



Normalized solutions for the fractional Hardy critical Schrödinger system

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Abstract. In this paper, we consider the existence and multiplicity of normalized solutions for the following fractional Schrödinger system with Hardy critical coupled term:

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = |u|^{p-2} u + \frac{\alpha v}{2_{s,\tau}^*} \frac{|u|^{\alpha-2} |v|^\beta u}{|x|^\tau}, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + \lambda_2 v = |v|^{p-2} v + \frac{\beta v}{2_{s,\tau}^*} \frac{|u|^\alpha |v|^{\beta-2} v}{|x|^\tau}, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a^2, \quad \int_{\mathbb{R}^N} v^2 dx = b^2, \end{cases}$$

where $N \geq 3s$, $s \in (0, 1)$, $\tau \in (0, 2s)$, $a, b > 0$, $v > 0$, and the exponents α, β, p satisfy

$$\alpha > 1, \beta > 1, \alpha + \beta = 2_{s,\tau}^* := \frac{2(N-\tau)}{N-2s}, \quad 2 < p < 2_s^* := \frac{2N}{N-2s},$$

the parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ appear as unknown Lagrange multipliers. In the case of $2 < p < 2 + \frac{4s}{N}$, we get the existence of a normalized ground state solution and a mountain-pass type normalized solution by employing the minimax principle, the Pohozaev manifold analysis, homotopy-stable family and the mountain pass theorem. In the case of $2 + \frac{4s}{N} \leq p < 2_s^*$, we obtain the existence of a mountain-pass type normalized solution by the Pohozaev manifold analysis and the mountain pass theorem.

Keywords: Hardy nonlinearity, fractional Schrödinger system, normalized solution, critical exponent, Pohozaev manifold.

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1 Introduction and main results

In this paper, we focus on the normalized solutions for the following fractional Schrödinger system with Hardy–Sobolev critical coupled term:

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = |u|^{p-2} u + \frac{\alpha v}{2_{s,\tau}^*} \frac{|u|^{\alpha-2} |v|^\beta u}{|x|^\tau}, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + \lambda_2 v = |v|^{p-2} v + \frac{\beta v}{2_{s,\tau}^*} \frac{|u|^\alpha |v|^{\beta-2} v}{|x|^\tau}, & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

having prescribed L^2 - norm

$$\int_{\mathbb{R}^N} u^2 dx = a^2, \quad \int_{\mathbb{R}^N} v^2 dx = b^2, \quad (1.2)$$

where $N \geq 3s$, $s \in (0, 1)$, $a, b > 0$, $v > 0$, the exponents p, α, β satisfy

$$\alpha > 1, \quad \beta > 1, \quad \alpha + \beta = 2_{s,\tau}^*, \quad 2 < p < 2_s^*,$$

and $\lambda_1, \lambda_2 \in \mathbb{R}$ appear as Lagrange multipliers which are part of the unknowns. Actually, $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent and $2_{s,\tau}^* := \frac{2(N-\tau)}{N-2s}$ is the fractional critical Hardy–Sobolev exponent. Up to normalization, the fractional Laplacian $(-\Delta)^s$ is defined for $u \in S(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $S(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing smooth functions, P.V. stands for the principle value of the integral and $C_{N,s}$ is the positive normalization constant. As a worthy tool in investigating numerous phenomena in engineering and physics, the fractional Laplacian appears in many diverse domains, including optimization, phase transitions, power science, anomalous diffusion, biology, water waves and so on. We refer to [1, 5, 10] for more related backgrounds.

Throughout this paper, $L^r(\mathbb{R}^N)$ denotes the Lebesgue space endowed with the norm $\|u\|_r = (\int_{\mathbb{R}^N} |u|^r dx)^{\frac{1}{r}}$ for any $1 \leq r < \infty$. $H^s(\mathbb{R}^N)$ is the fractional Hilbert space defined as

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^N) \right\},$$

and the standard inner product and norm are given respectively by

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv \right) dx, \quad \|u\|_{H^s}^2 = \langle u, u \rangle = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + \|u\|_2^2,$$

where

$$\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

Define

$$H := H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N),$$

which is equipped with the norm $\|(u, v)\|_H^2 := \|u\|_{H^s}^2 + \|v\|_{H^s}^2$.

In a variational point of view, normalized solutions to system (1.1) can be found as critical points of the energy functional $I(u, v) : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \rightarrow \mathbb{R}$,

$$I(u, v) := \frac{1}{2} (\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2) - \frac{1}{p} \int_{\mathbb{R}^N} (|u|^p + |v|^p) dx - \frac{v}{\alpha + \beta} \int_{\mathbb{R}^N} |x|^{-\tau} |u|^\alpha |v|^\beta dx.$$

It is standard to check that the energy functional is of class C^1 , and the critical points of $I(u, v)$ are restricted on the constraint $\mathcal{T}(a, b)$, where

$$\mathcal{T}(a, b) := \{(u, v) \in H : \|u\|_2^2 = a^2, \|v\|_2^2 = b^2\}.$$

The system (1.1) with the L^2 -norm constraints $\int_{\mathbb{R}^N} |u|^2 dx = a^2$ and $\int_{\mathbb{R}^N} |v|^2 dx = b^2$ describes the dynamics of coupled nonlinear waves:

$$\begin{cases} (-\Delta)^s \Psi_1 = i \frac{\partial \Psi_1}{\partial t} + |\Psi_1|^{p-2} \Psi_1 + \frac{\alpha v}{\alpha + \beta} |\Psi_1|^{\alpha-2} \Psi_1 |\Psi_2|^\beta, \\ (-\Delta)^s \Psi_2 = i \frac{\partial \Psi_2}{\partial t} + |\Psi_2|^{p-2} \Psi_2 + \frac{\beta v}{\alpha + \beta} |\Psi_1|^\alpha |\Psi_2|^{\beta-2} \Psi_2, \end{cases} \quad (1.3)$$

where i is the imaginary unit. System (1.3) describes several physical phenomena, such as the propagation of mutually incoherent waves packets in nonlinear optics, see [15, 30]. Physically, the solution Ψ_1 and Ψ_2 represent condensate amplitudes corresponding to different condensates. An important and well-known feature of this model is conservation of mass: the L^2 -norms $|\Psi_1(t, \cdot)|_2$, $|\Psi_2(t, \cdot)|_2$ of solutions are independent of $t \in \mathbb{R}$. These norms represent conserved physical quantities with direct physical interpretations. In the context of Bose-Einstein condensates, they correspond to the number of atoms in each component of the mixture. In nonlinear optics, they denote the power carried by each propagating mode. Precisely for this reason, searching for solutions with prescribed masses is not only natural but also physically imperative.

In recent years, numerous research can be found on the study for normalized solutions of the nonlinear Schrödinger equation with critical exponents, we refer to [7, 11, 13, 23, 26, 27, 33, 34] and the references therein. Meanwhile, to search for the existence of normalized solutions for nonlinear systems with critical exponents, Li and Zou [21] considered the following Schrödinger system with critical and subcritical nonlinearities.

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2}, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{q-2} v + \beta r_2 |u|^{r_1} |v|^{r_2-2} v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a_1^2, \quad \int_{\mathbb{R}^N} |v|^2 dx = a_2^2, \end{cases}$$

where $p, r_1 + r_2 < 2^*$ and $q \leq 2^*$. By studying the geometry of the Pohozaev manifold and the associated minimization problem, they got the existence of the positive normalized ground state solution under some assumptions on parameters. Bartsch et al. [3] considered the following nonlinear Schrödinger system with critical exponent:

$$\begin{cases} -\Delta u + \lambda_1 u = |u|^{2^*-2} u + \nu \alpha |u|^{\alpha-2} u |v|^\beta, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = |v|^{2^*-2} v + \nu \beta |u|^\alpha |v|^{\beta-2} v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \quad \int_{\mathbb{R}^N} |v|^2 dx = b^2, \end{cases}$$

where $N = 3, 4$. When $2 < \alpha + \beta < 2 + \frac{4}{N}$, they got a normalized ground state solution for $0 < \nu < \nu_0$, and when $2 + \frac{4}{N} \leq \alpha + \beta < 2^*$ they obtained the existence of a threshold $\nu_1 \geq 0$ such that a normalized ground state solution exists for $\nu > \nu_1$.

Li, Liu and Zou [22] studied the following Schrödinger system with Sobolev critical coupling term:

$$\begin{cases} -\Delta u + \lambda_1 u = \omega_1 |u|^{p-2} u + \frac{\alpha v}{2^*} |u|^{\alpha-2} u |v|^\beta, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \omega_2 |v|^{p-2} v + \frac{\beta v}{2^*} |u|^\alpha |v|^{\beta-2} v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \quad \int_{\mathbb{R}^N} |v|^2 dx = b^2, \end{cases}$$

where $N \geq 3, a, b > 0, \omega_1, \omega_2, v \in \mathbb{R} \setminus \{0\}, 2 < p \leq 2^*$ and $\alpha + \beta = 2^*$. They mainly present several existence and multiplicity results under explicit conditions on a, b for the focusing case $\omega_1, \omega_2 > 0$ and attractive case $v > 0$. Two normalized solutions under L^2 -subcritical case and a mountain-type solution under L^2 -critical and supercritical case are obtained respectively. Furthermore, they explore the existence and nonexistence of normalized solution when $p = 2^*$.

In order to investigate normalized solutions for the fractional systems with critical nonlinearity, Zuo et al. [36] studied the following fractional Sobolev critical nonlinear Schrödinger coupled systems,

$$\begin{cases} (-\Delta)^s u = \mu_1 u + |u|^{2_s^*-2} u + \eta_1 |u|^{p-2} u + \gamma \alpha |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \mu_2 v + |v|^{2_s^*-2} v + \eta_2 |v|^{q-2} v + \gamma \beta |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N, \\ \|u\|_{L^2}^2 = m_1^2 \text{ and } \|v\|_{L^2}^2 = m_2^2, \end{cases}$$

where $N > 2s, p, q$ and $\alpha + \beta \in (2 + \frac{4s}{N}, 2_s^*]$. By using the methods of scaling transformation, classification discussion and concentration-compactness principle, they obtained a positive normalized solution whose associated Lagrange multipliers are negative. In addition, they investigated the nonexistence of positive normalized solution under Sobolev-critical case. Guo et al. [14] considered the fractional Choquard system with a local perturbation, they obtained the existence of positive normalized solution when $p, q < 2_s^*$, and the nonexistence of positive normalized solution when $p = q = 2_s^*$. For more results with regard to Schrödinger systems, we refer to [2–4, 6, 8, 21, 28, 31, 32] and the references therein.

Motivated by the papers mentioned above, we shall consider the normalized solutions for the fractional Schrödinger system with Hardy critical term. From a variational perspective, the L^2 -critical exponent $\bar{p} = 2 + \frac{4s}{N}$ plays a crucial role in (1.1), which influences the structure of $I(u, v)$ on the constraint $\mathcal{T}(a, b)$. We mainly focus on the problem (1.1) from two different scenarios: (1) L^2 -subcritical case $2 < p < 2 + \frac{4s}{N}$; (2) L^2 -critical and L^2 -supercritical case $2 + \frac{4s}{N} \leq p < 2_s^*$.

Definition 1.1. We say that $(u, v) \in H$ is a weak solution of system (1.1) if

$$\begin{aligned} & \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi dx + \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \psi dx + \lambda_1 \int_{\mathbb{R}^N} u \phi dx + \lambda_2 \int_{\mathbb{R}^N} v \psi dx \\ &= \int_{\mathbb{R}^N} |u|^{p-2} u \phi dx + \int_{\mathbb{R}^N} |v|^{p-2} v \psi dx + \frac{\alpha v}{2_{s,\tau}^*} \int_{\mathbb{R}^N} \frac{|u|^{\alpha-2} |v|^\beta u}{|x|^\tau} \phi dx + \frac{\beta v}{2_{s,\tau}^*} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^{\beta-2} v}{|x|^\tau} \psi dx, \end{aligned}$$

for any $(\phi, \psi) \in H$.

Definition 1.2. A solution (u_0, v_0) is called a normalized ground state to system (1.1) if it satisfies

$$I'_{\mathcal{T}(a,b)}(u_0, v_0) = 0, \quad \text{and} \quad I(u_0, v_0) = \inf\{I(u, v) : I'_{\mathcal{T}(a,b)}(u, v) = 0, (u, v) \in \mathcal{T}(a, b)\}.$$

A solution (u_0, v_0) is called a normalized excited state to system (1.1) if it satisfies

$$I|'_{\mathcal{T}(a,b)}(u_0, v_0) = 0, \quad \text{and} \quad I(u_0, v_0) > \inf \left\{ I(u, v) : I|'_{\mathcal{T}(a,b)}(u, v) = 0, (u, v) \in \mathcal{T}(a, b) \right\}.$$

The normalized ground state energy $m_g(a, b)$ can be defined by

$$m_g(a, b) := \inf_{(u,v) \in \mathcal{T}(a,b)} I(u, v)$$

To establish our results for the system (1.1), we first introduce some relevant results concerning the following scalar equation

$$(-\Delta)^s u + \lambda u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \quad \|u\|_2^2 = a^2, \quad (1.4)$$

which has received a lot of attention recently [20, 24, 32, 35].

As we all known, normalized solutions of (1.4) can be obtained as critical points of the following energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx, \quad \text{in } H^s(\mathbb{R}^N), \quad (1.5)$$

on the constraint $S(a) = \{u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = a^2\}$. The least energy is defined by

$$e(a) := \inf_{u \in S(a)} E(u) = \inf \left\{ E(u) : u \in H^s(\mathbb{R}^N), \|u\|_2^2 = a^2 \text{ and } \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \gamma_{p,s} \|u\|_p^p \right\}, \quad (1.6)$$

where the parameter $\gamma_{p,s} := \frac{N(p-2)}{2ps}$.

In addition, there are following results.

Lemma 1.3 ([24]). *Let $N \geq 3s$, $p \in (2, 2_s^*) \setminus \{2 + \frac{4s}{N}\}$. Then the scalar equation (1.4) has a unique positive normalized solution u_1 with Lagrange multiplier $\lambda > 0$, and*

(1) *if $p < 2 + \frac{4s}{N}$, then*

$$e(a) = \inf_{\|u\|_2^2 = a^2} E(u) = E(u_1) < 0$$

(2) *if $p > 2 + \frac{4s}{N}$, then*

$$e(a) = \inf_{\|u\|_2^2 = a^2} \max_{t \in \mathbb{R}} E(e^{\frac{N}{2}t} u(e^t x)) = \max_{t \in \mathbb{R}} E(e^{\frac{N}{2}t} u_1(e^t x)) = E(u_1) > 0.$$

Moreover, the least energy $e(a)$ is strictly decreasing with respect to $a > 0$.

Nextly, the following definition of the homotopy-stable family and the generalized mini-max theorem will help us to construct a Palais–Smale sequence.

Definition 1.4 ([12, Definition 5.1]). Let X be a topological space and Y be a closed subset of X . We shall say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with extended boundary Y if for any set Z in $\mathcal{F}$ and any $\gamma \in C([0, 1] \times X, X)$ satisfying $\gamma(t, x) = x$ for all $(t, x) \in (\{0\} \times X) \cup ([0, 1] \times Y)$ we have that $\gamma(\{1\} \times Z) \in \mathcal{F}$.

Lemma 1.5 ([12, Theorem 5.2]). *Let φ be a C^1 -functional on a complete connected C^1 -Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with an extended closed boundary Y . Set*

$$m := m(\varphi, \mathcal{F}),$$

which is the minimax level of φ associated with the homotopy-stable family $\mathcal{F}$. Let F be a closed subset of X satisfying

$$(Z \cap F) \setminus Y \neq \emptyset, \quad \text{for every } Z \in \mathcal{F}, \quad (1.7)$$

and

$$\sup_{x \in Y} \varphi(x) \leq m \leq \inf_{x \in F} \varphi(x). \quad (1.8)$$

Then, for any sequence of sets $\{Z_n\}_n$ in $\mathcal{F}$ such that $\lim_n \sup_{Z_n} \varphi = m$, there exists a sequence $\{x_n\}_n$ in X such that

$$\lim_{n \rightarrow +\infty} \varphi(x_n) = m, \quad \lim_{n \rightarrow +\infty} \|d\varphi(x_n)\| = 0, \quad \lim_{n \rightarrow +\infty} \text{dist}(x_n, F) = 0, \quad \lim_{n \rightarrow +\infty} \text{dist}(x_n, Z_n) = 0.$$

To obtain the existence of normalized solutions to system (1.1), we introduce the following Pohozaev identity:

$$P(u, v) := \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 - \gamma_{p,s} \int_{\mathbb{R}^N} (|u|^p + |v|^p) dx - \nu \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx = 0, \quad (1.9)$$

as is shown in [3, 17, 28], normalized solutions for system (1.1) satisfy (1.9). Then we set the Pohozaev manifold:

$$\mathcal{P}(a, b) := \{(u, v) \in \mathcal{T}(a, b) : P(u, v) = 0\},$$

and $\mathcal{P}(a, b) \neq \emptyset$. The Pohozaev identity implies that all the solutions of system (1.1) belong to $\mathcal{P}(a, b)$, so that if (u, v) is a minimizer of the following minimization problem

$$m(a, b) := \inf_{(u, v) \in \mathcal{P}(a, b)} I(u, v),$$

then (u, v) is a normalized ground state solution of system (1.1).

We have the following results.

Theorem 1.6. *Let $N \geq 4s$, when $2 < p < 2 + \frac{4s}{N}$, there exists a constant $C_0 > 0$, such that if ν satisfies*

$$0 < \nu^{\frac{2-p\gamma_{p,s}}{2s^*, \tau-2}} < C_0 \left(a^{p(1-\gamma_{p,s})} + b^{p(1-\gamma_{p,s})} \right)^{-1} \quad (1.10)$$

then we have

- (1) $I|_{\mathcal{T}(a, b)}$ has a critical point of local minimum type (u_+, v_+) at negative level

$$m(a, b) = \inf_{\mathcal{P}^+(a, b)} I(u, v) < 0,$$

which is also a normalized ground state of system (1.1).

- (2) $I|_{\mathcal{T}(a, b)}$ has a second critical point of mountain pass type (u_-, v_-) at positive level

$$l(a, b) = \inf_{\mathcal{P}^-(a, b)} I(u, v) > 0,$$

which is also a normalized excited state of system (1.1).

- (3) Both (u_+, v_+) and (u_-, v_-) are composed of positive, radially symmetric functions, and solve system (1.1) for suitable $\lambda_{1,+}, \lambda_{2,+} > 0$ and $\lambda_{1,-}, \lambda_{2,-} > 0$.

Theorem 1.7. Let $N \geq 3s$, if one of the following conditions holds:

- (1) when $2 + \frac{4s}{N} < p < 2_s^*$, there exists $a_0 > 0$ such that $a, b \leq a_0$, where a_0 is defined by

$$a_0 = a_0(v) := \left[\frac{1}{\gamma_{p,s} C(N, P, s)} \left(\frac{2p\gamma_{p,s} v^{-(N-2s)/(2s-\tau)} (2s-\tau)}{2(N-\tau)(p\gamma_{p,s} - 2)} S_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}} \right)^{\frac{2-p\gamma_{p,s}}{2}} \right]^{\frac{1}{p(1-\gamma_{p,s})}}, \quad (1.11)$$

and it satisfies

$$e(a_0) = \frac{2s-\tau}{2(N-\tau)} v^{-\frac{N-2s}{2s-\tau}} S_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}};$$

- (2) when $2 + \frac{4s}{N} < p < 2_s^*$, there exists $v_1 = v_1(a, b, \alpha, \beta) > 0$ such that $v > v_1$;
 (3) when $p = 2 + \frac{4s}{N}$, we assume that

$$a^{\frac{4s}{N}} + b^{\frac{4s}{N}} < \left(\frac{1}{2} + \frac{s}{N} \right) C^{-1}(N, p, s), \quad (1.12)$$

where $C(N, p, s)$ is the best constant in the fractional Gagliardo–Nirenberg inequality, there exists $v_1 = v_1(a, b, \alpha, \beta) > 0$ such that $v > v_1$.

Then $I|_{\mathcal{T}(a,b)}$ has a critical point of mountain pass type $(\hat{u}, \hat{v})$ at positive level $m(a, b) > 0$, which is also a normalized ground state of system (1.1) with $\lambda_1, \lambda_2 > 0$.

We outline the ideas and methods employed in this paper to obtain the main results. For the L^2 -subcritical case where $2 < p < 2 + \frac{4s}{N}$, in order to obtain the first local minimum type normalized ground state, we establish the estimate of the normalized ground state level $m(a, b)$ to restore the compactness loss caused by critical growth. To obtain the second solution, we construct a Palais–Smale sequence at level $l(a, b)$ by using the generalized minimax theorem. And the energy estimate is established to ensure the boundedness of Palais–Smale sequence. For the L^2 -critical and L^2 -supercritical case where $2 + \frac{4s}{N} \leq p < 2_s^*$, we analyze the Pohozaev manifold and apply the mountain pass theorem to obtain the existence of a normalized ground state of system (1.1).

This paper is organized as follows. In Section 2, we give some preliminary results which will be used later. In Section 3, we analyse the several properties of the energy levels. In Section 4, we establish the compactness results. In Section 5, we give the proof of Theorem 1.6 and Theorem 1.7.

Throughout the paper, we use $\|u\|_p$ to denote the L^p -norm of u , $A \sim B$ represents $C_1 B \leq A \leq C_2 B$ for some positive constants $C_1, C_2 > 0$, “ $\rightarrow$ ” and “ $\rightharpoonup$ ” denote the strong convergence and weak convergence in corresponding space respectively, u_n^- denotes the negative part of u_n . C appears as a constant which may vary from line to line.

2 Preliminaries

In this section, we give some preliminary results which will be used throughout the paper. Firstly, let us recall the following fractional Sobolev embedding.

Lemma 2.1 ([29]). Let $N \geq 3s$ and $0 < s < 1$. There exists the best Hardy–Sobolev constant defined by

$$\mathcal{S} := \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_{s,\tau}^*}}{|x|^\tau} dx\right)^{\frac{2}{2_{s,\tau}^*}}}, \quad (2.1)$$

and $\mathcal{S}_{\alpha,\beta}$ is defined by

$$\mathcal{S}_{\alpha,\beta} := \inf_{u,v \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2}{\left(\int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx\right)^{\frac{2}{2_{s,\tau}^*}}}, \quad (2.2)$$

then we obtain that

$$\mathcal{S}_{\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_{s,\tau}^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_{s,\tau}^*}} \right] \mathcal{S},$$

and we have the Hardy–Sobolev inequality

$$\int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} \leq \mathcal{S}_{\alpha,\beta}^{-\frac{2_{s,\tau}^*}{2}} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2_{s,\tau}^*}{2}}. \quad (2.3)$$

Lemma 2.2 ([16] Fractional Gagliardo–Nirenberg inequality)). Let $N \geq 3s$, $0 < s < 1$, and $p \in (2, 2_{s,\tau}^*)$. Then there exists a constant $C(N, p, s) > 0$ such that

$$\|u\|_p \leq C(N, p, s) \|(-\Delta)^{\frac{s}{2}} u\|_2^{\gamma_{p,s}} \|u\|_2^{1-\gamma_{p,s}}, \quad \forall u \in H^s(\mathbb{R}^N), \quad (2.4)$$

where $\gamma_{p,s} := \frac{N(p-2)}{2ps}$.

We introduce a L^2 -norm invariant transformation $t \star u(x) := e^{\frac{N}{2}t} u(e^t x)$ and

$$(u, v) \in \mathcal{T}(a, b) \mapsto t \star (u, v) := (t \star u, t \star v) \in \mathcal{T}(a, b).$$

Then we define the fiber map

$$\begin{aligned} \Phi_{(u,v)}(t) &:= I(t \star (u, v)) = \frac{1}{2} e^{2st} (\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2) \\ &\quad - \frac{1}{p} e^{ps\gamma_{p,s}t} \int_{\mathbb{R}^N} (|u|^p + |v|^p) dx - \frac{\nu}{\alpha + \beta} e^{2_{s,\tau}^*st} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx. \end{aligned} \quad (2.5)$$

Through a direct computation, we obtain that $\Phi'_{(u,v)}(t) = sP(t \star (u, v))$, where $P(u, v)$ is defined in (1.9). Therefore, we define

$$\mathcal{P}(a, b) = \{(u, v) \in \mathcal{T}(a, b) : \Phi'_{(u,v)}(0) = 0\}.$$

To obtain the exact location and types of the critical points for $I(u, v)$, we consider to split the manifold $\mathcal{P}(a, b)$ into three parts

$$\mathcal{P}(a, b) = \mathcal{P}^+(a, b) \cup \mathcal{P}^-(a, b) \cup \mathcal{P}^0(a, b),$$

where

$$\begin{aligned} \mathcal{P}^+(a, b) &:= \{(u, v) \in \mathcal{P}(a, b) : \Phi''_{(u,v)}(0) > 0\}, \\ \mathcal{P}^-(a, b) &:= \{(u, v) \in \mathcal{P}(a, b) : \Phi''_{(u,v)}(0) < 0\}, \\ \mathcal{P}^0(a, b) &:= \{(u, v) \in \mathcal{P}(a, b) : \Phi''_{(u,v)}(0) = 0\}. \end{aligned}$$

We define

$$h(\rho) := \frac{1}{2}\rho^2 - A\rho^{p\gamma_{p,s}} - B\rho^{2_{s,\tau}^*},$$

where

$$A := \frac{1}{p}C(N, p, s) \left(a^{p(1-\gamma_{p,s})} + b^{p(1-\gamma_{p,s})} \right),$$

$$B := \frac{\nu}{2_{s,\tau}^*} S_{\alpha,\beta}^{-\frac{2_{s,\tau}^*}{2}}.$$

Then, for any $u, v \in H^s(\mathbb{R}^N)$,

$$I(u, v) \geq h \left(\left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{1}{2}} \right). \quad (2.6)$$

Furthermore, we have

$$\begin{aligned} h'(\rho) &= \rho - p\gamma_{p,s}A\rho^{p\gamma_{p,s}-1} - 2_{s,\tau}^*B\rho^{2_{s,\tau}^*-1} \\ &= \rho^{p\gamma_{p,s}-1} (g(\rho) - p\gamma_{p,s}A), \end{aligned}$$

where

$$g(\rho) = \rho^{2-p\gamma_{p,s}} - 2_{s,\tau}^*B\rho^{2_{s,\tau}^*-p\gamma_{p,s}}.$$

Let

$$\rho_* = \left(\frac{2 - p\gamma_{p,s}}{2_{s,\tau}^* (2_{s,\tau}^* - p\gamma_{p,s}) B} \right)^{\frac{1}{2_{s,\tau}^*-2}},$$

we can easily obtain that $g(\rho)$ is increasing on $(0, \rho_*)$ and decreasing on $(\rho_*, +\infty)$. Through direct computations, (1.10) gives that $g(\rho_*) > p\gamma_{p,s}A$ and $h(\rho_*) > 0$, which imply that $h(\rho)$ has only two critical points $0 < \rho_1 < \rho_* < \rho_2$, and

$$h(\rho_1) = \min_{0 < \rho < \rho_*} h(\rho) < 0,$$

$$h(\rho_2) = \max_{\rho > 0} h(\rho) > 0.$$

Thus, there exists $0 < R_0 < R_1$ such that $h(R_0) = h(R_1) = 0$, and $h(\rho) > 0$ if $\rho \in (R_0, R_1)$. According to [28, Lemma 5.2], we can prove that $\mathcal{P}^0(a, b) = \emptyset$, and it is a smooth manifold of codimension 2 in $\mathcal{T}(a, b)$. The fact will be used in the following fact.

When $2 < p < 2 + \frac{4s}{N}$, we have the following lemma.

Lemma 2.3. *For any $(u, v) \in \mathcal{T}(a, b)$, the function $\Phi_{u,v}(t)$ has exactly two critical points $t_+(u, v) < t_-(u, v) \in \mathbb{R}$ and two zero points $c(u, v) < d(u, v) \in \mathbb{R}$ with $t_+(u, v) < c(u, v) < t_-(u, v) < d(u, v)$. Furthermore,*

(1) $t \star (u, v) \in \mathcal{P}^+(a, b)$ and $t \star (u, v) \in \mathcal{P}^-(a, b)$, then either $t = t_+(u, v)$ or $t = t_-(u, v)$;

(2) $\left(\|(-\Delta)^{\frac{s}{2}}(t \star u)\|_2^2 + \|(-\Delta)^{\frac{s}{2}}(t \star v)\|_2^2 \right)^{\frac{1}{2}} \leq R_0$ for every $t \leq c(u, v)$ and

$$\begin{aligned} &I(t_+(u, v) \star (u, v)) \\ &:= \min \left\{ I(t \star (u, v)) : t \in \mathbb{R}, \left(\|(-\Delta)^{\frac{s}{2}}(t \star u)\|_2^2 + \|(-\Delta)^{\frac{s}{2}}(t \star v)\|_2^2 \right)^{\frac{1}{2}} \leq R_0 \right\} \\ &< 0; \end{aligned}$$

(3) $\Phi_{(u,v)}(t)$ is strictly decreasing and concave on $(t_-(u,v), +\infty)$ and

$$\Phi_{(u,v)}(t_-(u,v)) = \max \Phi_{(u,v)}(t); \quad (2.7)$$

(4) the maps $(u,v) \mapsto t_+(u,v)$ and $(u,v) \mapsto t_-(u,v)$ are of class C^1 .

Proof. We first prove that $\Phi_{(u,v)}(t)$ has exactly two critical points. According to (2.6), we have

$$\begin{aligned} \Phi_{(u,v)}(t) = I(t \star u) &\geq h \left(\left(\|(-\Delta)^{\frac{s}{2}}(t \star u)\|_2^2 + \|(-\Delta)^{\frac{s}{2}}(t \star v)\|_2^2 \right)^{\frac{1}{2}} \right) \\ &= h \left(e^t \left(\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \|(-\Delta)^{\frac{s}{2}}v\|_2^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

Since $h(\rho) > 0$ on an open interval (R_0, R_1) and $\Phi_{(u,v)}(-\infty) = 0^-$, $\Phi_{(u,v)}(+\infty) = -\infty$, we deduce that $\Phi_{(u,v)}(t)$ has at least two critical points $t_+(u,v) < t_-(u,v)$, more precisely, $t_+(u,v)$ is a local minimum point at a negative level and $t_-(u,v)$ is a global maximum point at a positive level. On the other hand, it is standard to prove that $\Phi_{(u,v)}(t)$ has at most two critical points as in [21, Remark 3.1]. Hence, it has exactly two critical points.

Then we have

$$\Phi_{(u,v)}(t_+(u,v)) = I(t_+(u,v) \star (u,v)) < 0$$

and

$$\left(\|(-\Delta)^{\frac{s}{2}}(t \star u)\|_2^2 + \|(-\Delta)^{\frac{s}{2}}(t \star v)\|_2^2 \right)^{\frac{1}{2}} \leq R_0,$$

this proves (2). Since the minimality $\Phi_{(u,v)}''(t_+(u,v)) \geq 0$ and $\Phi_{(u,v)}'(t_+(u,v)) = 0$, we obtain that $t_+(u,v) \star u \in \mathcal{P}^+(a,b)$. Similarly, we obtain that $t_-(u,v) \star u \in \mathcal{P}^-(a,b)$, this proves (1).

Then, according to the monotonicity of $\Phi_{(u,v)}(t)$ and $\Phi_{(u,v)}(+\infty) = -\infty$, $\Phi_{(u,v)}(-\infty) = 0^-$, we deduce that $\Phi_{(u,v)}(t)$ has two zero points. Since $\Phi_{(u,v)}(t)$ is a C^2 -function, we can easily check that $\Phi_{(u,v)}(t)$ has exactly two zero points. Therefore $\Phi_{(u,v)}(t)$ is strictly decreasing on $(t_-(u,v), +\infty)$, this proves (3).

Finally, we prove (4). We use the implicit function theorem for C^1 function $\psi = \Phi_{(u,v)}'(t)$ and use the facts that $\psi = 0$, $\partial_t \psi = \Phi_{(u,v)}''(t_+(u,v)) > 0$. Since $\mathcal{P}^0(a,b) = \emptyset$, it is not possible to pass with continuity from $\mathcal{P}^+(a,b)$ to $\mathcal{P}^-(a,b)$. Thus, $u \mapsto t_+(u,v)$ is of class C^1 . Similarly, we obtain that $u \mapsto t_-(u,v)$ is of class C^1 . $\square$

When $2 + \frac{4s}{N} \leq p < 2_s^*$, we have the following lemma.

Lemma 2.4. For every $(u,v) \in \mathcal{T}(a,b)$, $\Phi_{(u,v)}(t)$ has exactly one critical point $t_-(u,v)$. Moreover,

(1) $\mathcal{P}(a,b) = \mathcal{P}^-(a,b)$;

(2) $t \star (u,v) \in \mathcal{P}(a,b)$ if and only if $t = t_-(u,v)$;

(3) $\Phi_{(u,v)}(t)$ is strictly decreasing and concave on $(t_-(u,v), +\infty)$ and

$$\Phi_{(u,v)}(t_-(u,v)) = \max_{t \in \mathbb{R}} \Phi_{(u,v)}(t) > 0;$$

(4) the map $(u,v) \mapsto t_-(u,v)$ is of class C^1 .

Proof. Let $(u, v) \in \mathcal{T}(a, b)$, by (2.5) we have

$$\begin{aligned} \Phi'_{(u,v)}(t) &= se^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 - \gamma_{p,s} e^{(p\gamma_{p,s}-2)st} (\|u\|_p^p + \|v\|_p^p) \right) \\ &\quad - se^{2st} \nu e^{(2_{s,\tau}^*-2)st} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx. \end{aligned}$$

Let

$$W(t) = \gamma_{p,s} e^{(p\gamma_{p,s}-2)st} (\|u\|_p^p + \|v\|_p^p) + \nu e^{(2_{s,\tau}^*-2)st} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx,$$

in view of $p\gamma_{p,s} > 2$ and $2_{s,\tau}^* > 2$, we have $W(t)$ is increasing on $\mathbb{R}$, $W(-\infty) = 0^+$ and $W(+\infty) = +\infty$. Thus, $\Phi_{(u,v)}(t)$ has exactly one critical point $t_-(u, v)$. The following proof is similar to the proof of Lemma 2.3, we omit it here. $\square$

3 The properties of the energy levels

Define

$$\begin{aligned} m(a, b) &= \inf_{(u,v) \in \mathcal{P}(a,b)} I(u, v), \\ l(a, b) &= \inf_{(u,v) \in \mathcal{P}^-(a,b)} I(u, v). \end{aligned}$$

3.1 The case $2 < p < 2 + \frac{4s}{N}$

In the L^2 -subcritical case $2 < p < 2 + \frac{4s}{N}$, we establish the following several properties of the energy levels $m(a, b)$ and $l(a, b)$.

3.1.1 The properties of $m(a, b)$

For any $R > 0$, define

$$A_R(a, b) := \left\{ (u, v) \in \mathcal{T}(a, b) : \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{1}{2}} < R \right\}.$$

Lemma 3.1. *The following statements are true.*

- (1) $m(a, b) = \inf_{A_{R_0}(a,b)} I(u, v) = \inf_{A_{\rho^*}(a,b)} I(u, v) < 0$.
- (2) For any $0 < a_1 \leq a$ and $0 < b_1 \leq b$, we have $m(a, b) \leq m(a_1, b_1) + \varepsilon$ for arbitrary $\varepsilon > 0$.

Proof. (1) From Lemma 2.3, we have $\mathcal{P}^+(a, b) \subset A_{R_0}$ and

$$m(a, b) = \inf_{(u,v) \in \mathcal{P}(a,b)} I(u, v) = \inf_{(u,v) \in \mathcal{P}^+(a,b)} I(u, v) < 0.$$

Obviously $m(a, b) \geq \inf_{A_{R_0}(a,b)} I(u, v)$. On the other hand, for any $(u, v) \in A_{R_0}(a, b)$, there exists $t_+(u, v)$ such that $t_+(u, v) \star (u, v) \in \mathcal{P}^+(a, b)$ and

$$m(a, b) \leq I(t_+(u, v) \star (u, v)) \leq I(u, v),$$

then we obtain that $m(a, b) \leq \inf_{A_{R_0}(a, b)} I(u, v)$. It follows that $m(a, b) = \inf_{A_{R_0}(a, b)} I(u, v)$. According to the definition of $A_{\rho_*}(a, b)$, similar to the proof before, we can also obtain that

$$m(a, b) = \inf_{A_{\rho_*}(a, b)} I(u, v). \quad (3.1)$$

(2) Let $(u, v) \in A_{\rho_*}(a_1, b_1)$ be such that

$$I(u, v) \leq m(a_1, b_1) + \frac{\varepsilon}{2}. \quad (3.2)$$

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying

$$0 \leq \phi \leq 1 \quad \text{and} \quad \phi(x) = \begin{cases} 0, & \text{if } |x| \geq 2, \\ 1, & \text{if } |x| \leq 1. \end{cases} \quad (3.3)$$

For any $\delta > 0$, define $u_\delta(x) = u(x)\phi(\delta x)$ and $v_\delta(x) = v(x)\phi(\delta x)$. Then we obtain that $(u_\delta, v_\delta) \rightarrow (u, v)$ in H as $\delta \rightarrow 0$. Hence, we can fix a $\delta > 0$ small enough such that

$$I(u_\delta, v_\delta) \leq I(u, v) + \frac{\varepsilon}{4} \quad \text{and} \quad \left(\|(-\Delta)^{\frac{s}{2}} u_\delta\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\delta\|_2^2 \right)^{1/2} < \rho_* - \eta, \quad (3.4)$$

for a small $\eta > 0$. Now let $\vartheta \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp}(\vartheta) \subset \{x \in \mathbb{R}^N : \frac{4}{\delta} \leq |x| \leq \frac{4}{\delta} + 1\}$. Set

$$\vartheta_a = \frac{\sqrt{a^2 - \|u_\delta\|_2^2}}{\|\vartheta\|_2} \vartheta \quad \text{and} \quad \vartheta_b = \frac{\sqrt{b^2 - \|v_\delta\|_2^2}}{\|\vartheta\|_2} \vartheta.$$

Then for any $\lambda < 0$, noting that

$$(\text{supp}(u_\delta) \cup \text{supp}(v_\delta)) \cap (\text{supp}(\lambda \star \vartheta_a) \cup \text{supp}(\lambda \star \vartheta_b)) = \emptyset,$$

and we have $(u_\delta + \lambda \star \vartheta_a, v_\delta + \lambda \star \vartheta_b) \in \mathcal{T}(a, b)$. As $I(\lambda \star (\vartheta_a, \vartheta_b)) \rightarrow 0$ and $(\|(-\Delta)^{\frac{s}{2}} \lambda \star \vartheta_a\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \lambda \star \vartheta_b\|_2^2)^{1/2} \rightarrow 0$ as $\lambda \rightarrow -\infty$, we infer that

$$I(\lambda \star (\vartheta_a, \vartheta_b)) \leq \frac{\varepsilon}{4} \quad \text{and} \quad \left(\|(-\Delta)^{\frac{s}{2}} \lambda \star \vartheta_a\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \lambda \star \vartheta_b\|_2^2 \right)^{1/2} \leq \frac{\eta}{2}. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\left(\|(-\Delta)^{\frac{s}{2}} (u_\delta + \lambda \star \vartheta_a)\|_2^2 + \|(-\Delta)^{\frac{s}{2}} (v_\delta + \lambda \star \vartheta_b)\|_2^2 \right)^{1/2} < \rho_*.$$

Therefore, by (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned} m(a, b) &\leq I(u_\delta + \lambda \star \vartheta_a, v_\delta + \lambda \star \vartheta_b) \\ &= I(u_\delta, v_\delta) + I(\lambda \star \vartheta_a, \lambda \star \vartheta_b) \\ &\leq I(u, v) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &\leq m(a_1, b_1) + \varepsilon. \end{aligned} \quad \square$$

3.1.2 The properties of $l(a, b)$

Similar to the standard way in [20], we set $\mathcal{T}_r^s(a, b) = \mathcal{T}(a, b) \cap H_{rad}^s$, and define

$$\Gamma_1 := \left\{ \gamma \in C([0, 1], \mathbb{R} \times \mathcal{T}_r^s(a, b)) : \gamma(0) \in (0, \mathcal{P}^+(a, b)), \gamma(1) \in (0, I^{2m(a, b)}) \right\},$$

where

$$I^{2m(a, b)} := \{(u, v) \in H : I(u, v) \leq 2m(a, b)\}.$$

Lemma 3.2. Let $J(s, (u, v)) := I(s \star (u, v))$ and

$$\sigma_1 := \inf_{\gamma \in \Gamma_1} \max_{t \in [0, 1]} J(\gamma(t)), \quad (3.6)$$

then $l(a, b) = \sigma_1$.

Proof. For any $(u, v) \in \mathcal{P}^-(a, b)$, let $\gamma(t) := (0, ((1-t)t_+(u, v) + tt_0) \star (u, v))$ with $t_0 > 0$ large enough. Obviously $\gamma \in \Gamma_1$ and

$$\sigma_1 \leq \max_{t \in [0, 1]} J(\gamma(t)) \leq \max_{t \in \mathbb{R}} \Phi_{(u, v)}(t) = I(u, v),$$

which indicates $\sigma_1 \leq l(a, b)$. On the other hand, for any $\gamma = (\gamma_1, \gamma_2) \in \Gamma_1$, we can assume that $\gamma_1 = 0$. Since $\gamma_2(0) \in \mathcal{P}^+(a, b)$, we have $t_-(\gamma_2(0)) > t_+(\gamma_2(0)) = 0$; since $I(\gamma_2(1)) \leq 2m(a, b) < 0$, we have $t_-(\gamma_2(1)) < 0$, otherwise, if $t_-(\gamma_2(1)) \geq 0$, then $I(\gamma_2(1)) = \Phi_{(\gamma_2(1))}(0) \geq \Phi_{(\gamma_2(1))}(t_-(\gamma_2(1))) > 0$, which is impossible. Then we can infer that there exists $\tau_\gamma \in (0, 1)$ such that $t_-(\gamma_2(\tau_\gamma)) = 0$, that is $\gamma_2(\tau_\gamma) \in \mathcal{P}^-(a, b)$ and hence

$$\max_{t \in [0, 1]} J(\gamma(t)) \geq I(\gamma_2(\tau_\gamma)) \geq l(a, b),$$

which gives $l(a, b) \leq \sigma_1$.

Thus we obtain that $l(a, b) = \sigma_1$. We can also conclude that

$$\gamma_2([0, 1]) \cap \mathcal{P}^-(a, b) \neq \emptyset, \quad \forall \gamma \in \Gamma_1. \quad \square$$

Lemma 3.3. The following statements hold.

(1) For any $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$, there is

$$l(a, b) \leq l(a_1, b_1).$$

(2) Set

$$\tilde{\Gamma} := \left\{ \gamma(t) \in C([0, 1], \tilde{\mathcal{T}}(a, b)) : \gamma(0) \in \tilde{V}(a, b), \gamma(1) \in I^{2m(a, b)} \right\}, \quad (3.7)$$

with

$$\tilde{\mathcal{T}}(a, b) := \bigcup_{\substack{c \in [a/2, a] \\ d \in [b/2, b]}} \mathcal{T}(c, d), \quad \tilde{V}(a, b) := \bigcup_{\substack{c \in [a/2, a] \\ d \in [b/2, b]}} V(c, d),$$

and

$$V(c, d) := \left\{ (u, v) \in \mathcal{T}(c, d) : I'|_{\mathcal{T}(c, d)}(u, v) = 0, I(u, v) = m(c, d) \right\}.$$

Then

$$l(a, b) \leq \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0, 1]} I(\gamma(t)).$$

Proof. (1) Similar to [18, Lemma 3.2] we can get the conclusion, but for the complement of the arguments, we still give the proof. It is equivalent to proof that for any arbitrary $\varepsilon > 0$, one has

$$l(a, b) \leq l(a_1, b_1) + \varepsilon.$$

By the definition of $l(a_1, b_1)$, there exists $(u, v) \in \mathcal{P}^-(a, b)$ such that

$$I(u, v) \leq l(a_1, b_1) + \frac{\varepsilon}{2}. \quad (3.8)$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be radial and such that

$$0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 0, & \text{if } |x| \geq 2, \\ 1, & \text{if } |x| \leq 1. \end{cases} \quad (3.9)$$

For any small $\delta > 0$, we define $u_\delta(x) := u(x)\varphi(\delta x)$, $v_\delta(x) := v(x)\varphi(\delta x)$. Since $(u_\delta, v_\delta) \rightarrow (u, v)$ in H as $\delta \rightarrow 0^+$, by Lemma 2.3, one has $\lim_{\delta \rightarrow 0^+} t_-(u_\delta, v_\delta) = t_-(u, v)$ and hence

$$t_-(u_\delta, v_\delta) \star (u_\delta, v_\delta) \rightarrow t_-(u, v) \star (u, v) \quad \text{in } H \text{ as } \delta \rightarrow 0^+.$$

As a consequence, we can fix a $\delta > 0$ small enough such that

$$I(t_-(u_\delta, v_\delta) \star (u_\delta, v_\delta)) \leq I(u, v) + \frac{\varepsilon}{4}. \quad (3.10)$$

Now take $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp}(\psi) \subset B(0, \frac{4}{\delta})^c$ and set

$$\psi_a = \frac{\sqrt{a^2 - \|u_\delta\|_2^2}}{\|\psi\|_2} \psi \quad \text{and} \quad \psi_b = \frac{\sqrt{b^2 - \|v_\delta\|_2^2}}{\|\psi\|_2} \psi.$$

Then for any $\varsigma \leq 0$, one has

$$(\text{supp}(u_\delta) \cup \text{supp}(u_\delta)) \cap (\text{supp}(\varsigma \star \psi_a) \cup \text{supp}(\varsigma \star \psi_b)) = \emptyset,$$

and hence

$$(\tilde{u}_\varsigma, \tilde{v}_\varsigma) := (u_\delta + \varsigma \star \psi_a, v_\delta + \varsigma \star \psi_b) \in \mathcal{T}(a, b).$$

Let $t_\varsigma := t_-(\tilde{u}_\varsigma, \tilde{v}_\varsigma)$ which is given in Lemma 2.3. From $P(t_\varsigma \star (\tilde{u}_\varsigma, \tilde{v}_\varsigma)) = 0$, we have

$$e^{(2-p\gamma_{p,s})st_\varsigma} (\|(-\Delta)^{\frac{s}{2}} \tilde{u}_\varsigma\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \tilde{v}_\varsigma\|_2^2) = \gamma_{p,s} (\|\tilde{u}_\varsigma\|_p^p + \|\tilde{v}_\varsigma\|_p^p) + ve^{(2_{s,\tau}^* - p\gamma_{p,s})st_\varsigma} \int_{\mathbb{R}^N} \frac{|\tilde{u}_\varsigma|^\alpha |\tilde{v}_\varsigma|^\beta}{|x|^\tau} dx,$$

we see that $\limsup_{\varsigma \rightarrow -\infty} t_\varsigma < +\infty$, since $(\tilde{u}_\varsigma, \tilde{v}_\varsigma) \rightarrow (u_\delta, v_\delta) \neq (0, 0)$ as $\varsigma \rightarrow -\infty$. Then one has that $t_\varsigma + \varsigma \rightarrow -\infty$ as $\varsigma \rightarrow -\infty$ and that for $\varsigma < -1$ small enough

$$I((t_\varsigma + \varsigma) \star (\psi_a, \psi_b)) < \frac{\varepsilon}{4}. \quad (3.11)$$

Now using (3.8), (3.10) and (3.11), we obtain

$$\begin{aligned} l(a, b) &\leq I(t_\varsigma \star (\tilde{u}_\varsigma, \tilde{v}_\varsigma)) \\ &= I(t_\varsigma \star (u_\delta, v_\delta)) + I((t_\varsigma + \varsigma) \star (\psi_a, \psi_b)) \\ &\leq I(t_-(u_\delta, v_\delta) \star (u_\delta, v_\delta)) + \frac{\varepsilon}{4} \\ &\leq I(u, v) + \frac{\varepsilon}{2} \leq l(a_1, b_1) + \varepsilon, \end{aligned}$$

which completes the proof.

(2) Similar to the proof of Lemma 3.2, we take a $\gamma(t) \in \tilde{\Gamma}$. Since $\gamma(0) \in V(c, d)$ for some $c \in [a/2, a]$ and $d \in [b/2, b]$, we have $t_-(\gamma(0)) > t_+(\gamma(0)) = 0$; since $I(\gamma(1)) < 2m(a, b) < 0$, we have $t_-(\gamma(1)) < 0$. So there exists a $t_\gamma \in (0, 1)$ such that $t_-(\gamma(t_\gamma)) = 0$, i.e. $\gamma(t_\gamma) \in \mathcal{P}^-(c, d)$ for some $c \in [a/2, a]$ and $d \in [b/2, b]$. Then

$$l(a, b) \leq l(c, d) \leq I(\gamma(t_\gamma)) \leq \max_{t \in [0, 1]} I(\gamma(t)).$$

Thus, we obtain $l(a, b) \leq \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0, 1]} I(\gamma(t))$. $\square$

3.1.3 The relations between $m(a, b)$ and $l(a, b)$

Now, we fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for the Hardy–Sobolev constant S . If necessary multiplying U by a positive constant, we may assume that

$$(-\Delta)^{\frac{s}{2}} U = \frac{U^{2_{s,\tau}^* - 1}}{|x|^\tau}, \quad x \in \mathbb{R}^N, \quad (3.12)$$

testing the equation with U and using (2.1) show that

$$\|U\|_2^2 = \int_{\mathbb{R}^N} \frac{|U|^{2_{s,\tau}^*}}{|x|^\tau} = S^{\frac{N-\tau}{2s-\tau}}. \quad (3.13)$$

In view of [25, Theorem 1.1], we obtain that

$$\frac{C_1}{1 + |x|^{N-2s}} \leq U \leq \frac{C_2}{1 + |x|^{N-2s}}. \quad (3.14)$$

For any $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} U\left(\frac{|x|}{\varepsilon}\right) \quad (3.15)$$

is also a minimizer for S satisfying (3.12) and (3.13).

Next, we let

$$\eta_\varepsilon(x) = \varphi U_\varepsilon, \quad (3.16)$$

where $\varphi \in C_0^\infty(\mathbb{R}^N)$ is a radial cut-off function satisfying

$$0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 0, & \text{if } |x| \geq 2, \\ 1, & \text{if } |x| \leq 1. \end{cases} \quad (3.17)$$

Then as shown in [9, 19], we have the following estimates.

Lemma 3.4. As $\varepsilon \rightarrow 0^+$, we have

$$\|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2 = S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \quad \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} = S^{\frac{N-\tau}{2s}} - O(\varepsilon^{N-\tau}),$$

$$\| \eta_\varepsilon \|_2^2 = \begin{cases} O(\varepsilon^{2s}) & \text{if } N > 4s, \\ O(\varepsilon^{2s} |\log \varepsilon|) & \text{if } N = 4s, \\ O(\varepsilon^{N-2s}) & \text{if } N < 4s. \end{cases}$$

$$\| \eta_\varepsilon \|_p^p = \begin{cases} O\left(\varepsilon^{N - \frac{(N-2s)p}{2}}\right) & \text{if } p > \frac{N}{N-2s}, \\ O\left(\varepsilon^{\frac{N}{2}} |\log \varepsilon|\right) & \text{if } p = \frac{N}{N-2s}, \\ O\left(\varepsilon^{\frac{N-2s}{2} p}\right) & \text{if } p < \frac{N}{N-2s}. \end{cases}$$

For $p > 2$ and $\alpha, \beta > 1$, we have

$$(a+b)^p \geq a^p + b^p + pa^{p-1}b, \quad \forall a, b > 0,$$

$$(a_1 + b_1)^\alpha (a_2 + b_2)^\beta \geq a_1^\alpha a_2^\beta + b_1^\alpha b_2^\beta + \alpha a_1^{\alpha-1} a_2^\beta b_1 + \beta a_1^\alpha a_2^{\beta-1} b_2, \quad \forall a_1, a_2, b_1, b_2 > 0.$$

Thus we can obtain that for any $u, v > 0, t > 0$

$$\begin{aligned} & I(u + t\sqrt{\alpha}\eta_\varepsilon, v + t\sqrt{\beta}\eta_\varepsilon) \\ & \leq I(u, v) + I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) + t \int_{\mathbb{R}^N} \left(\sqrt{\alpha}(-\Delta)^{\frac{s}{2}}u + \sqrt{\beta}(-\Delta)^{\frac{s}{2}}v \right) \cdot (-\Delta)^{\frac{s}{2}}\eta_\varepsilon dx \\ & \quad - \int_{\mathbb{R}^N} \left(|u|^{p-1}t\sqrt{\alpha}\eta_\varepsilon + |v|^{p-1}t\sqrt{\beta}\eta_\varepsilon \right) dx - \frac{\alpha v}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|u|^{\alpha-1}|v|^\beta}{|x|^\tau} t\sqrt{\alpha}\eta_\varepsilon dx \\ & \quad - \frac{\beta v}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|u|^\alpha|v|^{\beta-1}}{|x|^\tau} t\sqrt{\beta}\eta_\varepsilon dx. \end{aligned} \quad (3.18)$$

Lemma 3.5. For small $\varepsilon > 0$,

(1) There exists $t_* > 0$ independent of ε such that

$$\max_{t \geq t_*} I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) \leq 2m(a, b);$$

$$(2) \max_{t > 0} I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) < \frac{2s-\tau}{2(N-\tau)} \nu^{-\frac{N-2s}{2s-\tau}} S_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + O(\varepsilon^{N-2s}) - C\|\eta_\varepsilon\|_p^p.$$

Proof. (1) Observe that

$$I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) \leq \frac{\alpha + \beta}{2} t^2 \|(-\Delta)^{\frac{s}{2}}\eta_\varepsilon\|_2^2 - \frac{\nu\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{2_{s, \tau}^*} t^{2_{s, \tau}^*} \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s, \tau}^*}}{|x|^\tau} dx. \quad (3.19)$$

In view of Lemma 3.4, we have $\|(-\Delta)^{\frac{s}{2}}\eta_\varepsilon\|_2^2 \rightarrow S_{\frac{N}{2s}}^N$ and $\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s, \tau}^*}}{|x|^\tau} \rightarrow S_{\frac{N-\tau}{2s-\tau}}^{\frac{N-\tau}{2s-\tau}}$. Therefore, there exists a $t_* > 0$ independent of ε such that $\max_{t \geq t_*} I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) \leq 2m(a, b)$.

(2) In view of (3.19), if $t \rightarrow 0$ we have $I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) < \frac{1}{2} \frac{2s-\tau}{2(N-\tau)} \nu^{-\frac{N-2s}{2s-\tau}} S_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}$. What is more, for a small $t_0 > 0$,

$$\begin{aligned} \max_{t \geq t_0} I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) & \leq \max_{t > 0} \left(\frac{\alpha + \beta}{2} t^2 \|(-\Delta)^{\frac{s}{2}}\eta_\varepsilon\|_2^2 - \frac{\nu\alpha^{\alpha/2}\beta^{\beta/2}}{2_{s, \tau}^*} t^{2_{s, \tau}^*} \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s, \tau}^*}}{|x|^\tau} dx \right) - C\|\eta_\varepsilon\|_p^p \\ & = \frac{2s-\tau}{2(N-\tau)} \left[\frac{(\alpha + \beta) \|(-\Delta)^{\frac{s}{2}}\eta_\varepsilon\|_2^2}{\left(\nu\alpha^{\alpha/2}\beta^{\beta/2} \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s, \tau}^*}}{|x|^\tau} dx \right)^{2/2_{s, \tau}^*}} \right]^{\frac{N-\tau}{2s-\tau}} - C\|\eta_\varepsilon\|_p^p \\ & = \frac{2s-\tau}{2(N-\tau)} \nu^{-\frac{N-2s}{2s-\tau}} \left(\left(\frac{\alpha}{\beta} \right)^{\beta/2_{s, \tau}^*} + \left(\frac{\beta}{\alpha} \right)^{\alpha/2_{s, \tau}^*} \right)^{\frac{N-\tau}{2s-\tau}} \\ & \quad \times \left(\frac{S_{\frac{N}{2s}}^N + O(\varepsilon^{N-2s})}{\left(S_{\frac{N-\tau}{2s-\tau}}^{\frac{N-\tau}{2s-\tau}} - O(\varepsilon^{N-\tau}) \right)^{\frac{2}{2_{s, \tau}^*}}} \right)^{\frac{N-\tau}{2s-\tau}} - C\|\eta_\varepsilon\|_p^p \end{aligned}$$

$$\begin{aligned}
&= \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} \left(\left(\frac{\alpha}{\beta} \right)^{\beta/2_{s,\tau}^*} + \left(\frac{\beta}{\alpha} \right)^{\alpha/2_{s,\tau}^*} \right)^{\frac{N-\tau}{2s-\tau}} S^{\frac{N-\tau}{2s-\tau}} + O\left(\varepsilon^{N-2s}\right) \\
&\quad - C \|\eta_\varepsilon\|_p^p \\
&= \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} S_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}} + O\left(\varepsilon^{N-2s}\right) - C \|\eta_\varepsilon\|_p^p.
\end{aligned}$$

Thus we conclude

$$\max_{t>0} I\left(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon\right) < \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} S_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}} + O\left(\varepsilon^{N-2s}\right) - C \|\eta_\varepsilon\|_p^p. \quad \square$$

We let

$$a_\varepsilon^2 := a^2 - 22_{s,\tau}^* t_*^2 \|\eta_\varepsilon\|_2^2, \quad b_\varepsilon^2 := b^2 - 22_{s,\tau}^* t_*^2 \|\eta_\varepsilon\|_2^2.$$

Obviously, $a_\varepsilon \rightarrow a$ and $b_\varepsilon \rightarrow b$ as $\varepsilon \rightarrow 0$. If $(u_\varepsilon, v_\varepsilon) \in V(a_\varepsilon, b_\varepsilon)$ is the positive solution for system (1.1), according to [3], we can find a sequence $\theta_\varepsilon \in \mathbb{R}^N$, such that

$$\begin{aligned}
a_\varepsilon^2 &\leq \|u_\varepsilon(\cdot - \theta_\varepsilon) + t\sqrt{\alpha}\eta_\varepsilon\|_2^2 \leq a^2, \quad \forall 0 \leq t \leq t_*, \\
b_\varepsilon^2 &\leq \|v_\varepsilon(\cdot - \theta_\varepsilon) + t\sqrt{\beta}\eta_\varepsilon\|_2^2 \leq b^2, \quad \forall 0 \leq t \leq t_*.
\end{aligned}$$

Lemma 3.6. *If $N \geq 4s$, we have*

$$0 < l(a, b) < m(a, b) + \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} S_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}}.$$

Proof. For any $(u, v) \in P^-(a, b)$, we can deduce from Lemma 2.3 that 0 is the unique maximum point of $\Phi_{(u,v)}$. Thus, we have

$$\begin{aligned}
I(u, v) &= \max_{t \in \mathbb{R}} I(t \star (u, v)) \\
&\geq \max_{t \in \mathbb{R}} h\left(\left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2\right)^{\frac{1}{2}}\right) \\
&= \max_{t \in \mathbb{R}} h(t) > h(\rho_*) > 0,
\end{aligned}$$

which indicates $l(a, b) > 0$. Let $\gamma_\varepsilon(t) := (u_\varepsilon(\cdot - \theta_\varepsilon) + t\sqrt{\alpha}\eta_\varepsilon, v_\varepsilon(\cdot - \theta_\varepsilon) + t\sqrt{\beta}\eta_\varepsilon)$. In view of (3.7) and Lemma 3.5, $\gamma_\varepsilon \in \tilde{\Gamma}$ for $\varepsilon > 0$ small. Since $(u_\varepsilon, v_\varepsilon) \in V(a_\varepsilon, b_\varepsilon)$ is the positive solution for system (1.1), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_\varepsilon \cdot (-\Delta)^{\frac{s}{2}} \eta_\varepsilon dx &= -\lambda_1 \int_{\mathbb{R}^N} u_\varepsilon \eta_\varepsilon dx + \int_{\mathbb{R}^N} |u_\varepsilon|^{p-2} u_\varepsilon \eta_\varepsilon dx \\
&\quad + \frac{\alpha v}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|u_\varepsilon|^{\alpha-2} |v_\varepsilon|^\beta u_\varepsilon}{|x|^\tau} \eta_\varepsilon dx,
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_\varepsilon \cdot (-\Delta)^{\frac{s}{2}} \eta_\varepsilon dx &= -\lambda_2 \int_{\mathbb{R}^N} v_\varepsilon \eta_\varepsilon dx + \int_{\mathbb{R}^N} |v_\varepsilon|^{p-2} v_\varepsilon \eta_\varepsilon dx \\
&\quad + \frac{\beta v}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^{\beta-2} v_\varepsilon}{|x|^\tau} \eta_\varepsilon dx.
\end{aligned} \tag{3.21}$$

According to (3.18), (3.20), (3.21), and Lemma 3.5, we obtain that

$$\begin{aligned} l(a, b) &\leq \max_{t \in [0, t_*]} I(\gamma_\varepsilon(t)) \leq I(u_\varepsilon, v_\varepsilon) + \max_{t > 0} I(t\sqrt{\alpha}\eta_\varepsilon, t\sqrt{\beta}\eta_\varepsilon) \\ &\leq m(a_\varepsilon, b_\varepsilon) + \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + O(\varepsilon^{N-2s}) - C \|\eta_\varepsilon\|_p^p. \end{aligned}$$

Now we estimate the upper boundary of $m(a_\varepsilon, b_\varepsilon)$. If $(u_0, v_0) \in V(a, b)$ is the positive solution for system (1.1), let $w_\varepsilon = \frac{a_\varepsilon}{a} u_0$ and $z_\varepsilon = \frac{b_\varepsilon}{b} v_0$. Obviously $\|w_\varepsilon\|_2 = a_\varepsilon$, $\|z_\varepsilon\|_2 = b_\varepsilon$ and

$$\left(\left\| (-\Delta)^{\frac{s}{2}} w_\varepsilon \right\|_2^2 + \left\| (-\Delta)^{\frac{s}{2}} z_\varepsilon \right\|_2^2 \right)^{1/2} \leq \left(\left\| (-\Delta)^{\frac{s}{2}} u_0 \right\|_2^2 + \left\| (-\Delta)^{\frac{s}{2}} v_0 \right\|_2^2 \right)^{1/2} < \rho_*,$$

which implies that $(w_\varepsilon, z_\varepsilon) \in A_{\rho_*}(a_\varepsilon, b_\varepsilon)$. In view of (3.1), we obtain that

$$\begin{aligned} m(a_\varepsilon, b_\varepsilon) &\leq I(w_\varepsilon, z_\varepsilon) = m(a, b) + I(w_\varepsilon, z_\varepsilon) - I(u_0, v_0) \\ &\leq m(a, b) + \frac{1}{p} \left(1 - \left(\frac{a_\varepsilon}{a} \right)^p \right) \|u_0\|_p^p + \frac{1}{p} \left(1 - \left(\frac{b_\varepsilon}{b} \right)^p \right) \|v_0\|_p^p \\ &\quad + \frac{v}{2_{s, \tau}^*} \left(1 - \left(\frac{a_\varepsilon}{a} \right)^\alpha \left(\frac{b_\varepsilon}{b} \right)^\beta \right) \int_{\mathbb{R}^N} \frac{u_0^\alpha v_0^\beta}{|x|^\tau} dx \\ &\leq m(a, b) + C_1 (a^2 - a_\varepsilon^2) + C_2 (b^2 - b_\varepsilon^2) \\ &\leq m(a, b) + C \|\eta_\varepsilon\|_2^2. \end{aligned} \tag{3.22}$$

Combining the above two inequalities, we deduce from Lemma 3.4 that

$$\begin{aligned} l(a, b) &\leq m(a, b) + \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + O(\varepsilon^{N-2s}) - C \|\eta_\varepsilon\|_p^p + C \|\eta_\varepsilon\|_2^2 \\ &= m(a, b) + \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} - O(\varepsilon^{N-(N-2s)p/2}) + O(\varepsilon^{2s}) \\ &< m(a, b) + \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}, \end{aligned}$$

where we use the fact that $N - (N - 2s)p/2 < 2s$ since $2 < p < 2 + 4s/N$ and $N \geq 4s$. $\square$

3.2 The case $2 + \frac{4s}{N} \leq p < 2_s^*$

In the L^2 -critical and L^2 -supercritical case $2 + \frac{4s}{N} \leq p < 2_s^*$, the energy level $m(a, b)$ have the following several properties.

Lemma 3.7. *Let $N \geq 3s$. Then*

(1) *for any $0 \leq a_1 \leq a$ and $0 < b_1 \leq b$, there is $m(a, b) \leq m(a_1, b_1)$;*

(2) *we have $0 < m(a, b) < \frac{2s-\tau}{2(N-\tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}$.*

Proof. (1) The proof is similar to the proof of Lemma 3.2 (1).

(2) For any $(u, v) \in \mathcal{P}(a, b)$, there is

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 &= \gamma_{p,s} (\|u\|_p^p + \|v\|_p^p) + \nu \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx \\ &\leq C_1 \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{p\gamma_{p,s}}{2}} \\ &\quad + C_2 \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2_s^* \tau}{2}}, \end{aligned}$$

which implies $\inf_{\mathcal{P}(a,b)} \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 > 0$. So we have

$$\begin{aligned} m(a,b) &= \inf_{\mathcal{P}(a,b)} I(u,v) = \inf_{\mathcal{P}(a,b)} I(u,v) - \frac{1}{p\gamma_{p,s}} P(u,v) \\ &= \frac{p\gamma_{p,s} - 2}{2p\gamma_{p,s}} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \frac{2_{s,\tau}^* - p\gamma_{p,s}}{2_{s,\tau}^* p\gamma_{p,s}} \nu \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx \\ &\geq C \inf_{\mathcal{P}(a,b)} \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 > 0. \end{aligned}$$

Take $u_\varepsilon = \frac{\sqrt{\alpha}c}{\|\eta_\varepsilon\|_2} \eta_\varepsilon$ and $v_\varepsilon = \frac{\sqrt{\beta}c}{\|\eta_\varepsilon\|_2} \eta_\varepsilon$ for a small constant $c > 0$. Let $t_\varepsilon := t_-(u_\varepsilon, v_\varepsilon)$ and then $t_\varepsilon \star (u_\varepsilon, v_\varepsilon) \in \mathcal{P}(\sqrt{\alpha}c, \sqrt{\beta}c)$. We obtain that

$$\begin{aligned} m(a,b) &\leq m(\sqrt{\alpha}c, \sqrt{\beta}c) \leq I(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) \\ &= \frac{1}{2} e^{2st_\varepsilon} \left(\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2 \right) - \frac{\nu}{2_{s,\tau}^*} e^{2_{s,\tau}^* st_\varepsilon} \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx \\ &\quad - \frac{1}{p} e^{p\gamma_{p,s} t_\varepsilon} (\|u_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p) \\ &\leq \max_{s>0} \left(\frac{\alpha + \beta}{2} r^2 \left\| (-\Delta)^{\frac{s}{2}} \eta_\varepsilon \right\|_2^2 - \frac{\nu \alpha^{\alpha/2} \beta^{\beta/2}}{2_{s,\tau}^*} r^{2_{s,\tau}^*} \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx \right) - C \|\eta_\varepsilon\|_p^p \|\eta_\varepsilon\|_2^{-p} e^{ps\gamma_{p,s} t_\varepsilon} \\ &= \frac{2s - \tau}{2(N - \tau)} \left[\frac{(\alpha + \beta) \|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2}{\left(\nu \alpha^{\alpha/2} \beta^{\beta/2} \int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx \right)^{2/2_{s,\tau}^*}} \right]^{\frac{N-\tau}{2s-\tau}} - C \|\eta_\varepsilon\|_p^p \|\eta_\varepsilon\|_2^{-p} e^{ps\gamma_{p,s} t_\varepsilon} \\ &= \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}} + O(\varepsilon^{N-2s}) - C \|\eta_\varepsilon\|_p^p \|\eta_\varepsilon\|_2^{-p} e^{ps\gamma_{p,s} t_\varepsilon}, \end{aligned} \quad (3.23)$$

where $r = \frac{e^{st_\varepsilon} c}{\|\eta_\varepsilon\|_2}$. Then we need to check that $e^{st_\varepsilon} \geq C \|\eta_\varepsilon\|_2$ for some constant $C > 0$ as $\varepsilon \rightarrow 0$.

If $p = 2 + \frac{4s}{N}$, by definition $P(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) = 0$ and (1.12), we have

$$\begin{aligned} e^{(2_{s,\tau}^* - 2)st_\varepsilon} &= \frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} - \gamma_{p,s} \frac{\|u_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} \\ &\geq \left(1 - \frac{2\mathcal{C}(N, p, s)}{p} \left(a^{4s/N} + b^{4s/N} \right) \right) \frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} \\ &\geq C \frac{\|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2 \|\eta_\varepsilon\|_2^{2_{s,\tau}^* - 2}}{\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx}. \end{aligned} \quad (3.24)$$

By Lemma 3.4, we have $\|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2 \sim 1$ and $\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx \sim 1$ which in turn together with (3.24) gives that $e^{st_\varepsilon} \geq C \|\eta_\varepsilon\|_2$.

If $p > 2 + \frac{4s}{N}$, by $P(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) = 0$, we have

$$e^{(2_{s,\tau}^* - 2)st_\varepsilon} \nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx \leq \|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2,$$

then it follows that

$$e^{st_\varepsilon} \leq \left(\frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} \right)^{1/(2_{s,\tau}^* - 2)}. \quad (3.25)$$

Using the fact that $p\gamma_{p,s} > 2$,

$$\begin{aligned} & e^{(2_{s,\tau}^* - 2)st_\varepsilon} \\ &= \frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} - \gamma_{p,s} \frac{\|u_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} e^{(p\gamma_{p,s} - 2)t_\varepsilon} \\ &\geq \frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} - \gamma_{p,s} \frac{\|u_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} \left(\frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_2^2}{\nu \int_{\mathbb{R}^N} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^\tau} dx} \right)^{\frac{p\gamma_{p,s} - 2}{2_{s,\tau}^* - 2}} \\ &= \frac{\|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2 \|\eta_\varepsilon\|_2^{2_{s,\tau}^* - 2}}{\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx} \left(C_1 - C_2 \left\| (-\Delta)^{\frac{s}{2}} \eta_\varepsilon \right\|_2^{2 \frac{p\gamma_{p,s} - 2_{s,\tau}^*}{2_{s,\tau}^* - 2}} \left(\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} dx \right)^{\frac{2 - p\gamma_{p,s}}{2_{s,\tau}^* - 2}} \frac{\|\eta_\varepsilon\|_p^p}{\|\eta_\varepsilon\|_2^{p(1 - \gamma_{p,s})}} \right). \end{aligned} \quad (3.26)$$

Using Lemma 3.4, we obtain $\|(-\Delta)^{\frac{s}{2}} \eta_\varepsilon\|_2^2 \sim 1$, $\int_{\mathbb{R}^N} \frac{|\eta_\varepsilon|^{2_{s,\tau}^*}}{|x|^\tau} \sim 1$ and that

$$\frac{\|\eta_\varepsilon\|_p^p}{\|\eta_\varepsilon\|_2^{p(1 - \gamma_{p,s})}} = \begin{cases} O\left(\varepsilon^{\frac{2N - Np + 2ps}{2} - \frac{(N - 2s)(2ps - Np + 2N)}{4s}}\right) & \text{if } 3s \leq N < 4s, \\ O\left(|\log \varepsilon|^{-\frac{4 - p}{2}}\right) & \text{if } N = 4s, \\ O(1) & \text{if } N > 4s. \end{cases}$$

In view of (3.26), we conclude that $e^{st_\varepsilon} \geq C \|\eta_\varepsilon\|_2$.

Substituting $e^{st_\varepsilon} \geq C \|\eta_\varepsilon\|_2$ into (3.23), we obtain

$$m(a, b) \leq \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N - 2s}{2s - \tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N - \tau}{2s - \tau}} + O\left(\varepsilon^{N - 2s}\right) - C \frac{\|\eta_\varepsilon\|_p^p}{\|\eta_\varepsilon\|_2^{p(1 - \gamma_{p,s})}}.$$

According to $2 + \frac{4s}{N} \leq p < 2_s^*$, we infer that $m(a, b) < \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N - 2s}{2s - \tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N - \tau}{2s - \tau}}$ for any $\varepsilon > 0$ small. $\square$

Nextly, we show that $m(a, b) \neq e(a)$ and $m(a, b) \neq e(b)$.

Lemma 3.8. *For any $a, b > 0$, the following energy estimate holds*

$$m(a, b) < \min\{e(a), e(b)\},$$

under one of the following conditions:

- (1) when $2 + \frac{4s}{N} < p < 2_s^*$, $a \leq a_0$ and $b \leq a_0$, where a_0 is defined in (1.11);
- (2) when $2 + \frac{4s}{N} \leq p < 2_s^*$, there exists $v_1 = v_1(a, b, \alpha, \beta) > 0$ such that $\nu > v_1$.

Proof. Firstly we give a simple proof for (1). Recalling the definition of a_0 in (1.11), by the properties of $e(a)$ in Lemma 2.1, a_0 satisfies

$$e(a_0) = \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}.$$

Note that $e(a)$ is strictly decreasing with respect to $a > 0$, we can see from Lemma 3.7 that

$$m(a, b) < \min\{e(a), e(b)\},$$

since $a \leq a_0$ and $b \leq a_0$. Thus condition (1) is established.

Similar to [21], we give the proof of (2). Let u and v be the positive solution of (1.1) with parameters a and b . It is easy to see that

$$E(t \star u) \rightarrow 0 \text{ and } E(t \star v) \rightarrow 0 \text{ as } t \rightarrow -\infty.$$

Therefore, there exists a $t_0 < -1$ which is independent of ν , such that

$$m(a, b) < \max_{t < t_0} I(t \star (u, v)) < \max_{t < t_0} (E(t \star u) + E(t \star v)) < \min\{e(a), e(b)\}.$$

If $t \geq t_0$, then the intersection term can be bounded from below:

$$\int_{\mathbb{R}^N} \frac{|t \star u|^\alpha |t \star v|^\beta}{|x|^\tau} dx = e^{\left(\frac{N}{2}(\alpha+\beta) - (N-\tau)\right)t} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^\tau} dx \geq C e^{\left(\frac{N}{2}(\alpha+\beta) - (N-\tau)\right)t_0}.$$

Consequently, we have

$$\begin{aligned} \max_{t \geq t_0} I(t \star (u, v)) &< \max_{t \geq t_0} (E(t \star u) + E(t \star v)) - C e^{\left(\frac{N}{2}(\alpha+\beta) - (N-\tau)\right)t_0} \nu \\ &< e(a) + e(b) - C e^{\left(\frac{N}{2}(\alpha+\beta) - (N-\tau)\right)t_0} \nu, \end{aligned}$$

and the last term is strictly smaller than $e(a)$ and $e(b)$ provided ν is sufficiently large. Thus, we have $m(a, b) < \max_{t \geq t_0} I(t \star (u, v)) < \min\{e(a), e(b)\}$. $\square$

4 Compactness results

To obtain the compactness of Palais–Smale sequences, we need to introduce the following compactness results.

Proposition 4.1. Suppose $2 < p < 2_s^*$, in addition, when $p = 2 + \frac{4s}{N}$, let

$$a^{\frac{4s}{N}} + b^{\frac{4s}{N}} < \left(\frac{1}{2} + \frac{s}{N}\right) C^{-1}(s, N, p).$$

If

$$m(a, b) \leq m(a_1, b_1) \text{ for any } 0 < a_1 \leq a, 0 < b_1 \leq b,$$

let $\{(u_n, v_n)\} \subset \mathcal{T}(a, b)$ be a sequence consisting of radial symmetric functions such that

$$I'(u_n, v_n) + \lambda_{1,n} u_n + \lambda_{2,n} v_n \rightarrow 0 \text{ for some } \lambda_{1,n}, \lambda_{2,n} \in \mathbb{R}, \quad (4.1)$$

$$P(u_n, v_n) \rightarrow 0, \quad u_n^-, v_n^- \rightarrow 0, \quad \text{a.e. in } \mathbb{R}^N, \quad (4.2)$$

with

$$I(u_n, v_n) \rightarrow c, \quad c \neq 0, \quad c \neq e(a), \quad c \neq e(b), \quad (4.3)$$

and

$$c < \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + \min \{0, e(a), e(b), m(a, b)\}. \quad (4.4)$$

Then there exists $u, v \in H_{rad}^s(\mathbb{R}^N)$, $u, v > 0$ and $\lambda_1, \lambda_2 > 0$ such that up to a subsequence $(u_n, v_n) \rightarrow (u, v)$ in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ and $(\lambda_{1,n}, \lambda_{2,n}) \rightarrow (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$.

Proof. We divide the proof into three main steps.

Step 1. Boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

When $2 < p < 2 + \frac{4s}{N}$, then $p\gamma_{p,s} < 2$. Since $P(u_n, v_n) \rightarrow 0$, we can see from the Gagliardo–Nirenberg inequality that for n large enough,

$$\begin{aligned} c + 1 &\geq I(u_n, v_n) - \frac{1}{2_{s,\tau}^*} P(u_n, v_n) \\ &= \frac{2s - \tau}{2(N - \tau)} (\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2) - \frac{2_{s,\tau}^* - p\gamma_{p,s}}{2_{s,\tau}^* p} (\|u_n\|_p^p + \|v_n\|_p^p) \\ &\geq \frac{2s - \tau}{2(N - \tau)} (\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2) - C (\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2)^{\frac{p\gamma_{p,s}}{2}}, \end{aligned}$$

for some $C > 0$, which implies that $\{(u_n, v_n)\}$ is bounded.

When $2 + \frac{4s}{N} < p < 2_{s,\tau}^*$, for n large enough, we have

$$\begin{aligned} c + 1 &\geq I(u_n, v_n) - \frac{1}{2} P(u_n, v_n) \\ &= \frac{p\gamma_{p,s} - 2}{2p} (\|u_n\|_p^p + \|v_n\|_p^p) + \frac{(2s - \tau)v}{2(N - \tau)} \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx. \end{aligned}$$

Since $p\gamma_{p,s} > 2$ and $\frac{(2s-\tau)v}{2(N-\tau)} > 0$, we obtain that $\|u_n\|_p^p + \|v_n\|_p^p$ and $\int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx$ are bounded. According the fact that $P(u_n, v_n) = o(1)$, we have

$$\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 = \gamma_{p,s} \int_{\mathbb{R}^N} (|u_n|^p + |v_n|^p) dx + v \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx + o(1),$$

which implies that $\{(u_n, v_n)\}$ is bounded.

When $p = 2 + \frac{4s}{N}$, we have

$$\begin{aligned} c + 1 &\geq I(u_n, v_n) - \frac{1}{2} P(u_n, v_n) \\ &= \frac{(2s - \tau)v}{2(N - \tau)} \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx. \end{aligned}$$

Obviously, $\int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx$ is bounded. According to Lemma 2.2 and (1.12), we obtain that

$$\begin{aligned} \|u_n\|_p^p + \|v_n\|_p^p &\leq C(N, p, s) (a^{\frac{4s}{N}} + b^{\frac{4s}{N}}) (\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2) \\ &< \left(\frac{1}{2} + \frac{s}{N} \right) (\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2), \end{aligned}$$

combining with $P(u_n, v_n) = o(1)$, we obtain that $\{(u_n, v_n)\}$ is bounded.

In view of (4.1) and take $(u_n, 0)$ and $(0, v_n)$ as test function, we obtain that

$$\begin{aligned}\lambda_{1,n} &= -\frac{1}{a^2} I'(u_n, v_n)(u_n, 0) + o(1) \\ &= -\frac{1}{a^2} \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 - \|u_n\|_p^p - \frac{\alpha v}{2_{s,\tau}^*} \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx \right) + o(1), \\ \lambda_{2,n} &= -\frac{1}{b^2} I'(u_n, v_n)(0, v_n) + o(1) \\ &= -\frac{1}{b^2} \left(\|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 - \|v_n\|_p^p - \frac{\beta v}{2_{s,\tau}^*} \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx \right) + o(1).\end{aligned}$$

Obviously, $\lambda_{1,n}$ and $\lambda_{2,n}$ are also bounded. Thus, we can conclude that there exists $u, v \in H^s(\mathbb{R}^N)$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that up to a subsequence

$$\begin{aligned}(u_n, v_n) &\rightharpoonup (u, v) \quad \text{in } H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N), \\ (u_n, v_n) &\rightarrow (u, v) \quad \text{in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \text{ for } 2 < p < 2_{s,\tau}^*, \\ (u_n, v_n) &\rightarrow (u, v) \quad \text{a.e. in } \mathbb{R}^N, \\ (\lambda_{1,n}, \lambda_{2,n}) &\rightarrow (\lambda_1, \lambda_2) \quad \text{in } \mathbb{R}^2.\end{aligned}$$

Then (4.1), (4.2) and (4.3) give that

$$I'(u, v) + \lambda_1 u + \lambda_2 v = 0, \quad (4.5)$$

$$u \geq 0, \quad v \geq 0, \quad (4.6)$$

which indicate that (u, v) is the weak solution of system (1.1) and hence

$$P(u, v) = 0.$$

Step 2. The weak limit $u \neq 0$ and $v \neq 0$.

Without loss of generality, we assume that $u = 0$ by contradiction. If $v = 0$, from $P(u_n, v_n) = o(1)$ we obtain that

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 &= v \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx + o(1) \\ &\leq v \mathcal{S}_{\alpha,\beta}^{-\frac{2_{s,\tau}^*}{2}} \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \right)^{\frac{2_{s,\tau}^*}{2}} + o(1).\end{aligned}$$

Assuming $\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \rightarrow l \geq 0$, we can conclude that $l \leq v \mathcal{S}_{\alpha,\beta}^{-\frac{2_{s,\tau}^*}{2}} l^{\frac{2_{s,\tau}^*}{2}}$, which indicates that $l = 0$ or $l \geq v^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}}$. Consequently, we obtain that $c = 0$ or

$$c = \lim_{n \rightarrow \infty} I(u_n, v_n) = \frac{l}{2} - \frac{l}{2_{s,\tau}^*} = \frac{2s-\tau}{2(N-\tau)} l \geq \frac{2s-\tau}{2(N-\tau)} v^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha,\beta}^{\frac{N-\tau}{2s-\tau}},$$

both of them are contradictions. If $v \neq 0$, according to maximum principle, we know that

$$\begin{cases} (-\Delta)^{\frac{s}{2}} v + \lambda_2 v = v^{p-1}, & \text{in } \mathbb{R}^N \\ v > 0. \end{cases}$$

In view of [24, Theorem 1.2], we obtain $\lambda_2 > 0$. Let $\bar{v}_n = v_n - v$, there exist

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 &= \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 + o(1), \\ \|v_n\|_p^p &= \|v\|_p^p + o(1), \\ \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^\tau} dx &= \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |\bar{v}_n|^\beta}{|x|^\tau} dx + o(1). \end{aligned}$$

Combined with the fact that $P(u_n, v_n) = o(1)$, we have

$$\begin{aligned} o(1) &= P(u_n, v_n) = \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 - \nu \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |\bar{v}_n|^\beta}{|x|^\tau} dx + P(u, v) + o(1) \\ &= \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 - \nu \int_{\mathbb{R}^N} \frac{|u_n|^\alpha |\bar{v}_n|^\beta}{|x|^\tau} dx. \end{aligned}$$

Similarly, there exists $\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \rightarrow 0$ or $\liminf_{n \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \geq \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}$. If $\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \rightarrow 0$, then

$$\begin{aligned} &\|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 + \lambda_2 \|\bar{v}_n\|_2^2 \\ &= (I'(u_n, v_n) + \lambda_{1,n} u_n + \lambda_{2,n} v_n)(0, \bar{v}_n) - (I'(u, v) + \lambda_1 u + \lambda_2 v)(0, \bar{v}_n) + o(1) \\ &= o(1), \end{aligned}$$

namely, $v_n \rightarrow v$ in $H^s(\mathbb{R}^N)$. Then from Lemma 1.3, we obtain

$$c = \lim_{n \rightarrow \infty} I(u_n, v_n) = \lim_{n \rightarrow \infty} \frac{2s - \tau}{2(N - \tau)} \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \right) + E(v) = e(b),$$

which is a contradiction. If $\liminf_{n \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \geq \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}$ we have

$$c = \lim_{n \rightarrow \infty} \frac{2s - \tau}{2(N - \tau)} \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \right) + E(v) \geq \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + e(b),$$

which is also a contradiction. Consequently, the limit $u \neq 0$ and $v \neq 0$.

Step 3. $u_n \rightarrow u$ and $v_n \rightarrow v$ in $H^s(\mathbb{R}^N)$.

Let $(\bar{u}_n, \bar{v}_n) = (u_n - u, v_n - v)$. There is

$$o(1) = P(u_n, v_n) = \|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 - \nu \int_{\mathbb{R}^N} \frac{|\bar{u}_n|^\alpha |\bar{v}_n|^\beta}{|x|^\tau} dx + o(1).$$

Similarly, we can obtain

$$\|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \rightarrow 0,$$

or

$$\liminf_{n \rightarrow \infty} \left(\|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \right) \geq \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}.$$

If the second case occurs, it indicates that

$$\begin{aligned} c &\geq \lim_{n \rightarrow \infty} \frac{2s - \tau}{2(N - \tau)} \left(\|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \right) + m(\|u\|_2, \|v\|_2) \\ &\geq \frac{2s - \tau}{2(N - \tau)} \nu^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}} + m(a, b), \end{aligned}$$

which is a contradiction. Thus, $\|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 \rightarrow 0$, by maximum principle, (u, v) is a positive solution of (4.5), and we can obtain $\lambda_1, \lambda_2 > 0$ through [24, Theorem 1.2]. Noting that

$$\begin{aligned} & \|(-\Delta)^{\frac{s}{2}} \bar{u}_n\|_2^2 + \lambda_1 \|\bar{u}_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \bar{v}_n\|_2^2 + \lambda_2 \|\bar{v}_n\|_2^2 \\ &= (I'(u_n, v_n) + \lambda_{1,n} u_n + \lambda_{2,n} v_n)(\bar{u}_n, \bar{v}_n) - (I'(u, v) + \lambda_1 u + \lambda_2 v)(\bar{u}_n, \bar{v}_n) + o(1) \\ &= o(1), \end{aligned}$$

which implies that $(u_n, v_n) \rightarrow (u, v)$ in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$. $\square$

5 Proof of the theorems

Proof of Theorem 1.6 (1) and (3). Firstly, we construct a Palais–Smale sequence for $I|_{\mathcal{T}(a,b)}$ at level $m(a, b)$. By the symmetric decreasing rearrangement, it is easy to verify that

$$m(a, b) = \inf_{A_{R_0}(a,b) \cap H_{\text{rad}}^s} I(u, v),$$

where H_{rad}^s denotes the subspace of radial symmetric functions in $H = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

Choose a minimizing sequence $(\tilde{u}_n, \tilde{v}_n) \in A_{R_0}(a, b) \cap H_{\text{rad}}^s$ and we assume $\tilde{u}_n \geq 0, \tilde{v}_n \geq 0$. Furthermore, by Lemma 2.3, for each n , there exists $t_+(\tilde{u}_n, \tilde{v}_n)$ such that $t_+(\tilde{u}_n, \tilde{v}_n) \star (\tilde{u}_n, \tilde{v}_n) \in \mathcal{P}^+(a, b)$ and $I(t_+(\tilde{u}_n, \tilde{v}_n) \star (\tilde{u}_n, \tilde{v}_n)) \leq I(\tilde{u}_n, \tilde{v}_n)$. Thus, we can assume $t_+(\tilde{u}_n, \tilde{v}_n) \star (\tilde{u}_n, \tilde{v}_n) \in \mathcal{P}^+(a, b) \cap H_{\text{rad}}^s$. To simplify the notation, we denote $t_+(\tilde{u}_n, \tilde{v}_n) \star (\tilde{u}_n, \tilde{v}_n)$ by $(\hat{u}_n, \hat{v}_n)$.

By Ekeland's variational principle, there exists a radial symmetric Palais–Smale sequence (u_n, v_n) for $I|_{\mathcal{T}(a,b) \cap H_{\text{rad}}^s}$ with the property $\|(u_n, v_n) - (\hat{u}_n, \hat{v}_n)\|_H \rightarrow 0$ as $n \rightarrow \infty$, which implies that

$$P(u_n, v_n) = P(\hat{u}_n, \hat{v}_n) + o(1) \rightarrow 0 \quad \text{and} \quad u_n^-, v_n^- \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Next, we apply Proposition 4.1 with $c = m(a, b)$. To verify conditions (4.3) and (4.4), since $e(a) < 0$ and $e(b) < 0$, we only need to prove $m(a, b) < \min\{e(a), e(b)\}$. Without loss of generality, we prove $m(a, b) < e(a)$. Let u_0 be the unique positive normalized solution of the scalar equation (1.4). From $h(R_0) = 0$, we have $\frac{1}{2}R_0^2 > \frac{1}{p}\mathcal{C}(N, p, s)a^{p(1-\gamma_{p,s})}R_0^{p\gamma_{p,s}}$, which simplifies to $R_0^{2-p\gamma_{p,s}} > \frac{2}{p}\mathcal{C}(N, p, s)a^{p(1-\gamma_{p,s})}$. Combining with the Gagliardo–Nirenberg inequality, we get

$$\|(-\Delta)^{\frac{s}{2}} u_0\|_2^2 = \gamma_{p,s} \|u_0\|_p^p \leq \gamma_{p,s} \mathcal{C}(N, p, s) a^{p(1-\gamma_{p,s})} \|(-\Delta)^{\frac{s}{2}} u_0\|_2^{p\gamma_{p,s}} < R_0^{2-p\gamma_{p,s}} \|(-\Delta)^{\frac{s}{2}} u_0\|_2^{p\gamma_{p,s}},$$

which gives $\|(-\Delta)^{\frac{s}{2}} u_0\|_2 < R_0$. Let $v_0 = \frac{b}{a} u_0$, then $t \star v_0$ satisfies $\|t \star v_0\|_2 = b$ and $\|(-\Delta)^{\frac{s}{2}} t \star v_0\|_2 \rightarrow 0$ as $t \rightarrow -\infty$. Thus, $(u_0, t \star v_0) \in A_{R_0}(a, b)$ and

$$m(a, b) \leq I(u_0, t \star v_0) \leq E(u_0) + \frac{e^{2st}}{2} \|(-\Delta)^{\frac{s}{2}} v_0\|_2^2 - \frac{e^{p\gamma_{p,s}st}}{p} \|v_0\|_p^p < e(a).$$

By Proposition 4.1, there exist $u, v \in H_{\text{rad}}^s(\mathbb{R}^N)$, $u > 0, v > 0$ and $\lambda_1, \lambda_2 > 0$ such that up to a subsequence:

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N),$$

$$(\lambda_{1,n}, \lambda_{2,n}) \rightarrow (\lambda_1, \lambda_2) \quad \text{in } \mathbb{R}^2.$$

Thus, (u, v) is a normalized ground state of system (1.1). $\square$

Nextly, we need to obtain a Palais–Smale sequence $\{(u_n, v_n)\} \subset \mathcal{T}(a, b) \cap H_{rad}^s$ for $I|_{\mathcal{T}(a, b)}$ at level $l(a, b)$ to prove Theorem 1.6 (2) and (3).

Lemma 5.1. *There exists a Palais–Smale sequence $(u_n, v_n) \subset \mathcal{T}(a, b) \cap H_{rad}^s$ for $I|_{\mathcal{T}(a, b)}$ at level $l(a, b)$, with $P(u_n, v_n) \rightarrow 0$ and $u_n^-, v_n^- \rightarrow 0$ a.e. in $\mathbb{R}^N$ as $n \rightarrow \infty$.*

Proof. In view of the Definition 1.4, it is easy to check that $\mathcal{F} = \{Z = \gamma([0, 1]) : \gamma \in \Gamma_1\}$ is a homotopy stable family of compact subsets of $X = \mathbb{R} \times \mathcal{T}_r^s(a, b)$ with boundary $Y = (0, \mathcal{P}^+(a, b)) \cup (0, I^{2m(a, b)})$. Set $F = (0, \mathcal{P}^-(a, b))$, then from Lemma 3.2, we can get

$$Z \cap F \setminus Y \neq \emptyset, \quad \forall Z \in \mathcal{F}.$$

Moreover, we have

$$\sup I(Y) \leq 0 < \sigma_1 \leq \inf I(F).$$

Thus the assumptions in Lemma 1.5 are satisfied. Therefore, taking a minimizing sequence $\{\gamma_n = (0, \beta_n)\} \subset \Gamma_1$ with $\beta_n \geq 0$ a.e. in $\mathbb{R}^N$, there exists a Palais–Smale sequence $\{(s_n, w_n, z_n)\} \subset \mathbb{R} \times \mathcal{T}_r^s(a, b)$ for $J|_{\mathbb{R} \times \mathcal{T}_r^s(a, b)}$ at level σ_1 , that is

$$\partial_s J(s_n, w_n, z_n) \rightarrow 0 \quad \text{and} \quad \partial_{(u, v)} J(s_n, w_n, z_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (5.1)$$

with the additional property

$$|s_n| + \text{dist}_H((w_n, z_n), \beta_n([0, 1])) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.2)$$

Let $(u_n, v_n) = s_n \star (w_n, z_n)$. The first condition in (5.1) implies $P(u_n, v_n) \rightarrow 0$, and the second condition implies

$$\begin{aligned} & \left\| dI|_{\mathcal{T}(a, b)}(u_n, v_n) \right\| \\ &= \sup \left\{ |dI(u_n, v_n)[(\phi, \psi)]| : (\phi, \psi) \in T_{(u_n, v_n)} \mathcal{T}(a, b), \|(\phi, \psi)\|_H \leq 1 \right\} \\ &= \sup \left\{ |dI(s_n \star (w_n, z_n))[s_n \star (-s_n) \star (\phi, \psi)]| : (\phi, \psi) \in T_{(u_n, v_n)} \mathcal{T}(a, b), \|(\phi, \psi)\|_H \leq 1 \right\} \\ &= \sup \left\{ \left| \partial_{(u, v)} J(s_n, w_n, z_n)[(-s_n) \star (\phi, \psi)] \right| : (\phi, \psi) \in T_{(u_n, v_n)} \mathcal{T}(a, b), \|(\phi, \psi)\|_H \leq 1 \right\} \\ &\leq C \left\| \partial_{(u, v)} J(s_n, w_n, z_n) \right\| \sup \left\{ \|(-s_n) \star (\phi, \psi)\|_H : (\phi, \psi) \in T_{(u_n, v_n)} \mathcal{T}(a, b), \|(\phi, \psi)\|_H \leq 1 \right\} \\ &= C \left\| \partial_{(u, v)} J(s_n, w_n, z_n) \right\| \left(e^{-2s_n} \left(\|(-\Delta)^{\frac{s}{2}} \phi\|_2^2 + \|(-\Delta)^{\frac{s}{2}} \psi\|_2^2 \right) + e^{-Ns_n} (\|\phi\|_p^p + \|\psi\|_p^p) \right) \\ &\leq C \left\| \partial_{(u, v)} J(s_n, w_n, z_n) \right\| \|(\phi, \psi)\|_H \\ &\leq C \left\| \partial_{(u, v)} J(s_n, w_n, z_n) \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, (5.2) implies that $u_n^-, v_n^- \rightarrow 0$ a.e. in $\mathbb{R}^N$. □

Proof of Theorem 1.6 (2) and (3). By Lemma 5.1, there exists a Palais–Smale sequence $(u_n, v_n) \subset \mathcal{T}(a, b) \cap H_{rad}^s$ for $I|_{\mathcal{T}(a, b)}$ at level $l(a, b) > 0$. In view of Lemma 3.3, we know that for any $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$, there is

$$l(a, b) \leq l(a_1, b_1).$$

What's more, we conclude that for $N \geq 4s$,

$$l(a, b) < m(a, b) + \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}}.$$

in Lemma 3.6. Thus, according to Proposition 4.1, we can deduce that (u_-, v_-) is a mountain pass type normalized point which is composed of positive, radially symmetric functions, and solves system (1.1) for $\lambda_{1,-}, \lambda_{2,-} > 0$. $\square$

In the L^2 -critical and L^2 -supercritical case, we construct a Palais–Smale sequence at level $m(a, b)$ to prove Theorem 1.7.

Lemma 5.2. *When $2 + \frac{4s}{N} \leq p < 2_s^*$, there exists a Palais–Smale sequence $(u_n, v_n) \subset \mathcal{T}(a, b) \cap H_{rad}^s$ for $I|_{\mathcal{T}(a,b)}$ at level $m(a, b)$, with $P(u_n, v_n) \rightarrow 0$ and $u_n^-, v_n^- \rightarrow 0$ a.e. in $\mathbb{R}^N$ as $n \rightarrow \infty$.*

Proof. Noting that $m(a, b) = \inf_{\gamma \in \Gamma_2} \max_{t \in [0,1]} J(\gamma(t))$ with $J(s, (u, v)) := I(s \star (u, v))$ and

$$\Gamma_2 := \{ \gamma \in C([0, 1], \mathbb{R} \times \mathcal{T}_r(a, b)) : \gamma(0) \in (0, A_\xi(a, b)), \gamma(1) \in (0, I^0) \},$$

for a small $\xi > 0$. Then similar to Lemma 5.1, we can obtain the required Palais–Smale sequence. $\square$

Proof of Theorem 1.7. By Lemma 3.7 and Lemma 5.2, we obtain that there exists a Palais–Smale sequence $(u_n, v_n) \subset \mathcal{T}(a, b) \cap H_{rad}^s$ for $I|_{\mathcal{T}(a,b)}$ at level

$$0 < m(a, b) < \frac{2s - \tau}{2(N - \tau)} v^{-\frac{N-2s}{2s-\tau}} \mathcal{S}_{\alpha, \beta}^{\frac{N-\tau}{2s-\tau}},$$

and for any $0 \leq a_1 \leq a$ and $0 < b_1 \leq b$, there is $m(a, b) \leq m(a_1, b_1)$. In view of Lemma 3.8, we have $m(a, b) < \min\{e(a), e(b)\}$ if one of the following three conditions hold:

- (1) when $2 + \frac{4s}{N} < p < 2_s^*$, $a \leq a_0$ and $b \leq a_0$, where a_0 is defined in (1.11);
- (2) when $2 + \frac{4s}{N} < p < 2_s^*$, there exists $\nu_1 = \nu_1(a, b, \alpha, \beta) > 0$ such that $\nu > \nu_1$.
- (3) when $p = 2 + \frac{4s}{N}$, we assume that

$$a^{\frac{4s}{N}} + b^{\frac{4s}{N}} < \left(\frac{1}{2} + \frac{s}{N} \right) C^{-1}(N, p, s),$$

there exists $\nu_1 = \nu_1(a, b, \alpha, \beta) > 0$ such that $\nu > \nu_1$.

Thus, according to Proposition 4.1, we deduce that system (1.1) has a mountain pass type normalized solution with $\lambda_1, \lambda_2 > 0$ when $2 + \frac{4s}{N} \leq p < 2_s^*$. $\square$

Statements and declarations

No potential conflict of interest was reported by the author(s).

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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