nu|^{q_\nu-2}\nabla u_\nu - |\nabla u|^{q_\nu-2}\nabla u\right)(\nabla u_\nu-\nabla u)\,dx + \int_\Omega |\nabla u|^{q_\nu-2}\nabla u(\nabla u_\nu-\nabla u)\,dx\right) \\ &\geq C\left(I_1 + \frac{1}{2^{q_\nu-1}}\int_\Omega |\nabla u_\nu-\nabla u|^{q_\nu} + \int_\Omega |\nabla u|^{q_\nu-2}\nabla u(\nabla u_\nu-\nabla u)\,dx\right). \end{aligned}$$

Hence, by (2.20) and (2.24), passing to the limit in the above relation we arrive at

$$\lim_{\nu\to\infty}\int_\Omega |\nabla u_\nu - \nabla u|^{q_\nu}\,dx = 0. \tag{2.25}$$

Thus, we conclude, by (2.23) and (2.25), that $u_\nu \to u$ in X_ν as $\nu \to \infty$.

Now, by Hölder's inequality

$$0 \leq \int_\Omega |\nabla u_\nu - \nabla u|^p\,dx \leq \left(\int_\Omega |\nabla u_\nu - \nabla u|^{p_\nu}\,dx\right)^{\frac{p}{p_\nu}} |\Omega|^{1-\frac{p}{p_\nu}} \to 0,$$

$$0 \leq \int_\Omega |\nabla u_\nu - \nabla u|^q\,dx \leq \left(\int_\Omega |\nabla u_\nu - \nabla u|^{q_\nu}\,dx\right)^{\frac{q}{q_\nu}} |\Omega|^{1-\frac{q}{q_\nu}} \to 0,$$

then, $u_\nu \to u$ in $\widehat{X}$. Since $\widehat{X} \hookrightarrow W_0^{1,\alpha}(\Omega)$, we conclude that $u_\nu \to u$ in $W_0^{1,\alpha}(\Omega)$.

ii) Case $2 \leq p_\nu \leq p$ $\forall \nu \in \mathbb{N}$: The proof for this case is similar to the previous one and will be omitted here. □

3 Main result

Definition 3.1. We say that $u \in X$ is a weak solution of (1.1) if

$$M(\mathcal{A}(u)) \int_\Omega a\left(|\nabla u|^{p(b(u))}\right) |\nabla u|^{p(b(u))-2} \nabla u \cdot \nabla v\, dx = \int_\Omega f(x,u) v\, dx$$

for all $v \in X$.

Our main result is stated in the following:

Theorem 3.2. *If (M_0), (a_0), (a_1) and (F_1) hold, then problem (1.1) has a weak solution.*

Proof. According to Lemma 2.2, we can find a weak solution $u = u_\lambda$ for each $\lambda \in \mathbb{R}$ to the nonlocal $p(\lambda)\&q(\lambda)$ elliptic problem

$$\left| \begin{array}{c} u \in X_\lambda = W_0^{1,p(\lambda)}(\Omega) \cap W_0^{1,\gamma(\lambda)}(\Omega) \\ M(\mathcal{A}(u)) \displaystyle\int_\Omega a\left(|\nabla u|^{p(\lambda)}\right) |\nabla u|^{p(\lambda)-2} \nabla u \cdot \nabla v\, dx = \int_\Omega f(x,u) v\, dx, \quad \forall v \in X_\lambda. \end{array} \right. \tag{3.1}$$

By (a_0), (F_1) and Hölder's inequality, we derive

$$\begin{aligned} m_0 k_0 \int_\Omega |\nabla u_\lambda|^{p(\lambda)}\, dx + m_0 k_2 H(k_3) \int_\Omega |\nabla u_\lambda|^{q(\lambda)}\, dx &\le \|f(.,u_\lambda)\|_{\alpha'} \|u_\lambda\|_\alpha \\ \le \widehat{C}_4 \left(\|u_\lambda\|_\alpha + \|u_\lambda\|_\alpha^\alpha\right) &\le \widehat{C}_5 \left(\|\nabla u_\lambda\|_\alpha + \|\nabla u_\lambda\|_\alpha^\alpha\right). \end{aligned}$$

Then we proceed by contradiction, repeating the arguments given in Part 1 of Lemma 2.5, to obtain that $(u_\lambda)_\lambda$ is bounded in X_λ, that is there exists $\widehat{C}_6 > 0$ such that

$$\|u_\lambda\|_{X_\lambda} \le \widehat{C}_6. \tag{3.2}$$

This implies, by Hölder's inequality, the continuous embedding $X_\lambda \hookrightarrow W_0^{1,p(\lambda)}(\Omega)$ and (3.2)

$$\begin{aligned} \|\nabla u_\lambda\|_\alpha &\le \|\nabla u_\lambda\|_{p(\lambda)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}} \le \widehat{C}_7 \|u_\lambda\|_{X_\lambda} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}} \\ &\le \widehat{C}_7 \widehat{C}_6 |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}} \le \widehat{C}_7 \widehat{C}_6 \left(\max_{p \in [\alpha,\beta]} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p}} + 1 \right) =: C_9. \end{aligned}$$

The above inequality and the boundedness of b, show that there exists $K \in \mathbb{R}$ such that

$$b(u_\lambda) \in [-K, K].$$

Thus, we can define the fixed point operator $\mathcal{P} : [-K,K] \to [-K,K]$, given by $\mathcal{P}(\lambda) = b(u_\lambda)$. We will prove that $\mathcal{P}$ is continuous. In fact, if $\lambda_\nu \to \lambda$ as $\nu \to \infty$, by the continuity of p, we have $p(\lambda_\nu) \to p(\lambda)$. It follows from Lemma 2.5, considering $p_\nu = p(\lambda_\nu)$, that

$$u_{\lambda_\nu} \to u_\lambda \quad \text{in } W_0^{1,\alpha}(\Omega) \quad \text{as } \nu \to \infty.$$

Since b is continuous, we infer that $b(u_{\lambda_\nu}) \to b(u_\lambda)$ as $\nu \to \infty$. So, $\mathcal{P}$ is continuous. Due to the Brouwer's fixed point theorem there exists $\lambda_0 \in [-K,K]$, verifying $\mathcal{P}(\lambda_0) = \lambda_0$, that is u_{λ_0} is then a weak solution of the problem (1.1). This completes the proof of Theorem 3.2. □

Acknowledgements

The author would like to thank the referee for the careful reading of the manuscript and for the valuable comments and suggestions that improved the presentation of this work. This research was supported by Universidad Nacional Mayor de San Marcos, R. R. N° 0007064-2026-R and Project N° B26142061.

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