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# Existence and multiplicity of weak solutions for a nonlocal Robin system with variable exponents

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**Abstract.** This paper studies a nonlocal system of elliptic equations involving variable-exponent $p_j(x)$-Laplacian operators subject to Robin boundary conditions. Each of the $n$ equations contains a nonlocal source term of the form $\lambda f_j(x,u_j)\big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\big)^{h_j}$, where $\mathbb{D}\subset\mathbb{R}^N$ is a bounded smooth domain, $\lambda>0$ is a parameter, and each $p_j$ is a continuous variable exponent satisfying $1<p_j^-\le p_j(x)\le p_j^+<N$. Working in the framework of variable-exponent Sobolev spaces, we first establish the existence of at least one nontrivial weak solution. We then apply a three-critical-points theorem of Bonanno–Marano type to prove that the energy functional has at least three critical points, hence that the system has at least two nontrivial weak solutions, for a suitable range of the parameter $\lambda$. A key step is the verification of the Palais–Smale condition, obtained under a structural growth condition linking the exponents $\beta_j$, the nonlocal exponents $h_j$ and the lower bounds $p_j^-$; this condition guarantees subcritical growth and compactness of the associated energy functional. An illustrative example set in the unit ball of $\mathbb{R}^3$ confirms the applicability of the abstract results.

**Keywords:** elliptic equations, Robin boundary conditions, variable exponent Sobolev spaces, nonlocal term, multiplicity of solutions.

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## 1 Introduction

Problems involving the $p(x)$-Laplacian operator with Robin boundary conditions arise naturally in the study of differential equations with spatially dependent nonlinearities. The $p(x)$-Laplacian extends the classical $p$-Laplacian by allowing the exponent to vary with position, which makes it particularly suitable for modeling materials and processes exhibiting nonhomogeneous behavior. Robin boundary conditions, combining Dirichlet and Neumann effects,

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provide a flexible framework for describing interactions between the domain and its boundary, and they frequently appear in physical and engineering applications such as heat transfer with convection, diffusion through semi-permeable membranes, and contact mechanics.

The mathematical foundation for variable-exponent Lebesgue and Sobolev spaces was systematically developed in the early 2000s, notably by Fan and Zhao [10], who established essential embedding and compactness properties crucial for variational methods. Around the same time, Růžička [16] demonstrated the relevance of these spaces in modeling electrorheological fluids, sparking broad interest in nonstandard growth problems. For further details on the theoretical background of Lebesgue and Sobolev spaces with variable exponents, the reader is referred to the foundational works [6, 11, 12, 16].

Subsequently, scalar elliptic problems with $p(x)$-Laplacian and Robin boundary conditions were investigated by several authors. Deng [5] studied the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) & \text{in } \mathbb{D}, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} + a(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\mathbb{D}, \end{cases}$$

and established bifurcation-type results for positive solutions using sub-supersolution and variational techniques. Later, Allaoui [1] employed the mountain pass theorem to prove existence under different growth assumptions.

A significant step was taken by Kefi [13], who considered a Robin problem with an indefinite weight $V(x)$ and proved via Ekeland's variational principle, the existence of nontrivial weak solutions under the condition $1 < q(x) < p(x) < N$. This work highlighted the role of sign-changing potentials and laid groundwork for more complex systems.

More recently, attention has turned to systems of equations. In particular, Khaleghi and Safari [15] addressed a Steklov-type system involving the $p(x)$-Laplacian:

$$\begin{cases} -\Delta_{p(x)} u_i = a(x)|u_i|^{p(x)-2}u_i & \text{in } \mathbb{D}, \\ |\nabla u_i|^{p(x)-2}\dfrac{\partial u_i}{\partial \nu} = F_{u_i}(x,u) & \text{on } \partial\mathbb{D}, \end{cases}$$

and proved multiplicity of solutions using a three-critical-points theorem. However, their boundary condition is of Neumann–Steklov type, lacking the reactive Robin term $a_j(x)|u_j|^{p_j(x)-2}u_j$.

Our work builds directly on the Robin framework of Kefi [13], but introduces two major innovations: a nonlocal source term of the form $\lambda f_j(x,u_j)\big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\big)^{h_j}$, which models global feedback (as seen in population dynamics or thermoelasticity), and a system with componentwise variable exponents $p_j(x)$, allowing each equation to obey its own heterogeneous law. In particular, we focus on the analysis of a new family of $p_j(x)$-Laplacian system described as follows:

$$\begin{cases} -\Delta_{p_j(x)} u_j = \lambda f_j(x,u_j)\Big(\displaystyle\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j} & \text{in } \mathbb{D}, \\ |\nabla u_j|^{p_j(x)-2}\dfrac{\partial u_j}{\partial \nu} + a_j(x)|u_j|^{p_j(x)-2}u_j = 0 & \text{on } \partial\mathbb{D}, \end{cases} \qquad j = 1,\dots,n, \tag{1.1}$$

where $u = (u_1, u_2, \dots, u_n)$ is the unknown vector function with components $u_j\colon \mathbb{D} \to \mathbb{R}$, and $\mathbb{D} \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\mathbb{D}$. The $p_j(x)$-Laplace operator is defined as

$$\Delta_{p_j(x)} u_j := \operatorname{div}\big(|\nabla u_j|^{p_j(x)-2}\nabla u_j\big),$$

with $p_j \in C(\overline{\mathbb{D}})$ satisfying

$$1 < p_j^- := \inf_{x\in\overline{\mathbb{D}}} p_j(x) \le \sup_{x\in\overline{\mathbb{D}}} p_j(x) =: p_j^+ < N.$$

Moreover, $h_j$ is a positive constant for each $j$, the function $a_j \in L^\infty(\partial\mathbb{D})$ is such that $\inf_{x\in\partial\mathbb{D}} a_j(x) > 0$, and $\lambda > 0$ is a positive parameter.

For each $j = 1, \dots, n$, the function $f_j\colon \mathbb{D}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

$$F_j(x,t) = \int_0^t f_j(x,s)\,ds, \qquad (x,t) \in \mathbb{D}\times\mathbb{R},$$

so that $F_j(x,\cdot) \in C^1(\mathbb{R})$ and $F_j(x,0) = 0$ for all $x \in \mathbb{D}$. The flux density $f_j$ satisfies the growth conditions:

(H0) there exist constants $\hat{c}_f, m_1, \delta_0 > 0$ (independent of $j$) such that, for every $j = 1, \dots, n$,

(i) $|f_j(x,t)| \le \hat{c}_f |t|^{\beta_j(x)-1}$ for all $(x,t) \in \mathbb{D}\times\mathbb{R}$;

(ii) $f_j(x,t)\,t \ge m_1 |t|^{\alpha_j(x)}$ for all $x \in \mathbb{D}$ and $|t| \le \delta_0$;

(iii) $f_j(x,t)\,t \ge 0$ for all $(x,t) \in \mathbb{D}\times\mathbb{R}$,

where the exponent functions $\alpha_j, \beta_j \in C_+(\overline{\mathbb{D}}) := \{\zeta \in C(\overline{\mathbb{D}}) :\ \zeta(x) > 1\ \forall x \in \overline{\mathbb{D}}\}$ satisfy:

(H) $\beta_j(x) \le \alpha_j(x)$ for all $x \in \overline{\mathbb{D}}$, and $\max\limits_{j=1,\dots,n} \{(h_j+1)\beta_j^+\} < \min\limits_{i=1,\dots,n} \{p_i^-\}$,

where $\beta_j^+ = \sup_{x\in\overline{\mathbb{D}}} \beta_j(x)$.

**Remark 1.1.** (a) Conditions (H0)(i) and (H0)(ii) give $m_1 |t|^{\alpha_j(x)} \le \hat{c}_f |t|^{\beta_j(x)}$ for $|t| \le \delta_0$; letting $t \to 0$ shows that they are compatible only if $\beta_j(x) \le \alpha_j(x)$, which is the first requirement in (H).

(b) By (H0)(iii), $f_j(x,s) \ge 0$ for $s > 0$ and $f_j(x,s) \le 0$ for $s < 0$, hence $F_j(x,t) \ge 0$ for all $(x,t)$. Consequently $\int_{\mathbb{D}} F_j(x,u_j)\,dx \ge 0$, the power $\big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\big)^{h_j}$ is well defined for every real $h_j > 0$, and the second term of the energy functional introduced in Section 3 is nonnegative.

(c) Integrating (H0)(i) and (H0)(ii), we obtain

$$|F_j(x,t)| \le \frac{\hat{c}_f}{\beta_j(x)} |t|^{\beta_j(x)} \quad \text{for all } (x,t), \qquad F_j(x,t) \ge \frac{m_1}{\alpha_j(x)} |t|^{\alpha_j(x)} \quad \text{for } |t| \le \delta_0.$$

Moreover, (H0)(i) gives $f_j(x,0) = 0$, so $u = 0$ is a (trivial) solution of (1.1).

To the best of our knowledge, no existing work combines Robin boundary conditions, variable exponents, nonlocal terms, and multiplicity analysis in a system setting. By employing variational methods in product variable exponent Sobolev spaces and applying critical point theorems of Bonanno–Candito–D'Aguì [4] and Bonanno–Marano [3], we establish both existence and multiplicity of weak solutions under a natural subcritical condition:

$$\max_{j=1,\dots,n} \{(h_j+1)\beta_j^+\} < \min_{i=1,\dots,n} \{p_i^-\}.$$

Thus, our results extend the scalar Robin theory [1, 5, 13] to a new class of nonlocal, heterogeneous systems with rich solution structure.

**Remark 1.2.** Note that the global condition $\max_{j=1,\dots,n}\{(h_j+1)\beta_j^+\} < \min_{i=1,\dots,n}\{p_i^-\}$, implies the local condition for every component $j$, thus, we have $(h_j+1)\beta_j^+ < p_j^-$ for any $j=1,\dots,n$.

The remainder of the paper is organized as follows. In Section 2, we review the functional setting and recall the main compactness results. Section 3 is dedicated to the presentation of the required analytical tools and to the proof of the Palais–Smale condition under assumptions (H0)–(H). Finally, in Section 4, we establish the existence and multiplicity of solutions for system (1.1).

## 2 Functional setting and variational structure

In what follows, for any $r \in C_+(\overline{\mathbb{D}})$ (i.e., $r\colon \overline{\mathbb{D}} \to (1,\infty)$ is continuous), we define for every $s>0$

$$[s]^r := \max\{s^{r^-}, s^{r^+}\}, \qquad [s]_r := \min\{s^{r^-}, s^{r^+}\},$$

where $r^- := \min_{x\in\overline{\mathbb{D}}} r(x)$ and $r^+ := \max_{x\in\overline{\mathbb{D}}} r(x)$. Then the following properties hold for all $s,\alpha>0$:

(i) $[s]^{\frac{1}{r}} = \max\{s^{\frac{1}{r^+}}, s^{\frac{1}{r^-}}\}$,

(ii) $[s]_{\frac{1}{r}} = \min\{s^{\frac{1}{r^+}}, s^{\frac{1}{r^-}}\}$,

(iii) $[s]_{\frac{1}{r}} = b \iff s = [b]^r, \quad [s]^{1/r} = b \iff s = [b]_r$,

(iv) $[s]_r[\alpha]_r \le [s\alpha]_r \le [s\alpha]^r \le [s]^r[\alpha]^r$.

We denote the variable exponent Lebesgue space [9] by

$$L^{p_j(x)}(\mathbb{D}) = \Big\{u\colon \mathbb{D}\to\mathbb{R} :\ u \text{ is measurable and } \int_{\mathbb{D}} |u(x)|^{p_j(x)}\,dx < \infty\Big\}.$$

We define its Luxemburg norm as

$$|u|_{p_j(x)} := \inf\Big\{\mu>0 :\ \int_{\mathbb{D}} \Big|\frac{u(x)}{\mu}\Big|^{p_j(x)} dx \le 1\Big\}.$$

For every $u \in L^{p_j(x)}(\mathbb{D})$ and $v \in L^{p_j'(x)}(\mathbb{D})$, where $L^{p_j'(x)}(\mathbb{D})$ is the conjugate space of $L^{p_j(x)}(\mathbb{D})$, one has the following Hölder type inequality

$$\Big|\int_{\mathbb{D}} uv\,dx\Big| \le \Big(\frac{1}{p_j^-} + \frac{1}{p_j'^-}\Big)|u|_{p_j(x)}|v|_{p_j'(x)}.$$

Here we display some facts which will be used later.

**Proposition 2.1** ([9]). *For every $u \in L^{p_j(x)}(\mathbb{D})$, one has*

$$\big[|u|_{p_j(x)}\big]_{p_j} \le \int_{\mathbb{D}} |u(x)|^{p_j(x)}\,dx \le \big[|u|_{p_j(x)}\big]^{p_j}.$$

**Proposition 2.2** ([10]). *Let $p$ and $q$ be two functions in $C_+(\overline{\mathbb{D}})$ satisfying $q(x)\le p(x)$ a.e. on $\mathbb{D}$. Then $L^{p(x)}(\mathbb{D}) \hookrightarrow L^{q(x)}(\mathbb{D})$ and $|u|_{q(x)} \le c_q|u|_{p(x)}$, where $c_q$ is a positive constant.*

For any function $f$ in the space $L^{p(x)}(\mathbb{D})$ and $g$ in the conjugate space $L^{p'(x)}(\mathbb{D})$, the Hölder-type inequality is satisfied as follow:

**Proposition 2.3** (Hölder type inequality [2, 8]). *Let $p, q, s \geq 1$ be measurable functions defined on $\mathbb{D}$ and*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \text{for a.e. } x \in \mathbb{D}.$$

*If $f \in L^{p(x)}(\mathbb{D})$ and $g \in L^{q(x)}(\mathbb{D})$, then $fg \in L^{s(x)}(\mathbb{D})$ and*

$$\|fg\|_{s(x)} \leq 2\|f\|_{p(x)}\|g\|_{q(x)}. \tag{2.1}$$

*Similarly, if $\frac{1}{s(x)} + \frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for a.e. $x \in \mathbb{D}$, then*

$$\int_{\mathbb{D}} |f(x)g(x)h(x)|\, dx \leq 3\|f\|_{s(x)}\|g\|_{p(x)}\|h\|_{q(x)}. \tag{2.2}$$

For $p \in \{p_j,\ j = 1, \ldots, n\}$, the Sobolev space with variable exponent is denoted by $W^{1,p(x)}(\mathbb{D})$, and defined as

$$W^{1,p(x)}(\mathbb{D}) := \left\{u \in L^{p(x)}(\mathbb{D}) :\ |\nabla u| \in L^{p(x)}(\mathbb{D})\right\}.$$

The norm on the space $W^{1,p(x)}(\mathbb{D})$ is given by $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. Throughout this paper, for any function $u_j \in W^{1,p_j(x)}(\mathbb{D})$, we denote

$$\|u_j\|_{a_j} := \inf\left\{\mu > 0 :\ \int_{\mathbb{D}} \left|\frac{\nabla u_j}{\mu}\right|^{p_j(x)} dx + \int_{\partial\mathbb{D}} a_j(x)\left|\frac{u_j}{\mu}\right|^{p_j(x)} d\sigma \leq 1\right\}.$$

In view of assumptions on the function $a_j(x)$, it is easy to see that $\|u_j\|_{a_j}$ represents a norm on the space $X_j := W^{1,p_j(x)}(\mathbb{D})$. Furthermore, $\|u_j\|_{a_j}$ and $\|u_j\|_{1,p_j(x)}$ are equivalent [5]. In this paper, we will use the norm $\|u_j\|_{a_j}$ and its associated modular $\Gamma^{a_j}_{p_j(x)} : X_j \to \mathbb{R}$ given by

$$\Gamma^{a_j}_{p_j(x)}(u_j) = \int_{\mathbb{D}} |\nabla u_j|^{p_j(x)}\, dx + \int_{\partial\mathbb{D}} a_j(x)|u_j|^{p_j(x)}\, d\sigma.$$

Furthermore, one has:

**Proposition 2.4** ([5, Proposition 2.4]). *For $u_j \in X_j$, we have*

$$\left[\|u_j\|_{a_j}\right]_{p_j} \leq \Gamma^{a_j}_{p_j(x)}(u_j) \leq \left[\|u_j\|_{a_j}\right]^{p_j}.$$

In what follows, we set

$$X := \prod_{j=1}^{n} W^{1,p_j(x)}(\mathbb{D}) := \prod_{j=1}^{n} X_j, \quad \text{where } X_j = W^{1,p_j(x)}(\mathbb{D}),$$

endowed with the norm

$$\|u\| = \|(u_1, \ldots, u_n)\| = \sum_{j=1}^{n} \|u_j\|_{a_j}$$

for $u = (u_1, \ldots, u_n) \in X$. The space $X$ thus defined is a reflexive and separable Banach space. As a consequence of Proposition 2.2, if $q(x) \leq p(x)$ a.e. on $\mathbb{D}$, we have $W^{1,p(x)}(\mathbb{D}) \hookrightarrow W^{1,q(x)}(\mathbb{D})$. Thanks to the strict inequality $\beta_j^+ < p_j^-$ $(j = 1, \ldots, n)$, Proposition 2.2 yields, for each $j$, $W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow W^{1,\beta_j^+}(\mathbb{D})$. Therefore the product space $X$ satisfies the continuous embedding

$$X = \prod_{j=1}^{n} W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow \prod_{j=1}^{n} W^{1,\beta_j^+}(\mathbb{D}).$$

For the sake of clarity, we introduce throughout the paper the following notation for the critical Sobolev exponent associated with the $j$-th direction:

$$p_j^*(x) := \begin{cases} \dfrac{N\,p_j(x)}{N - p_j(x)} & \text{if } p_j(x) < N, \\ +\infty & \text{if } p_j(x) \geq N. \end{cases}$$

**Remark 2.5.** Under assumption (H), we have $\beta_j(x) \leq \beta_j^+ < p_j^- \leq p_j(x) < p_j^*(x)$ for all $x \in \overline{\mathbb{D}}$ and all $j = 1, \ldots, n$. Consequently, by the compact embedding theorem for variable exponent Sobolev spaces [11], for each $j = 1, \ldots, n$ the embedding

$$W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow L^{\beta_j(x)}(\mathbb{D})$$

is continuous and compact.

**Lemma 2.6** ([7]). *Let $\mathbb{D} \subset \mathbb{R}^N$ be a measurable set with $0 < |\mathbb{D}| < +\infty$. Suppose that $g\colon \mathbb{D} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition*

$$|g(x,t)| \leq \theta(x) + c|t|^{p(x)/q(x)} \quad \textit{for almost every } (x,t) \in \mathbb{D} \times \mathbb{R},$$

*where*

- $p(\cdot), q(\cdot) \in C_+(\overline{\mathbb{D}})$ *(i.e., continuous positive functions on the closure $\overline{\mathbb{D}}$),*
- $c > 0$ *is a constant,*
- $\theta \in L^{q(\cdot)}(\mathbb{D})$ *and $\theta(x) > 0$ a.e. in $\mathbb{D}$.*

*Then the Nemytskii operator $N_g$ defined by $N_g(u)(x) = g\big(x, u(x)\big)$, $x \in \mathbb{D}$, maps $L^{p(\cdot)}(\mathbb{D})$ into $L^{q(\cdot)}(\mathbb{D})$ and is continuous and bounded.*

## 3 Analytical tools and study of problem (1.1)

The energy functional associated with the system is the functional $J\colon X \to \mathbb{R}$ $\big($where $X := \prod_{j=1}^n W^{1,p_j(\cdot)}(\mathbb{D})\big)$ defined by:

$$J(u) = \sum_{j=1}^n \int_{\mathbb{D}} \frac{1}{p_j(x)} |\nabla u_j|^{p_j(x)} dx + \sum_{j=1}^n \int_{\partial\mathbb{D}} \frac{a_j(x)}{p_j(x)} |u_j|^{p_j(x)} d\sigma - \lambda \sum_{j=1}^n \frac{1}{h_j+1} \Big( \int_{\mathbb{D}} F_j(x,u_j)\,dx \Big)^{h_j+1}.$$

Then $J(u)$, can be written as $J(u) := A(u) - \lambda B(u)$, where

$$A(u) := \sum_{j=1}^n \int_{\mathbb{D}} \frac{1}{p_j(x)} |\nabla u_j|^{p_j(x)} dx + \sum_{j=1}^n \int_{\partial\mathbb{D}} \frac{a_j(x)}{p_j(x)} |u_j|^{p_j(x)} d\sigma,$$

$$B(u) := \sum_{j=1}^n \frac{1}{h_j+1} \Big( \int_{\mathbb{D}} F_j(x,u_j)\,dx \Big)^{h_j+1}.$$

It is easy to see that $A$ and $B$ are well defined (recall that $B \geq 0$ by Remark 1.1(b)), moreover $J$ is of class $C^1$, and $u \in X \setminus \{0_{\mathbb{R}^n}\}$ is said weak solution of the problem (1.1), if $J'(u)[v] = 0$, for all $v \in X$.

**Definition 3.1.** Let $A$ and $B$ be two continuously Gâteaux differentiable functionals defined on a real Banach space $X$, and let $r \in \mathbb{R}$. The functional $J := A - B$ is said to satisfy the *Palais–Smale condition cut-off upper at level r*, denoted by $(PS)^{[r]}$, if every sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ such that

- $J(u_k)$ is bounded,
- $\lim_{k\to+\infty} \|J'(u_k)\|_{X^*} = 0$,
- $A(u_k) < r$ for all $k \in \mathbb{N}$,

admits a convergent subsequence. If $r = +\infty$, the functional $J := A - B$ is said to satisfy the standard Palais–Smale condition.

Our principal existence result is established through the following theorem:

**Theorem 3.2** ([4, Theorem 3.1])**.** *Let $X$ be a real Banach space, and let $A, B\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals satisfying $\inf_{u\in X} A(u) = A(0) = B(0) = 0$. Assume there exist a constant $r > 0$ and a point $\bar{u} \in X$ such that $0 < A(\bar{u}) < r$ and*

$$\frac{\sup_{u\in A^{-1}(]-\infty,r[)} B(u)}{r} < \frac{B(\bar{u})}{A(\bar{u})}.$$

*Then, for every*

$$\lambda \in \Lambda := \left]\frac{A(\bar{u})}{B(\bar{u})}, \frac{r}{\sup_{u\in A^{-1}(]-\infty,r[)} B(u)}\right[,$$

*if the functional $J_\lambda := A - \lambda B$ satisfies the $(PS)^{[r]}$-condition, then there exists $u_\lambda \in A^{-1}(]0, r[)$ such that $J_\lambda(u_\lambda) \leq J_\lambda(u)$ for all $u \in A^{-1}(]0, r[)$, and moreover $J'_\lambda(u_\lambda) = 0$.*

Essentially, this theorem establishes the existence of critical points for the functional $J_\lambda = A - \lambda B$, provided that the above prescribed hypotheses are satisfied.

The primary tool employed to establish the multiplicity result is a three critical points theorem of Bonanno–Marano type, which builds upon the results presented in [3].

**Theorem 3.3** ([3, Theorem 3.6])**.** *Consider a reflexive real Banach space $X$, and let $A\colon X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semi-continuous functional. Assume that the Gâteaux derivative of $A$ has a continuous inverse on $X^*$. Furthermore, let $B\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that*

$(a_0)$ $\inf_X A = A(0) = B(0) = 0$.

*Suppose that there exist $r > 0$ and $\bar{u} \in X$, with $r < A(\bar{u})$, such that*

$(a_1)$ $\dfrac{\sup_{A(u)<r} B(u)}{r} < \dfrac{B(\bar{u})}{A(\bar{u})}$,

$(a_2)$ *for any $\lambda \in \Lambda_r := \left]\dfrac{A(\bar{u})}{B(\bar{u})}, \dfrac{r}{\sup_{A(u)<r} B(u)}\right[$, $J_\lambda := A - \lambda B$ is coercive.*

*Then, for any $\lambda \in \Lambda_r$, $A - \lambda B$ has at least three distinct critical points in $X$.*

In the sequel, and in order to ensure the conditions required by our theorems, we present some necessary intermediate results.

**Proposition 3.4.** *For each $j_0 \in \{1,\dots,n\}$, we have*

$$\frac{1}{p_{j_0}^+}\left[\|u_{j_0}\|_{a_{j_0}}\right]_{p_{j_0}} \le A(u) \le \frac{1}{p^-}\sum_{j=1}^{n}\left[\|u_j\|_{a_j}\right]^{p_j}.$$

*Proof.* By using, Proposition 2.4, we can deduce that, for each $j_0 \in \{1,\dots,n\}$, one has

$$\begin{aligned}\frac{1}{p_{j_0}^+}\left[\|u_{j_0}\|_{a_{j_0}}\right]_{p_{j_0}} &\le \int_{\mathbb{D}}\frac{1}{p_{j_0}(x)}|\nabla u_{j_0}|^{p_{j_0}(x)}dx + \int_{\partial\mathbb{D}}\frac{a_{j_0}(x)}{p_{j_0}(x)}|u_{j_0}|^{p_{j_0}(x)}d\sigma \le A(u)\\ &= \sum_{j=1}^{n}\int_{\mathbb{D}}\frac{1}{p_j(x)}|\nabla u_j|^{p_j(x)}dx + \sum_{j=1}^{n}\int_{\partial\mathbb{D}}\frac{a_j(x)}{p_j(x)}|u_j|^{p_j(x)}d\sigma\\ &\le \frac{1}{p^-}\sum_{j=1}^{n}\Gamma^{a_j}_{p_j(x)}(u_j) \le \frac{1}{p^-}\sum_{j=1}^{n}\left[\|u_j\|_{a_j}\right]^{p_j},\end{aligned}$$

where $p(x) := \min_{j=1,\dots,n} p_j(x)$. □

**Lemma 3.5.** *(i) $A \in C^1(X,\mathbb{R})$, coercive, convex and sequentially weakly lower semicontinuous. Moreover, $A'$ is a homeomorphism from $X$ to its dual $X^*$.*

*(ii) $B \in C^1(X,\mathbb{R})$ and $B'$ is compact.*

*Proof.* (i) We observe that for any $u = (u_1,\dots,u_n) \in X := \prod_{j=1}^{n} W^{1,p_j(\cdot)}(\mathbb{D})$ with $\|u\| \to \infty$, there exists some index $j_0 \in \{1,\dots,n\}$ such that $\|u_{j_0}\|_{a_{j_0}} \to \infty$. Without loss of generality, we may assume that $\|u_{j_0}\|_{a_{j_0}} \ge 1$ whenever $\|u\| \ge 1$. As a result, by Proposition 3.4 we obtain the following estimate:

$$\begin{aligned}A(u) &\ge \int_{\mathbb{D}}\frac{1}{p_{j_0}(x)}|\nabla u_{j_0}|^{p_{j_0}(x)}dx + \int_{\partial\mathbb{D}}\frac{1}{p_{j_0}(x)}a_{j_0}(x)|u_{j_0}|^{p_{j_0}(x)}d\sigma\\ &\ge \frac{1}{p_{j_0}^+}\Gamma^{a_{j_0}}_{p_{j_0}(x)}(u_{j_0}) \ge \frac{1}{p_{j_0}^+}\|u_{j_0}\|^{p_{j_0}^-}_{a_{j_0}}.\end{aligned} \tag{3.1}$$

Since $p_{j_0}^- > 1$, it follows that the functional $A$ is coercive. Moreover, since it is the sum of sequentially weakly lower semicontinuous functionals, $A$ itself is sequentially weakly lower semicontinuous.

In addition, $A$ is of class $C^1$ on $X$, and for all $v = (v_1, v_2, \dots, v_n) \in X$, we have

$$A'(u)(v) = \sum_{j=1}^{n}\int_{\mathbb{D}}|\nabla u_j|^{p_j(x)-2}\nabla u_j\cdot\nabla v_j\,dx + \sum_{j=1}^{n}\int_{\partial\mathbb{D}}a_j(x)|u_j|^{p_j(x)-2}u_jv_j\,d\sigma.$$

To demonstrate that $A'$ is a homeomorphism, we proceed in two steps:

*Step 1.* We prove that $A'\colon X \to X^*$ is coercive, strictly monotone. To establish the coercivity, it is easy to see by employing similar argument as in inequality (3.1), that, for any $u \in X$ with $\|u\| \ge 1$, we have:

$$\langle A'(u), u\rangle \ge \|u_{j_0}\|^{p_{j_0}^-}_{a_{j_0}}, \tag{3.2}$$

which verifies the coercivity of $A'$. Furthermore, by utilizing the assumption on $a_j$ for $j \in \{1,\dots,n\}$, we can ascertain that $A'$ is strictly monotone. In fact:

$$\begin{aligned}\langle A'(u) - A'(v), u - v\rangle &= \sum_{j=1}^{n}\int_{\mathbb{D}}\left(|\nabla u_j|^{p_j(x)-2}\nabla u_j - |\nabla v_j|^{p_j(x)-2}\nabla v_j\right)\cdot(\nabla u_j - \nabla v_j)\,dx\\ &\quad + \sum_{j=1}^{n}\int_{\partial\mathbb{D}}a_j(x)\left(|u_j|^{p_j(x)-2}u_j - |v_j|^{p_j(x)-2}v_j\right)(u_j - v_j)\,d\sigma > 0.\end{aligned}$$

This asserts that, $A'$ is strictly monotone. Note that this result is a direct consequence of the elementary inequality [17] namely, for arbitrary $p > 1$, one has:

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \geq \begin{cases} c_p|a-b|^p & \text{if } p \geq 2, \\ c_p \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

where $c_p > 0$ depends only on $p$.

*Step 2.* The operator $A'$ is of $(S_+)$-type, in fact, let $u_k \rightharpoonup u$ in $X$, and $\limsup_{k\to\infty}\langle A'(u_k) - A'(u), u_k - u\rangle \leq 0$. The strict monotonicity of $A'$ implies that

$$0 \leq \liminf_{k\to\infty}\langle A'(u_k) - A'(u), u_k - u\rangle \leq \limsup_{k\to\infty}\langle A'(u_k) - A'(u), u_k - u\rangle \leq 0,$$

therefore, $\lim_{k\to\infty}\langle A'(u_k) - A'(u), u_k - u\rangle = 0$. Since $\inf_{x\in\partial\mathbb{D}} a_j(x) > 0$, thus $\limsup_{k\to\infty}\langle I'(u_k) - I'(u), u_k - u\rangle \leq 0$, where $I' \colon X \to X^*$ is defined as

$$I(u) = \sum_{j=1}^{n}\int_{\mathbb{D}} \frac{1}{p_j(x)}|\nabla u_j|^{p_j(x)}dx, \qquad I'(u)(v) = \sum_{j=1}^{n}\int_{\mathbb{D}} |\nabla u_j|^{p_j(x)-2}\nabla u_j \cdot \nabla v_j\, dx.$$

In the sequel, for $j_0 \in \{1,\dots,n\}$ and $u_{j_0} \in W^{1,p_{j_0}(x)}(\mathbb{D})$, put:

$$K'(u_{j_0})(v_{j_0}) = \int_{\mathbb{D}} |\nabla u_{j_0}|^{p_{j_0}(x)-2}\nabla u_{j_0} \cdot \nabla v_{j_0}\, dx, \quad \text{for any } v_{j_0} \in W^{1,p_{j_0}(x)}(\mathbb{D}).$$

It is well known that, $K'(u_{j_0})$, for $j_0 = 1,\dots,n$, is of $(S_+)$-type [9], consequently $u_{j_0,k} \to u_{j_0}$ in $W^{1,p_{j_0}(x)}(\mathbb{D})$. Thus, $(u_{1,k},\dots,u_{n,k}) \to (u_1,\dots,u_n)$ in $X = \prod_{j=1}^{n} W^{1,p_j(x)}(\mathbb{D})$, hence, the operator $A'$ is a mapping of $(S_+)$-type. The rest of the proof is a consequence of the Minty–Browder theorem, thus, by Theorem 26.A of [18], the operator $A$ admits a continuous inverse on $X^*$. For further details, we refer the reader to Lemma 3.5 in [14].

(ii) Recall that

$$B(u) = \sum_{j=1}^{n}\frac{1}{h_j+1}\Big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j+1},$$

then

$$B'(u)(v) = \sum_{j=1}^{n}\Big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j}\int_{\mathbb{D}} f_j(x,u_j)v_j\,dx \quad \text{for any } v = (v_1,\dots,v_n) \in X.$$

(Since $F_j$ depends only on $(x,u_j)$, the derivative of the $j$-th term of $B$ acts only on $v_j$; hence the critical points of $J$ are exactly the weak solutions of (1.1).) Define

$$\psi_j'(u)(v) = \Big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j}\int_{\mathbb{D}} f_j(x,u_j)v_j\,dx.$$

Then $B' = \sum_{j=1}^{n}\psi_j'$. It suffices to show that each $\psi_j'$ is compact. For this purpose, consider $(u_k)_k \subset X$ a sequence that converges weakly to $u$ in $X$, i.e., $u_k \rightharpoonup u$. By Remark 2.5, the embedding $W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow L^{\beta_j(x)}(\mathbb{D})$ is compact. Thus, passing to a subsequence (still denoted $(u_k)_k$), we have

$$u_k \to u \text{ strongly in } \prod_{j=1}^{n} L^{\beta_j(x)}(\mathbb{D}) \quad \text{and} \quad u_k(x) \to u(x) \text{ for almost every } x \in \mathbb{D}.$$

The continuity of $F_j(x,\cdot)$ (recall $F_j(x,t) = \int_0^t f_j(x,s)\,ds$ and $f_j$ is a Carathéodory function) ensures $F_j(x,u_{j,k}) \to F_j(x,u_j)$ for almost every $x \in \mathbb{D}$. Moreover, the growth condition on $f_j$ yields $|F_j(x,t)| \le \frac{\hat{c}_f}{\beta_j(x)}|t|^{\beta_j(x)}$. Hence, by Lemma 2.6 (applied with $q \equiv 1$), the Nemytskii operator $u_j \mapsto F_j(\cdot,u_j)$ is continuous from $L^{\beta_j(x)}(\mathbb{D})$ into $L^1(\mathbb{D})$, and therefore

$$\int_{\mathbb{D}} F_j(x,u_{j,k})\,dx \to \int_{\mathbb{D}} F_j(x,u_j)\,dx \quad \text{as } k \to +\infty. \tag{3.3}$$

The Nemytskii operator $N_{f_j}(u_j)(x) = f_j(x,u_j(x))$, where $f_j$ is a Carathéodory function satisfying the given growth, maps continuously $L^{\beta_j(x)}(\mathbb{D}) \to L^{\frac{\beta_j(x)}{\beta_j(x)-1}}(\mathbb{D})$. Thus, $N_{f_j}(u_{j,k}) \to N_{f_j}(u_j)$ in $L^{\frac{\beta_j(x)}{\beta_j(x)-1}}(\mathbb{D})$. Using the Hölder inequality in variable-exponent spaces and the compact embedding, for any $v \in X$ we estimate

$$\begin{aligned}\Big|\int_{\mathbb{D}} f_j(x,u_{j,k})v_j\,dx - \int_{\mathbb{D}} f_j(x,u_j)v_j\,dx\Big| &\le \int_{\mathbb{D}} |f_j(x,u_{j,k}) - f_j(x,u_j)||v_j|\,dx \\ &\le 2\big|N_{f_j}(u_{j,k}) - N_{f_j}(u_j)\big|_{\frac{\beta_j(x)}{\beta_j(x)-1}} |v_j|_{\beta_j(x)} \\ &\le 2c_j\big\|N_{f_j}(u_{j,k}) - N_{f_j}(u_j)\big\|_{\frac{\beta_j(x)}{\beta_j(x)-1}} \|v_j\|_{a_j},\end{aligned} \tag{3.4}$$

where $c_j$ is the constant of the compact embedding $W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow L^{\beta_j(x)}(\mathbb{D})$, with $1 < \beta_j(x) < p_j^*(x)$. Combining the results from equations (3.3) and (3.4), we conclude that $\psi_j'(u_k) \to \psi_j'(u)$ in $X^*$, establishing that $B'$ is completely continuous and thus compact. □

**Lemma 3.6.** *For every $\lambda > 0$, $J$ satisfies the Palais–Smale condition in $X$.*

*Proof.* Let $\{u_k = (u_{1,k},\dots,u_{n,k})\}_{k\in\mathbb{N}} \subset X$ be a Palais–Smale sequence for $J$, that is,

$$J(u_k) \text{ is bounded in } \mathbb{R} \quad \text{and} \quad \|J'(u_k)\|_{X^*} \to 0 \text{ as } k \to \infty. \tag{3.5}$$

Our goal is to prove that $\{u_k\}$ has a strongly convergent subsequence in $X$. Using hypothesis (H0), the continuous embedding $W^{1,p_j(x)}(\mathbb{D}) \hookrightarrow L^{\beta_j(x)}(\mathbb{D})$, and Proposition 2.1, we have:

$$\begin{aligned}\langle B'(u),u\rangle &= \sum_{j=1}^n \Big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j} \int_{\mathbb{D}} f_j(x,u_j)u_j\,dx \le \sum_{j=1}^n \frac{\hat{c}_f^{\,h_j+1}}{(\beta_j^-)^{h_j}}\Big(\int_{\mathbb{D}} |u_j|^{\beta_j(x)}dx\Big)^{h_j+1} \\ &\le \sum_{j=1}^n \frac{\hat{c}_f^{\,h_j+1}}{(\beta_j^-)^{h_j}}\Big(\big[|u_j|_{\beta_j(x)}\big]^{\beta_j}\Big)^{h_j+1} \le \sum_{j=1}^n \frac{\hat{c}_f^{\,h_j+1}\big([c_j]^{\beta_j}\big)^{h_j+1}}{(\beta_j^-)^{h_j}}\Big(\big[\|u_j\|_{a_j}\big]^{\beta_j}\Big)^{h_j+1}.\end{aligned}$$

Thus, for sufficiently large $k$, by inequality (3.2), we have

$$\langle J'(u_k),u_k\rangle \ge \|u_{j_0,k}\|_{a_{j_0}}^{p_{j_0}^-} - \lambda\sum_{j=1}^n \frac{\hat{c}_f^{\,h_j+1}\big([c_j]^{\beta_j}\big)^{h_j+1}}{(\beta_j^-)^{h_j}}\|u_{j,k}\|_{a_j}^{(h_j+1)\beta_j^+},$$

for each fixed $j_0 = 1,\dots,n$. On the other hand, since $\|J'(u_k)\|_{X^*} \to 0$, we have $|\langle J'(u_k),u_k\rangle| \le \|J'(u_k)\|_{X^*}\|u_k\|_X$. Set $\varepsilon_k := \|J'(u_k)\|_{X^*} \to 0$ and $C_j := \frac{\hat{c}_f^{\,h_j+1}([c_j]^{\beta_j})^{h_j+1}}{(\beta_j^-)^{h_j}}$. Then, for every $j_0 = 1,\dots,n$, we have:

$$\|u_{j_0,k}\|_{a_{j_0}}^{p_{j_0}^-} \le \lambda\sum_{j=1}^n C_j\|u_{j,k}\|_{a_j}^{(h_j+1)\beta_j^+} + \varepsilon_k\|u_k\|_X. \tag{3.6}$$

Suppose, for contradiction, that $\{u_k\}$ is not bounded in $X$. Since $\|u_k\|_X = \sum_{j=1}^n \|u_{j,k}\|_{a_j}$, passing to a subsequence we may fix an index $j^* \in \{1,\dots,n\}$ and set $M_k := \|u_{j^*,k}\|_{a_{j^*}} = \max_{1\le j\le n} \|u_{j,k}\|_{a_j}$, so that $M_k \to \infty$ and $\|u_k\|_X \le nM_k$. Applying (3.6) with $j_0 = j^*$ and using $\|u_{j,k}\|_{a_j} \le M_k$ for every $j$, one has:

$$M_k^{p_{j^*}^-} \le \lambda \sum_{j=1}^n C_j M_k^{(h_j+1)\beta_j^+} + n\varepsilon_k M_k. \tag{3.7}$$

Dividing by $M_k^{p_{j^*}^-} > 0$:

$$1 \le \lambda \sum_{j=1}^n C_j M_k^{(h_j+1)\beta_j^+ - p_{j^*}^-} + n\varepsilon_k M_k^{1-p_{j^*}^-}. \tag{3.8}$$

By assumption (H), $(h_j+1)\beta_j^+ < \min_i p_i^- \le p_{j^*}^-$ for every $j = 1,\dots,n$, so each exponent $(h_j+1)\beta_j^+ - p_{j^*}^-$ is strictly negative; also $p_{j^*}^- > 1$, so $1 - p_{j^*}^- < 0$. Since $M_k \to \infty$ and $\varepsilon_k \to 0$, the right-hand side tends to 0 as $k \to \infty$, contradicting the left-hand side being identically $\ge 1$. Hence the sequence $\{u_k\}$ is bounded in $X$. Since $X$ is reflexive, up to a subsequence, still denoted by $\{u_k\}$, we have $u_k \rightharpoonup u$ weakly in $X$ for some $u \in X$. The nonlinear operator $B' \colon X \to X^*$ is compact, so $B'(u_k) \to B'(u)$ strongly in $X^*$. From the decomposition $J'(u_k) = A'(u_k) - \lambda B'(u_k)$ in $X^*$, we obtain $\langle A'(u_k) - A'(u), u_k - u\rangle \to 0$. As $A' \colon X \to X^*$ is of $(S_+)$ type, one has $u_k \to u$ strongly in $X$. Thus the functional $J$ satisfies the Palais–Smale condition. □

## 4 Proofs of main theorems

Our main existence result is as follows.

**Theorem 4.1.** *Assume that (H0) and (H) are fulfilled, furthermore, suppose that there exist two constants $r > 0$ and $\delta \in (0, \delta_0]$ such that*

$$\sum_{j=1}^n \frac{|a_j|_\infty}{p_j^-} [\delta]^{p_j} |\partial\mathbb{D}| < r \tag{4.1}$$

*and $\xi_\delta < \eta_r$, where*

$$\xi_\delta := \frac{\displaystyle\sum_{j=1}^n \frac{|a_j|_\infty}{p_j^-} [\delta]^{p_j} |\partial\mathbb{D}|}{\displaystyle\sum_{j=1}^n \frac{1}{h_j+1} \Big(\frac{m_1}{\alpha_j^+} |\mathbb{D}|\, [\delta]_{\alpha_j}\Big)^{h_j+1}} \quad \text{and} \quad \eta_r := \frac{r}{\displaystyle\sum_{j=1}^n \frac{\hat{c}_f^{\,h_j+1} \big([c_j]^{\beta_j}\big)^{h_j+1}}{(h_j+1)(\beta_j^-)^{h_j+1}} \Big(\Big[[p_j^+ r]^{\frac{1}{p_j}}\Big]^{\beta_j}\Big)^{h_j+1}}.$$

*Then, for any $\lambda \in ]\xi_\delta, \eta_r[$, system (1.1) has at least one non-trivial weak solution.*

*Proof.* Our goal is to prove the existence of a solution by invoking Theorem 3.2. To do so, we need to confirm that every hypothesis of this theorem holds. First, note that the required regularity conditions on the functions $A$ and $B$ are satisfied. Additionally, we have $\inf_{u\in X} A(u) = A(0) = B(0) = 0$. Furthermore, by Lemma 3.6, the functional $J$ fulfills the $(PS)^{[r]}$ condition for every $r > 0$. We define the vector $V_\delta = (v_\delta, \dots, v_\delta) \in X$ whose components are the constant function $v_\delta(x) = \delta$, $x \in \overline{\mathbb{D}}$. It can be established that:

$$\sum_{j=1}^n \frac{\inf_{x\in\partial\mathbb{D}} a_j(x)}{p_j^+} [\delta]_{p_j} |\partial\mathbb{D}| \le A(V_\delta) = \sum_{j=1}^n \int_{\partial\mathbb{D}} \frac{a_j(x)}{p_j(x)} |v_\delta|^{p_j(x)} d\sigma \le \sum_{j=1}^n \frac{|a_j|_\infty}{p_j^-} [\delta]^{p_j} |\partial\mathbb{D}|,$$

therefore, $A(V_\delta) < r$. Moreover, since $0 < \delta \leq \delta_0$, Remark 1.1(c) gives $F_j(x,\delta) \geq \frac{m_1}{\alpha_j(x)}\delta^{\alpha_j(x)}$. Consequently,

$$B(V_\delta) \geq \sum_{j=1}^{n} \frac{1}{h_j+1}\Big(\int_{\mathbb{D}} \frac{m_1}{\alpha_j^+}\delta^{\alpha_j(x)}dx\Big)^{h_j+1} \geq \sum_{j=1}^{n} \frac{1}{h_j+1}\Big(\frac{m_1}{\alpha_j^+}|\mathbb{D}|\,[\delta]_{\alpha_j}\Big)^{h_j+1}.$$

We can deduce that:

$$\frac{B(V_\delta)}{A(V_\delta)} \geq \frac{\displaystyle\sum_{j=1}^{n} \frac{1}{h_j+1}\Big(\frac{m_1}{\alpha_j^+}|\mathbb{D}|\,[\delta]_{\alpha_j}\Big)^{h_j+1}}{\displaystyle\sum_{j=1}^{n} \frac{|a_j|_\infty}{p_j^-}[\delta]^{p_j}|\partial\mathbb{D}|} = \frac{1}{\xi_\delta}.$$

Furthermore, by Proposition 3.4, for each $j = 1,\dots,n$ and $u = (u_1,\dots,u_n) \in A^{-1}(]-\infty, r[)$, we have $\frac{1}{p_j^+}\big[\|u_j\|_{a_j}\big]_{p_j} \leq A(u) < r$. Therefore,

$$\|u_j\|_{a_j} \leq \big[p_j^+ A(u)\big]^{\frac{1}{p_j}} \leq \big[p_j^+ r\big]^{\frac{1}{p_j}}.$$

Hence, by (H0), we deduce

$$\begin{aligned} B(u) &= \sum_{j=1}^{n} \frac{1}{h_j+1}\Big(\int_{\mathbb{D}} F_j(x,u_j)\,dx\Big)^{h_j+1} \leq \sum_{j=1}^{n} \frac{1}{h_j+1}\Big(\int_{\mathbb{D}} \frac{\hat{c}_f}{\beta_j(x)}|u_j|^{\beta_j(x)}dx\Big)^{h_j+1} \\ &\leq \sum_{j=1}^{n} \frac{\hat{c}_f^{\,h_j+1}}{(h_j+1)(\beta_j^-)^{h_j+1}}\Big(\int_{\mathbb{D}} |u_j|^{\beta_j(x)}dx\Big)^{h_j+1} \leq \sum_{j=1}^{n} \frac{\hat{c}_f^{\,h_j+1}\big([c_j]^{\beta_j}\big)^{h_j+1}}{(h_j+1)(\beta_j^-)^{h_j+1}}\Big(\big[\|u_j\|_{a_j}\big]^{\beta_j}\Big)^{h_j+1}. \end{aligned} \tag{4.2}$$

Finally, one has

$$\sup_{A(u)<r} B(u) \leq \sum_{j=1}^{n} \frac{\hat{c}_f^{\,h_j+1}\big([c_j]^{\beta_j}\big)^{h_j+1}}{(h_j+1)(\beta_j^-)^{h_j+1}}\Big(\Big[\big[p_j^+ r\big]^{\frac{1}{p_j}}\Big]^{\beta_j}\Big)^{h_j+1} = \frac{r}{\eta_r}.$$

Hence $\dfrac{\sup_{A(u)<r} B(u)}{r} \leq \dfrac{1}{\eta_r} < \dfrac{1}{\xi_\delta} \leq \dfrac{B(V_\delta)}{A(V_\delta)}$, and we have confirmed that the conditions of Theorem 3.2 are satisfied with $\bar{u} = V_\delta$. Therefore, for each

$$\lambda \in ]\xi_\delta, \eta_r[ \subseteq \Big]\frac{A(V_\delta)}{B(V_\delta)},\ \frac{r}{\sup_{u\in A^{-1}(]-\infty,r[)} B(u)}\Big[,$$

$J$ admits at least one non-zero critical point (since $u_\lambda \in A^{-1}(]0,r[)$), which corresponds to a weak solution of the system. □

Our multiplicity result is as follows:

**Theorem 4.2.** *Assume that (H0) and (H) are fulfilled, furthermore, suppose that there exist two constants $r > 0$ and $\delta \in (0, \delta_0]$ such that*

$$r < \sum_{j=1}^{n} \frac{\inf_{x\in\partial\mathbb{D}} a_j(x)}{p_j^+}[\delta]_{p_j}|\partial\mathbb{D}| \tag{4.3}$$

*and $\xi_\delta < \eta_r$. Then, for any $\lambda \in ]\xi_\delta, \eta_r[$, the functional $J$ has at least three distinct critical points; consequently, system (1.1) admits at least two nontrivial weak solutions.*

*Proof.* By proceeding in a similar manner to the proof of Theorem 4.1, one can easily verify that $A(V_\delta) > r$, and that $\frac{\sup_{A(u)\le r} B(u)}{r} \le \frac{1}{\eta_r} < \frac{1}{\xi_\delta} \le \frac{B(V_\delta)}{A(V_\delta)}$, thus the condition $(a_1)$ of Theorem 3.3 is satisfied, and it remains to establish condition $(a_2)$. In fact, for $\|u\|$ great enough, by Proposition 3.4, together with inequality (4.2), one has

$$J(u) \ge \frac{1}{p_{j_0}^+}\|u_{j_0}\|_{a_{j_0}}^{p_{j_0}^-} - \lambda \sum_{j=1}^{n} \frac{\hat{c}_f^{\,h_j+1}\big([c_j]^{\beta_j}\big)^{h_j+1}}{(h_j+1)(\beta_j^-)^{h_j+1}} \Big( \big[\|u_j\|_{a_j}\big]^{\beta_j} \Big)^{h_j+1}.$$

Since $\max_{j=1,\dots,n}\{(h_j+1)\beta_j^+\} < \min_{i=1,\dots,n}\{p_i^-\}$, we deduce that $J$ is coercive, thus, all conditions of Theorem 3.3 hold and $J$ has at least three distinct critical points. Since $f_j(x,0) = 0$ by (H0)(i), $u = 0$ is one of them; hence at least two of these critical points are nontrivial weak solutions of system (1.1). □

**Example 4.3.** Assume $n = 1$, $f(x,u) := |u|^{\alpha-2}u$, with $\alpha = \frac{6}{5}$, and consider the following nonlinear problem in the unit ball $\mathbb{D} = B(0,1) \subset \mathbb{R}^3$:

$$\begin{cases} -\Delta_{\frac{5}{2}} u = \lambda\,|u|^{-\frac{4}{5}}u\Big(\displaystyle\int_{B(0,1)} \frac{|u|^{\frac{6}{5}}}{\frac{6}{5}}\,dx\Big) & \text{in } B(0,1), \\ |\nabla u|^{\frac{1}{2}}\dfrac{\partial u}{\partial \nu} + |u|^{\frac{1}{2}}u = 0 & \text{on } \partial B(0,1), \end{cases}$$

where $\lambda > 0$. Here $F(x,u) = \frac{5}{6}|u|^{\frac{6}{5}} \ge 0$, and (H0) holds with $\hat{c}_f = m_1 = 1$, $\alpha = \beta = \frac{6}{5}$ and any $\delta_0 > 0$, since $|f(x,u)| = |u|^{\frac{1}{5}}$ and $f(x,u)u = |u|^{\frac{6}{5}} \ge 0$. The exponents satisfy the condition

$$\text{(H)} \qquad (h+1)\beta < p < 3$$

with the values $p = \frac{5}{2}$, and $h = 1$. A direct computation gives

$$(h+1)\beta = 2\cdot\frac{6}{5} = \frac{12}{5} = 2.4 < 2.5 = \frac{5}{2} < 3,$$

thus, condition (H) is fulfilled. Let $\delta > 0$ and $r > 0$ satisfy

$$r > \frac{8\pi}{5}\,\delta^{\frac{5}{2}} \tag{4.4}$$

(this is condition (4.1)). Define

$$\xi_\delta := \frac{324}{125\pi}\,\delta^{\frac{1}{10}}, \qquad \text{and} \qquad \eta_r := \frac{72\,r^{\frac{1}{25}}}{25\big(\frac{5}{2}\big)^{\frac{24}{25}} c^{\frac{12}{5}}},$$

where $c > 0$ is the best constant of the embedding $W^{1,\frac{5}{2}}(\mathbb{D}) \hookrightarrow L^{\frac{6}{5}}(\mathbb{D})$. If, in addition, $\xi_\delta < \eta_r$ (which always holds for $r$ large enough, since $\eta_r \to +\infty$ as $r \to +\infty$), then, by Theorem 4.1, for every $\lambda \in (\xi_\delta, \eta_r)$ the problem admits at least one nontrivial weak solution in $W^{1,\frac{5}{2}}(B(0,1))$. Moreover, since $\xi_\delta \to 0$ as $\delta \to 0^+$ and $\eta_r \to +\infty$ as $r \to +\infty$, the problem has a nontrivial weak solution for every $\lambda > 0$.

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## References


[1] M. Allaoui, Existence of solutions for a Robin problem involving the $p(x)$-Laplacian, *Appl. Math. E-Notes* **14**(2014), 107–115. MR3272704

[2] S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions. Existence, uniqueness, localization, blow-up*, Atlantis Studies in Differential Equations, Atlantis Press, Paris, 2015. https://doi.org/10.2991/978-94-6239-112-3; MR3328376; Zbl 1410.35001

[3] G. Bonanno, S. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.* **89**(2010), 1–10. https://doi.org/10.1080/00036810903397438; MR2604276; Zbl 1194.58008

[4] G. Bonanno, P. Candito, G. D'Aguì, Variational methods on finite dimensional Banach spaces and discrete problems, *Adv. Nonlinear Stud.* **14**(2014), No. 4, 915–939. https://doi.org/10.1515/ans-2014-0406

[5] S. G. Deng, Positive solutions for Robin problem involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **360**(2009), 548–560. https://doi.org/10.1016/j.jmaa.2009.06.032; MR2561253; Zbl 1181.35099

[6] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8; MR2790542; Zbl 1222.46002

[7] X. Fan, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.* **339**(2008), No. 2, 1395–1412. https://doi.org/10.1016/j.jmaa.2007.08.003; MR2377096; Zbl 1136.46025

[8] X. Fan, X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.* **59**(2004), 173–188. https://doi.org/10.1016/j.na.2004.07.009

[9] Z. L. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **12**(2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5; MR1954585; Zbl 1146.35353

[10] X. L. Fan, D. Zhao, On the generalized Orlicz–Sobolev space $W^{k,p(x)}(\mathbb{D})$, *J. Gansu Educ. College* **12**(1998), No. 1, 1–6.

[11] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\mathbb{D})$ and $W^{m,p(x)}(\mathbb{D})$, *J. Math. Anal. Appl.* **263**(2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617; MR1866056; Zbl 1028.46041

[12] P. Harjulehto, P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, Vol. 2236, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-15100-3; MR3931352; Zbl 1436.46002

[13] K. Kefi, On the Robin problem with indefinite weight in Sobolev spaces with variable exponents, *Z. Anal. Anwend.* **37**(2018), 25–38. https://doi.org/10.4171/ZAA/1600

[14] K. Kefi, J. Liu, M. Hamdani, Existence and multiplicity analysis of a system of nonlinear elliptic equations: theoretical results and applications, *Georgian Math. J.* **33**(2026), No. 3, 527–539. https://doi.org/10.1515/gmj-2025-2085

[15] A. Khaleghi, F. Safari, Multiplicity of weak solutions for the Steklov systems involving the $p(x)$-Laplacian operator, *Filomat* **38**(2024), No. 21, 7541–7549. https://doi.org/10.2298/FIL2421541K

[16] M. Růžička, *Electrorheological fluids: Modeling and mathematical theory*, Springer, Berlin, 2000. https://doi.org/10.1007/BFb0104029; MR1810360; Zbl 0962.76001

[17] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$ (in French) [Regularity of the solution of a nonlinear equation in $\mathbb{R}^N$], in: P. Bénilan, J. Robert (Eds.), *Journées d'analyse non linéaire*, Lecture Notes in Mathematics, Vol. 665, Springer, Berlin, 1978, pp. 205–227. https://doi.org/10.1007/BFb0061807

[18] E. Zeidler, *Nonlinear functional analysis and its applications, II/B. Nonlinear monotone operators*, Springer, Berlin, 1990. https://doi.org/10.1007/978-1-4612-0981-2; MR1033498; Zbl 0684.47029