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Well-posedness and exponential stability results for a shear beam model with dual-phase-lag thermoelasticity

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Abstract. In this work, we consider a one-dimensional linear thermoelastic shear beam model coupled with a heat equation, where the heat conduction is given by the generalized dual-phase-lag model. We establish the well-posedness of the system by using the semigroup theory. Then, the exponential stability is obtained by applying the Gearhart–Herbst–Prüss–Huang theorem under the hypothesis $2\tau_\theta > \tau_q$. In addition, we conduct several numerical tests to illustrate our theoretical results.

Keywords: shear beam model, dual-phase-lag, well-posedness, exponential stability, semigroup theory.

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1 Introduction

The shear beam model is an improvement over the Euler–Bernoulli beam model. This improvement is achieved by incorporating the shear distortion effect while neglecting rotary inertia. Unlike the Euler–Bernoulli and Rayleigh beam models, the shear model has two dependent variables for the dynamic of the beam, which are represented by the coupled equations

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x = 0, \\ -b\psi_{xx} + \kappa(\varphi_x+\psi) = 0, \end{cases} \tag{1.1}$$

where φ is the transverse displacement, ψ is the angular rotation of the neutral axis and $\rho_1 = \rho\tilde{A}$, $\kappa = k'G\tilde{A}$, $b = EI$, with ρ is the density, $\tilde{A}$ is the cross-sectional area, k' is the

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transverse shear factor, G is the shear modulus, I is the second moment of area of the cross-section and E is the Young modulus. The system (1.1) has only one phase velocity $\sqrt{\frac{\kappa}{\rho_1}}$, this wave speed corresponds to one of the classical Timoshenko system, where the Timoshenko system is given in [22] by the system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \end{cases} \tag{1.2}$$

where $\rho_2 = \rho I$. It is well known that the system (1.2) has two wave speeds $\nu_1 = \sqrt{\frac{\kappa}{\rho_1}}$ and $\nu_2 = \sqrt{\frac{b}{\rho_2}}$; and by adding the linear dampings in both equations of (1.2) (as shown, for example, in [19]), the exponential stability is achieved without any condition on the coefficients. But if the damping effect is acting on only one equation of (1.2) (for example, see [6,15,21]), then the system is exponentially stable if and only if the wave speeds are equal and this is not necessarily always true. For this matter, Almeida Júnior and Ramos [2] showed that this hypothesis is not required anymore for exponential stability of the Elishakoff's truncated model given by

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \mu\psi_t = 0, \end{cases} \tag{1.3}$$

where the system has just only wave speed $\nu = \sqrt{\frac{b}{\rho_2\left(1+\frac{b\rho_1}{\kappa\rho_2}\right)}} = \frac{\nu_1\nu_2}{\sqrt{\nu_1^2+\nu_2^2}}$. So, they proved that the system (1.3) is exponentially stable regardless to any relationship between the coefficients and concluded that the dissipative truncated versions of the Timoshenko system are advantageous over the classical Timoshenko equations in the stabilization context. To read more about truncated versions of the Timoshenko system, see [3,5,9,18]. Later, the authors of [4] considered the following shear beam model with a frictional damping acting on the transverse displacement

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu\varphi_t = 0, & \text{in } (0,L) \times (0,+\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, & \text{in } (0,L) \times (0,+\infty), \end{cases} \tag{1.4}$$

with the following Dirichlet–Neumann boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \quad \forall t > 0, \tag{1.5}$$

and proved the existence and uniqueness of solutions, by exploiting the Faedo–Galerkin method, and then used the multiplier method to show that the energy of the system decays exponentially regardless to any relationship between coefficients. Ramos et al. [17] considered the following shear beam model

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & \text{in } (0,L) \times (0,+\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\psi_t = 0, & \text{in } (0,L) \times (0,+\infty), \end{cases} \tag{1.6}$$

with the same boundary conditions (1.5). Using semigroup techniques, they demonstrated that the system (1.6) is not exponentially stable. Additionally, the well-posedness was discussed. Concerning stabilization via heat dissipation, Ahmima et al. [1] studied a one-dimensional thermoelastic truncated Timoshenko system of type III with the Dirichlet–Neumann–Neumann boundary conditions. They established the well-posedness by using semigroup theory and with the help of some non-traditional operators and proved the exponential stability result irrespective of the coefficients of the system. Also, Benmoussa et al.

[8] studied the thermoelastic shear beam model, where the thermal effect acting on the vertical displacement. They proved the well-posedness result by the use of the Faedo–Galerkin method and showed the exponential stability by utilizing the multiplier method and without imposing any relationship between the coefficients. In their work, the heat conduction is governed by the classical Fourier's law

$$q(x,\, t) = -\kappa\theta_x(x,\, t),$$

where q denotes the heat flux, θ represents the temperature deviation from the environmental temperature and κ is the heat conductivity. Fourier's theory of heat conduction is widely known for its instantaneous propagation of thermal disturbances throughout the medium, independently of the distance from the point of disturbance. This theory has faced criticism from a physical perspective due to the paradox of the infinite speed of the thermal signal propagation. To address this shortcoming, researchers have proposed several alternative theories of heat conduction.

In 1995, Tzou [23] proposed a modification to Fourier's law known as the dual-phase-lag (DPL) model. This proposal involved a theory of thermal flux incorporating a delay, and it was primarily based on a new constitutive equation

$$q\big(x,\ t+\tau_q\big) = -\kappa\theta_x(x,\ t+\tau_\theta) \tag{1.7}$$

which suggests that a temperature gradient across a material at position x at time $t+\tau_\theta$ causes a heat flux to flow at a different time $t+\tau_q$. The delay time τ_θ is the phase lag of the temperature gradient and τ_q represents the phase lag of the heat flux. Taylor's series expansion of (1.7) retaining terms up to the second-order in τ_q and the first-order in τ_θ leads to the following generalized heat conduction law

$$\frac{\tau_q^2}{2}q_{tt} + \tau_q q_t + q = -\kappa(\tau_\theta\theta_{xt} + \theta_x). \tag{1.8}$$

In this regard, Messaoudi et al. [13] considered a one-dimensional truncated Timoshenko system coupled with a heat equation, where the heat flux is given by the generalized dual-phase-lag model. Precisely, they studied the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, & \text{in } (0,1)\times(0,+\infty),\\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x+\psi) + \gamma\left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = 0, & \text{in } (0,1)\times(0,+\infty),\\ \left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa(\tau_\theta\theta_{xt}+\theta_x)_x + \gamma\theta^0\psi_{xt} = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases}$$

together with initial and boundary conditions. They established the well-posedness of the problem by employing the semigroup theory and certain nonclassical differential operators. Furthermore, by using the multiplier method, they demonstrated that the single heat control suffices to exponentially stabilize the entire system without requiring any relation between the system coefficients.

In this paper, we are concerned by the following thermoelastic shear beam model with dual-phase-lag model

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \delta\left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = 0, & \text{in } (0,1)\times(0,\infty), \\ -b\psi_{xx} + k(\varphi_x+\psi) = 0, & \text{in } (0,1)\times(0,\infty), \\ \left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa(\tau_\theta\theta_{xt} + \theta_x)_x + \delta\theta^0\varphi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1.9}$$

associated with the following boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad \forall t \geq 0 \tag{1.10}$$

and the initial conditions

$$(\varphi, \varphi_t, \theta, \theta_t, \theta_{tt})(x,0) = (\varphi_0, \varphi_1, \theta_0, \theta_1, \theta_2)(x), \quad \forall x \in (0,1). \tag{1.11}$$

The choice of the shear beam model coupled with the dual-phase-lag heat conduction theory is dictated by critical physical considerations and modern engineering applications. Explicitly, the shear beam model successfully resolves the physical paradox of "equal wave speeds" required for the exponential stability in classical Timoshenko systems, providing a mathematically robust and physically realizable framework for stabilization without restrictive coefficient conditions [14, 16]. From a mechanical perspective, in high-frequency vibrations and micro-scale structures, classical theories underestimate deflections; thus, the shear model provides highly accurate predictions for deep beams and laminated composite structures [20]. Furthermore, incorporating the DPL model eliminates the non-physical paradox of infinite thermal propagation speed inherent in Fourier's law, which is essential for analyzing ultra-fast laser processing and microelectronic cooling under rapid transient loads [24]. Consequently, investigating system (1.9) offers profound insights into the coupling of mechanical shearing and non-Fourier heat dissipation, which is vital for optimizing aerospace components and precision nanotechnology devices [11].

This paper is organized as follows. In Section 2, we establish the existence and uniqueness of the solution of system (1.9)–(1.11). In Section 3, under the hypothesis $2\tau_\theta > \tau_q$ and by using the Gearhart–Herbst–Prüss–Huang theorem, we show the exponential decay of the solution energy. Section 4 is devoted to the numerical study of our problem.

2 Well-posedness

In this section, we use the semi-group theory to prove the existence of a unique solution of problem (1.9)–(1.11). First, from the third equation of (1.9) and the boundary conditions (1.10), we get

$$\frac{\tau_q^2}{2}\int_0^1 \theta_{ttt}dx + \tau_q\int_0^1 \theta_{tt}dx + \int_0^1 \theta_t dx = 0. \tag{2.1}$$

The equation (2.1) is a linear differential equation of the third order, with a general solution is in the form

$$\int_0^1 \theta(x,t)dx = e^{-\frac{t}{\tau_q}}\left[\mu_1\cos\left(\frac{t}{\tau_q}\right) + \mu_2\sin\left(\frac{t}{\tau_q}\right)\right] + \mu_3$$

and, by using the initial conditions (1.11), we find that

$$\mu_1 = -\int_0^1 \left(\frac{\tau_q^2}{2}\theta_2 + \tau_q\theta_1\right)(x)dx,$$

$$\mu_2 = \frac{\tau_q^2}{2}\int_0^1 \theta_2(x)dx$$

and

$$\mu_3 = \int_0^1 \left(\frac{\tau_q^2}{2}\theta_2 + \tau_q\theta_1 + \theta_0\right)(x)dx.$$

Therefore, by setting

$$\bar{\theta}(x,t) = \theta(x,t) - \left(e^{-\frac{t}{\tau_q}}\left[\mu_1\cos\left(\frac{t}{\tau_q}\right) + \mu_2\sin\left(\frac{t}{\tau_q}\right)\right] + \mu_3\right),$$

we conclude that $(\varphi,\psi,\bar{\theta})$ satisfies (1.9), (1.10) and the initial conditions

$$\bar{\theta}(x,0) = \bar{\theta}_0(x) := \theta_0(x) - \int_0^1 \theta_0(x)dx,$$

$$\bar{\theta}_t(x,0) = \bar{\theta}_1(x) := \theta_1(x) - \int_0^1 \left[\tau_q\theta_2(x) + \theta_1(x)\right]dx,$$

$$\bar{\theta}_{tt}(x,0) = \bar{\theta}_2(x) := \theta_2(x) + \int_0^1 \theta_2(x)dx.$$

Moreover, we have

$$\int_0^1 \bar{\theta}(x,t)dx = 0, \quad \forall t \geq 0.$$

Then, we can apply Poincaré's inequality for $\bar{\theta}$. In the sequel, we work with $(\varphi,\psi,\bar{\theta})$, but for simplify, we write (φ,ψ,θ). Next, from $(1.9)_2$, we get

$$\psi = -k\mathcal{S}^{-1}\varphi_x, \tag{2.2}$$

such that

$$\mathcal{S} = -b\partial_{xx} + kI.$$

By substituting (2.2) into (1.9), we obtain the following new system

$$\begin{cases} \rho_1\varphi_{tt} - k(\mathcal{B}\varphi_x)_x + \delta\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa(\tau_\theta\theta_{xt} + \theta_x)_x + \delta\theta^0\varphi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{2.3}$$

where the differential operator $\mathcal{B}$ is defined by

$$\mathcal{B} = I - k\mathcal{S}^{-1}. \tag{2.4}$$

Let $\mathcal{U} = (\varphi,\phi,\theta,\omega,\eta)^T$, where $\phi = \varphi_t$, $\omega = \theta_t$ and $\eta = \theta_{tt}$. Then, we can write (2.3) in the form of an abstract Cauchy problem

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U}, \ \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0 = (\varphi_0,\varphi_1,\theta_0,\theta_1,\theta_2)^T, \end{cases} \tag{2.5}$$

such that the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\mathcal{U} = \begin{pmatrix} \phi \\ -\frac{1}{\rho_1}\left[-k(\mathcal{B}\varphi_x)_x + \delta\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)_x\right] \\ \omega \\ \eta \\ -\dfrac{2}{\tau_q^2}\left[\tau_q\eta + \omega - \kappa(\tau_\theta\omega_x + \theta_x)_x + \delta\theta^0\phi_x\right] \end{pmatrix}, \tag{2.6}$$

where $\mathcal{H}$ is the Hilbert space given by

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1), \tag{2.7}$$

with

$$L_*^2(0,1) = \left\{\varphi \in L^2(0,1), \int_0^1 \varphi(x)dx = 0\right\},$$
$$H_*^1(0,1) = \left\{\varphi \in H^1(0,1), \int_0^1 \varphi(x)dx = 0\right\}$$

and equipped with the inner product

$$\langle \mathcal{U}, \mathcal{U}^* \rangle_{\mathcal{H}} = \rho_1\theta^0\langle \phi, \overline{\phi^*}\rangle + k\theta^0\left\langle \mathcal{B}^{\frac{1}{2}}\varphi_x, \mathcal{B}^{\frac{1}{2}}\overline{\varphi_x^*}\right\rangle + \left\langle \left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right), \overline{\left(\frac{\tau_q^2}{2}\eta^* + \tau_q\omega^* + \theta^*\right)}\right\rangle$$
$$+ \frac{\kappa\tau_q^2}{2\tau_\theta}\left\langle (\tau_\theta\omega_x + \theta_x), \overline{(\tau_\theta\omega_x^* + \theta_x^*)}\right\rangle + \kappa\left[\frac{\tau_q}{2\tau_\theta}(2\tau_\theta - \tau_q) + \tau_\theta\right]\langle \theta_x, \overline{\theta_x^*}\rangle,$$

for $\mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T, \mathcal{U}^* = (\varphi^*, \phi^*, \theta^*, \omega^*, \eta^*)^T \in \mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \begin{Bmatrix} \mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T \in \mathcal{H} : \varphi \in H^2(0,1) \cap H_0^1(0,1); \phi \in H_0^1(0,1); \\ \theta, \omega, \eta \in H_*^1(0,1); \tau_\theta\omega + \theta \in H_*^2(0,1) \end{Bmatrix}, \tag{2.8}$$

where

$$H_*^2(0,1) = \left\{\varphi \in H^2(0,1) \cap H_*^1(0,1) : \varphi_x(0) = \varphi_x(1) = 0\right\}.$$

Remark 2.1.

- The operator $\mathcal{S}$ is bijective from $H^2(0,1) \cap H_0^1(0,1)$ (resp. $H_*^2(0,1) \cap H_*^1(0,1)$, $H_0^1(0,1)$) into $L^2(0,1)$ (resp. $L_*^2(0,1)$, $H^{-1}(0,1)$) and, in all cases, $\mathcal{S}^{-1}$ is bounded with $\|\mathcal{S}^{-1}\| < \frac{1}{k}$.
- The operator $\mathcal{B} : L^2(0,1) \longrightarrow L^2(0,1)$ is a positive self-adjoint bounded operator.

The following theorem is essential in establishing the main result of this section.

Theorem 2.2 (see [12, Theorem 1.2.4, p. 3])**.** *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Theorem 2.3. *The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{T}(t) = e^{\mathcal{A}t}$ of contractions on the Hilbert space $\mathcal{H}$.*

The proof of this latter theorem will be established through the following two lemmas.

Lemma 2.4. *If $2\tau_\theta \geq \tau_q$, then the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is dissipative.*

Proof. For any $\mathcal{U} \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = & -\theta^0 \left\langle -k(\mathcal{B}\varphi_x)_x + \delta\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)_x, \overline{\phi}\right\rangle + k\theta^0\langle \mathcal{B}\phi_x, \overline{\varphi_x}\rangle \\
& + \left\langle \kappa(\tau_\theta\omega_x + \theta_x)_x - \delta\theta^0\phi_x, \overline{\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)}\right\rangle \\
& + \frac{\kappa\tau_q^2}{2\tau_\theta}\left\langle (\tau_\theta\eta_x + \omega_x), \overline{(\tau_\theta\omega_x + \theta_x)}\right\rangle + \kappa\left[\frac{\tau_q}{2\tau_\theta}(2\tau_\theta - \tau_q) + \tau_\theta\right]\langle \omega_x, \overline{\theta_x}\rangle.
\end{aligned}$$

By a simple calculation, we get

$$\begin{aligned}
\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = & \, 2k\theta^0 i \operatorname{Im}\langle \mathcal{B}\phi_x, \overline{\varphi_x}\rangle + 2\delta\theta^0 i \operatorname{Im}\left\langle \left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right), \overline{\phi_x}\right\rangle \\
& + 2\kappa i \operatorname{Im}\left\langle (\tau_\theta\omega_x + \theta_x), \overline{\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)_x}\right\rangle + 2\kappa\tau_\theta i \operatorname{Im}\langle \omega_x, \overline{\theta_x}\rangle \\
& - \frac{\kappa\tau_q}{2}(2\tau_\theta - \tau_q)\|\omega_x\|_2^2 - \kappa\|\theta_x\|_2^2.
\end{aligned}$$

Therefore,

$$\operatorname{Re}\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = -\frac{\kappa\tau_q}{2}(2\tau_\theta - \tau_q)\|\omega_x\|_2^2 - \kappa\|\theta_x\|_2^2 \leq 0.$$

So, the operator $\mathcal{A}$ is dissipative. □

Lemma 2.5. *Let $R(\mathcal{A})$ be the range of the operator $\mathcal{A}$. Then, $R(\mathcal{A}) = \mathcal{H}$.*

Proof. We will prove that, for each $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, there is $\mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T \in D(\mathcal{A})$ such that $\mathcal{A}\mathcal{U} = \mathcal{F}$; that is,

$$\begin{cases}
\phi = f_1, \\
k(\mathcal{B}\varphi_x)_x - \delta\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)_x = \rho_1 f_2, \\
\omega = f_3, \\
\eta = f_4, \\
-\tau_q\eta - \omega + \kappa(\tau_\theta\omega_x + \theta_x)_x - \delta\theta^0\phi_x = \frac{\tau_q^2}{2}f_5.
\end{cases} \tag{2.9}$$

To solve the system (2.9), we use the Fourier series. So, taking into account the definition of $\mathcal{H}$, we can assume that

$$f_i = \sum_{n=1}^{+\infty} f_n^i \sin(n\pi x), \qquad \text{for } i = 1, 2$$

and

$$f_i = \frac{f_0^i}{2} + \sum_{n=1}^{+\infty} f_n^i \cos(n\pi x), \quad \text{for } i = 3, 4, 5,$$

with $f_0^i = 0$ for $i = 3,4,5$ (because $\int_0^1 f_i(x)dx = 0$), and

$$\sum_{n=1}^{+\infty} n^2 \left| f_n^i \right|^2 < \infty \quad \text{for } i = 1,3,4, \tag{2.10}$$

$$\sum_{n=1}^{+\infty} \left| f_n^i \right|^2 < \infty \quad \text{for } i = 2,5. \tag{2.11}$$

Now, we look for $\varphi, \phi, \theta, \omega$ and η in the form

$$(\varphi, \phi) = \sum_{n=1}^{+\infty} (\varphi_n, \phi_n) \sin(n\pi x) \quad \text{and} \quad (\theta, \omega, \eta) = \sum_{n=1}^{+\infty} (\theta_n, \omega_n, \eta_n) \cos(n\pi x). \tag{2.12}$$

From (2.9)$_1$, (2.9)$_3$ and (2.9)$_4$, respectively, we have

$$\phi_n = f_n^1, \quad \omega_n = f_n^3 \quad \text{and} \quad \eta_n = f_n^4.$$

Thus,

$$\sum_{n=1}^{+\infty} (n\pi)^2 |\phi_n|^2 = \sum_{n=1}^{+\infty} (n\pi)^2 \left| f_n^1 \right|^2 < \infty, \qquad \sum_{n=1}^{+\infty} (n\pi)^2 |\omega_n|^2 = \sum_{n=1}^{+\infty} (n\pi)^2 |f_n^3|^2 < \infty$$

and

$$\sum_{n=1}^{+\infty} (n\pi)^2 |\eta_n|^2 = \sum_{n=1}^{+\infty} (n\pi)^2 \left| f_n^4 \right|^2 < \infty.$$

Consequently, we get

$$\phi \in H_0^1(0,1) \quad \text{and} \quad \omega, \eta \in H_*^1(0,1). \tag{2.13}$$

By substituting (2.9)$_1$, (2.9)$_3$ and (2.9)$_4$ into (2.9)$_5$, we arrive at

$$-\sum_{n=1}^{+\infty} (n\pi)^2 (\tau_\theta \omega_n + \theta_n) \cos(n\pi x) = \frac{1}{\kappa} \sum_{n=1}^{+\infty} \left[\frac{\tau_q^2}{2} f_n^5 + \tau_q f_n^4 + f_n^3 + \delta\theta^0 (n\pi) f_n^1 \right] \cos(n\pi x).$$

So,

$$-(\tau_\theta \omega_n + \theta_n) = \frac{1}{\kappa (n\pi)^2} \left[\frac{\tau_q^2}{2} f_n^5 + \tau_q f_n^4 + f_n^3 + \delta\theta^0 (n\pi) f_n^1 \right]$$

and

$$\sum_{n=1}^{+\infty} (n\pi)^4 |\tau_\theta \omega_n + \theta_n|^2 = \frac{1}{\kappa^2} \sum_{n=1}^{+\infty} \left| \frac{\tau_q^2}{2} f_n^5 + \tau_q f_n^4 + f_n^3 + \delta\theta^0 (n\pi) f_n^1 \right|^2.$$

We use the inequality $\left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2$ to find that

$$\sum_{n=1}^{+\infty} (n\pi)^4 |\tau_\theta \omega_n + \theta_n|^2 \leq \frac{\tau_q^4}{\kappa^2} \sum_{n=1}^{+\infty} |f_n^5|^2 + \frac{4\tau_q^2}{\kappa^2} \sum_{n=1}^{+\infty} \left| f_n^4 \right|^2 + \frac{4}{\kappa^2} \sum_{n=1}^{+\infty} |f_n^3|^2 + \frac{4\left(\delta\theta^0\right)^2}{\kappa^2} \sum_{n=1}^{+\infty} (n\pi)^2 \left| f_n^1 \right|^2.$$

The Comparison Test and (2.10) imply that

$$\sum_{n=1}^{+\infty} \left| f_n^4 \right|^2 < \infty, \qquad \sum_{n=1}^{+\infty} |f_n^3|^2 < \infty. \tag{2.14}$$

Thus, (2.10), (2.11) and (2.14) give

$$\sum_{n=1}^{+\infty}(n\pi)^4|\tau_\theta\omega_n+\theta_n|^2<\infty.$$

Hence,

$$(\tau_\theta\omega+\theta)\in H^2_*(0,1) \tag{2.15}$$

and consequently, recalling (2.13),

$$\theta\in H^1_*(0,1).$$

On the other hand, applying the operator $\mathcal{S}$ to $(2.9)_2$ yields

$$\begin{aligned}
&-bk\sum_{n=1}^{+\infty}(n\pi)^4\varphi_n\sin(n\pi x)\\
&\quad=\sum_{n=1}^{+\infty}\left[\rho_1 b(n\pi)^2 f_n^2+\rho_1 k f_n^2-\delta b(n\pi)^3\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right]\sin(n\pi x)\\
&\qquad-\sum_{n=1}^{+\infty}\left[\delta k(n\pi)\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right]\sin(n\pi x).
\end{aligned}$$

So,

$$\begin{aligned}
-bk\varphi_n&=\frac{1}{(n\pi)^4}\left[\rho_1 b(n\pi)^2 f_n^2+\rho_1 k f_n^2-\delta b(n\pi)^3\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right]\\
&\quad-\frac{1}{(n\pi)^4}\left[\delta k(n\pi)\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
(bk)^2\sum_{n=1}^{+\infty}(n\pi)^4|\varphi_n|^2&\le\sum_{n=1}^{+\infty}\frac{2}{(n\pi)^4}\left|\rho_1 b(n\pi)^2 f_n^2+\rho_1 k f_n^2-\delta b(n\pi)^3\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right|^2\\
&\quad+\sum_{n=1}^{+\infty}\frac{2}{(n\pi)^4}\left|\delta k(n\pi)\left(\frac{\tau_q^2}{2}\eta_n+\tau_q\omega_n+\theta_n\right)\right|^2.
\end{aligned}$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned}
\sum_{n=1}^{+\infty}(n\pi)^4|\varphi_n|^2&\le\frac{1}{b^2}\sum_{n=1}^{+\infty}\left[\frac{6\rho_1^2}{(n\pi)^4}|f_n^2|^2+\frac{\delta^2}{(n\pi)^2}\left(\frac{3\tau_q^4}{2}|\eta_n|^2+6\left(\tau_q^2|\omega_n|^2+|\theta_n|^2\right)\right)\right]\\
&\quad+\frac{3}{k^2}\sum_{n=1}^{+\infty}\left[2\rho_1^2|f_n^2|^2+\delta^2(n\pi)^2\left(\frac{3\tau_q^4}{2}|\eta_n|^2+6\left(\tau_q^2|\omega_n|^2+|\theta_n|^2\right)\right)\right]
\end{aligned}$$

and using (2.10), (2.11) and that $\eta_n=f_n^4$, $\omega_n=f_n^3$ and $\theta\in H^1_*(0,1)$, we find that

$$\sum_{n=1}^{+\infty}(n\pi)^4|\varphi_n|^2<\infty.$$

Thus, $\varphi\in H^2(0,1)$ and by (2.12) we get $\varphi\in H^2(0,1)\cap H^1_0(0,1)$. Hence, there is $\mathcal{U}\in D(\mathcal{A})$ such that $\mathcal{A}\mathcal{U}=\mathcal{F}$. Consequently, $R(\mathcal{A})=\mathcal{H}$. □

Lemma 2.6. *The domain of $\mathcal{A}$ is dense in $\mathcal{H}$.*

Proof. To show that $D(\mathcal{A})$ is dense in $\mathcal{H}$, it suffices to check that, for $\mathcal{F} \in \mathcal{H}$,

$$\langle \mathcal{F}, \mathcal{U} \rangle_{\mathcal{H}} = 0, \quad \forall \mathcal{U} \in D(\mathcal{A}) \Longrightarrow \mathcal{F} = 0_{\mathcal{H}}.$$

Let $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ be such that

$$\langle \mathcal{F}, \mathcal{U} \rangle_{\mathcal{H}} = 0, \tag{2.16}$$

for any $\mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T \in D(\mathcal{A})$. Since $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is surjective, then there is some $\mathcal{U}_\mathcal{F} = (\varphi_\mathcal{F}, \phi_\mathcal{F}, \theta_\mathcal{F}, \omega_\mathcal{F}, \eta_\mathcal{F})^T \in D(\mathcal{A})$ such that $\mathcal{A}\mathcal{U}_\mathcal{F} = \mathcal{F}$, that is

$$\begin{cases} \phi_\mathcal{F} = f_1, \\ k(\mathcal{B}\varphi_{\mathcal{F}x})_x - \delta\left(\dfrac{\tau_q^2}{2}\eta_\mathcal{F} + \tau_q\omega_\mathcal{F} + \theta_\mathcal{F}\right)_x = \rho_1 f_2, \\ \omega_\mathcal{F} = f_3, \\ \eta_\mathcal{F} = f_4, \\ -\tau_q\eta_\mathcal{F} - \omega_\mathcal{F} + \kappa(\tau_\theta\omega_{\mathcal{F}x} + \theta_{\mathcal{F}x})_x - \delta\theta^0\phi_{\mathcal{F}x} = \dfrac{\tau_q^2}{2} f_5. \end{cases} \tag{2.17}$$

Substituting (2.17) in (2.16), it becomes

$$\begin{aligned} 0 = & -\theta^0\left\langle -k(\mathcal{B}\varphi_{\mathcal{F}x})_x + \delta\left(\frac{\tau_q^2}{2}\eta_\mathcal{F} + \tau_q\omega_\mathcal{F} + \theta_\mathcal{F}\right)_x, \overline{\phi}\right\rangle + k\theta^0\langle \mathcal{B}\phi_{\mathcal{F}x}, \overline{\varphi_x}\rangle \\ & + \left\langle \kappa(\tau_\theta\omega_{\mathcal{F}x} + \theta_{\mathcal{F}x})_x - \delta\theta^0\phi_{\mathcal{F}x}, \overline{\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)}\right\rangle \\ & + \frac{\kappa\tau_q^2}{2\tau_\theta}\left\langle(\tau_\theta\eta_{\mathcal{F}x} + \omega_{\mathcal{F}x}), \overline{(\tau_\theta\omega_x + \theta_x)}\right\rangle + \kappa\left[\frac{\tau_q}{2\tau_\theta}(2\tau_\theta - \tau_q) + \tau_\theta\right]\langle \omega_{\mathcal{F}x}, \overline{\theta_x}\rangle. \end{aligned} \tag{2.18}$$

Firstly, we choose $\mathcal{U} = \mathcal{U}_\mathcal{F} \in D(\mathcal{A})$ in (2.16), we thus obtain

$$0 = \langle \mathcal{F}, \mathcal{U}_\mathcal{F} \rangle_{\mathcal{H}} = \langle \mathcal{A}\mathcal{U}_\mathcal{F}, \mathcal{U}_\mathcal{F} \rangle_{\mathcal{H}}.$$

So, we have

$$\mathrm{Re}\langle \mathcal{A}\mathcal{U}_\mathcal{F}, \mathcal{U}_\mathcal{F} \rangle_{\mathcal{H}} = -\frac{\kappa\tau_q}{2}(2\tau_\theta - \tau_q)\|\omega_{\mathcal{F}x}\|_2^2 - \kappa\|\theta_{\mathcal{F}x}\|_2^2 \le 0,$$

which implies that $\omega_{\mathcal{F}x} = \theta_{\mathcal{F}x} = 0$ and the Poincaré–Wirtinger inequality yields $\omega_\mathcal{F} = \theta_\mathcal{F} = 0$. Secondly, by taking $\mathcal{U} = (0, 0, 0, 0, \eta)^T \in D(\mathcal{A})$ in (2.18), we arrive at

$$\langle \phi_{\mathcal{F}x}, \overline{\eta} \rangle = 0, \quad \forall \eta \in H_*^1(0, 1). \tag{2.19}$$

If we choose $\eta = \eta^* - \int_0^1 \eta^*(x)dx$ in (2.19), where $\eta^* \in C_0^\infty(0, 1)$, we get

$$\langle \phi_{\mathcal{F}x}, \overline{\eta_x^*} \rangle = 0, \quad \forall \eta^* \in C_0^\infty(0, 1).$$

This implies that $\phi_\mathcal{F} = 0$. Similarly, if $\mathcal{U} = (0, 0, \theta, \omega, 0)^T \in D(\mathcal{A})$ in (2.18), we find that

$$\left\langle \eta_\mathcal{F}, \overline{(\tau_\theta\omega + \theta)_{xx}} \right\rangle = 0, \quad \forall (\theta, \omega) \in E,$$

where $E = \{(\theta, \omega) \in [H^1_*(0,1)]^2; \tau_\theta \omega + \theta \in H^2_*(0,1)\}$. Since $g : (\theta, \omega) \mapsto \tau_\theta \omega + \theta$ is surjective from E into $H^2_*(0,1)$ (Indeed, $\forall \varpi \in H^2_*(0,1)$, $\exists (\theta, \omega) = \left(\theta, \frac{1}{\tau_\theta}(\varpi - \theta)\right) \in E : g(\theta\,, \omega) = \varpi$) and ∂_{xx} is bijective from $H^2_*(0,1)$ into $L^2_*(0,1)$, it follows that

$$\langle \eta_{\mathcal{F}}, \overline{\xi} \rangle = 0, \quad \forall \xi \in L^2_*(0,1)$$

and since $\{\chi_x : \chi \in C_0^\infty(0,1)\} \subset L^2_*(0,1)$, then, in particular, we have

$$\langle \eta_{\mathcal{F}}, \overline{\chi_x} \rangle = 0, \quad \forall \chi \in C_0^\infty(0,1).$$

This leads to $\eta_{\mathcal{F}x} = 0$ and thanks to Poincaré–Wirtinger inequality, we get $\eta_{\mathcal{F}} = 0$. Now, for $\mathcal{U} = (0, \phi, 0, 0, 0)^T \in D(\mathcal{A})$ in (2.18), we arrive at

$$\left\langle (\mathcal{B}\varphi_{\mathcal{F}x})_x, \overline{\phi} \right\rangle = 0, \quad \forall \phi \in H^1_0(0,1).$$

Therefore, $(\mathcal{B}\varphi_{\mathcal{F}x})_x = 0$ and it follows that from the fact that $\mathcal{B}\varphi_{\mathcal{F}x} \in H^1_*(0,1)$ and the Poincaré–Wirtinger inequality that $\mathcal{B}\varphi_{\mathcal{F}x} = 0$. Hence, we get $\varphi_{\mathcal{F}x} = 0$ and by using the Poincaré inequality, we find that $\varphi_{\mathcal{F}} = 0$. Consequently, $\mathcal{U}_{\mathcal{F}} \equiv 0$ and $\mathcal{F} = 0_{\mathcal{H}}$. This completes the proof of Lemma 2.6. □

Remark 2.7. It easily checked that $\ker(\mathcal{A}) = \{0_{\mathcal{H}}\}$ and through the proof of Lemma 2.6, we notice that $\|\mathcal{U}\|_{\mathcal{H}} \leq C\|\mathcal{F}\|_{\mathcal{H}}$, and thus 0 belongs to the resolvent of $\mathcal{A}$.

Proof of Theorem 2.3. Thanks to Lemmas 2.4, 2.5, 2.6 and Theorem 2.2, the conclusion follows. □

Then, the well-posedness result for the auxiliary problem (2.5) is given by

Theorem 2.8. *Let $\mathcal{U}_0 \in \mathcal{H}$ and assume that $2\tau_\theta \geq \tau_q$. Then, there exists a unique solution $\mathcal{U} \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.5). Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$, then $\mathcal{U} \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Now, we consider the following Hilbert space

$$H = H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times H^1_*(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \tag{2.20}$$

equipped with the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_H = {} & \rho_1 \theta^0 \left\langle \phi, \overline{\tilde{\phi}} \right\rangle + b\theta^0 \left\langle \psi_x, \overline{\tilde{\psi}_x} \right\rangle + k\theta^0 \left\langle \varphi_x + \psi, \overline{\tilde{\varphi}_x + \tilde{\psi}} \right\rangle \\ & + \left\langle \left(\frac{\tau_q^2}{2}\eta + \tau_q \omega + \theta \right), \overline{\left(\frac{\tau_q^2}{2}\tilde{\eta} + \tau_q \tilde{\omega} + \tilde{\theta} \right)} \right\rangle \\ & + \frac{\kappa \tau_q^2}{2\tau_\theta} \left\langle (\tau_\theta \omega_x + \theta_x), \overline{(\tau_\theta \tilde{\omega}_x + \tilde{\theta}_x)} \right\rangle + \kappa \left[\frac{\tau_q}{2\tau_\theta}(2\tau_\theta - \tau_q) + \tau_\theta \right] \left\langle \theta_x, \overline{\tilde{\theta}_x} \right\rangle, \end{aligned} \tag{2.21}$$

for $U = (\varphi, \phi, \psi, \theta, \omega, \eta)^T, \tilde{U} = (\tilde{\varphi}, \tilde{\phi}, \tilde{\psi}, \tilde{\theta}, \tilde{\omega}, \tilde{\eta})^T \in H$ and define the following mapping

$$\begin{cases} P : \mathcal{H} \longrightarrow H \\ \mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T \mapsto P\mathcal{U} = U = (\varphi, \phi, \psi, \theta, \omega, \eta)^T, \end{cases}$$

with ψ is the unique solution of

$$\begin{cases} \mathcal{S}\psi = -k\varphi_x, \\ \psi(0) = \psi(1) = 0. \end{cases} \tag{2.22}$$

Then, P is an isometry from $\mathcal{H}$ into H. Indeed, for any $\mathcal{U} = (\varphi, \phi, \theta, \omega, \eta)^T \in \mathcal{H}$, we have

$$\begin{aligned} \|\mathcal{U}\|_{\mathcal{H}}^2 = \rho_1\theta^0\langle\phi,\overline{\phi}\rangle + k\theta^0\langle\mathcal{B}\varphi_x,\overline{\varphi_x}\rangle + \left\langle\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right),\overline{\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)}\right\rangle \\ + \frac{\kappa\tau_q^2}{2\tau_\theta}\left\langle(\tau_\theta\omega_x+\theta_x),\overline{(\tau_\theta\omega_x+\theta_x)}\right\rangle + \kappa\left[\frac{\tau_q}{2\tau_\theta}(2\tau_\theta-\tau_q)+\tau_\theta\right]\langle\theta_x,\overline{\theta_x}\rangle \end{aligned} \tag{2.23}$$

and using the definition of the operator $\mathcal{B}$ and (2.22), we get

$$\begin{aligned} k\theta^0\langle\mathcal{B}\varphi_x,\varphi_x\rangle &= k\theta^0\left\langle\varphi_x - k\mathcal{S}^{-1}\varphi_x,\varphi_x\right\rangle \\ &= k\theta^0\langle\varphi_x+\psi,\varphi_x\rangle \\ &= k\theta^0\langle\varphi_x+\psi,\varphi_x+\psi\rangle + \theta^0\langle-k\varphi_x-k\psi,\psi\rangle \\ &= k\theta^0\langle\varphi_x+\psi,\varphi_x+\psi\rangle + \theta^0\langle(\mathcal{S}-kI)\psi,\psi\rangle \\ &= k\theta^0\langle\varphi_x+\psi,\varphi_x+\psi\rangle + b\theta^0\langle\psi_x,\psi_x\rangle. \end{aligned} \tag{2.24}$$

By substitution (2.24) into (2.23), we find that $\|\mathcal{U}\|_{\mathcal{H}}^2 = \|U\|_H^2$, where $U = (\varphi, \phi, \psi, \theta, \omega, \eta)^T$.

The isomorphism P from $\mathcal{H}$ into $H_p = P(\mathcal{H})$ allows us to define a C_0-semigroup $(T(t))_{t\geq 0}$ on H_p given by

$$T(t) = P\mathcal{T}P^{-1}\ ,\ \forall t \geq 0,$$

and its infinitesimal generator A is

$$\begin{aligned} A(\varphi,\phi,\psi,\theta,\omega,\eta)^T &= P\mathcal{A}P^{-1}(\varphi,\phi,\psi,\theta,\omega,\eta)^T \\ &= \begin{pmatrix} \phi \\ -\frac{1}{\rho_1}\left[-k(\varphi_x+\psi)_x + \delta\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)_x\right] \\ -k\mathcal{S}^{-1}\phi_x \\ \omega \\ \eta \\ -\frac{2}{\tau_q^2}\left[\tau_q\eta+\omega-\kappa(\tau_\theta\omega_x+\theta_x)_x+\delta\theta^0\phi_x\right] \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} D(A) &= \left\{U \in H_p : P^{-1}U = \mathcal{U} \in D(\mathcal{A})\right\} \\ &= \left\{\begin{array}{r} U = (\varphi,\phi,\psi,\theta,\omega,\eta)^T \in H_p : \varphi,\psi \in H^2(0,1)\cap H_0^1(0,1); \phi \in H_0^1(0,1); \\ \theta,\omega,\eta \in H_*^1(0,1); \tau_\theta\omega+\theta \in H_*^2(0,1) \end{array}\right\}. \end{aligned}$$

See [10, p. 59]. Moreover, $(T(t))_{t\geq 0}$ is the C_0-semigroup associated with the problem (1.9)–(1.11). Therefore, the main result of well-posedness is given by

Theorem 2.9. *Let* $(\varphi_0, \varphi_1, \theta_0, \theta_1, \theta_2) \in \mathcal{H}$. *Then, there exists a unique solution* $(\varphi, \varphi_t, \psi, \theta, \theta_t, \theta_{tt}) \in C(\mathbb{R}^+, H_p)$ *of problem* (1.9)–(1.11). *Furthermore, if* $(\varphi_0, \varphi_1, \theta_0, \theta_1, \theta_2) \in D(\mathcal{A})$, *then,* $(\varphi, \varphi_t, \psi, \theta, \theta_t, \theta_{tt}) \in C(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H_p)$.

Remark 2.10. Since P is isometric from $\mathcal{H}$ into H, we have, for any $U = (\varphi, \phi, \psi, \theta, \omega, \eta)^T \in H$,

$$\langle AU, U\rangle_H = \left\langle P\mathcal{A}P^{-1}U, U\right\rangle_H = \left\langle \mathcal{A}P^{-1}U, P^{-1}U\right\rangle_{\mathcal{H}} = \langle\mathcal{A}\mathcal{U},\mathcal{U}\rangle_{\mathcal{H}},$$

where $\mathcal{U} = P^{-1}U \in \mathcal{H}$. Therefore, the result obtained in Lemma 2.4 entails

$$\operatorname{Re}\langle AU, U\rangle_H = -\frac{\kappa\tau_q}{2}(2\tau_\theta-\tau_q)\|\tau_\theta\omega_x+\theta_x\|_2^2 - \kappa\|\theta_x\|_2^2. \tag{2.25}$$

Moreover, we have $\rho(\mathcal{A}) = \rho(A)$, which implies that $0 \in \rho(A)$.

3 Exponential stability

In this section, we establish the exponential decay of the system (1.9)–(1.11) under the hypothesis

$$2\tau_\theta > \tau_q. \tag{3.1}$$

For this purpose, we first state a theorem that gives the necessary and sufficient conditions for the exponential stability of the strongly continuous semigroup of contractions associated with a dissipative system on a Hilbert space. See [7], p.7.

Theorem 3.1 (Gearhart–Herbst–Prüss–Huang). *Let $\rho(A)$ be the resolvent set of the operator A and e^{At} be a C_0-semigroup of contractions on a Hilbert space H. Then e^{At} is exponentially stable if and only if*

$$i\mathbb{R} \equiv \{i\lambda : \lambda \in \mathbb{R}\} \subset \rho(A) \tag{3.2}$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \left\| (i\lambda I - A)^{-1} \right\|_{\mathcal{L}(H)} < \infty. \tag{3.3}$$

The main stability result of this manuscript is the following theorem.

Theorem 3.2. *Assume that (3.1) holds. Then, the C_0-semigroup associated with the problem (1.9)–(1.11) is exponentially stable.*

In order to prove the above theorem, we only need to verify the following two propositions.

Proposition 3.3. *The imaginary axis belongs to the resolvent set of A.*

Proof. Assume, by contradiction, that (3.2) is not true. Since $0 \in \rho(A)$, there exists $\lambda \in \mathbb{R}^*$ such that $i\lambda \in \sigma(A)$. Consequently, there exists a sequence of vector functions $(U_n)_{n\in\mathbb{N}^*}$ such that, for any $n \in \mathbb{N}^*$, $U_n = (\varphi_n, \phi_n, \psi_n, \theta_n, \omega_n, \eta_n)^T \in D(A)$, $\|U_n\|_H = 1$ and satisfies

$$\|(i\lambda I - A)U_n\|_H \longrightarrow 0, \tag{3.4}$$

(see [12, Proof of Theorem 2.2.1], p. 25, for details regarding this sequence). This is equivalent to

$$\begin{cases} i\lambda\varphi_n - \phi_n \longrightarrow 0 \quad \text{in } H_0^1(0,1), \\ i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x + \delta\left(\dfrac{\tau_q^2}{2}\eta_n + \tau_q\omega_n + \theta_n\right)_x \longrightarrow 0 \quad \text{in } L^2(0,1), \\ i\lambda\psi_n + k\mathcal{S}^{-1}\phi_{nx} \longrightarrow 0 \quad \text{in } H_0^1(0,1), \\ i\lambda\theta_n - \omega_n \longrightarrow 0 \quad \text{in } H_*^1(0,1), \\ i\lambda\omega_n - \eta_n \longrightarrow 0 \quad \text{in } H_*^1(0,1), \\ \tau_q\left(i\lambda\dfrac{\tau_q}{2} + 1\right)\eta_n + \omega_n - \kappa(\tau_\theta\omega_{nx} + \theta_{nx})_x + \delta\theta^0\phi_{nx} \longrightarrow 0 \quad \text{in } L_*^2(0,1). \end{cases} \tag{3.5}$$

First, it results from (3.4) that

$$\mathrm{Re}\langle(i\lambda I - A)U_n, U_n\rangle_H \longrightarrow 0$$

and by using (2.25), we arrive at

$$\frac{\kappa\tau_q}{2}(2\tau_\theta - \tau_q)\|\omega_{nx}\|_2^2 + \kappa\|\theta_{nx}\|_2^2 \longrightarrow 0.$$

This yields

$$\begin{aligned} \theta_n &\longrightarrow 0 \quad \text{in } H^1_*(0,1), \\ \omega_n &\longrightarrow 0 \quad \text{in } H^1_*(0,1). \end{aligned} \tag{3.6}$$

In addition, $(3.5)_5$ implies that

$$\eta_n \longrightarrow 0 \quad \text{in } H^1_*(0,1).$$

Consequently, (3.5) reduces to

$$\begin{cases} i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x \longrightarrow 0 & \text{in } L^2(0,1), \\ i\lambda\psi_n + k\mathcal{S}^{-1}\phi_{nx} \longrightarrow 0 & \text{in } H^1_0(0,1), \\ -\kappa(\tau_\theta\omega_{nx} + \theta_{nx})_x + \delta\theta^0\phi_{nx} \longrightarrow 0 & \text{in } L^2_*(0,1). \end{cases} \tag{3.7}$$

Now, by taking the inner product of $-\kappa(\tau_\theta\omega_{nx} + \theta_{nx})_x + \delta\theta^0\phi_{nx}$ with $k\overline{\mathcal{B}\varphi_{nx}}$ in $L^2(0,1)$, we obtain

$$\begin{aligned} \left\langle -\kappa(\tau_\theta\omega_{nx} + \theta_{nx})_x + \delta\theta^0\phi_{nx}, k\overline{\mathcal{B}\varphi_{nx}} \right\rangle = & -\kappa\left\langle (\tau_\theta\omega_{nx} + \theta_{nx}), \overline{i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x} \right\rangle \\ & + i\lambda\rho_1\kappa\left\langle (\tau_\theta\omega_{nx} + \theta_{nx}), \overline{\phi_n} \right\rangle - i\lambda\delta\theta^0\rho_1\|\phi_n\|_2^2 \\ & + \delta\theta^0\left\langle \phi_n, \overline{i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x} \right\rangle. \end{aligned} \tag{3.8}$$

Since $\|U_n\|_H = 1$, then $(\mathcal{B}\varphi_{nx})_{n\in\mathbb{N}^*}$ and $(\phi_n)_{n\in\mathbb{N}^*}$ are bounded in $L^2(0,1)$. So, the Cauchy–Schwarz inequality, (3.6), $(3.7)_1$ and $(3.7)_3$ lead to

$$\begin{aligned} \left\langle -\kappa(\tau_\theta\omega_{nx} + \theta_{nx})_x + \delta\theta^0\phi_{nx}, k\overline{\mathcal{B}\varphi_{nx}} \right\rangle &\longrightarrow 0, \\ \left\langle (\tau_\theta\omega_{nx} + \theta_{nx}), \overline{i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x} \right\rangle &\longrightarrow 0, \\ \left\langle (\tau_\theta\omega_{nx} + \theta_{nx}), \overline{\phi_n} \right\rangle &\longrightarrow 0, \\ \left\langle \phi_n, \overline{i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x} \right\rangle &\longrightarrow 0. \end{aligned}$$

Hence, by taking the limit in (3.8), we get

$$\phi_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{3.9}$$

In light of the definition of the operator $\mathcal{S}$, we find that

$$\left\|\mathcal{S}^{-1}\phi_{nx}\right\|_{H^1_0} \le \left\|\mathcal{S}^{-1}\right\| \|\phi_{nx}\|_{H^{-1}} \le \left\|\mathcal{S}^{-1}\right\| \|\phi_n\|_2,$$

in other words, $\mathcal{S}^{-1} \circ \partial_x : L^2(0,1) \longrightarrow H^1_0(0,1)$ is a bounded operator. Thus, we have

$$\mathcal{S}^{-1}\phi_{nx} \longrightarrow 0 \quad \text{in } H^1_0(0,1).$$

Therefore, $(3.7)_2$ implies that

$$\psi_n \longrightarrow 0 \quad \text{in } H^1_0(0,1).$$

Next, the definition of the operator $\mathcal{B}$ gives

$$\langle i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x, \overline{\varphi_n} \rangle = \langle i\lambda\rho_1\phi_n, \overline{\varphi_n} \rangle + k\left(\|\varphi_{nx}\|_2^2 - k\left\langle \mathcal{S}^{-1}\varphi_{nx}, \overline{\varphi_{nx}} \right\rangle \right)$$

and taking into account Remark 2.1, the Cauchy–Schwarz inequality leads to

$$\langle i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x, \overline{\varphi_n} \rangle \ge \langle i\lambda\rho_1\phi_n, \overline{\varphi_n} \rangle + k\left(1 - k\left\|\mathcal{S}^{-1}\right\|\right) \|\varphi_{nx}\|_2^2. \tag{3.10}$$

It is easy to verify that

$$\langle i\lambda\rho_1\phi_n - k(\mathcal{B}\varphi_{nx})_x, \overline{\varphi_n}\rangle \longrightarrow 0,$$
$$\langle i\lambda\rho_1\phi_n, \overline{\varphi_n}\rangle \longrightarrow 0,$$

hence, the fact that $1 - k\|\mathcal{S}^{-1}\| > 0$ and (3.10) entail

$$\varphi_n \longrightarrow 0 \quad \text{in } H_0^1(0,1).$$

According to the above calculations, we conclude that $\|U_n\|_H \longrightarrow 0$, which contradicts the fact that $\|U_n\|_H = 1$ and the proof is complete. □

For $\lambda \in \mathbb{R}^*$, we now consider the following resolvent equation associated with the operator A

$$i\lambda U - AU = \mathcal{F}, \tag{3.11}$$

where $U = (\varphi, \phi, \psi, \theta, \omega, \eta)^T \in D(A)$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)^T \in H_p$. Using the components of U and $\mathcal{F}$, (3.11) takes the form

$$\begin{cases} i\lambda\varphi - \phi = f_1, \\ i\lambda\rho_1\phi - k(\varphi_x + \psi)_x + \delta\left(\dfrac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)_x = \rho_1 f_2, \\ -i\lambda b\psi_{xx} + i\lambda k(\varphi_x + \psi) = -bf_{3xx} + k(f_{1x} + f_3), \\ i\lambda\theta - \omega = f_4, \\ i\lambda\omega - \eta = f_5, \\ i\lambda\left(\dfrac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right) - \kappa(\tau_\theta\omega_x + \theta_x)_x + \delta\theta^0\phi_x = \dfrac{\tau_q^2}{2}f_6 + \tau_q f_5 + f_4. \end{cases} \tag{3.12}$$

Proposition 3.4. *Let A be the operator defined above, then*

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty.$$

To prove Proposition 3.4, we need the following lemmas

Lemma 3.5. *Under the above notations and the hypothesis (3.1), we have*

$$\|\tau_\theta\omega_x + \theta_x\|_2^2 + \|\theta_x\|_2^2 \le m\|\mathcal{F}\|_H\|U\|_H, \tag{3.13}$$

for some constant $m > 0$.

Proof. Estimate (3.13) follows directly from (2.25), (3.11) and Cauchy–Schwarz inequality. □

Lemma 3.6. *Under the above notations and the hypothesis (3.1), we have, for any $\varepsilon > 0$,*

$$\left\|\left(\frac{\tau_q^2}{2}\eta + \tau_q\omega + \theta\right)\right\|_2^2 \le m\|\mathcal{F}\|_H\|U\|_H + \varepsilon\|\phi\|_2^2 + m\left(1 + \frac{1}{\varepsilon}\right)\left(\|\tau_\theta\omega_x + \theta_x\|_2^2 + \|\theta_x\|_2^2\right), \tag{3.14}$$

where $m > 0$ is a constant.

Proof. Multiplying (3.12)$_6$ by $\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}$ and integrating over $(0,1)$, we arrive at

$$\begin{aligned}\int_0^1\left(\frac{\tau_q^2}{2}f_6+\tau_q f_5+f_4\right)\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}dx &= i\lambda\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}dx\\ &\quad-\kappa\int_0^1(\tau_\theta\omega_x+\theta_x)_x\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}dx\\ &\quad+\delta\theta^0\int_0^1\phi_x\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}dx.\end{aligned}$$

Using (3.12)$_4$, (3.12)$_5$ and integration by parts, it follows that

$$\begin{aligned}\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2 &= -\int_0^1\left(\frac{\tau_q^2}{2}f_6+\tau_q f_5+f_4\right)\overline{\left(\frac{\tau_q^2}{2}\omega+\tau_q\theta\right)}dx\\ &\quad+\int_0^1\overline{\theta}\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)dx\\ &\quad-\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{\left(\frac{\tau_q^2}{2}f_5+\tau_q f_4\right)}dx\\ &\quad+\kappa\int_0^1(\tau_\theta\omega_x+\theta_x)\overline{\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)}dx\\ &\quad-\delta\theta^0\int_0^1\phi\overline{\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)}dx.\end{aligned}$$

So, the Cauchy–Schwarz, Poincaré and Young inequalities give

$$\begin{aligned}\frac{1}{2}\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2 &\le c_p\left\|\left(\frac{\tau_q^2}{2}f_6+\tau_q f_5+f_4\right)\right\|_2\left\|\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)\right\|_2+\frac{c_p^2}{2}\|\theta_x\|_2^2\\ &\quad+c_p\left\|\left(\frac{\tau_q^2}{2}f_{5x}+\tau_q f_{4x}\right)\right\|_2\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2\\ &\quad+\frac{\kappa}{2}\|(\tau_\theta\omega_x+\theta_x)\|_2^2+\frac{\kappa}{2}\left\|\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)\right\|_2^2+\varepsilon\|\phi\|_2^2\\ &\quad+\frac{(\delta\theta^0)^2}{4\varepsilon}\left\|\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)\right\|_2^2.\end{aligned}\tag{3.15}$$

Finally, from the two equalities

$$\left(\frac{\tau_q^2}{2}\omega_x+\tau_q\theta_x\right)=\frac{\tau_q^2}{2\tau_\theta}(\tau_\theta\omega_x+\theta_x)+\frac{\tau_q}{2\tau_\theta}(2\tau_\theta-\tau_q)\theta_x,\tag{3.16}$$

$$\left(\frac{\tau_q^2}{2}f_{5x}+\tau_q f_{4x}\right)=\frac{\tau_q^2}{2\tau_\theta}(\tau_\theta f_{5x}+f_{4x})+\frac{\tau_q}{2\tau_\theta}(2\tau_\theta-\tau_q)f_{4x},\tag{3.17}$$

we conclude (3.14). □

Lemma 3.7. *Under the above notations, we have, for any $\mu > 0$ and for some constant $m > 0$,*

$$
\begin{aligned}
\|\phi\|_2^2 \le\ & m\|\mathcal{F}\|_H\|U\|_H + \mu\|(\varphi_x+\psi)\|_2^2 \\
& + m\left(1+\frac{1}{\mu}\right)\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2 + m\|(\tau_\theta\omega_x+\theta_x)\|_2^2.
\end{aligned} \tag{3.18}
$$

Proof. Integrating the last equation of (3.12) over $(0,x)$, multiplying it by $\rho_1\overline{\phi}$ and then integrating on $(0,1)$, we obtain

$$
\begin{aligned}
\rho_1\delta\theta^0\|\phi\|_2^2 =\ & \rho_1\int_0^1\overline{\phi}\int_0^x\left(\frac{\tau_q^2}{2}f_6+\tau_q f_5+f_4\right)(y)dydx + \rho_1\kappa\int_0^1\overline{\phi}(\tau_\theta\omega_x+\theta_x)dx \\
& - i\lambda\rho_1\int_0^1\overline{\phi}\int_0^x\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)(y)dydx.
\end{aligned} \tag{3.19}
$$

On the other hand, from $(3.12)_2$, we have

$$
\begin{aligned}
-i\lambda\rho_1\int_0^1\overline{\phi}\int_0^x\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)(y)dydx =\ & -k\int_0^1\overline{(\varphi_x+\psi)}\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)dx \\
& + \rho_1\int_0^1\overline{f_2}\int_0^x\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)(y)dydx \\
& + \delta\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2.
\end{aligned} \tag{3.20}
$$

By substituting (3.20) into (3.19), we get

$$
\begin{aligned}
\rho_1\delta\theta^0\|\phi\|_2^2 =\ & \rho_1\int_0^1\overline{\phi}\int_0^x\left(\frac{\tau_q^2}{2}f_6+\tau_q f_5+f_4\right)(y)dydx + \rho_1\kappa\int_0^1\overline{\phi}(\tau_\theta\omega_x+\theta_x)dx \\
& - k\int_0^1\overline{(\varphi_x+\psi)}\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)dx + \delta\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2 \\
& + \rho_1\int_0^1\overline{f_2}\int_0^x\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)(y)dydx.
\end{aligned}
$$

Therefore, the Cauchy–Schwarz and Young inequalities yield (3.18). □

Lemma 3.8. *Under the above notations, we have*

$$
\|\psi_x\|_2^2 + \|(\varphi_x+\psi)\|_2^2 \le m\left(1+\frac{1}{|\lambda|}\right)\|\mathcal{F}\|_H\|U\|_H + m\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2 + m\|\phi\|_2^2, \tag{3.21}
$$

for some constant $m > 0$.

Proof. Multiplying $(3.12)_3$ by $\overline{\psi}$, integrating over $(0,1)$ and using integration by parts, we obtain

$$
\begin{aligned}
& i\lambda b\|\psi_x\|_2^2 + i\lambda k\|(\varphi_x+\psi)\|_2^2 \\
&\quad = b\int_0^1 f_{3x}\overline{\psi_x}dx + k\int_0^1(f_{1x}+f_3)\overline{\psi}dx - i\lambda k\int_0^1(\varphi_x+\psi)_x\overline{\varphi}dx.
\end{aligned} \tag{3.22}
$$

On the other hand, multiplying $(3.12)_2$ by $\overline{i\lambda\varphi}$ and integrating over $(0,1)$, we arrive at

$$\begin{aligned} &-i\lambda k\int_0^1(\varphi_x+\psi)_x\overline{\varphi}dx\\ &\quad=i\lambda\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{(\varphi_x+\psi)}dx-i\lambda\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{\psi}dx\\ &\qquad+i\lambda\rho_1\int_0^1\phi(\overline{i\lambda\varphi})dx+i\lambda\rho_1\int_0^1 f_2\overline{\varphi}dx. \end{aligned}\tag{3.23}$$

By substituting (3.23) into (3.22), we find that

$$\begin{aligned} i\lambda b\|\psi_x\|_2^2+i\lambda k\|(\varphi_x+\psi)\|_2^2=b\int_0^1 f_{3x}\overline{\psi_x}dx+k\int_0^1(f_{1x}+f_3)\overline{\psi}dx+i\lambda\rho_1\int_0^1 f_2\overline{\varphi}dx\\ +i\lambda\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{(\varphi_x+\psi)}dx\\ -i\lambda\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{\psi}dx\\ +i\lambda\rho_1\int_0^1\phi(\overline{i\lambda\varphi})dx. \end{aligned}$$

Using the first equation of (3.12), we get

$$\begin{aligned} b\|\psi_x\|_2^2+k\|(\varphi_x+\psi)\|_2^2=\frac{b}{i\lambda}\int_0^1 f_{3x}\overline{\psi_x}dx+\frac{k}{i\lambda}\int_0^1(f_{1x}+f_3)\overline{\psi}dx+\rho_1\int_0^1 f_2\overline{\varphi}dx\\ +\rho_1\int_0^1\phi\overline{f_1}dx+\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{(\varphi_x+\psi)}dx\\ -\delta\int_0^1\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\overline{\psi}dx+\rho_1\|\phi\|_2^2. \end{aligned}$$

Thanks to the Cauchy–Schwarz, Poincaré and Young inequalities, estimate (3.21) follows. □

Proof of Proposition 3.4. First, by substituting (3.13) into (3.14), we have, for any $\varepsilon>0$,

$$\left\|\left(\frac{\tau_q^2}{2}\eta+\tau_q\omega+\theta\right)\right\|_2^2\le m_\varepsilon\|\mathcal{F}\|_H\|U\|_H+\varepsilon\|\phi\|_2^2,\tag{3.24}$$

where the constant $m_\varepsilon>0$ depends on ε. Next, by substituting (3.13) and (3.24) into (3.18), we find that, for any ϵ, $\mu>0$,

$$\|\phi\|_2^2\le m_{\epsilon,\mu}\|\mathcal{F}\|_H\|U\|_H+\mu\|(\varphi_x+\psi)\|_2^2+\epsilon\left(1+\frac{1}{\mu}\right)\|\phi\|_2^2,$$

where $m_{\epsilon,\mu}$ is a positive constant depending on ϵ and μ. Hence choosing $\epsilon=\frac{\mu}{2(1+\mu)}$, we get, for any $\zeta=2\mu>0$,

$$\|\phi\|_2^2\le m_\zeta\|\mathcal{F}\|_H\|U\|_H+\zeta\|(\varphi_x+\psi)\|_2^2,\tag{3.25}$$

where the constant $m_\zeta>0$ depends on ζ. Now, by substituting (3.24) and (3.25) into (3.21), we obtain, for any $\tilde{\varepsilon}$, $\tilde{\zeta}>0$,

$$\|\psi_x\|_2^2+\|(\varphi_x+\psi)\|_2^2\le m_{\tilde{\varepsilon},\tilde{\zeta}}\left(1+\frac{1}{|\lambda|}\right)\|\mathcal{F}\|_H\|U\|_H+\tilde{\varepsilon}\|\phi\|_2^2+\tilde{\zeta}\|(\varphi_x+\psi)\|_2^2,\tag{3.26}$$

where $m_{\tilde{\epsilon},\tilde{\zeta}}$ is a positive constant that depends on $\tilde{\epsilon}$ and $\tilde{\zeta}$. Finally, using (3.13), (3.24), (3.25) and (3.26); and choosing ϵ, $\tilde{\epsilon}$, ζ, $\tilde{\zeta}$ small enough, we arrive at

$$\|U\|_H^2 \le C\left(1+\frac{1}{|\lambda|}\right)\|\mathcal{F}\|_H\|U\|_H,$$

where C is a positive constant. When λ goes to infinity, we get

$$\|U\|_H \le C\|\mathcal{F}\|_H,$$

with this, we have finished the proof of Proposition 3.4. □

Proof of Theorem 3.2. The proof follows directly from the above two propositions. □

4 Numerical approximation

In this section, we propose a finite-element approximation to the system (1.9) with the boundary conditions (1.10) and under the initial conditions (1.11). To obtain the weak formulation, we multiply, respectively, the equations of (1.9) by test functions $\xi,\eta \in H_0^1(0,1)$ and $\chi \in H_*^1(0,1)$, then integrate by parts to find the following

$$\begin{cases} \rho_1(\varphi_{tt},\xi) + k(\varphi_x+\psi,\xi_x) - \delta\left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta, \xi_x\right) = 0, \\ b(\psi_x,\eta_x) + k(\varphi_x+\psi,\eta) = 0, \\ \left(\dfrac{\tau_q^2}{2}\theta_{ttt} + \tau_q\theta_{tt} + \theta_t, \chi\right) + \kappa(\tau_\theta\theta_{tx} + \theta_x, \chi_x) - \delta\theta^0(\varphi_t,\chi_x) = 0, \end{cases} \tag{4.1}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(0,1)$. Let us partition the interval $(0,1)$ into subintervals $I_i = (x_{i-1},x_i)$ of length $h = 1/m$ with $0 = x_0 < x_1 < \cdots < x_m = 1$ and define

$$P_h = \left\{u \in H^1(0,1), u|_{I_i} \text{ is a linear polynomial}\right\},$$

with its subspaces

$$P_h^0 = P_h \cap H_0^1(0,1) \quad \text{and} \quad P_h^* = P_h \cap H_*^1(0,1).$$

For a given final time T and a positive integer N, let $\Delta t = T/N$ be the time step and $t_n = n\Delta t$, $n = 0,\dots,N$. The approximate solution for (4.1) is to find φ_h^n, $\psi_h^n \in P_h^0$ and $\theta_h^n \in P_h^*$, such that, for all ξ_h, $\eta_h \in P_h^0$ and $\chi_h \in P_h^*$,

$$\begin{cases} \rho_1(\varphi_{htt}^n,\xi_h) + k(\varphi_{hx}^n+\psi_h^n,\xi_{hx}) - \delta\left(\dfrac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n, \xi_{hx}\right) = 0, \\ b(\psi_{hx}^n,\eta_{hx}) + k(\varphi_{hx}^n+\psi_h^n,\eta_h) = 0, \\ \left(\dfrac{\tau_q^2}{2}\theta_{httt}^n + \tau_q\theta_{htt}^n + \theta_{ht}^n, \chi_h\right) + \kappa(\tau_\theta\theta_{hxt}^n + \theta_{hx}^n, \chi_{hx}) - \delta\theta^0(\varphi_{ht}^n,\chi_{hx}) = 0, \end{cases} \tag{4.2}$$

where the notations φ_h^0, φ_{ht}^0, θ_h^0, θ_{ht}^0 and θ_{htt}^0 are adequate approximations to φ_0, φ_1, θ_0, θ_1 and θ_2, respectively. Then, the discrete energy is given by

$$\begin{aligned} E_h^n = &\frac{1}{2}\left(\|\frac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n\|_2^2 + \kappa\tau\|\theta_{hx}^n\|_2^2 + \frac{\kappa\tau_q^2}{2\tau_\theta}\|\tau_\theta\theta_{hxt}^n + \theta_{hx}^n\|_2^2\right) \\ &+ \frac{1}{2}\theta^0\left(\rho_1\|\varphi_{ht}^n\|_2^2 + k\|\varphi_{hx}^n + \psi_h^n\|_2^2 + b\|\psi_{hx}^n\|_2^2\right), \end{aligned} \tag{4.3}$$

where

$$\tau = \left(\frac{2\tau_\theta - \tau_q}{2\tau_\theta}\right)\tau_q + \tau_\theta > 0.$$

The next result is a discrete version of the energy decay property satisfied by the solution of system (1.9).

Lemma 4.1. *For $n = 1, 2, \ldots, N$, the discrete energy satisfies the following stability property*

$$\frac{E_h^n - E_h^{n-1}}{\Delta t} \leq 0. \tag{4.4}$$

Proof. Choosing $\xi_h = \theta^0 \varphi_{ht}^n$, $\eta_h = \theta^0 \psi_{ht}^n$ and $\chi_h = \frac{\tau_q^2}{2}\theta_{htt}^n + \tau_q \theta_{ht}^n + \theta_h^n$ in (4.2) leads to

$$\begin{cases} \dfrac{\rho_1 \theta^0}{\Delta t}\left(\varphi_{ht}^n - \varphi_{ht}^{n-1}, \varphi_{ht}^n\right) + k\theta^0(\varphi_{hx}^n + \psi_h^n, \varphi_{htx}^n) - \delta\theta^0\left(\dfrac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n, \varphi_{htx}^n\right) = 0, \\ \dfrac{b\theta^0}{\Delta t}\left(\psi_{hx}^n, \psi_{hx}^n - \psi_{hx}^{n-1}\right) + k\theta^0(\varphi_{hx}^n + \psi_h^n, \psi_{ht}^n) = 0, \\ \dfrac{1}{\Delta t}\left(\dfrac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n - \left(\dfrac{\tau_q^2}{2}\theta_{htt}^{n-1} + \tau_q\theta_{ht}^{n-1} + \theta_h^{n-1}\right), \dfrac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n\right) \\ \quad + \kappa\left(\tau_\theta\theta_{hxt}^n + \theta_{hx}^n, \dfrac{\tau_q^2}{2}\theta_{httx}^n + \tau_q\theta_{htx}^n + \theta_{hx}^n\right) + \delta\theta^0\left(\varphi_{htx}^n, \dfrac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n\right) = 0. \end{cases} \tag{4.5}$$

Adding the latter equations and thanks to the following identity

$$(a - b, a) = \frac{1}{2}(\|a - b\|_2^2 + \|a\|_2^2 - \|b\|_2^2),$$

we find that

$$\begin{aligned} &\frac{\rho_1\theta^0}{2\Delta t}\left(\|\varphi_{ht}^n\|_2^2 - \|\varphi_{ht}^{n-1}\|_2^2\right) + k\theta^0(\varphi_{hx}^n + \psi_h^n, \varphi_{htx}^n + \psi_{ht}^n) + \frac{b\theta^0}{2\Delta t}\left(\|\psi_{hx}^n\|_2^2 - \|\psi_{hx}^{n-1}\|\right) \\ &\quad + \frac{1}{2\Delta t}\left(\left\|\frac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n\right\|_2^2 - \left\|\frac{\tau_q^2}{2}\theta_{htt}^{n-1} + \tau_q\theta_{ht}^{n-1} + \theta_h^{n-1}\right\|_2^2\right) \\ &\quad + \kappa\left(\tau_\theta\theta_{hxt}^n + \theta_{hx}^n, \frac{\tau_q^2}{2}\theta_{httx}^n + \tau_q\theta_{htx}^n + \theta_{hx}^n\right) \\ &= -\frac{\rho_1\theta^0}{2\Delta t}\|\varphi_{ht}^n - \varphi_{ht}^{n-1}\|_2^2 - \frac{b\theta^0}{2\Delta t}\|\psi_{hx}^n - \psi_{hx}^{n-1}\|_2^2 \\ &\quad - \frac{1}{2\Delta t}\left\|\frac{\tau_q^2}{2}\theta_{htt}^n + \tau_q\theta_{ht}^n + \theta_h^n - \left(\frac{\tau_q^2}{2}\theta_{htt}^{n-1} + \tau_q\theta_{ht}^{n-1} + \theta_h^{n-1}\right)\right\|_2^2. \end{aligned}$$

Next, we have that

$$\begin{aligned} &k\theta^0(\varphi_{hx}^n + \psi_h^n, \varphi_{htx}^n + \psi_{ht}^n) \\ &\quad = \frac{k\theta^0}{2\Delta t}\left(\|\varphi_{hx}^n + \psi_h^n\|_2^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|_2^2 + \|\varphi_{hx}^n + \psi_h^n - (\varphi_{hx}^{n-1} + \psi_h^{n-1})\|_2^2\right) \end{aligned}$$

and

$$\begin{aligned}
&\kappa\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\frac{\tau_q^2}{2}\theta^n_{httx}+\tau_q\theta^n_{htx}+\theta^n_{hx}\right)\\
&\quad=\frac{\kappa\tau_q^2\tau_\theta}{2\Delta t}\left(\theta^n_{hxt},\theta^n_{htx}-\theta^{n-1}_{htx}\right)+\frac{\kappa\tau_q^2}{2\Delta t}\left(\theta^n_{hx},\theta^n_{htx}-\theta^{n-1}_{htx}\right)\\
&\qquad+\kappa\tau_\theta\tau_q\|\theta^n_{htx}\|_2^2+\kappa(\tau_\theta+\tau_q)(\theta^n_{hxt},\theta^n_{hx})+\kappa\|\theta^n_{hx}\|_2^2\\
&\quad=\frac{\kappa\tau_q^2}{2\tau_\theta\Delta t}\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\tau_\theta\theta^n_{htx}+\theta^n_{hx}-\left(\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx}\right)\right)-\frac{\kappa\tau_q^2}{2\tau_\theta\Delta t}\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\theta^n_{hx}-\theta^{n-1}_{hx}\right)\\
&\qquad+\kappa\tau_\theta\tau_q\|\theta^n_{htx}\|_2^2+\kappa(\tau_\theta+\tau_q)(\theta^n_{hxt},\theta^n_{hx})+\kappa\|\theta^n_{hx}\|_2^2\\
&\quad=\frac{\kappa\tau_q^2}{2\tau_\theta\Delta t}\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\tau_\theta\theta^n_{htx}+\theta^n_{hx}-\left(\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx}\right)\right)-\frac{\kappa\tau_q^2}{2\tau_\theta}(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\theta^n_{hxt})\\
&\qquad+\kappa\tau_\theta\tau_q\|\theta^n_{htx}\|_2^2+\kappa(\tau_\theta+\tau_q)(\theta^n_{hxt},\theta^n_{hx})+\kappa\|\theta^n_{hx}\|_2^2\\
&\quad=\frac{\kappa\tau_q^2}{2\tau_\theta\Delta t}\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\tau_\theta\theta^n_{htx}+\theta^n_{hx}-\left(\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx}\right)\right)+\kappa\left((\tau_\theta+\tau_q)-\frac{\tau_q^2}{2\tau_\theta}\right)(\theta^n_{hx},\theta^n_{hxt})\\
&\qquad+\kappa\left(\tau_\theta\tau_q-\frac{\tau_q^2}{2}\right)\|\theta^n_{htx}\|_2^2+\kappa\|\theta^n_{hx}\|_2^2,
\end{aligned}$$

which gives

$$\begin{aligned}
&\kappa\left(\tau_\theta\theta^n_{hxt}+\theta^n_{hx},\frac{\tau_q^2}{2}\theta^n_{httx}+\tau_q\theta^n_{htx}+\theta^n_{hx}\right)\\
&\quad=\frac{\kappa\tau_q^2}{4\Delta t\tau_\theta}\Big(\|\tau_\theta\theta^n_{htx}+\theta^n_{hx}\|_2^2-\|\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx}\|_2^2\\
&\qquad+\|\tau_\theta\theta^n_{htx}+\theta^n_{hx}-(\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx})\|_2^2\Big)+\tau'\|\theta^n_{htx}\|_2^2+\kappa\|\theta^n_{hx}\|_2^2\\
&\qquad+\frac{\kappa}{2\Delta t}\tau\Big(\|\theta^n_{hx}\|_2^2-\|\theta^{n-1}_{hx}\|_2^2+\|\theta^n_{hx}-\theta^{n-1}_{hx}\|_2^2\Big),
\end{aligned}$$

where $\tau'=\kappa(\tau_q\tau_\theta-\frac{\tau_q^2}{2})>0$ and $\tau=(\frac{2\tau_\theta-\tau_q}{2\tau_\theta})\tau_q+\tau_\theta>0$.

These results together yield

$$\begin{aligned}
\frac{E^n_h-E^{n-1}_h}{\Delta t}&=-\frac{\rho_1\theta^0}{2\Delta t}\|\varphi^n_{ht}-\varphi^{n-1}_{ht}\|_2^2-\frac{b\theta^0}{2\Delta t}\|\psi^n_{hx}-\psi^{n-1}_{hx}\|_2^2\\
&\quad-\frac{1}{2\Delta t}\left\|\frac{\tau_q^2}{2}\theta^n_{htt}+\tau_q\theta^n_{ht}+\theta^n_h-\left(\frac{\tau_q^2}{2}\theta^{n-1}_{htt}+\tau_q\theta^{n-1}_{ht}+\theta^{n-1}_h\right)\right\|_2^2\\
&\quad-\frac{k\theta^0}{2\Delta t}\|\varphi^n_{hx}+\psi^n_h-(\varphi^{n-1}_{hx}+\psi^{n-1}_h)\|_2^2-\frac{\kappa\tau_q^2}{4\Delta t\tau_\theta}\|\tau_\theta\theta^n_{htx}+\theta^n_{hx}-(\tau_\theta\theta^{n-1}_{htx}+\theta^{n-1}_{hx})\|_2^2\\
&\quad-\frac{\kappa}{2\Delta t}\tau\|\theta^n_{hx}-\theta^{n-1}_{hx}\|_2^2-\tau'\|\theta^n_{htx}\|_2^2-\kappa\|\theta^n_{hx}\|_2^2\le 0,
\end{aligned}$$

and the lemma is proved recalling the definition of the discrete energy (4.3). □

4.1 Numerical simulation

The solution at the time step t_n is given by the following matrix system

$$\begin{cases} \rho_1 \mathbf{M}\varphi^n_{htt} + k\mathbf{R}\varphi^n_{hx} = -k\mathbf{G}\psi^{n-1}_h + \delta\mathbf{G}\left(\frac{\tau_q^2}{2}\theta^{n-1}_{htt} + \tau_q\theta^{n-1}_{ht} + \theta^{n-1}_h\right), \\ (b\mathbf{R} + k\mathbf{M})\psi^n_h = k\mathbf{G}\varphi^n_h, \\ \frac{\tau_q^2}{2}\mathbf{M}\theta^n_{httt} + \tau_q\mathbf{M}\theta^n_{htt} + (\mathbf{M} + \kappa\tau_\theta\mathbf{R})\theta^n_{ht} + \kappa\mathbf{R}\theta^n_h = \delta\theta^0\mathbf{G}\varphi^n_{ht}, \end{cases} \tag{4.6}$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{R}$, $\mathbf{G}$ are the stiffness matrices.

To solve the coupled system (4.6), we use a second-order Newmark method for the first equation and a third-order Newmark method for the third equation (see [13] and references therein). We divide the spatial interval $[0,1]$ into $m = 50$ subintervals, where the spatial step size $h = 0.02$. The temporal interval $[0,T] = [0,100]$ with a time-step size $\Delta t = 10^{-2}$.

We run our code for N time steps ($N = T/\Delta t$), using the following initial conditions:

$$\varphi^0_h(x) = \sin(\pi x), \quad \varphi^0_{ht}(x) = \frac{1}{2}\sin(\pi x)$$

$$\theta^0_h(x) = \cos(\pi x), \quad \theta^0_{ht}(x) = \cos(\pi x), \quad \theta^0_{htt}(x) = \cos(\pi x)$$

and we investigate the decay of discrete energy by considering the following data

$$\rho_1 = 10, \quad k = 5, \quad \delta = \kappa = 0.1, \quad b = 20,$$

$$\tau_q = 1.4, \quad \tau_\theta = 1.5, \quad \theta^0 = 1.$$

In Figure 4.1, we plot the cross section cuts for the approximate solution (φ, ψ, θ) at different points ($x = 0.3$, $x = 0.5$ and $x = 0.9$). Next, the Figure 4.2 shows the behavior of the discrete energy curves (4.2) using Simpson method to approximate the norm L^2 in (4.3). The Figure 4.2 shows that the energy of the system (1.9) decreases exponentially.

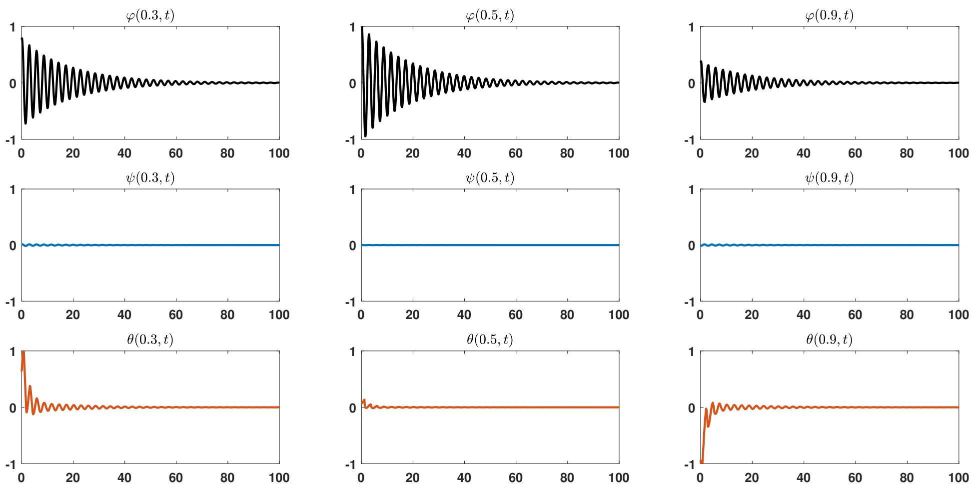

Figure 4.1: Damping cross section waves of the solutions

As a conclusion, we observed that the numerical solution converges to zero and an exponential decay seems to be reached which is compatible with the theoretical results.

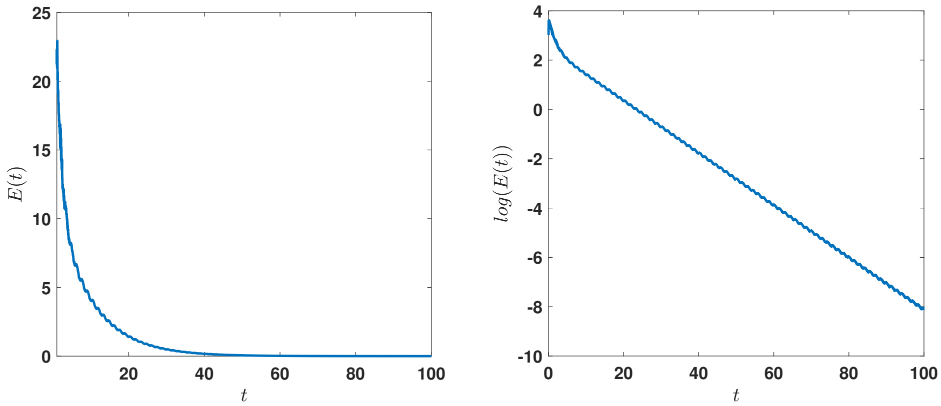

Figure 4.2: The evolution of the discrete energy in natural and log.

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Authors' contributions

All the are participate equally in the paper.

Conflict of interest

There is no conflict of Interest.

Availability of data

There is no data available.

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