Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **32**, 1–20; https://doi.org/10.14232/ejqtde.2026.1.32 www.math.u-szeged.hu/ejqtde/

# Multiple normalized solutions for a class of dipolar Gross–Pitaevskii equation with a mass subcritical perturbation

**Yalin Shen**[1], **Yichen Zhang**[2] and **Thin Van Nguyen**[✉3,4]

[1]School of Mathematical Sciences, Jiangsu University, Zhenjiang 212013, P. R. China
[2]School of Mathematical Sciences, Beihang University, Beijing 100191, P. R. China
[3]Thai Nguyen University of Education, Department of Mathematics, Luong Ngoc Quyen street, Phan Dinh Phung Ward, Thai Nguyen, Vietnam
[4]Thang Long Institute of Mathematics and Applied Sciences, Thang Long University, Dinh Cong Ward, Hanoi, Vietnam

Received 24 January 2026, appeared 4 August 2026
Communicated by Maria Alessandra Ragusa

**Abstract.** In this paper, we study the existence of multiple normalized solutions to the following dipolar Gross–Pitaevskii equation with a mass subcritical perturbation

$$\begin{cases} -\frac{1}{2}\Delta u + \mu u + V(\varepsilon x)u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2)u + \lambda_3 |u|^{p-2}u = 0, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2, \end{cases}$$

where $a, \varepsilon > 0$, $2 < p < \frac{10}{3}$, $\mu \in \mathbb{R}$ is the Lagrange multiplier, $\lambda_3 < 0$, $(\lambda_1, \lambda_2) \in \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 < \frac{4\pi}{3}\lambda_2 \le 0 \text{ or } \lambda_1 < -\frac{8\pi}{3}\lambda_2 \le 0\right\}$, $V(x)$ is an external potential, $*$ denotes convolution, $K(x) = \frac{1-3\cos^2\theta(x)}{|x|^3}$ and $\theta(x)$ is the angle between the dipole axis determined by $(0,0,1)$ and the vector $x$. Under suitable assumptions of $V$, we use variational methods to prove that the number of normalized solutions is not less than the number of global minimum points of $V$ if $\varepsilon > 0$ is sufficiently small.

**Keywords:** dipolar Gross–Pitaevskii equation, multiple normalized solutions, variational methods.

**2020 Mathematics Subject Classification:** 35A15, 35J20, 35J60.

## 1 Introduction

In 1995, Cornell et al. achieved the first Bose–Einstein condensate in their experiments on dilute alkali gases. Since this groundbreaking experiment, condensates of other atoms have also been produced. In recent years, dipolar Bose–Einstein condensates, i.e., condensates composed of particles with permanent electric or magnetic dipole moments, have attracted much attention, see [3, 4]. By using the Gross–Pitaevskii approximation to describe dipolar

✉Corresponding author. Email: Thinmath@gmail.com

Bose–Einstein condensates, we obtain the following three-dimensional dimensionless dipolar Gross–Pitaevskii equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V(x)\psi + \lambda_1|\psi|^2\psi + \lambda_2(K*|\psi|^2)\psi + \lambda_3|\psi|^{p-2}\psi, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

where $\psi : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{C}$ is the wave function, $i$ is the imaginary unit, $*$ denotes convolution, $2 < p < 6$, $V(x)$ is an external potential, $\lambda_i (i = 1,2,3)$ describe the strengths of the three nonlinearities, $K(x) = \frac{1-3\cos^2\theta(x)}{|x|^3}$ and $\theta(x)$ is the angle between the dipole axis determined by $(0,0,1)$ and the vector $x$. Moreover, equation (1.1) with $\lambda_3 \neq 0$ models quantum fluctuations in the condensate (the so-called Lee–Huang–Yang correction), see [12]. For more physical background, we refer to [6,7,13] and the references therein.

To our knowledge, rigorous mathematical research on problem (1.1) with initial condition $\psi_0(x) = \psi(0,x) \in H^1(\mathbb{R}^3)$ was first conducted by Carles et al. [9], where $V(x) = \frac{|x|^2}{2}$ and $\lambda_3 = 0$. They proved that for $\lambda_1 \geq \frac{4\pi}{3}\lambda_2 \geq 0$, the problem admits a unique global solution. This case is called the stable regime because the solution does not blow up in finite time. In the unstable regime ($\lambda_1 < \frac{4\pi}{3}\lambda_2$), the solution may blow up in finite time. Later, Antonelli and Sparber [2] studied the stationary equation of problem (1.1), that is,

$$-\frac{1}{2}\Delta u + \mu u + V(x)u + \lambda_1|u|^2u + \lambda_2(K*|u|^2)u + \lambda_3|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

where $\mu \in \mathbb{R}$ and $u(x)$ is a time-independent function satisfying $\psi(x,t) = e^{i\mu t}u(x)$. They proved the existence of solutions to (1.2) for $\mu > 0$, $\lambda_3 = 0$ and $V \equiv 0$ in $\mathbb{R}^3$, under the conditions that either $\lambda_1 < \frac{4\pi}{3}\lambda_2$ and $\lambda_2 > 0$, or $\lambda_1 < -\frac{8\pi}{3}\lambda_2$ and $\lambda_2 < 0$. Based on the terminology introduced in [2,9] and the definition of the dipole kernel $K(x)$, the coordinate plane $(\lambda_1, \lambda_2)$ is divided into the stable regime

$$\begin{cases} \lambda_1 - \frac{4\pi}{3}\lambda_2 \geq 0, & \lambda_2 \geq 0, \quad \text{or} \\ \lambda_1 + \frac{8\pi}{3}\lambda_2 \geq 0, & \lambda_2 \leq 0 \end{cases}$$

and the unstable regime

$$\begin{cases} \lambda_1 - \frac{4\pi}{3}\lambda_2 < 0, & \lambda_2 > 0, \quad \text{or} \\ \lambda_1 + \frac{8\pi}{3}\lambda_2 < 0, & \lambda_2 < 0, \end{cases}$$

which affect the sign (see [10]) of the nonlinear potential energy

$$N(u) := \int_{\mathbb{R}^3} \lambda_1|u|^4 + \lambda_2(K*|u|^2)|u|^2dx. \tag{1.3}$$

In recent years, problem (1.2) with prescribed mass $\int_{\mathbb{R}^3}|u|^2dx = a^2$ and $a > 0$ has been considered in the stable or unstable regimes, see [5,8,14–16,19]. For $V \equiv 0$ in $\mathbb{R}^3$, $\lambda_3 > 0$ and $p \in (4,6]$, Luo and Stylianou [15] proved the nonexistence of the ground state solution in the stable regime. For $V(x) = \frac{|x|^2}{2}$ and $\lambda_3 = 0$, Carles and Hajaiej [8] proved the existence of a unique non-negative global minimizer with Steiner symmetry in the stable regime. For $V(x) = \frac{\gamma|x|^2}{2}$, $\lambda_3 = 0$ and $\gamma \geq 0$, Bellazzini and Jeanjean [5] proved the existence of a ground state solution in the unstable regime, and showed that any ground state solution is orbitally unstable. For $V \equiv 0$ in $\mathbb{R}^3$, Luo and Stylianou proved the existence of ground state solutions and local minimizers in the unstable regime under the following two cases: $\lambda_3 < 0$ and $p = 5$, see [14]; $\lambda_3 > 0$ and $p \in (4, 6]$, see [15]. For $V \equiv 0$ in $\mathbb{R}^3$, $\lambda_3 < 0$ and $2 < p < \frac{10}{3}$, Luo and Yang [16] studied the existence of ground state solutions in the unstable regime. Moreover, they proved that any ground state is a local minimizer and is stable under the Cauchy flow. Very recently, for the unstable regime with $V \equiv 0$ in $\mathbb{R}^3$ and $2 < p < \frac{10}{3}$, Wu and Tang [19] considered the existence of multiple normalized solutions by replacing $\lambda_3$ with $-h(\epsilon x) < 0$, where $\epsilon > 0$ and $h$ satisfies some technical assumptions. To present a clear overview of the above research progress, we summarize these results in the following table:

| regime | $V$ | $\lambda_3$ | $p$ | classifications of solutions | reference |
|---|---|---|---|---|---|
| stable | $V \equiv 0$ | $\lambda_3 > 0$ | $p \in (4, 6]$ | no ground state | [15] |
| stable | $V \not\equiv 0$ | $\lambda_3 = 0$ | — | ground state | [8] |
| unstable | $V \not\equiv 0$ | $\lambda_3 = 0$ | — | ground state | [5] |
| unstable | $V \equiv 0$ | $\lambda_3 < 0$ | $p = 5$ | ground state | [14] |
| unstable | $V \equiv 0$ | $\lambda_3 < 0$ | $p \in (2, \frac{10}{3})$ | local minimizer/ground state | [16] |
| unstable | $V \equiv 0$ | $\lambda_3 > 0$ | $p \in (4, 6]$ | local minimizer | [15] |
| unstable | $V \equiv 0$ | $\lambda_3 < 0$ | $p \in (2, \frac{10}{3})$ | multiple normalized solutions | [19] |

In light of the above discussion, we focus on the existence of multiple normalized solutions of problem (1.2) with prescribed mass when $V \not\equiv 0$ in $\mathbb{R}^3$ and $\lambda_3 \neq 0$, which has not been studied in the existing literature. For this purpose, in present paper, we consider the following dipolar Gross–Pitaevskii equation

$$\begin{cases} -\frac{1}{2}\Delta u + \mu u + V(\varepsilon x)u + \lambda_1|u|^2u + \lambda_2(K * |u|^2)u + \lambda_3|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2, \end{cases} \tag{1.4}$$

where $a, \varepsilon > 0$, $2 < p < \frac{10}{3}$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^2 \times \mathbb{R}^-$ and $\mu \in \mathbb{R}$ denotes the Lagrange multiplier. Moreover, the potential $V$ satisfies the following assumptions:

$(V_1)$ $V \in L^\infty(\mathbb{R}^3)$, $V(x) \geq 0$ for all $x \in \mathbb{R}^3$.

$(V_2)$ $V_\infty = \lim_{|x|\to+\infty} V(x) > V_0 := \min_{x\in\mathbb{R}^3} V(x) = 0$.

$(V_3)$ $V^{-1}(\{0\}) = \{c_1, c_2, \ldots, c_l\}$ with $c_1 = 0$ and $c_j \neq c_s$ if $j \neq s$.

Solutions of (1.4) are the critical points of the energy function $J_\varepsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$ constrained to

$$S(a) := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = a^2 \right\}, \tag{1.5}$$

where

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx + \frac{1}{2}N(u) + \frac{2\lambda_3}{p}\int_{\mathbb{R}^3} |u|^p dx.$$

Here $N(u)$ is defined in (1.3) and $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with norm

$$\|u\|_{H^1(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}, \qquad \forall u \in H^1(\mathbb{R}^3).$$

Clearly, $J_\varepsilon \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. If $u$ is a critical point of $J_\varepsilon|_{S(a)}$, then it follows from the Lagrange multiplier rule that $(u, \mu)$ is a solution of problem (1.4) for some $\mu \in \mathbb{R}$.

We point out that conditions $(V_1)$–$(V_3)$ were introduced by [1,20] to study the following local nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \lambda u + g(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.6}$$

where $a, \varepsilon > 0$, $V$ denotes the potential, $\lambda \in \mathbb{R}$ is the Lagrange multiplier and $g$ is a function. Alves [1] proved that the numbers of normalized solutions is at least the number of global minimum points of $V$ when $\varepsilon$ is small enough and $g$ is a continuous function with $L^2$-subcritical growth. Later, Xu et al. [20] extended the result of [1] to problem (1.6) with mixed nonlinearities, where $g(u) = \beta|u|^{q-2}u + |u|^{p-2}u$ and $2 < q < 2 + \frac{4}{N} < p < \frac{2N}{N-2}$. Inspired by [19,20], our main result is as follows.

**Theorem 1.1.** *Let $(\lambda_1, \lambda_2) \in \mathcal{B} := \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 < \frac{4\pi}{3}\lambda_2 \le 0 \text{ or } \lambda_1 < -\frac{8\pi}{3}\lambda_2 \le 0\right\}$, $\lambda_3 < 0$ and $2 < p < \frac{10}{3}$. Suppose that $(V_1)$–$(V_3)$ hold. Then there exist $\bar{\varepsilon}, V_* > 0$ and $\bar{a} > 0$, such that for $\|V\|_\infty < V_*$, $\varepsilon \in (0, \bar{\varepsilon})$ and $a \in (0, \bar{a}]$, problem (1.4) has at least $l$ couples $(u_j, \mu_j) \in H^1(\mathbb{R}^3) \times \mathbb{R}^+$ of weak solutions satisfying $\int_{\mathbb{R}^3} |u_j|^2 dx = a^2$ and negative energies for $j = 1, 2, \ldots, l$.*

**Remark 1.2.** The key difficulty in studying the multiplicity of normalized solutions to equation (1.4) arises from the nonlocal dipolar interaction term $(K * |u|^2)u$, which complicates both energy estimates and compactness analysis. In this paper, we overcome these difficulties by carrying out a careful analysis of this nonlocal term. Moreover, in contrast to local nonlinear problems, where cutoff functions are employed, our approach simplifies the construction of the energy functional by utilizing only the properties of the set defined in (3.2).

**Remark 1.3.** An example satisfying conditions $(V_1)$–$(V_3)$ is the following function $V : \mathbb{R}^3 \to \mathbb{R}$,

$$V(x) = 1 - \frac{1}{1 + |x - k_1||x - k_2| \cdots |x - k_l|},$$

where $0 = |k_1| < |k_2| < \cdots < |k_l|$.

The paper is organized as follows. In Section 2, we give some preliminary results. Section 3 is devoted to proving the existence of normalized solutions of the autonomous case of problem (1.4). Finally, we prove Theorem 1.1 in Section 4.

## 2 Preliminaries

In this section, we give some preliminary results. For convenience, let $\|\cdot\|_q := \|\cdot\|_{L^q(\mathbb{R}^3)}$ for $q \in [1, +\infty]$. We start with the following Gagliardo–Nirenberg inequality, see [17].

**Lemma 2.1.** *For any $u \in H^1(\mathbb{R}^3)$ and $q \in (2, 6)$, there exists a constant $C_q > 0$ such that*

$$\|u\|_q \le C_q \|\nabla u\|_2^{\gamma_q} \|u\|_2^{1-\gamma_q},$$

*where $\gamma_q := \frac{3(q-2)}{2q}$.*

Next, we give some estimates for the nonlocal term $N(u)$.

**Lemma 2.2** ([19, Lemma 2.1]). *Let $u \in H^1(\mathbb{R}^3)$, then $|N(u)| \le \Lambda \|u\|_4^4$, where $\Lambda := \max\{|\lambda_1 - \frac{4\pi}{3}\lambda_2|, |\lambda_1 + \frac{8\pi}{3}\lambda_2|\}$. Moreover, if $(\lambda_1, \lambda_2) \in \mathcal{B}$, then $N(u) < 0$.*

**Lemma 2.3** ([19, Lemma 2.2])**.** *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then up to a subsequence,*

$$\int_{\mathbb{R}^3} (K * |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} (K * |v_n|^2)|v_n|^2 dx + \int_{\mathbb{R}^3} (K * |u|^2)|u|^2 dx + o_n(1),$$

*where $v_n := u_n - u$ and $o_n(1) \to 0$ as $n \to +\infty$. In addition, one has*

$$N(u_n) = N(v_n) + N(u) + o_n(1).$$

Finally, we introduce a function $g(a,r) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g(a,r) := \frac{1}{2} - \frac{\Lambda}{2} C_4^4 ra - \frac{2|\lambda_3|}{p} C_p^p a^{p(1-\gamma_p)} r^{p\gamma_p - 2}, \tag{2.1}$$

which satisfies the following properties.

**Lemma 2.4.** *Let $2 < p < \frac{10}{3}$ and $\lambda_3 < 0$. Then for any $a > 0$, the function $g(a,r)$ has a unique global maximum point. Moreover, the maximum value satisfies*

$$\max_{r>0} g(a,r) \begin{cases} > 0 & \text{if } a < \bar{a}, \\ = 0 & \text{if } a = \bar{a}, \\ < 0 & \text{if } a > \bar{a}, \end{cases}$$

*where $\bar{a}$ is defined in (2.2).*

*Proof.* In view of $2 < p < \frac{10}{3}$, we have $0 < p\gamma_p < 2$. Combining this with (2.1) yields $g(a,r) \to -\infty$ as $r \to 0^+$ and $g(a,r) \to -\infty$ as $r \to +\infty$ for any $a > 0$. Furthermore, we define $f_a(r) := g(a,r)$, then it follows that

$$f_a'(r) = \frac{\partial(g(a,r))}{\partial r} = -\frac{\Lambda}{2} C_4^4 a - \frac{2|\lambda_3|}{p}(p\gamma_p - 2) C_p^p a^{p(1-\gamma_p)} r^{p\gamma_p - 3}.$$

By letting $f_a'(r) = 0$, one has

$$r_a = \left( \frac{4|\lambda_3|(2 - p\gamma_p) C_p^p a^{(1-\gamma_p)p - 1}}{\Lambda C_4^4 p} \right)^{\frac{1}{3 - p\gamma_p}}.$$

It follows that $r_a$ is the only global maximum point of $g(a,r)$. Then we have

$$\begin{aligned}
\max_{r>0} g(a,r) &= g(a, r_a) \\
&= \frac{1}{2} - \frac{\Lambda}{2} C_4^4 r_a a - \frac{2|\lambda_3|}{p} C_p^p a^{p(1-\gamma_p)} r_a^{p\gamma_p - 2} \\
&= \frac{1}{2} - \frac{\Lambda}{2} C_4^4 a \left( \frac{4|\lambda_3|(2 - p\gamma_p) C_p^p a^{(1-\gamma_p)p-1}}{\Lambda C_4^4 p} \right)^{\frac{1}{3-p\gamma_p}} \\
&\quad - \frac{2|\lambda_3|}{p} C_p^p a^{p(1-\gamma_p)} \left( \frac{4|\lambda_3|(2 - p\gamma_p) C_p^p a^{(1-\gamma_p)p-1}}{\Lambda C_4^4 p} \right)^{\frac{p\gamma_p - 2}{3 - p\gamma_p}} \\
&= \frac{1}{2} - \frac{1}{2} (\Lambda C_4^4)^{\frac{p\gamma_p - 2}{p\gamma_p - 3}} \left( \frac{4|\lambda_3|(2 - p\gamma_p) C_p^p}{p} \right)^{\frac{1}{3 - p\gamma_p}} \cdot a^{\frac{4}{3}} \\
&\quad - 2 \left( \frac{C_p^p |\lambda_3|}{p} \right)^{\frac{1}{3-p\gamma_p}} \left( \frac{4(2 - p\gamma_p)}{\Lambda C_4^4} \right)^{\frac{p\gamma_p - 2}{3 - p\gamma_p}} \cdot a^{\frac{4}{3}} \\
&= \frac{1}{2} - \alpha a^{\frac{4}{3}},
\end{aligned}$$

where $\gamma_p$ is defined in Lemma 2.1 and

$$\alpha := \frac{1}{2}(\Lambda C_4^4)^{\frac{p\gamma_p-2}{p\gamma_p-3}} \left(\frac{4|\lambda_3|(2-p\gamma_p)C_p^p}{p}\right)^{\frac{1}{3-p\gamma_p}} + 2\left(\frac{C_p^p|\lambda_3|}{p}\right)^{\frac{1}{3-p\gamma_p}} \left(\frac{4(2-p\gamma_p)}{\Lambda C_4^4}\right)^{\frac{p\gamma_p-2}{3-p\gamma_p}}.$$

Thus, we can take

$$\bar{a} := \left(\frac{1}{2\alpha}\right)^{\frac{3}{4}} > 0, \tag{2.2}$$

which concludes the proof. □

**Lemma 2.5.** *Let $2 < p < \frac{10}{3}$, $\lambda_3 < 0$. Suppose that $(a_1, r_1) \in (0, \bar{a}) \times (0, +\infty)$ satisfies $g(a_1, r_1) \geq 0$. Then for any $a_2 \in (0, a_1]$, there holds*

$$g(a_2, r_2) \geq 0 \quad \text{if } r_2 \in \left[\frac{a_2}{a_1} r_1, r_1\right]. \tag{2.3}$$

*Proof.* By (2.1), one has $a \mapsto g(a, r)$ is decreasing for $a > 0$. Therefore, we have

$$g(a_2, r_1) \geq g(a_1, r_1) \geq 0. \tag{2.4}$$

Furthermore, in view of $a_2 \in (0, a_1]$, one has

$$\begin{aligned} g\left(a_2, \frac{a_2}{a_1} r_1) - g(a_1, r_1\right) &= \frac{1}{2} - \frac{\Lambda}{2} C_4^4 \frac{a_2^2}{a_1} r_1 - \frac{2|\lambda_3|}{p} C_p^p a_2^{p(1-\gamma_p)} \left(\frac{a_2}{a_1} r_1\right)^{p\gamma_p - 2} \\ &\quad - \left(\frac{1}{2} - \frac{\Lambda}{2} C_4^4 r_1 a_1 - \frac{2|\lambda_3|}{p} C_p^p a_1^{p(1-\gamma_p)} r_1^{p\gamma_p-2}\right) \\ &= \frac{\Lambda}{2} C_4^4 a_1 r_1 \left(1 - \frac{a_2^2}{a_1^2}\right) + \frac{2|\lambda_3|}{p} C_p^p r_1^{p\gamma_p-2} a_1^{(1-\gamma_p)p} \left[1 - \left(\frac{a_2}{a_1}\right)^{p-2}\right] \\ &\geq 0. \end{aligned} \tag{2.5}$$

Then it follows from Lemma 2.4 and (2.4)–(2.5) that (2.3) holds, which concludes the proof. □

## 3 Existence result for the autonomous problem

In this section, we establish the existence of normalized solutions of the autonomous case of problem (1.4), that is,

$$\begin{cases} -\frac{1}{2}\Delta u + \mu u + \omega u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^{p-2} u = 0, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2, \end{cases} \tag{3.1}$$

where $a > 0$, $\omega \geq 0$, $2 < p < \frac{10}{3}$, $(\lambda_1, \lambda_2) \in \mathcal{B}$, $\lambda_3 < 0$ and $\mu \in \mathbb{R}$ denotes the Lagrange multiplier. Solutions of problem (3.1) are the critical points of the corresponding energy functional $J_\omega : H^1(\mathbb{R}^3) \to \mathbb{R}$ constrained to (1.5), where

$$J_\omega(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \omega |u|^2 dx + \frac{1}{2} N(u) + \frac{2\lambda_3}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Our main result in this section is as follows.

**Theorem 3.1.** *Let $(\lambda_1, \lambda_2) \in \mathcal{B}$, $\lambda_3 < 0$ and $2 < p < \frac{10}{3}$. Then there exist $V_* > 0$, $\bar{a} > 0$ and $\bar{r} > 0$ such that for any $\omega < V_*$ and $a \in (0, \bar{a})$, the functional $J_\omega$ has an interior local minimizer on*

$$W_{\bar{r}}^a := \{u \in S(a) : \|\nabla u\|_2 < \bar{r}\}. \tag{3.2}$$

*Moreover, equation (3.1) has a positive normalized solution $u_a$ with $J_\omega(u_a) < 0$ and the corresponding Lagrange multiplier $\mu_a > 0$.*

To prove the above theorem, we divide the proof into the following lemmas.

**Lemma 3.2.** *For any $u \in S(a)$, there holds*

$$J_\omega(u) \geq \|\nabla u\|_2^2 g(a, \|\nabla u\|_2).$$

*Proof.* For any $u \in S(a)$, it follows from (2.1), Lemmas 2.1 and 2.2 that

$$\begin{aligned}
J_\omega(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \omega |u|^2 dx + \frac{1}{2}N(u) + \frac{2\lambda_3}{p}\int_{\mathbb{R}^3} |u|^p dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\Lambda}{2}\|u\|_4^4 + \frac{2\lambda_3}{p}\int_{\mathbb{R}^3} |u|^p dx \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\Lambda}{2}C_4^4 \|\nabla u\|_2^3 a - \frac{2|\lambda_3|}{p} C_p^p a^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p} \\
&= \|\nabla u\|_2^2 \left[\frac{1}{2} - \frac{\Lambda}{2}C_4^4 \|\nabla u\|_2 a - \frac{2|\lambda_3|}{p} C_p^p a^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p - 2}\right] \\
&= \|\nabla u\|_2^2 g(a, \|\nabla u\|_2).
\end{aligned}$$

This completes the proof. ☐

In the following, we define

$$W_{\bar{r}}^a := \{u \in S(a) : \|\nabla u\|_2 < \bar{r}\},$$

where

$$\bar{r} := r_{\bar{a}} = \left(\frac{4|\lambda_3|(2 - p\gamma_p)C_p^p \bar{a}^{(1-\gamma_p)p-1}}{\Lambda C_4^4 p}\right)^{\frac{1}{3-p\gamma_p}} > 0. \tag{3.3}$$

Obviously, it follows from Lemmas 2.4 and 3.2 that $J_\omega$ is bounded from below in $W_{\bar{r}}^a$ for any $a \in (0, \bar{a})$, where $\bar{a}$ is given by (2.2). Then one can define the local minimization problem by

$$\Upsilon_\omega(a) := \inf_{u \in W_{\bar{r}}^a} J_\omega(u) \quad \text{for any } a \in (0, \bar{a}).$$

Moreover, $\Upsilon_\omega(a)$ satisfies the following properties.

**Lemma 3.3.** *Let $2 < p < \frac{10}{3}$ and $\lambda_3 < 0$. For any $a \in (0, \bar{a})$, there exists $V_* > 0$ such that*

$$\Upsilon_\omega(a) < 0 < \inf_{u \in \partial W_{\bar{r}}^a} J_\omega(u), \quad \text{if } \omega < V_*.$$

*Proof.* Given $u_0 \in W_{\bar{r}}^a \cap L^\infty(\mathbb{R}^N)$ a nonnegative function, we set $u_t(x) = t^{\frac{3}{2}} u_0(tx)$ for all $t > 0$. Then we have

$$\int_{\mathbb{R}^3} |u_t|^2 dx = a^2, \quad \int_{\mathbb{R}^3} |\nabla u_t|^2 dx = t^2 \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^3} |u_t|^p dx = t^{\gamma_p p} \int_{\mathbb{R}^3} |u_0|^p dx.$$

Combining these with Lemmas 2.1 and 2.2, one has

$$\begin{aligned} J_\omega(u_t) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_t|^2 dx + \int_{\mathbb{R}^3} \omega |u_t|^2 dx + \frac{1}{2}N(u_t) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u_t|^p dx \\ &< \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx + \omega a^2 - \frac{2|\lambda_3|}{p} t^{p\gamma_p} \int_{\mathbb{R}^3} |u_0|^p dx \\ &= t^{p\gamma_p}\left[\frac{t^{2-p\gamma_p}}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u_0|^p dx\right] + \omega a^2. \end{aligned}$$

Then by $0 < p\gamma_p < 2$ and $u_0 \in W_{\bar{r}}^a$, there exists $t_0 > 0$ small enough such that

$$t_0^{p\gamma_p}\left[\frac{t_0^{2-p\gamma_p}}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u_0|^p dx\right] =: \kappa < 0.$$

Furthermore, we can fix a constant $V_* := V_*(a) > 0$ such that

$$V_* \bar{a}^2 + \kappa < 0.$$

Then based on the above facts, we have if $\omega < V_*$,

$$J_\omega(u_{t_0}) < 0.$$

It follows that $\Upsilon_\omega(a) < 0$ if $\omega < V_*$. On the other hand, if $u \in \partial W_{\bar{r}}^a$, one has $u \in S(a)$ and $\|\nabla u\|_2 = \bar{r}$. Together these with Lemmas 2.4 and 3.2 yield

$$J_\omega(u) \geq \|\nabla u\|_2^2 g(a, \|\nabla u\|_2) = \bar{r}^2 g(a, \bar{r}) > 0 \quad \text{for any } a \in (0, \bar{a}),$$

which implies $\inf_{u \in \partial W_{\bar{r}}^a} J_\omega(u) > 0$ for any $a \in (0, \bar{a})$. We complete the proof. □

From now on, we fix $\omega < V_*$, which is defined in Lemma 3.3.

**Lemma 3.4.** *$\Upsilon_\omega(a)$ is continuous with regard to $a \in (0, \bar{a})$.*

*Proof.* For any $a \in (0, \bar{a})$, let $\{a_n\} \subset (0, \bar{a})$ satisfy $a_n \to a$ as $n \to +\infty$. Then by the definition of $\Upsilon_\omega(a_n)$ and Lemma 3.3, there exists $\{u_n\} \subset W_{\bar{r}}^{a_n}$ satisfying for all $n \in \mathbb{N}^+$,

$$J_\omega(u_n) - \frac{1}{n} < \Upsilon_\omega(a_n) \quad \text{and} \quad J_\omega(u_n) < 0. \tag{3.4}$$

We claim $\Upsilon_\omega(a_n) \to \Upsilon_\omega(a)$ as $n \to +\infty$. Once this claim is true, then it follows from the definition of continuity that $\Upsilon_\omega(a)$ is continuous with regard to $a \in (0, \bar{a})$. Next, we use three steps to prove the above claim.

**Step 1.** Let $w_n := \frac{a}{a_n} u_n$, then $\{w_n\} \subset W_{\bar{r}}^a$.

Indeed, it is clear that $w_n \in S(a)$. Next, we need to verify $\|\nabla w_n\|_2 < \bar{r}$. When $a \leq a_n$, we can easily obtain $\|\nabla w_n\|_2 < \bar{r}$. On the other hand, when $a_n < a < \bar{a}$. By Lemma 2.4 and (3.3), one has $g(a, \bar{r}) > 0$ for all $a \in (0, \bar{a})$. Then it follows from Lemma 2.5 that

$$g(a_n, r) \geq 0 \quad \text{if } r \in \left[\frac{a_n}{a}\bar{r}, \bar{r}\right]. \tag{3.5}$$

Furthermore, combining (3.4) and Lemma 3.2, one has $0 > J_\omega(u_n) \geq \|\nabla u_n\|_2^2 g(a, \|\nabla u_n\|_2)$. This, together with (3.5), gives $\|\nabla u_n\|_2 < \frac{a_n}{a}\bar{r}$. Therefore, $\|\nabla w_n\|_2 = \frac{a}{a_n}\|\nabla u_n\|_2 < \bar{r}$.

**Step 2.** There holds $\Upsilon_\omega(a) \le \Upsilon_\omega(a_n) + o_n(1)$.

By Step 1, $\lim_{n\to\infty} a_n = a$, $\{u_n\} \subset W_{\bar{r}}^{a_n}$, Lemmas 2.1 and 3.3, we have

$$\begin{aligned}
\Upsilon_\omega(a) \le J_\omega(w_n) &= J_\omega(u_n) + (J_\omega(w_n) - J_\omega(u_n)) \\
&= J_\omega(u_n) + \frac{1}{2}\left[\left(\frac{a}{a_n}\right)^2 - 1\right]\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \left[\left(\frac{a}{a_n}\right)^2 - 1\right]\int_{\mathbb{R}^3} \omega |u_n|^2 dx \\
&\quad + \frac{1}{2}\left[\left(\frac{a}{a_n}\right)^4 - 1\right] N(u_n) \\
&\quad - \frac{2|\lambda_3|}{p}\left[\left(\frac{a}{a_n}\right)^p - 1\right]\int_{\mathbb{R}^3} |u_n|^p dx \longrightarrow J_\omega(u_n) + o_n(1), \quad \text{as } n \to \infty.
\end{aligned}$$

This, together with (3.4), yields

$$\Upsilon_\omega(a) \le J_\omega(w_n) = J_\omega(u_n) + o_n(1) \le \Upsilon_\omega(a_n) + o_n(1).$$

**Step 3.** There holds $\Upsilon_\omega(a) \ge \Upsilon_\omega(a_n) + o_n(1)$.

Indeed, let $\zeta_n := \frac{a_n}{a} z_n$ and $\{u_n\} \subset W_{\bar{r}}^{a_n}$ satisfying for all $n \in \mathbb{N}^+$,

$$J_\omega(z_n) - \frac{1}{n} < \Upsilon_\omega(a) \quad \text{and} \quad J_\omega(z_n) < 0.$$

Then as the same proof in Steps 1–2, we can obtain

$$\Upsilon_\omega(a_n) \le J_\omega(\zeta_n) = J_\omega(z_n) + (J_\omega(\zeta_n) - J_\omega(z_n)) \le \Upsilon_\omega(a) + o_n(1).$$

This completes the proof. □

**Lemma 3.5.** *Let $0 < a_1 < a_2 < \bar{a}$. There holds*

$$\Upsilon_\omega(a_2) \le \Upsilon_\omega(a_1) + \Upsilon_\omega\left(\sqrt{a_2^2 - a_1^2}\right).$$

*Moreover, if $\Upsilon_\omega(a_1)$ or $\Upsilon_\omega(\sqrt{a_2^2 - a_1^2})$ is attained, then the above inequality is strict.*

*Proof.* Let $0 < a_1 < a_2 < \bar{a}$ and $v_n = \theta u_n$, where $1 < \theta < \frac{a_2}{a_1}$. Then by the same proof as in (3.4) and Lemma 3.4-Step 1, there exists $\{u_n\} \subset W_{\bar{r}}^{a_1}$ satisfying for all $n \in \mathbb{N}^+$,

$$J_\omega(u_n) < \Upsilon_\omega(a_1) + \frac{1}{n} \quad \text{and} \quad J_\omega(u_n) < 0$$

and $v_n \in W_{\bar{r}}^{\theta a_1}$. Combining these with Lemma 2.2 and $2 < p < \frac{10}{3}$, we deduce that

$$\begin{aligned}
\Upsilon_\omega(\theta a_1) \le J_\omega(v_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} \omega |v_n|^2 dx + \frac{1}{2} N(v_n) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |v_n|^p dx \\
&= \frac{\theta^2}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \theta^2 \int_{\mathbb{R}^3} \omega |u_n|^2 dx + \frac{\theta^4}{2} N(u_n) - \frac{2|\lambda_3|\theta^p}{p}\int_{\mathbb{R}^3} |u_n|^p dx \\
&= \theta^2 J_\omega(u_n) + \frac{\theta^4 - \theta^2}{2} N(u_n) + \frac{2|\lambda_3|(\theta^2 - \theta^p)}{p}\int_{\mathbb{R}^3} |u_n|^p dx \\
&< \theta^2 J_\omega(u_n) \\
&\le \theta^2 \Upsilon_\omega(a_1) + \theta^2 \frac{1}{n},
\end{aligned} \tag{3.6}$$

which implies that

$$\Upsilon_\omega(\theta a_1) \le \theta^2 \Upsilon_\omega(a_1), \tag{3.7}$$

as $n \to \infty$. Therefore, we have

$$\begin{aligned}\Upsilon_\omega(a_2) &= \frac{a_1^2}{a_2^2}\Upsilon_\omega\left(\frac{a_2}{a_1}a_1\right) + \frac{a_2^2-a_1^2}{a_2^2}\Upsilon_\omega\left(\frac{a_2}{\sqrt{a_2^2-a_1^2}}\sqrt{a_2^2-a_1^2}\right)\\ &\le \Upsilon_\omega(a_1) + \Upsilon_\omega\left(\sqrt{a_2^2-a_1^2}\right).\end{aligned}$$

If $\Upsilon_\omega(a_1)$ is attained, it follows from (3.6) that the inequality is strict. The proof is finished. □

**Lemma 3.6.** *Let $\{u_n\} \subset W_{\bar r}^a$ be a minimizing sequence with respect to $J_\omega$. Then, for some subsequence, either*

*(i) $\{u_n\}$ is strongly convergent, or*

*(ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \to +\infty$ such that the sequence $v_n(x) = u_n(x + y_n)$ is strongly convergent to a function $v \in W_{\bar r}^a$ with $J_\omega(v) = \Upsilon_\omega(a)$.*

*Proof.* Let $\{u_n\} \subset W_{\bar r}^a$ be a minimizing sequence such that

$$J_\omega(u_n) = \Upsilon_\omega(a) + o_n(1).$$

It follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore, there exists $u \in H^1(\mathbb{R}^3)$ such that, up to subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

Case 1. $u \not\equiv 0$ in $\mathbb{R}^3$. Let $v_n = u_n - u$, it follows from the Brezis–Lieb lemma [18] and Lemma 2.3 that

$$\|u_n\|_2^2 = \|v_n\|_2^2 + \|u\|_2^2 + o_n(1), \qquad \|\nabla u_n\|_2^2 = \|\nabla v_n\|_2^2 + \|\nabla u\|_2^2 + o_n(1), \tag{3.8}$$

$$\|u_n\|_p^p = \|v_n\|_p^p + \|u\|_p^p + o_n(1), \qquad N(u_n) = N(v_n) + N(u) + o_n(1). \tag{3.9}$$

Then we can see $\|\nabla u\|_2 < \bar r$ and $\|u\|_2 \le a$. Next, we claim $\|u\|_2 = a$. Suppose by contradiction that $\|u\|_2 = b < a$, it follows that $\|v_n\|_2 = \sqrt{a^2 - b^2}$. Then by (3.8), (3.9) and Lemma 3.5, one has

$$\begin{aligned}\Upsilon_\omega(a) + o_n(1) = J_\omega(u_n) &= J_\omega(v_n) + J_\omega(u) + o_n(1)\\ &\ge \Upsilon_\omega(\sqrt{a^2-b^2}) + \Upsilon_\omega(b) + o_n(1)\\ &> \Upsilon_\omega(a) + o_n(1),\end{aligned}$$

which is a contradiction. Thus, $\|u\|_2 = a$. It follows that $v_n \to 0$ in $L^2(\mathbb{R}^3)$ and $u \in W_{\bar r}^a$. Combining this with Fatou's lemma gives

$$\begin{aligned}\Upsilon_\omega(a) &= \lim_{n\to+\infty} J_\omega(u_n)\\ &= \lim_{n\to+\infty}\left(\frac12\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \int_{\mathbb{R}^3}\omega|u_n|^2dx + \frac12 N(u_n) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3}|u_n|^p dx\right)\\ &\ge J_\omega(u) \ge \Upsilon_\omega(a),\end{aligned}$$

which implies that $J_\omega(u) = \Upsilon_\omega(a)$ and $\lim_{n\to+\infty} J_\omega(u_n) = J_\omega(u)$. Therefore, we have $u_n \to u$ in $H^1(\mathbb{R}^3)$.

Case 2. $u \equiv 0$ in $\mathbb{R}^3$. We claim there are $\eta, R > 0$ and $\{y_n\} \in \mathbb{R}^3$ such that for all $n$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y_n)} |u_n|^2 dx \geq \eta > 0, \tag{3.10}$$

where $B_R(y_n) := \{x \in \mathbb{R}^3 : |x - y_n| < R\}$. Once this claim is true, then one has $|y_n| \to +\infty$. Otherwise, there exists $R_n > 0$ such that

$$\int_{B_{R_n}(0)} |u_n|^2 dx > \int_{B_R(y_n)} |u_n|^2 dx \geq \eta > 0.$$

However, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, and consequently, $u_n \to 0$ in $L^2(B_{R_n}(0))$, which is a contradiction. In addition, we let $v_n := u_n(x + y_n)$, it follows that $\|v_n\|_2 = \|u_n\|_2$, $\|\nabla v_n\|_2 = \|\nabla u_n\|_2$ and $J_\omega(v_n) = J_\omega(u_n)$. Therefore, there exists $v \in H^1(\mathbb{R}^3)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Combining this with (3.10), we have

$$\eta \leq \|u_n\|_{L^2(B_R(y_n))} = \|v_n\|_{L^2(B_R(y_n))} \leq \|v_n - v\|_{L^2(B_R(y_n))} + \|v\|_{L^2(B_R(y_n))},$$

which implies $v \not\equiv 0$ in $\mathbb{R}^3$. Then by the same argument as in Case 1, we have $v_n \to v$ in $H^1(\mathbb{R}^3)$.

Next, we prove the above claim. Assume by contradiction that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y_n)} |u_n|^2 dx = 0.$$

Combining this with [18, Lemma 1.21], we have $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $2 < p < 6$. This, together with Lemma 2.2, gives

$$\begin{aligned} J_\omega(u_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \omega |u_n|^2 dx + \frac{1}{2} N(u_n) - \frac{2|\lambda_3|}{p} \int_{\mathbb{R}^3} |u_n|^p dx \\ &\geq \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\Lambda}{2}\|u_n\|_4^4 + o_n(1) \\ &= \frac{1}{2}\|\nabla u_n\|_2^2 + o_n(1), \end{aligned}$$

which contradicts to $\Upsilon_\omega(a) < 0$. We complete the proof. □

*Proof of Theorem 3.1.* By Lemmas 3.3 and 3.6, there exists $u_a \in W_{\tilde{r}}^a \subset S(a)$ such that

$$J_\omega(u_a) = \inf_{u\in W_{\tilde{r}}^a} J_\omega(u) = \Upsilon_\omega(a) < 0 \text{ and } J_\omega|_{S(a)}'(u) = 0.$$

Note that $\|\nabla |u_a|\|_2 \leq \|\nabla u_a\|_2$. Then, $u_a$ is a nonnegative solution. As in the proof of [16, Theorem 1.1], the strong maximum principle implies that $u_a > 0$ in $\mathbb{R}^3$. Furthermore, by Lemmas 2.2 and 3.3, there exists $\mu_a \in \mathbb{R}$ such that

$$\begin{aligned} -\mu_a a^2 &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_a|^2 dx + \int_{\mathbb{R}^3} \omega |u_a|^2 dx + N(u_a) - |\lambda_3| \int_{\mathbb{R}^3} |u_a|^p dx \\ &= \Upsilon_\omega(a) + \frac{1}{2} N(u_a) - \frac{(p-2)|\lambda_3|}{p} \int_{\mathbb{R}^3} |u_a|^p dx \\ &< 0. \end{aligned}$$

Thus, $\mu_a > 0$. The proof is finished. □

**Corollary 3.7.** *Let $0 \le \omega_1 < \omega_2 \le V_*$, then $\Upsilon_{\omega_1}(a) < \Upsilon_{\omega_2}(a) < 0$.*

*Proof.* Let $u_{\mu_2} \in W_r^a$ such that $J_{\omega_2}(u_{\mu_2}) = \Upsilon_{\omega_2}(a)$. Combining this with Lemma 3.3, we have

$$\Upsilon_{\omega_1}(a) \le J_{\omega_1}(u_{\mu_2}) < J_{\omega_2}(u_{\mu_2}) = \Upsilon_{\omega_2}(a) < 0,$$

which completes the proof. □

## 4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We start with the following definitions. Let

$$J_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} V_\infty |u|^2 dx + \frac{1}{2}N(u) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u|^p dx$$

and

$$J_0(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}N(u) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u|^p dx, \quad \text{for all } u \in H^1(\mathbb{R}^3).$$

In what follows, we assume that $\|V\|_\infty < V_*$, where $V_*$ is given by Lemma 3.3. Combining this with Theorem 3.1, we can define the following minimization problems

$$\Upsilon_0(a) := \inf_{u \in W_r^a} J_0(u), \quad \Upsilon_\infty(a) := \inf_{u \in W_r^a} J_\infty(u), \quad \Upsilon_\varepsilon(a) := \inf_{u \in W_r^a} J_\varepsilon(u) \quad \text{for any } a \in (0, \bar{a}).$$

**Lemma 4.1.** $\limsup_{\varepsilon \to 0^+} \Upsilon_\varepsilon(a) \le \Upsilon_0(a)$ *and there is $\varepsilon_0 > 0$ such that $\Upsilon_\varepsilon(a) < \Upsilon_\infty(a)$ for any $\varepsilon \in (0, \varepsilon_0)$.*

*Proof.* By Theorem 3.1, there is $u_0 \in W_r^a$ such that $J_0(u_0) = \Upsilon_0(a)$. Then, we have

$$\Upsilon_\varepsilon(a) \le J_\varepsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|u_0|^2 dx + \frac{1}{2}N(u_0) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u_0|^p dx.$$

Letting $\varepsilon \to 0^+$, it follows from the Lebesgue dominated convergence theorem that

$$\limsup_{\varepsilon \to 0^+} \Upsilon_\varepsilon(a) \le \limsup_{\varepsilon \to 0^+} J_\varepsilon(u_0) = J_0(u_0) = \Upsilon_0(a).$$

On the other hand, combining Corollary 3.7 and $V_0 = 0 < V_\infty$, we have $\Upsilon_0(a) < \Upsilon_\infty(a) < 0$. Therefore, there is $\varepsilon_0 > 0$ such that $\Upsilon_\varepsilon(a) < \Upsilon_\infty(a)$ for any $\varepsilon \in (0, \varepsilon_0)$. We complete the proof. □

**Lemma 4.2.** *Suppose that $(V_1)$–$(V_3)$ hold. Let $\{u_n\} \subset W_r^a$ satisfy $J_\varepsilon(u_n) \to c < \Upsilon_\infty(a)$ as $n \to \infty$. If $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, then $u_\varepsilon \ne 0$.*

*Proof.* Suppose by contradiction that $u_\varepsilon = 0$. In view of $(V_1)$–$(V_3)$, for any $\delta > 0$, there exists $R > 0$ such that

$$V(x) \ge V_\infty - \delta \quad \text{for all } |x| \ge R.$$

Then, one has

$$\begin{aligned} c + o_n(1) = J_\varepsilon(u_n) &= J_\infty(u_n) + \int_{\mathbb{R}^3} (V(\varepsilon x) - V_\infty)|u_n|^2 dx \\ &\ge J_\infty(u_n) + \int_{B_{R/\varepsilon}(0)} (V(\varepsilon x) - V_\infty)|u_n|^2 dx - \delta \int_{B^c_{R/\varepsilon}(0)} |u_n|^2 dx. \end{aligned}$$

Due to $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \to 0$ in $L^2(B_{R/\varepsilon}(0))$. It follows that

$$c + o_n(1) \geq J_\infty(u_n) - \delta C + o_n(1) \geq \Upsilon_\infty(a) - \delta C + o_n(1), \tag{4.1}$$

for some $C > 0$. Since $\delta > 0$ is arbitrary, we have $c \geq \Upsilon_\infty(a)$, which is a contradiction. Therefore, $u_\varepsilon \neq 0$. This completes the proof. □

**Lemma 4.3.** *Assume that $(V_1)$–$(V_3)$ hold. For $a \in (0, \bar{a})$, let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\varepsilon$ restricts to $W_{\tilde{r}}^a$ with $c < \Upsilon_\infty(a) < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$. If $u_n \not\to u_\varepsilon$ in $H^1(\mathbb{R}^3)$, then decreasing $\varepsilon_0$ if necessary, there exist $\tau > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that*

$$\liminf_{n\to+\infty} \|u_n - u_\varepsilon\|_2^2 \geq \tau.$$

*Proof.* Let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\varepsilon$ restricts to $W_{\tilde{r}}^a$, then

$$J_\varepsilon(u_n) \to c \quad \text{and} \quad \|J_\varepsilon|'_{W_{\tilde{r}}^a}(u_n)\|_{H^{-1}(\mathbb{R}^3)} \to 0 \quad \text{as } n \to +\infty.$$

We define the functional $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx.$$

It follows from [18, Lemma 5.11 and Proposition 5.12] that there exists $\{\mu_n\} \subset \mathbb{R}$ satisfying

$$\|J'_\varepsilon(u_n) + \mu_n \Phi'(u_n)\|_{H^{-1}(\mathbb{R}^3)} \to 0 \quad \text{as } n \to +\infty. \tag{4.2}$$

In view of $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, one has $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Combining this with (4.2), we have $\{\mu_n\}$ is bounded. Then there is $\mu_\varepsilon \in \mathbb{R}$ such that $\mu_n \to \mu_\varepsilon$ as $n \to \infty$. Thus,

$$J'_\varepsilon(u_\varepsilon) + \mu_\varepsilon \Phi'(u_\varepsilon) = 0 \quad \text{in } H^{-1}(\mathbb{R}^3) \quad \text{and} \quad \|J'_\varepsilon(v_n) + \mu_\varepsilon \Phi'(v_n)\|_{H^{-1}(\mathbb{R}^3)} \to 0 \quad \text{as } n \to +\infty,$$

where $v_n = u_n - u_\varepsilon$. It follows that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 dx + N(v_n) - |\lambda_3| \int_{\mathbb{R}^3} |v_n|^p dx + \mu_\varepsilon \int_{\mathbb{R}^3} |v_n|^2 dx = o_n(1). \tag{4.3}$$

Furthermore, it follows from (4.2) and $N(u_n) < 0$ that

$$\begin{aligned} 0 > \Upsilon_\infty(a) > c &= \lim_{n\to+\infty} J_\varepsilon(u_n) \\ &= \lim_{n\to+\infty} [J_\varepsilon(u_n) - \langle J'_\varepsilon(u_n), u_n\rangle - \mu_n a^2 + o_n(1)] \\ &= \lim_{n\to+\infty} \left[-\frac{1}{2}N(u_n) + \left(1 - \frac{2}{p}\right)|\lambda_3| \int_{\mathbb{R}^3} |u_n|^p dx - \mu_n a^2 + o_n(1)\right] \\ &\geq -\mu_\varepsilon a^2, \end{aligned}$$

which implies that

$$\mu_\varepsilon \geq -\frac{\Upsilon_\infty(a)}{a^2} \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Combining this with $(V_1)$ and (4.3), one has for all $\varepsilon \in (0, \varepsilon_1)$,

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + N(v_n) - |\lambda_3| \int_{\mathbb{R}^3} |v_n|^p dx - \frac{\Upsilon_\infty(a)}{a^2}\int_{\mathbb{R}^3} |v_n|^2 dx \leq o_n(1).$$

This, together with Lemma 2.2, gives

$$\min\left\{\frac{1}{2}, -\frac{\Upsilon_\infty(a)}{a^2}\right\}\|v_n\|^2_{H^1(\mathbb{R}^3)} \leq -N(v_n) + |\lambda_3|\int_{\mathbb{R}^3}|v_n|^p dx + o_n(1)$$
$$\leq \Lambda\|v_n\|_4^4 + |\lambda_3|\|v_n\|_p^p + o_n(1).$$

By $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, then there exists $\iota > 0$ independent of $\varepsilon$ such that $\|v_n\|_{H^1(\mathbb{R}^3)} \geq \iota$. Therefore, we have

$$\liminf_{n\to+\infty}(\Lambda\|v_n\|_4^4 + |\lambda_3|\|v_n\|_p^p) \geq \min\left\{\frac{1}{2}, -\frac{\Upsilon_\infty(a)}{a^2}\right\}\iota^2.$$

Combining this with Lemma 2.1 and Sobolev embedding inequality theorem, there exists $\tau > 0$ independent of $\varepsilon \in (0, \varepsilon_1)$ such that

$$\liminf_{n\to+\infty}\|u_n - u_\varepsilon\|_2^2 \geq \tau.$$

The proof is finished. □

With the above lemmas in hand, we can prove the following compactness result.

**Lemma 4.4.** *Assume that $(V_1)$–$(V_3)$ hold. For each $a \in (0, \bar{a})$ and $\varepsilon \in (0, \varepsilon_0)$, the functional $J_\varepsilon$ satisfies the $(PS)_c$ condition restricted to $W_{\bar{r}}^a$ for $c < \Upsilon_0(a) + \xi$, where*

$$0 < \xi \leq \min\left\{\Upsilon_\infty(a) - \Upsilon_0(a), \frac{\tau^2}{a}(\Upsilon_\infty(a) - \Upsilon_0(a))\right\}.$$

*Proof.* Let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\varepsilon$ restricted to $W_{\bar{r}}^a$ with $c < \Upsilon_0(a) + \xi$. Note that $c < \Upsilon_\infty(a) < 0$. Obviously, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows that there exists $u_\varepsilon \in H^1(\mathbb{R}^3)$ satisfying $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$. Combining this with Lemma 4.2 and $c < \Upsilon_\infty(a)$, we have $u_\varepsilon \neq 0$. Let $v_n := u_n - u_\varepsilon$. We claim $v_n \to 0$ in $H^1(\mathbb{R}^3)$. Once this claim is true, we complete the proof. Next, we prove the above claim. Suppose by contradiction that $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$. Then it follows from Lemma 4.3 that there exists $\tau > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\liminf_{n\to+\infty}\|u_n - u_\varepsilon\|_2^2 \geq \tau.$$

We set $\|u_\varepsilon\|_2 = b \in (0, a)$, $\|v_n\|_2 = d_n \in (0, a)$ and assume that $d_n \to d > 0$ as $n \to \infty$. Thus, $d^2 \geq \tau$ and $a^2 = b^2 + d^2$. Moreover, we have $\|\nabla v_n\|_2 \leq \|\nabla u_n\|_2 < \bar{r}$ and $\|\nabla u_\varepsilon\|_2 \leq \liminf_{n\to+\infty}\|\nabla u_n\|_2 < \bar{r}$. It follows that $v_n \in W_{\bar{r}}^{d_n}$ and $u_\varepsilon \in W_{\bar{r}}^b$. Then as the same proof in (4.1), one has for any $\delta > 0$,

$$\begin{aligned} c + o_n(1) = J_\varepsilon(u_n) &= J_\varepsilon(v_n) + J_\varepsilon(u_\varepsilon) + o_n(1) \\ &\geq J_\infty(v_n) - \delta C + J_0(u_\varepsilon) + o_n(1) \\ &\geq \Upsilon_\infty(d_n) - \delta C + \Upsilon_0(b) + o_n(1), \end{aligned}$$

where $C > 0$ independent of $\delta$, $\varepsilon$, $n$. Combining this with (3.7), we have

$$c + o_n(1) \geq \frac{d_n^2}{a^2}\Upsilon_\infty(a) + \frac{b^2}{a^2}\Upsilon_0(a) - \delta C + o_n(1).$$

Letting $n \to \infty$, it follows from the arbitrariness of $\delta$ that

$$
\begin{aligned}
c &\geq \frac{d^2}{a^2}\Upsilon_\infty(a) + \frac{b^2}{a^2}\Upsilon_0(a) \\
&= \Upsilon_0(a) + \frac{d^2}{a^2}[\Upsilon_\infty(a) - \Upsilon_0(a)] \\
&\geq \Upsilon_0(a) + \frac{\tau}{a^2}[\Upsilon_\infty(a) - \Upsilon_0(a)],
\end{aligned}
$$

contradicting to $c < \Upsilon_0 + \xi$. Therefore, $u_n \to u_\varepsilon$ in $H^1(\mathbb{R}^3)$. We complete the proof. □

In what follows, we fix $\xi_0$, $\rho_0$ such that:

- $\overline{B_{\xi_0}(c_i)} \cap \overline{B_{\xi_0}(c_j)} = \emptyset$ for $i \neq j$ with $i, j \in \{1, 2, \ldots, l\}$ and $c_i$, $c_j$ are defined in $(V_3)$;
- $\bigcup_{i=1}^{l} B_{\xi_0}(c_i) \subset B_{\rho_0}(0)$;
- $K_{\frac{\xi_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\xi_0}{2}}(c_i)}$.

Let us define the function $Q_\varepsilon : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ by

$$
Q_\varepsilon(u) := \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)|u^2|dx}{\int_{\mathbb{R}^3} |u^2|dx},
$$

where $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$
\chi(x) := \begin{cases} x, & \text{if } |x| \leq \rho_0, \\ \rho_0 \frac{x}{|x|}, & \text{if } |x| > \rho_0. \end{cases}
$$

**Lemma 4.5.** *Assume that $(V_1)$–$(V_3)$ hold. For $a \in (0, \bar{a})$, there is $0 < \varepsilon_1 < \varepsilon_0$ and $0 < \xi_1 < \xi$ such that if $u \in W_{\bar{r}}^a$ and $J_\varepsilon(u) \leq \Upsilon_0(a) + \xi_1$, then*

$$
Q_\varepsilon(u) \in K_{\frac{\xi_0}{2}}, \quad \textit{for all } \varepsilon \in (0, \varepsilon_1).
$$

*Proof.* Suppose by contradiction that there is $\xi_n \to 0$, $\varepsilon_n \to 0$ and $\{u_n\} \subset W_{\bar{r}}^a$ such that

$$
J_{\varepsilon_n}(u_n) \leq \Upsilon_0(a) + \xi_n \quad \text{and} \quad Q_{\varepsilon_n}(u_n) \notin K_{\frac{\xi_0}{2}}.
$$

This, together with $(V_1)$, gives

$$
\Upsilon_0(a) \leq J_0(u_n) \leq J_\varepsilon(u_n) \leq \Upsilon_0(a) + \xi_n,
$$

which implies that

$$
J_0(u_n) \to \Upsilon_0(a) \quad \text{as } n \to +\infty.
$$

Therefore, $\{u_n\} \subset W_{\bar{r}}^a$ is a minimizing sequence with respect to $\Upsilon_0(a)$. Then it follows from Lemma 3.6 that one has two cases:

(i) $u_n \to u$ in $H^1(\mathbb{R}^3)$ for some $u \in W_{\bar{r}}^a$, or

(ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \to +\infty$ such that the sequence $v_n(x) = u_n(x + y_n)$ is strongly convergent to a function $v \in W_{\bar{r}}^a$.

If case (i) holds. Then by the Lebesgue dominated convergence theorem, we have

$$Q_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon_n x)|u_n^2|dx}{\int_{\mathbb{R}^3} |u_n^2|dx} \to \frac{\int_{\mathbb{R}^3} \chi(0)|u^2|dx}{\int_{\mathbb{R}^3} |u^2|dx} = 0 \in K_{\frac{\xi_0}{2}} \quad \text{as } n \to +\infty,$$

which contradicting to $Q_{\varepsilon_n}(u_n) \notin K_{\frac{\xi_0}{2}}$.

If case (ii) holds. When $|\varepsilon_n y_n| \to +\infty$, then it follows from $v_n \to v$ in $H^1(\mathbb{R}^3)$ that

$$\begin{aligned} J_{\varepsilon_n}(u_n) = J_{\varepsilon_n}(v_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n x + \varepsilon_n y_n)|v_n|^2 dx + \frac{1}{2}N(v_n) \\ &\quad - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |v_n|^p dx \to J_\infty(v), \quad \text{as } n \to +\infty. \end{aligned} \tag{4.4}$$

Combining this with $J_{\varepsilon_n}(u_n) \le \Upsilon_0(a) + \xi_n$ and $v \in W_r^a$, one has

$$\Upsilon_\infty(a) \le J_\infty(v) \le \Upsilon_0(a),$$

which contradicts Corollary 3.7. On the other hand, when $|\varepsilon_n y_n| \to y_0$ for some $y_0$ in $\mathbb{R}^3$. Then similar to (4.4), one has $J_\varepsilon(u_n) \to J_{y_0}(v)$ as $n \to +\infty$. This combining with $J_{\varepsilon_n}(u_n) \le \Upsilon_0(a) + \xi_n$ and $v \in W_r^a$, we have

$$\Upsilon_{y_0}(a) \le J_{y_0}(v) \le \Upsilon_0(a).$$

This, together with Corollary 3.7 and $(V_1)$–$(V_3)$, gives $V(y_0) = V_0$ and $y_0 = c_i$ for some $i = 1, 2, \ldots, l$. Therefore, one has

$$Q_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon_n x + \varepsilon_n y_n)|v_n^2|dx}{\int_{\mathbb{R}^3} |v_n^2|dx} \to \frac{\int_{\mathbb{R}^3} \chi(y_0)|v^2|dx}{\int_{\mathbb{R}^3} |v^2|dx} = c_i \in K_{\frac{\xi_0}{2}} \quad \text{as } n \to +\infty,$$

which contradicts $Q_{\varepsilon_n}(u_n) \notin K_{\frac{\xi_0}{2}}$. We complete the proof. □

To obtain multiple normalized solutions, we define

$$N_\varepsilon^i := \{u \in W_r^a : |Q_\varepsilon(u) - c_i| \le \xi_0\},$$
$$\partial N_\varepsilon^i := \{u \in W_r^a : |Q_\varepsilon(u) - c_i| = \xi_0\}$$

and

$$\beta_\varepsilon^i := \inf_{u \in N_\varepsilon^i} J_\varepsilon(u), \qquad \tilde{\beta}_\varepsilon^i := \inf_{u \in \partial N_\varepsilon^i} J_\varepsilon(u).$$

**Lemma 4.6.** *Assume that $(V_1)$–$(V_3)$ hold. For $a \in (0, \bar{a})$, then there is $\varepsilon_2 \in (0, \varepsilon_1)$ satisfying*

$$\beta_\varepsilon^i < \Upsilon_0(a) + \xi_1 \quad \text{and} \quad \beta_\varepsilon^i < \tilde{\beta}_\varepsilon^i \quad \text{for all } \varepsilon \in (0, \varepsilon_2).$$

*Proof.* Let $u \in W_r^a$ satisfy $J_0(u) = \Upsilon_0(a)$. For $1 \le i \le l$, we set

$$\hat{u}_\varepsilon^i(x) := u\left(x - \frac{c_i}{\varepsilon}\right).$$

Obviously, $\hat{u}_\varepsilon^i \in W_r^a$ for all $\varepsilon > 0$ and $1 \le i \le l$. Moreover, it follows from the definition of $Q_\varepsilon$ that $Q_\varepsilon(\hat{u}_\varepsilon^i) \to c_i$ as $\varepsilon \to 0^+$. Therefore, $\hat{u}_\varepsilon^i \in N_\varepsilon^i$ for $\varepsilon$ small enough. Furthermore, we have

$$\begin{aligned} J_\varepsilon(\hat{u}_\varepsilon^i) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \hat{u}_\varepsilon^i|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|\hat{u}_\varepsilon^i|^2 dx + \frac{1}{2}N(\hat{u}_\varepsilon^i) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |\hat{u}_\varepsilon^i|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x + c_i)|u|^2 dx + \frac{1}{2}N(u) - \frac{2|\lambda_3|}{p}\int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

This, together with $(V_3)$, gives

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\hat{u}_\varepsilon^i) = J_{c_i}(u) = J_0(u) = \Upsilon_0(a).$$

Combining this with $\hat{u}_\varepsilon^i \in N_\varepsilon^i$ for $\varepsilon$ small enough, we have there is $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\beta_\varepsilon^i < \Upsilon_0(a) + \xi_1 \quad \text{for all } \varepsilon \in (0, \varepsilon_2). \tag{4.5}$$

On the other hand, for any $v \in \partial N_\varepsilon^i$, one has $v \in W_{\bar{r}}^a$ and $|Q_\varepsilon(v) - c_i| = \xi_0$. Then, $Q_\varepsilon(v) \notin K_{\frac{\xi_0}{2}}$. This, together with Lemma 4.5, gives

$$J_\varepsilon(v) > \Upsilon_0(a) + \xi_1 \quad \text{for all } v \in \partial N_\varepsilon^i \text{ and } \varepsilon \in (0, \varepsilon_2),$$

which implies that $\tilde{\beta}_\varepsilon^i = \inf_{v\in\partial N_\varepsilon^i} J_\varepsilon(v) \geq \Upsilon_0(a) + \xi_1$ for all $\varepsilon \in (0, \varepsilon_2)$. Then it follows from (4.5) that

$$\beta_\varepsilon^i < \tilde{\beta}_\varepsilon^i \quad \text{for all } \varepsilon \in (0, \varepsilon_2).$$

We complete the proof. □

*Proof of Theorem 1.1.* Let $\varepsilon_3 \in (0, \varepsilon_2)$. For all $i = 1, 2, \ldots, l$ and $\varepsilon \in (0, \varepsilon_3)$, we can use Ekeland's variational principle [11] to find a sequence $\{u_n^i\} \subset N_\varepsilon^i \subset W_{\bar{r}}^a$ such that

$$J_\varepsilon(u_n^i) \to \beta_\varepsilon^i \quad \text{and} \quad J_\varepsilon(v) - J_\varepsilon(u_n^i) \geq -\frac{1}{n}\|v - u_n^i\|_{H^1(\mathbb{R}^3)}, \quad \forall v \in N_\varepsilon^i \text{ with } v \neq u_n^i.$$

In view of Lemma 4.6, one has $\beta_\varepsilon^i < \tilde{\beta}_\varepsilon^i$. Thus, $u_n^i \in N_\varepsilon^i \setminus \partial N_\varepsilon^i$ for $n$ large enough. Next, for $\delta > 0$ small enough, we define a map $\Phi : (-\delta, \delta) \to W_{\bar{r}}^a$ by

$$\Phi(t) = a\frac{u_n^i + tv}{\|u_n^i + tv\|_2^2}$$

belonging to $C^1((-\delta, \delta), W_{\bar{r}}^a)$ and satisfying

$$\Phi(t) \in N_\varepsilon^i \setminus \partial N_\varepsilon^i \quad \forall t \in (-\delta, \delta), \quad \Phi(0) = u_n^i, \quad \Phi'(0) = v,$$

where $v \in T_{u_n^i} W_{\bar{r}}^a$. Then, we have

$$J_\varepsilon(\Phi(t)) - J_\varepsilon(u_n^i) \geq -\frac{1}{n}\|\Phi(t) - u_n^i\|_{H^1(\mathbb{R}^3)} \quad \forall t \in (-\delta, 0) \cup (0, \delta),$$

which concludes that

$$\begin{aligned}\frac{J_\varepsilon(\Phi(t)) - J_\varepsilon(\Phi(0))}{t} &= \frac{J_\varepsilon(\Phi(t)) - J_\varepsilon(u_n^i)}{t} \geq -\frac{1}{n}\left\|\frac{\Phi(t) - u_n^i}{t}\right\|_{H^1(\mathbb{R}^3)} \\ &= -\frac{1}{n}\left\|\frac{\Phi(t) - \Phi(0)}{t}\right\|_{H^1(\mathbb{R}^3)}.\end{aligned}$$

By $J_\varepsilon \in C^1(H^1(\mathbb{R}^3, \mathbb{R}))$ and letting $t \to 0^+$, one has

$$\left\langle J_\varepsilon'(u_n^i), v\right\rangle \geq -\frac{1}{n}\|v\|_{H^1(\mathbb{R}^3)}.$$

Replacing $v$ by $-v$, we infer that

$$\sup\left\{\left|\left\langle J_\varepsilon'(u_n^i), v\right\rangle\right| : \|v\|_{H^1(\mathbb{R}^3)} = 1\right\} \leq \frac{1}{n},$$

which implies that

$$J_\varepsilon(u_n^i) \to \beta_n^i \quad \text{and} \quad \|J_\varepsilon|'_{W_r^a}(u_n^i)\|_{H^{-1}(\mathbb{R}^3)} \to 0 \quad \text{as } n \to +\infty.$$

Then, $\{u_n^i\}$ is a $(PS)_{\beta_n^i}$ for $J_\varepsilon$ restricted to $W_r^a$. Combining this with Lemmas 4.4 and 4.6, we have there is $u^i \in H^1(\mathbb{R}^3)$ such that $u_n^i \to u^i$ in $H^1(\mathbb{R}^3)$. Thus,

$$u^i \in N_\varepsilon^i, \quad J_\varepsilon(u^i) = \beta_\varepsilon^i \quad \text{and} \quad J_\varepsilon|'_{W_r^a}(u^i) = 0.$$

Moreover, it follows from the definition of $Q_\varepsilon(u)$ that

$$Q_\varepsilon(u^i) \in \overline{B_{\xi_0}(c_i)}, \quad Q_\varepsilon(u^j) \in \overline{B_{\xi_0}(c_j)} \quad \text{and} \quad \overline{B_{\xi_0}(c_i)} \cap \overline{B_{\xi_0}(c_j)} = \emptyset \quad \text{for } i \neq j.$$

Then we have $u^i \neq u^j$ for $i \neq j$, where $1 \leq i, j \leq l$. Therefore, $J_\varepsilon|_{S(a)}$ has at least $l$ nontrivial critical points for all $\varepsilon \in (0, \varepsilon_3)$. Then there exists a Lagrange multiplier $\mu_i \in \mathbb{R}$ such that

$$\begin{aligned} -\mu_i a^2 &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_i|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|u_i|^2 dx + N(u_i) - |\lambda_3| \int_{\mathbb{R}^3} |u_i|^p dx \\ &= \beta_\varepsilon^i + \frac{1}{2}N(u_i) - \frac{(p-2)|\lambda_3|}{p}\int_{\mathbb{R}^3} |u_i|^p dx. \end{aligned} \tag{4.6}$$

In addition, it follows from Corollary 3.7, Lemmas 4.4 and 4.6 that $\beta_\varepsilon^i < 0$. This, together with (4.6), gives $\mu_i > 0$. We complete the proof. □

## Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. The research results of Nguyen Van Thin are supported by Thang Long University under grant number: DTV-2025-01.

## References

[1] C. Alves, On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$, *Z. Angew. Math. Phys.* **73**(2022), No. 3, Paper No. 97, 17 pp. https://doi.org/10.1007/s00033-022-01741-9; MR4411070; Zbl 1490.35176

[2] P. Antonelli, C. Sparber, Existence of solitary waves in dipolar quantum gases, *Phys. D.* **240**(2011), No. 4–5, 426–431. https://doi.org/10.1016/j.physd.2010.10.004; MR2748730; Zbl 1213.37102

[3] W. Bao, Y. Cai, Y, Mathematical theory and numerical methods for Bose–Einstein condensation, *Kinet. Relat. Models.* **6**(2013), No. 1, 1–135. https://doi.org/10.3934/krm.2013.6.1; MR3005624; Zbl 1266.82009

[4] W. Bao, Y. Cai, H. Wang, Efficient numerical method for computing ground states and dynamic of dipolar Bose–Einstein condensates, *J. Comput. Phys.* **229**(2010), No. 20, 7874–7892. https://doi.org/10.1016/j.jcp.2010.07.001; MR2674308; Zbl 1198.82036

[5] J. Bellazzini, L. Jeanjean, On dipolar quantum gases in the unstable regime, *SIAM J. Math. Anal.* **48**(2016), No. 3, 2028–2058. https://doi.org/10.1137/15M1015959; MR3510005; Zbl 1352.35157

[6] J. Bellazzini, L. Forcella, Asymptotic dynamic for dipolar quantum gases below the ground state energy threshold, *J. Funct. Anal.* **277**(2019), No. 6, 1958–1998. https://doi.org/10.1016/j.jfa.2019.04.005; MR3985521; Zbl 1420.35340

[7] J. Bellazzini, L. Forcella, Dynamical collapse of cylindrical symmetric dipolar Bose–Einstein condensates, *Calc. Var. Partial Differ. Equ.* **60**(2021), No. 6, 33, https://doi.org/10.1007/s00526-021-02096-1; MR4314145; Zbl 1479.35764

[8] R. Carles, H. Hajaiej, Complementary study of the standing waves solutions of the Gross–Pitaevskii equation in dipolar quantum gases, *Bull. Lond. Math. Soc.* **47**(2015), No. 3, 509–518. https://doi.org/10.1112/blms/bdv024; MR3354446

[9] R. Carles, P. Markowich, C. Sparber, On the Gross–Pitaevskii equation for trapped dipolar quantum gases, *Nonlinearity* **21**(2008), No. 11, 2569–2590. https://doi.org/10.1088/0951-7715/21/11/006; MR2448232; Zbl 1157.35102

[10] V. Dinh, On the instability of standing waves for 3D dipolar Bose–Einstein condensates, *Phys. D* **419**(2021), No. 132852, 12 pp. https://doi.org/10.1016/j.physd.2021.132856; MR4219990; Zbl 1508.35144

[11] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**(1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0; MR346619; Zbl 0286.49015

[12] B. Feng, L. Cao, J. Liu, Existence of stable standing waves for the Lee–Huang–Yang corrected dipolar Gross–Pitaevskii equation, *Appl. Math. Lett.* **115**(2021), 106952. https://doi.org/10.1016/j.aml.2020.106952; MR4191934; Zbl 1460.35153

[13] Y. He, X. Luo, Concentrating standing waves for the Gross–Pitaevskii equation in trapped dipolar quantum gases, *J. Differential Equations.* **266**(2019), No. 1, 600–629. https://doi.org/10.1016/j.jde.2018.07.047; MR3870573; Zbl 1403.35104

[14] Y. Luo, A. Stylianou, Ground states for a nonlocal mixed order cubic-quartic Gross–Pitaevskii equation, *J. Math. Anal. Appl.* **496**(2021), 124802. https://doi.org/10.1016/j.jmaa.2020.124802; MR4186680; Zbl 1462.35362

[15] Y. Luo, A. Stylianou, On 3D dipolar Bose–Einstein condensates involving quantum fluctuations and three-body interactions, *Discrete Contin. Dyn. Syst. Ser. B* **26**(2021), No. 6, 3455–3477, https://doi.org/10.3934/dcdsb.2020239; MR4235662; Zbl 1471.35259

[16] X. Luo, T. Yang, Stabilization in dipolar Gross–Pitaevskii theory by mass-subcritical perturbation, *J. Fixed Point Theory Appl.* **25**(2023), Paper No. 1, Paper No. 28, 22 pp. https://doi.org/10.1007/s11784-022-01018-8; MR4522811; Zbl 1505.35215

[17] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.* **87**(1983), No. 4, 567–576. https://doi.org/10.1007/BF01208265; MR691044; Zbl 0527.35023

[18] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser Boston Inc, Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[19] M. Wu, C. Tang, Multiplicity of normalized solutions for dipolar Gross–Pitaevskii equation with a mass subcritical perturbation, *J. Fixed Point Theory Appl.* **26**(2024), Paper No. 4, Paper No. 41, 25 pp. https://doi.org/10.1007/s11784-024-01130-x; MR4801760; Zbl 1553.35077

[20] L. Xu, C. Song, Q. Xie, Multiplicity of normalized solutions for Schrödinger equation with mixed nonlinearity, *Taiwanese J. Math.* **28**(2024), No. 3, 589–609. https://doi.org/10.11650/tjm/240202; MR4748165; Zbl 1540.35142