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Global structure of positive solutions for nonlinear $(k, n-k)$ conjugate boundary value problems

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Abstract. We are concerned with the global structure of positive solutions of the $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}u^{(n)}(t) = \lambda a(t) f(u(t)), & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le k-1, \\ u^{(j)}(1) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{P}$$

where $n \ge 2$, $1 \le k \le n-1$, $\lambda > 0$ is a parameter, $a \in C([0,1], (0,\infty))$ and $f \in C([0,\infty), [0,\infty))$. We obtain existence and multiplicity results for positive solutions of problem (P) under suitable growth conditions on f. The proof of main result is based upon bifurcation techniques.

Keywords: positive solutions, principal eigenvalue, bifurcation, conjugate boundary value problem.

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1 Introduction

In this paper, we consider the following $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}u^{(n)}(t) = \lambda a(t) f(u(t)), & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le k-1, \\ u^{(j)}(1) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{1.1}$$

where $n \ge 2$, $1 \le k \le n-1$, $\lambda > 0$ is a parameter, $a \in C([0,1], (0,\infty))$ and $f \in C([0,\infty), [0,\infty))$ will be specified later.

For the special case of $n = 2$ and $k = 1$, problem (1.1) reduces to the $(1, 2-1)$ conjugate boundary value problem

$$\begin{cases} -u''(t) = \lambda a(t) f(u(t)), & 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

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For $n = 4$ and $k = 2$, the deformations of an elastic beam whose both ends clamped are described by the following fourth-order problem

$$\begin{cases} u''''(t) = \lambda a(t) f(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases} \tag{1.3}$$

Existence and multiplicity of positive solutions of (1.2) and (1.3) have been extensively studied via different methods. For examples, Amann [2], Erbe and Wang [12] and Henderson and Wang [13] studied the existence and multiplicity of positive solutions of (1.2) by the fixed point theorem in cone. Agarwal and Chow [1], Cabada and Enguiça [4] proved the existence of positive solutions of (1.3) by using monotone iterative methods. Korman [17], Lazer and Meckenna [18] and Shen [22] established the existence and multiplicity of solutions of (1.3) by means of bifurcation techniques.

For general $(k, n-k)$ conjugate boundary value problems, Eloe and Henderson [11] studied the following problem

$$\begin{cases} (-1)^{n-k} y^{(n)}(t) = f(t, y), & 0 < t < 1, \\ y^{(i)}(0) = 0, & 0 \le i \le k-1, \\ y^{(j)}(1) = 0, & 0 \le j \le n-k-1. \end{cases} \tag{1.4}$$

They proved the existence of positive solutions of problem (1.4) by using the fixed point theorem in cone under the nonlinearity f is decreasing in y, for each fixed t.

Subsequently, Davis and Henderson [8] investigated the existence of three positive solutions for the $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k} y^{(n)}(t) = f(y), & 0 < t < 1, \\ y^{(i)}(0) = 0, & 0 \le i \le k-1, \\ y^{(j)}(1) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{1.5}$$

by employing the Leggett–Williams fixed point theorem, where the nonlinearity f is locally bounded.

It is worth noting that the works in [8,11] only established the existence and multiplicity of positive solutions without providing any information about the global structure of the positive solutions.

There also exist relevant results on the global structure of positive solutions. Specifically, Rynne [20] considered the following second-order elliptic problem

$$\begin{cases} \widetilde{L} u = \lambda a u + g(\cdot, u) u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $\widetilde{L}$ being a uniformly elliptic operator. Using global bifurcation theory, he analyzed the oscillating global structure of positive solutions of (1.6). Nevertheless, the nonlinearity g must satisfy a uniform boundedness condition

$$-\gamma^{--} \le \frac{g(x, \xi)}{a(x)} \le \gamma^{++}, \quad (x, \xi) \in \bar{\Omega} \times \mathbb{R},$$

where $\gamma^{--}, \gamma^{++} \in (0, \infty)$. Moreover, when g is sufficiently small, the bifurcating set of positive solutions of (1.6) is a single unbounded smooth curve, which is parameterized by a weighted norm and contains all positive solutions of (1.6). The proof of this result is strongly dependent on the self-adjointness of $a^{-1}\widetilde{L}$ (see [20, Theorem 5]).

Sim and Tanaka [23] showed the existence of three positive solutions for the one-dimensional p-Laplacian problem

$$\begin{cases} -\left(|y'|^{p-2}y'\right)' = \lambda a(t)f(y), \quad 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{1.7}$$

via bifurcation techniques. They obtained an S-shaped unbounded connected component. In their work, the principal eigenvalue admits an expression via the Rayleigh quotient, and the proof of their main result depends on the Sturm comparison theorem.

Now, a natural question is whether we can obtain global results similar to those in [21,23] for the $(k, n-k)$ conjugate boundary value problem (1.1)? We will give a positive answer by using bifurcation techniques. Throughout we assume

(H1) $a \in C([0,1],(0,\infty))$ and $0 < a_* \le a \le a^*$ on $[0,1]$ for some $a_*, a^* \in (0,\infty)$;

(H2) there exist $\alpha, f_0, f_1 \in (0,\infty)$ such that

$$\lim_{s\to 0^+} \frac{f(s) - f_0 s}{s^{1+\alpha}} = -f_1;$$

(H3) $f_\infty := \lim_{s\to\infty} \frac{f(s)}{s} = 0$;

(H4) there exist $\gamma^- \in (0, \frac{\lambda_1}{f_0})$, $\gamma^+ \in (0,\infty)$ and a sequence $\{\xi_j\}$ such that for any $\sigma \in \left(0, \frac{1}{2}\right)$, there exists $A(\sigma) \in (0,1)$ such that

$$0 < \xi_{2j-1} < A(\sigma)\xi_{2j} < \xi_{2j} < \xi_{2j+1}, \quad j = 1, 2, \dots$$

and

$$\frac{f(s)}{s} < \frac{f_0(k-1)!(n-k-1)!}{(\lambda_1 + \gamma^+ f_0)a^* \int_0^1 s^{n-k-1}(1-s)^k \,\mathrm{d}s}, \quad s \in (0, \xi_{2j-1}],$$

$$\frac{f(s)}{s} > \frac{f_0(k-1)!(n-k-1)!}{(\lambda_1 - \gamma^- f_0)a_*[A(\sigma)]^2 \int_\sigma^{1-\sigma} s^{n-k-1}(1-s)^k \,\mathrm{d}s}, \quad s \in [A(\sigma)\xi_{2j}, \xi_{2j}],$$

where λ_1 is the principal eigenvalue of eigenvalue problem corresponding to problem (1.1).

(H5) there exists $s_0 \in (0,\infty)$ such that

$$\frac{f(s)}{s} > \frac{f_0(k-1)!(n-k-1)!}{\lambda_1 a_*[A(\sigma)]^2 \int_\sigma^{1-\sigma} s^{n-k-1}(1-s)^k \mathrm{d}s}, \quad s \in [A(\sigma)s_0, s_0].$$

Let $X = C[0,1]$ with norm

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

The main results of this paper are the following

Theorem 1.1. *Assume that (H1)–(H4) hold. Then there exists an unbounded connected component $\mathcal{D}^+$ of the set of positive solutions of problem* (1.1) *in* $\mathbb{R}^+ \times X$*, such that* *(i)* $\mathcal{D}^+$ *joins* $\left(\frac{\lambda_1}{f_0}, 0\right)$ *to infinity;*

(ii) if $(\lambda, u) \in \mathcal{D}^+$ *and* $\|u\|_\infty = \xi_{2j-1}$*, then* $\lambda > \frac{\lambda_1}{f_0} + \gamma^+$*;*

(iii) if $(\lambda, u) \in \mathcal{D}^+$ *and* $\|u\|_\infty = \xi_{2j}$*, then* $\lambda < \frac{\lambda_1}{f_0} - \gamma^-$*.*

Corollary 1.2. *Assume that (H1)–(H4) hold. Then for all* $\lambda \in \left(\frac{\lambda_1}{f_0} - \gamma^-, \frac{\lambda_1}{f_0} + \gamma^+\right)$*, (1.1) has infinitely many positive solutions.*

Theorem 1.3. *Assume that (H1)–(H3) and (H5) hold. Then there exists an unbounded connected component* $\mathcal{D}^+$ *of the set of positive solutions of problem (1.1) in* $\mathbb{R}^+ \times X$*, such that*

(i) $\mathcal{D}^+$ *joins* $\left(\frac{\lambda_1}{f_0}, 0\right)$ *to infinity;*

(ii) there exists $\delta > 0$*, such that if* $(\lambda, u) \in \mathcal{D}^+$ *satisfies* $\left|\lambda - \frac{\lambda_1}{f_0}\right| + \|u\|_\infty \leq \delta$*, then* $\lambda > \frac{\lambda_1}{f_0}$*;*

(iii) there exists $(\lambda, u) \in \mathcal{D}^+$ *with* $\|u\|_\infty = s_0$*, which implies that* $\lambda < \frac{\lambda_1}{f_0}$*.*

Corollary 1.4. *Assume that (H1)–(H3) and (H5) hold. Then there exist* $\lambda_* \in \left(0, \frac{\lambda_1}{f_0}\right)$ *and* $\lambda^* \in \left(\frac{\lambda_1}{f_0}, \infty\right)$*, such that*

(i) (1.1) has at least one positive solution if $\lambda = \lambda_*$*;*

(ii) (1.1) has at least two positive solutions if $\lambda_* < \lambda \leq \frac{\lambda_1}{f_0}$*;*

(iii) (1.1) has at least three positive solutions if $\frac{\lambda_1}{f_0} < \lambda < \lambda^*$*;*

(iv) (1.1) has at least two positive solutions if $\lambda = \lambda^*$*;*

(v) (1.1) has at least one positive solution if $\lambda > \lambda^*$*.*

Remark 1.5. Compared with the studies in [8, 11, 21, 27, 28], the nonlinearity f in this paper need not be monotonic or bounded, but only needs to satisfy suitable growth conditions.

Remark 1.6. When n is odd, the boundary conditions of (1.1) are asymmetric and the corresponding differential operator is not selfadjoint; in this case, the principal eigenvalues of (1.1) and its adjoint problem are the same, but their corresponding eigenfunctions are different. To the best of our knowledge, few results in the existing literature focus on the investigation of positive solutions to (1.1) via its adjoint problem.

Remark 1.7. Since problem (1.1) does not have a variational structure, its principal eigenvalue cannot be expressed via the Rayleigh quotient as in [23]. Using Elias's theory [9] for the associated eigenvalue problem, we prove that the principal eigenvalue exists and is simple. See Section 2.

Remark 1.8. We can find some simple examples to illustrate our main results. For example, if we take

$$\hat{f}(s) = \begin{cases} 5s - 3s^3, & 0 \leq s \leq 1, \\ 200s - 198, & 1 < s \leq 48, \\ \dfrac{451296}{s}, & s > 48, \end{cases}$$

then the $(2, 4-2)$ conjugate boundary value problem

$$\begin{cases} u''''(t) = \lambda \hat{f}(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0, \end{cases} \tag{1.8}$$

has an S-shaped component of positive solutions.

In fact, for (1.8), $\hat{f}_0 = 5$, and $\hat{f}_1 = 3$ with $\alpha = 2$, $\lambda_1 = \omega_1^4$ (≈ 500.55), where ω_1 is the minimal positive root of $\cos\omega \cosh\omega = 1$. In this case,

$$\sigma = \frac{1}{4} \in \left(0, \frac{1}{2}\right), \quad A(\sigma) = \frac{1}{24} \in (0,1),$$

and

$$\min_{s\in[2,48]} \frac{\hat{f}(s)}{s} > 100.43,$$

which implies that (H5) with $s_0 = 48$ is satisfied. Thus, all conditions of Theorem 1.3 are fulfilled.

For other related results on the $(k, n-k)$ conjugate boundary value problems, see Cabada and Saavedra [5], Elias [10], Hao et al. [14], Jiang and Zhang [15], Yao [27], Yang [26], Zhang et al. [28] and the references therein.

The rest of this paper is arranged as follows. In Section 2, we provide some preliminary results. In Section 3, we show the existence of bifurcation from $\left(\frac{\lambda_1}{f_0}, 0\right)$ and the rightward direction of bifurcation. Section 4 is devoted to completing the proof of Theorem 1.1 and Theorem 1.3.

2 Preliminaries

Let $G(t,s)$ be the Green's function of the homogeneous problem associated with (1.1). As is well known from the papers [7,16,25,28], $G(t,s)$ can be expressed in the form

$$G(t,s) = \begin{cases} \dfrac{1}{(k-1)!(n-k-1)!} \displaystyle\int_0^{t(1-s)} x^{k-1}(x+s-t)^{n-k-1}dx, & 0 \le t \le s \le 1, \\ \dfrac{1}{(k-1)!(n-k-1)!} \displaystyle\int_0^{s(1-t)} x^{n-k-1}(x+t-s)^{k-1}dx, & 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.1 ([16,28]). *The function $G(t,s)$ defined as above has the following properties*

$$G(t,s) \le \beta s^{n-k-1}(1-s)^k, \quad 0 \le t,s \le 1,$$

$$\frac{\beta}{n-1} g(t) s^{n-k}(1-s)^k \le G(t,s) \le \alpha g(t) s^{n-k-1}(1-s)^{k-1}, \quad 0 \le t,s \le 1,$$

where

$$\beta = \frac{1}{(k-1)!(n-k-1)!}, \quad g(t) = t^k(1-t)^{n-k},$$
$$\alpha = \frac{1}{\min\{k, n-k\}(k-1)!(n-k-1)!}.$$

Remark 2.2. It is noted that the term $(-1)^{n-k}$ is particularly convenient, especially when determining the sign of the Green's function. As shown in [10, Theorem 0.13], the Green's function $G_n(t,s)$ of the following problem

$$\begin{cases} u^{(n)}(t) = f(t), & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le k-1, \\ u^{(j)}(1) = 0, & 0 \le j \le n-k-1. \end{cases}$$

satisfies

$$(-1)^{n-k} G_n(t,s) > 0 \quad \text{on } (0,1) \times (0,1).$$

In addition, the following two statements have been shown in [16].

Proposition 2.3 ([16]). *For any $\sigma \in (0, \frac{1}{2})$ there exists an $A(\sigma) \in (0,1)$ such that for all $s \in [0,1]$*

$$G(t,s) \ge \beta A(\sigma) s^{n-k-1}(1-s)^k, \quad \sigma \le t \le 1-\sigma. \tag{2.1}$$

Proposition 2.4 ([16]). *Let $u \in C^n[0,1]$ satisfy*

$$\begin{cases} (-1)^{n-k} u^{(n)}(t) \ge 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le k-1, \\ u^{(j)}(1) = 0, & 0 \le j \le n-k-1. \end{cases}$$

Then for any $\sigma \in (0, \frac{1}{2})$ there exists an $A(\sigma) \in (0,1)$ such that

$$u(t) \ge A(\sigma) \|u\|_\infty, \quad \sigma \le t \le 1-\sigma. \tag{2.2}$$

It is worth noting that inequalities of the form (2.1) or (2.2), as well as analogous inequalities for the corresponding Green's functions, are very useful in the study of positive solutions to boundary value problems; see [11, 12, 24, 27].

Let $L : D(L) \to X$ be defined by

$$Lu = (-1)^{n-k} u^{(n)}, \quad u \in D(L),$$

where

$$D(L) = \left\{ u \in C^n[0,1] \mid u^{(i)}(0) = u^{(j)}(1) = 0, \ i = 0,1,\dots,k-1; \ j = 0,1,\dots,n-k-1 \right\}.$$

Obviously, L is an n-th order, disconjugate ordinary differential operator on $[0,1]$ with separated boundary conditions at 0 and 1, see [10]. Consider the linear eigenvalue problem

$$\begin{cases} Lu(t) = \lambda a(t) u(t), & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le k-1, \\ u^{(j)}(1) = 0, & 0 \le j \le n-k-1. \end{cases} \tag{2.3}$$

From Elias's eigenvalue theory [9, Theorems 1–5], we can get the following results.

Lemma 2.5. *Assume that (H1) holds.*

(i) (2.3) *has an infinite sequence of positive eigenvalues*

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots.$$

(ii) $\lambda_k \to \infty$ *as* $k \to \infty$.

(iii) To each eigenvalue λ_k*, there corresponds an essentially unique eigenfunction* ϕ_k *which has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* 0*;* 0 *and* 1 *are also simple zeros of* ϕ_k*.*

To continue the development, we now consider the boundary value problems which is adjoint to (2.3). Let L^* denote the adjoint of L, and consider

$$\begin{cases} L^*y(t) = \lambda a(t)y(t), & 0 < t < 1, \\ y^{(i)}(1) = 0, & 0 \le i \le k-1, \\ y^{(j)}(0) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{2.4}$$

where $L^*y := (-1)^k y^{(n)}$. From Coppel [6], we know that L is disconjugate on $[0,1]$ if, and only if, L^* is disconjugate on $[0,1]$. Therefore, Lemma 2.5 is also valid for (2.4).

Lemma 2.5 states that the geometric multiplicity of each λ_k is 1. If n is even and L is self-adjoint operator, then the algebraic multiplicity of each λ_k is also 1, see Rynne [21]. However, if n is odd, the operator L is inherently non-self adjoint, so the standard results for self-adjoint higher-order operators do not apply directly.

To carry out the bifurcation analysis for positive solutions of (1.1), it is therefore essential to first establish that the algebraic multiplicity of the principal eigenvalue λ_1 is 1.

Lemma 2.6. *Assume that (H1) holds. Then, the algebraic multiplicity of* λ_1 *is* 1*.*

Proof. To show the algebraic multiplicity of λ_1 is 1, it is enough to prove

$$\ker(L - \lambda_1(a)I) = \ker(L - \lambda_1(a)I)^2.$$

Clearly,

$$\ker(L - \lambda_1(a)I) \subseteq \ker(L - \lambda_1(a)I)^2.$$

Suppose on the contrary that the algebraic multiplicity of λ_1 is greater that 1. Then, there exists $z \in \ker(L - \lambda_1(a)I)^2 \setminus \ker(L - \lambda_1(a)I)$, and subsequently

$$Lz(t) - \lambda_1 a(t)z(t) = \gamma a(t)\phi_1(t), \quad 0 < t < 1, \tag{2.5}$$

for some $\gamma \neq 0$, where ϕ_1 is the positive normalized eigenfunction of (2.3) corresponding to the principal eigenvalue λ_1.

Multiplying both sides of (2.5) by ψ_1 (the positive normalized eigenfunction of the adjoint problem (2.4) corresponding to the principal eigenvalue λ_1), integrating from 0 to 1, and using the adjoint property

$$\int_0^1 Lz(s)\psi_1(s)\,\mathrm{d}s = \int_0^1 z(s)L^*\psi_1(s)\,\mathrm{d}s = \lambda_1 \int_0^1 a(s)z(s)\psi_1(s)\,\mathrm{d}s,$$

we deduce that

$$0 = \gamma \int_0^1 a(s)\phi_1(s)\psi_1(s)\,\mathrm{d}s \neq 0,$$

which is a contradiction. □

Remark 2.7. Results concerning the algebraic multiplicity of eigenvalues for non-self adjoint problems can be found in Bi and Liu [3, Lemma 4.4]. Nevertheless, their argument does not incorporate the weight function a, which is essential to our problem.

3 Rightward bifurcation

Let

$$E = \Big\{ u \in C^{n-1}[0,1] \mid u^{(i)}(0) = u^{(j)}(1) = 0,\ i = 0,1,\dots,k-1;\ j = 0,1,\dots,n-k-1 \Big\},$$

with norm $\|u\| = \max\{\|u^{(i)}\|_\infty, i = 0,1,\dots,n-1\}$. Then E is a Banach space.

Lemma 3.1. *Assume that (H1) and (H2) hold. Let ϕ_1 be the positive normalized eigenfunction corresponding to the principal eigenvalue λ_1 of (2.3). Let $\{(\eta_n, u_n)\}$ be a sequence of positive solutions of (1.1) satisfying $\eta_n \to \frac{\lambda_1}{f_0}$ and $\|u_n\| \to 0$. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ_1 on $[0,1]$.*

Proof. Let $\{(\eta_n, u_n)\}$ be a sequence of positive solutions of problem (1.1), then

$$u_n(t) = \eta_n \int_0^1 G(t,s)a(s)f(u_n(s))\,\mathrm{d}s.$$

Set $v_n := \frac{u_n}{\|u_n\|}$, we have that

$$v_n(t) = \eta_n \int_0^1 G(t,s)a(s)\frac{f(u_n(s))}{u_n}v_n(s)\,\mathrm{d}s.$$

Since $\|v_n\| = 1$, by the Ascoli–Arzelà theorem, a subsequence of $\{v_n\}$ uniformly converges to a limit v with $\|v\| = 1$. We again denote the subsequence by $\{v_n\}$. By recalling (H2) and using Lebesgue's dominated convergence theorem, we obtain

$$v(t) = \frac{\lambda_1}{f_0}\int_0^1 G(t,s)a(s)f_0 v(s)\mathrm{d}s = \lambda_1 \int_0^1 G(t,s)a(s)v(s)\,\mathrm{d}s,$$

which implies that $v = \phi_1$. □

Lemma 3.2. *Assume that (H1)–(H3) hold. Then there exists an unbounded connected component $\mathcal{C}^+$ of the set of positive solutions of problem (1.1), which joins to $\left(\frac{\lambda_1}{f_0}, 0\right)$.*

Proof. We extend f to an odd function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(u) = \begin{cases} f(u), & u \geq 0, \\ -f(-u), & u < 0. \end{cases}$$

In what follows, we replace f with $\tilde{f}$. For simplicity, the modified function $\tilde{f}$ will still be denoted by f.

Condition (H2) implies that there exists $\zeta \in C(\mathbb{R}, \mathbb{R})$ such that

$$f(s) = f_0 s + \zeta(s),$$

with ζ satisfying

$$\lim_{|s|\to 0} \frac{\zeta(s)}{s} = 0.$$

Let us consider

$$Lu - \lambda a(t) f_0 u = \lambda a(t)\zeta(u) \tag{3.1}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. Then auxiliary problem (3.1) is equivalent to

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\left[\lambda a(s) f_0 u(s) + \lambda a(s)\zeta(u(s))\right] ds \\ &=: \lambda f_0 L^{-1}\left[a(\cdot)u(\cdot)\right](t) + \lambda L^{-1}[a(\cdot)\zeta(u(\cdot))](t). \end{aligned} \tag{3.2}$$

Then $L^{-1} : X \to E$ is compact. Obviously, we have that $\left\|L^{-1}[a(\cdot)\zeta(u(\cdot))]\right\| = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals.

Let S denote the set of functions in E which have no interior nodal (i.e., nondegenerate zeros) in $(0, 1)$, and

$$S^+ := \{u \in E \mid u > 0\} \quad \text{and} \quad S^- := \{u \in E \mid u < 0\},$$

then set $S = S^+ \cup S^-$.

By Lemma 2.6 and Rabinowitz global bifurcation theorem [19], there exists a continuum $\mathcal{C}^+ \subset \mathbb{R}^+ \times \mathcal{S}^+$ of solutions of (3.2) such that $\left(\frac{\lambda_1}{f_0}, 0\right) \in \mathcal{C}^+$ and $\mathcal{C}^+$ is either unbounded or joins $\left(\frac{\lambda_j}{f_0}, 0\right)$, where λ_j is an eigenvalue of problem (2.3) with $j \neq 1$. We will show that the second alternative cannot occur.

Suppose for contradiction that there exists $(\eta_n, u_n) \in \mathcal{C}^+$ and $u_n \neq 0$, such that $(\eta_n, u_n) \to \left(\frac{\lambda_j}{f_0}, 0\right)$, $n \to \infty$, $j \neq 1$. Multiply both sides of (3.1) by ψ_1 (the positive normalized eigenfunction of the adjoint problem (2.4) corresponding to the principal eigenvalue λ_1) and integrate from 0 to 1. Noting that

$$\int_0^1 L u_n(s)\psi_1(s)\, ds = \int_0^1 u_n(s) L^* \psi_1(s)\, ds, \tag{3.3}$$

we then deduce that

$$\left(\frac{\lambda_1}{f_0} - \eta_n\right) \int_0^1 a(s) f_0 \psi_1(s) \frac{u_n(s)}{\|u_n\|}\, ds = \eta_n \int_0^1 a(s) \frac{\zeta(u_n(s))}{\|u_n\|} \psi_1(s)\, ds. \tag{3.4}$$

Let $\tilde{\zeta}(s) = \max_{0 \le |t| \le s} |\zeta(t)|$. Then $\tilde{\zeta}$ is non-decreasing with respect to s, and it holds that

$$\lim_{|s| \to 0} \frac{\tilde{\zeta}(s)}{s} = 0. \tag{3.5}$$

(3.5) implies that

$$\eta_n \int_0^1 a(s) \frac{\zeta(u_n(s))}{\|u_n\|} \psi_1(s)\, ds \to 0, \quad n \to \infty.$$

It follows from Lemma 3.1 that $\frac{u_n}{\|u_n\|}$ converges to ϕ_1, thus we have

$$\left(\frac{\lambda_1}{f_0} - \eta_n\right) \int_0^1 a(s) f_0 \psi_1(s) \frac{u_n(s)}{\|u_n\|}\, ds \to \left(\lambda_1 - \lambda_j\right) \int_0^1 a(s)\psi_1(s)\phi_1(s)\, ds \neq 0, \quad n \to \infty,$$

which is a contradiction! □

Lemma 3.3. *Assume that (H1) and (H2) hold. Let $(\lambda, u) \in \mathcal{C}^+$, there exists $M > 0$ independent of u such that*

$$\|u\| \le \lambda M \|u\|_\infty.$$

Proof. From the boundary conditions of (1.1) and Rolle's theorem, we have

$$u^{(n-1)}(\tau) = 0 \quad \text{for some} \quad \tau \in (0,1).$$

Thus,

$$(-1)^{n-k} u^{(n-1)}(t) = \lambda \int_\tau^t a(s) f(u(s))\, ds.$$

Note that (H2) and (H3) imply

$$f(s) \le f^* s, \quad s \ge 0,$$

for some $f^* > 0$, we get

$$\|u^{(n-1)}\|_\infty \le \lambda a^* f^* \|u\|_\infty \le \lambda M_{n-1} \|u\|_\infty.$$

Moreover,

$$\|u^{(j)}\|_\infty \le \lambda M_j \|u\|_\infty, \quad j = 0, 1, \dots, n-2.$$

Let $M = \max\{M_0, M_1, \dots, M_{n-1}\}$. Then

$$\|u\| = \max_{0 \le j \le n-1} \|u^{(j)}\|_\infty \le \lambda M \|u\|_\infty,$$

where $M > 0$ is independent of u. □

Remark 3.4. Noting that

$$\mathcal{C}^+ \subset \mathbb{R} \times E,$$

since the S-shaped connected component of positive solutions to problem (1.1) cannot be derived under the topology of the space $\mathbb{R} \times E$, we have to embed $\mathcal{C}^+$ into $\mathbb{R} \times X$ to obtain a new connected component

$$\mathcal{D}^+ \subset \mathbb{R} \times X.$$

A similar argument is used in [24].

Lemma 3.5. *Assume that (H1)–(H3) hold. Then there exists $\delta > 0$ such that for any $(\lambda, u) \in \mathcal{D}^+$ satisfying $|\lambda - \frac{\lambda_1}{f_0}| + \|u\|_\infty \le \delta$, we have $\lambda > \frac{\lambda_1}{f_0}$. That is, $\mathcal{D}^+$ is rightward near the bifurcation point $(\frac{\lambda_1}{f_0}, 0)$.*

Proof. Assume to the contrary that there exists a sequence $\{(\eta_n, u_n)\}$ such that $(\eta_n, u_n) \in \mathcal{D}^+$, $\eta_n \to \frac{\lambda_1}{f_0}$, $\|u_n\|_\infty \to 0$ and $\eta_n \le \frac{\lambda_1}{f_0}$. Substituting $(\lambda, u) = (\eta_n, u_n)$ into (1.1), multiplying both sides by ψ_1 (the positive normalized eigenfunction of the adjoint problem (2.4) corresponding to the principal eigenvalue λ_1), once again using the adjoint property (3.3), and integrating it from 0 to 1, it may be deduced that

$$\int_0^1 a(s)\psi_1(s) \frac{f(u_n(s))}{\|u_n\|_\infty} ds = \frac{\lambda_1}{\eta_n} \int_0^1 a(s)\psi_1(s) \frac{u_n(s)}{\|u_n\|_\infty} ds.$$

Combining Lemma 3.1 and applying Lebesgue's dominated convergence theorem, we deduce that

$$\int_0^1 a(s)\psi_1(s) \frac{f(u_n(s)) - f_0 u_n(s)}{u_n(x)^{1+\alpha}} \left| \frac{u_n(s)}{\|u_n\|_\infty} \right|^{1+\alpha} ds = \frac{\lambda_1 - f_0 \lambda_n}{\lambda_n \|u_n\|_\infty^\alpha} \int_0^1 a(s)\psi_1(s) \left| \frac{u_n(s)}{\|u_n\|_\infty} \right| ds \ge 0,$$

and

$$\int_0^1 a(s)\psi_1(s) \frac{f(u_n(s)) - f_0 u_n(s)}{u_n(s)^{1+\alpha}} \left| \frac{u_n(s)}{\|u_n\|_\infty} \right|^{1+\alpha} ds \to -f_1 \int_0^1 a(s)\psi_1(s) |\phi_1(s)|^{1+\alpha} ds < 0.$$

This is a contradicton. Then, the $\mathcal{D}^+$ is rightward near $(\frac{\lambda_1}{f_0}, 0)$. □

4 Proof of the main result

Lemma 4.1. *Assume that (H1)–(H5) hold. If $(\lambda, u) \in \mathcal{D}^+$ such that $\|u\|_\infty = s_0$, then $\lambda < \frac{\lambda_1}{f_0}$.*

Proof. Let $(\lambda, u) \in \mathcal{D}^+$ with $\|u\|_\infty = s_0$.

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))\,\mathrm{d}s \\ &\geq \lambda \int_\sigma^{1-\sigma} G(t,s)a(s)\frac{f(u(s))}{u(s)}u(s)\,\mathrm{d}s \\ &\geq \lambda a_* \beta \|u\|_\infty [A(\sigma)]^2 \int_\sigma^{1-\sigma} s^{n-k-1}(1-s)^k \min_{u(s)\in[A(\sigma)s_0, s_0]} \frac{f(u(s))}{u(s)}\,\mathrm{d}s. \end{aligned}$$

By (H5), we have

$$\lambda \leq \frac{1}{a_*\beta[A(\sigma)]^2}\left[\min_{s\in[A(\sigma)s_0,s_0]}\frac{f(s)}{s}\int_\sigma^{1-\sigma} s^{n-k-1}(1-s)^k \mathrm{d}s\right]^{-1} < \frac{\lambda_1}{f_0}. \qquad \square$$

Lemma 4.2. *Assume that (H1)–(H5) hold. Then there exists $\lambda_* > 0$ such that*

$$\mathcal{D}^+ \cap ((0, \lambda_*) \times X) = \emptyset.$$

Proof. Let $\|u\|_\infty = u(t_0)$. By Lemma 3.3,

$$\|u\|_\infty = u(t_0) \leq \int_0^{t_0} |u'(s)|\,\mathrm{d}s \leq \lambda M_1 \|u\|_\infty,$$

that is, $\lambda \geq 1/M_1 > 0$. $\square$

Lemma 4.3. *Assume that (H1)–(H5) hold, and let I be a compact subinterval in $\mathbb{R}^+$. Then there exists $M_I > 0$ such that*

$$\sup\{\, \|u_n\|_\infty \mid (\eta_n, u_n) \in \mathcal{C} \text{ and } \eta_n \in I \,\} \leq M_I.$$

Proof. Suppose for contradiction that there exists a sequence of positive solutions $\{(\eta_n, u_n)\}$ of (1.1) satisfying $\eta_n \in I$ and $\|u_n\|_\infty \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|_\infty}$, then

$$\begin{cases} Lv_n(t) = \eta_n a(t)\frac{f(u_n(t))}{u_n(t)} v_n(t), & t \in (0,1), \\ v_n^{(i)}(0) = 0, & 0 \leq i \leq k-1, \\ v_n^{(j)}(1) = 0, & 0 \leq j \leq n-k-1. \end{cases} \tag{4.1}$$

Since $\|v_n\|_\infty = 1$, by the Ascoli–Arzelà theorem, we may assume that there exists v with $\|v\|_\infty = 1$ such that $\{v_n\}$ uniformly converges to v. However, (H3) and (4.1) imply that

$$v(t) \equiv 0, \quad t \in [0,1],$$

which in turn contradicts $\|v\|_\infty = 1$. $\square$

Lemma 4.4. *Assume that (H1)–(H5) hold. Then there exists $\{(\eta_n, u_n)\} \subset \mathcal{D}^+$ such that*

$$\eta_n \to \infty, \quad \|u_n\|_\infty \to \infty, \quad n \to \infty.$$

Proof. Since $\mathcal{D}^+$ is unbounded, there exists $\{(\eta_n, u_n)\} \subset \mathcal{D}^+$ such that

$$\eta_n + \|u_n\|_\infty \to \infty, \quad n \to \infty. \tag{4.2}$$

Next, we present the proof in steps.

Step 1. We show that $\sup\{\, \eta_n \mid (\eta_n, u_n) \in \mathcal{D}^+\} = \infty$.

Assume on the contrary that $\sup\{\, \eta_n \mid (\eta_n, u_n) \in \mathcal{D}^+\} < \infty$. Then Lemma 4.3 implies that $\|u_n\|_\infty$ is bounded, which contradicts (4.2).

Step 2. We show that $\sup\{\, \|u_n\|_\infty \mid (\eta_n, u_n) \in \mathcal{D}^+\} = \infty$.

Note that

$$u_n(t) = \eta_n \int_0^1 G(t,s)a(s)f(u_n(s))\,\mathrm{d}s.$$

Thus,

$$\begin{aligned}\|u_n\|_\infty &\geq \eta_n\beta \int_0^1 s^{n-k-1}(1-s)^k a(s)\frac{f(u_n(s))}{u_n(s)}u_n(s)\,\mathrm{d}s\\ &\geq \eta_n a_*\beta\|u_n\|_\infty A(\sigma)f_*(\|u_n\|_\infty)\int_\sigma^{1-\sigma} s^{n-k-1}(1-s)^k\,\mathrm{d}s,\end{aligned}$$

where $f_*(s) = \min_{r\in[A(\sigma)s,s]}\frac{f(r)}{r}$. By Step 1, $\eta_n \to \infty$, which means that $f_*(\|u_n\|_\infty) \to 0$. From (H3), it follow that $\|u_n\|_\infty \to \infty$. □

Proof of Theorem 1.1. Let $\mathcal{D}^+$ be the connected component defined by Remark 3.4. By Lemmas 3.2 and 3.5, $\mathcal{D}^+$ bifurcates from $\big(\frac{\lambda_1}{f_0}, 0\big)$ and goes rightward.

Let $(\lambda, u) \in \mathcal{D}^+$ with $\|u\| = \xi_{2j-1}$. Then

$$\begin{aligned}u(t) &= \lambda\int_0^1 G(t,s)a(s)f(u(s))\,\mathrm{d}s\\ &\leq \lambda\beta\int_0^1 s^{n-k-1}(1-s)^k a(s)\frac{f(u(s))}{u(s)}u(s)\,\mathrm{d}s\\ &\leq \lambda a^*\beta \max_{0<s<\kappa_{2j-1}}\frac{f(s)}{s}\int_0^1 s^{n-k-1}(1-s)^k\,\mathrm{d}s\|u\|_\infty,\end{aligned}$$

which implies

$$\lambda > \frac{\lambda_1}{f_0} + \gamma^+. \tag{4.3}$$

Similar to Lemma 4.1, it is easy to check that for any $(\lambda, u) \in \mathcal{D}^+$ with $\|u\| = \xi_{2j}$, we have

$$\lambda < \frac{\lambda_1}{f_0} - \gamma^-. \tag{4.4}$$

(4.3) and (4.4) imply that, under condition (H4), the connected component $\mathcal{D}^+$ oscillates about $\frac{\lambda_1}{f_0}$ in a certain sense as it approaches infinity. This completes the proof. □

Proof of Theorem 1.3. Let $\mathcal{C}$ be the connected component defined by Remark 3.4. By Lemmas 3.2 and 3.5, $\mathcal{D}^+$ bifurcates from $\big(\frac{\lambda_1}{f_0}, 0\big)$ and goes rightward. Let (η_n, u_n) be as in Lemma 4.4. Then there exists $(\lambda_0, u_0) \in \mathcal{D}^+$ such that $\|u_0\|_\infty = s_0$. It then follows from Lemma 4.1 that $\lambda_0 < \frac{\lambda_1}{f_0}$. From Lemmas 3.5, 4.1, 4.2 and 4.4, we know that $\mathcal{D}^+$ passes through the points $\big(\frac{\lambda_1}{f_0}, v_1\big)$ and $\big(\frac{\lambda_1}{f_0}, v_2\big)$ with $\|v_1\|_\infty < s_0 < \|v_2\|_\infty$. Furthermore, there exist $\overline{\lambda}$ and $\underline{\lambda}$ such that $0 < \underline{\lambda} < \frac{\lambda_1}{f_0} < \overline{\lambda}$, satisfying (i) and (ii) below:

(i) if $\lambda \in \left(\frac{\lambda_1}{f_0}, \overline{\lambda}\right]$, then there exist u and v such that (λ, u), $(\lambda, v) \in \mathcal{D}^+$ and $\|u\|_\infty < \|v\|_\infty < s_0$;

(ii) if $\lambda \in \left[\underline{\lambda}, \frac{\lambda_1}{f_0}\right]$, then there exist u and v such that (λ, u), $(\lambda, v) \in \mathcal{D}^+$ and $\|u\|_\infty < s_0 < \|v\|_\infty$.

Define

$$\lambda^* = \sup\{\overline{\lambda} \mid \overline{\lambda} \text{ satisfies (i)}\} \quad \text{and} \quad \lambda_* = \inf\{\underline{\lambda} \mid \underline{\lambda} \text{ satisfies (ii)}\}.$$

Then (1.1) has positive solutions u_{λ_*} at $\lambda = \lambda_*$ and u_{λ^*} at $\lambda = \lambda^*$, respectively. By Lemma 4.1, for each $\lambda > \frac{\lambda_1}{f_0}$, there exists w such that $(\lambda, w) \in \mathcal{C}$ and $\|w\|_\infty > s_0$. Thus, together with Lemma 4.4, completes the proof of Theorem 1.3. □

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