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Fractional differential equations with impulses-analysis of some errors

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Abstract. Fractional differential equations with impulses have been studied over the past decades. A methodology often used for the Caputo fractional derivative case is to change the problem into an equivalent integral equation in a space of piecewise continuous functions. One approach is to change the fractional derivative at each impulse point, thereby studying a sequence of problems, one on each subinterval. A second approach is to have one fixed fractional derivative on the whole interval. These approaches have often been confused. In the second case an integral equation was proposed by Fečkan, Zhou and Wang (*Commun. Nonlinear Sci. Numer. Simul.*, 2012). This paper and some successors have been cited a large number of times. However, it is shown here, with simple counter-examples, that the proposed piecewise continuous solutions of the integral equation do not give solutions of the fractional differential equation with an impulse. Moreover, it is proved that no solution of this type is actually possible. A similar approach for the non-instantaneous impulse case is also shown to be not correct. An impulse problem involving the Riemann–Liouville fractional derivative is also discussed.

Keywords: fractional derivatives, impulses.

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1 Introduction

Ordinary differential equations (ODEs) with impulses have been well studied and the concept of a solution does not give difficulties. It is not surprising that authors have considered fractional differential equations with impulses. Here matters are less clear. Two types of impulsive problems have been considered, so-called instantaneous, some jumps at some points, and non-instantaneous, jump effect lasting on some intervals.

For the Caputo fractional derivative case there have been two approaches. For instantaneous impulses, the first method is to change the initial point of the fractional derivative at each impulse point, thereby studying a sequence of problems, one on each subinterval, as for example in [23] and many other papers. The second method is the natural one of having a

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fractional derivative with a fixed initial point for the whole interval. This second method has been used by Fečkan, Wang and Zhou in a series of papers following their paper [10]. These approaches have often been confused. In fact, the authors of [23] claimed that the concept of solutions in the paper [10] was incorrect, while the authors of [10] pointed out in [9] that, in fact, the two different approaches were being used.

A recent paper Feng–Chen [11] stated that the first method was incorrect and the method of [10] was correct. This only repeats what is asserted in [10] including the same 'counter-example' as was given in [10]. We will prove that the opposite is true. We believe the first method is mathematically correct, though changing the fractional derivative at each impulse point lacks some motivation. However, for the second method, we show with simple examples that the 'solution' given in [10] is not a solution of the impulsive problem. In fact, we prove more: no solution of the type proposed in [10] is possible. This 'solution' has been used many times, for example [8,14,21,24,27], and there have been hundreds of citations of these papers.

The authors of [34] claim the formula for solutions obtained using the first method is not correct by making a comparison with the formula that is obtained by letting the impulses vanish, which is in fact the second method. In [34] a formula for solutions is given, which involves one parameter and would give uncountably many solutions; this seems to be impossible as essentially we show in Lemma 6.1. However, when the parameter is zero, their formula is the same as in [10] so it is not correct. The paper [34] also uses the version of Caputo derivative (recalled in Definition 2.8) which is not applicable unless functions are absolutely continuous (AC); clearly functions with impulses are not AC.

A possible misconception arises because the Caputo derivative of a constant (on the whole interval) is 0, but in [10] and in [34] this seems to be applied to a piecewise constant function and the Caputo derivative of such a function is not 0, as we will prove in Example 3.1.

We also show that the solution for non-instantaneous impulsive problem, using the second approach as given in Wang, Zhou, Lin [28], is not correct. This has been used by several authors, for example [24], [31]. Agarwal, Hristova, O'Regan [1,2] mention both approaches and show that the proposed integral equations give different solutions; these authors use the first method of changing the fractional derivative at impulse points.

We also treat a problem of order between 1 and 2 and show that, using one fixed fractional derivative on the whole interval, the method proposed in [26] is faulty.

A recent paper [32] has discussed an impulse problem involving the Riemann–Liouville fractional derivative. It is claimed that there are two solutions which leads to a one-parameter family of solutions. We give a straightforward proof that gives exactly one solution, which shows that this claim requires justification.

It is well known that many properties for ODEs do not carry over to the fractional differential equation case. The reason is essentially due to the non-locality of fractional derivatives. Impulsive problems are another example, typical techniques for impulsive ODEs do not work in the fractional case.

2 Definitions

2.1 Riemann–Liouville fractional integral

We always consider $\alpha \in (0,1)$. We will consider functions defined on an interval $[0,T]$ which is, by a simple change of variable, equivalent to any finite interval. This interval is often not written but is to be implicitly understood in some statements.

Definition 2.1. The Riemann–Liouville (R–L) fractional integral, with initial point (lower limit) 0, of order $\alpha \in (0,1)$ of a function $u \in L^1[0,T]$ is defined as an $L^1[0,T]$ function by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

Here Γ denotes the well-known Gamma function.

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so, by the well known results on convolutions, for example Hewitt–Stromberg [17, Theorem 21.31], $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is defined and finite for almost every (a.e.) t.

Interchanging the order of integration, using Fubini's theorem, shows that these fractional integral operators satisfy a semigroup property as follows:

Lemma 2.2 (Semigroup property)**.** *Let $\alpha, \beta > 0$ and $u \in L^1$. Then $I^\alpha I^\beta(u) = I^{\alpha+\beta}(u)$ as L^1 functions, in fact $I^\alpha I^\beta(u)(t) = I^{\alpha+\beta}(u)(t)$ for each t for which $I^{\alpha+\beta}|u|(t)$ exists (finite), that is for a.e. $t \in [0,T]$. If u is continuous, or if $u \in L^1$ and $\alpha + \beta \geq 1$, equality holds for all $t \in [0,T]$.*

The result is given in [7, Theorem 2.2] and in [22, (2.21)]. A detailed proof is given in [29, Lemma 3.4].

Fractional integrals have good properties in Hölder spaces. The monograph by Samko, Kilbas and Marichev [22, §3.1, 3.2, 13.4] contains important facts and proofs, many of which were first proved by Hardy and Littlewood [15] (1928).

Definition 2.3. For an interval $[0,T]$, the Hölder space denoted $H^\lambda = H^\lambda[0,T]$, $0 < \lambda \leq 1$, (also the notation $C^{0,\lambda}$ is often used) consists of all functions u defined on $[0,T]$ for which there is a positive real constant C_u, such that

$$|u(t) - u(s)| \leq C_u |t-s|^\lambda, \quad \text{for all } t, s \in [0,T]. \tag{2.1}$$

We recall the following useful properties of fractional integral.

Proposition 2.4. *Let $0 < \alpha < 1$.*

(1) For $1 \leq p \leq \infty$, I^α is a bounded operator from $L^p[0,T]$ into $L^p[0,T]$.

(2) For $0 < 1/p < \alpha < 1$, the fractional integral operator I^α is bounded from L^p into the Hölder space $H^{\alpha-1/p}$. Moreover, $I^\alpha f(t) \to 0$ as $t \to 0^+$, that is $(I^\alpha f)(0) = 0$. This also holds for $p = \infty$ taking $1/\infty = 0$.

(3) I^α maps $AC[0,T]$ into $AC[0,T]$.

(4) For $f \in C^1[0,T]$, $I^\alpha f \in C^1[0,T]$ if and only if $f(0) = 0$.

(5) I^α does not map $C[0,T]$ into $AC[0,T]$.

The important property (2) is due to Hardy and Littlewood [15]. The result (2) is also proved in [22, Theorem 3.6 and Corollary]. Property (5) was explicitly given in [6] and [29]. More details and references concerning (5) are given in [20] and [30].

2.2 Fractional derivatives

Fractional derivatives (abbreviated FD) are 'inverses' of fractional integrals. We write D for the usual derivative operator, that is, $Du = u'$, with the usual one sided derivative at endpoints of an interval.

The Riemann–Liouville (R–L) fractional derivative of order $\alpha \in (0,1)$ is defined for functions that are more than integrable as follows.

Definition 2.5. For $\alpha \in (0,1)$ the R–L fractional derivative $D^\alpha u$ of a function $u \in L^1$ is defined at a point $t \in [0,T]$ when $I^{1-\alpha}u$ is differentiable at t by

$$D^\alpha u(t) := D\, I^{1-\alpha}u(t).$$

Note that the definition depends on the initial point used in the fractional integral, for us this is always fixed to be 0. $I^{1-\alpha}u(t)$ must be defined on an interval $(t-\delta, t+\delta)$ (when $0 < t < T$), this requires $u(s)$ to be defined for a.e. $s \in [0, t+\delta)$ and we require $u \in L^1[0, t+\delta)$.

The condition $I^{1-\alpha}u \in AC[0,T]$ must be imposed if we want to relate solutions of R–L fractional differential equations defined for a.e. $t \in [0,T]$ with solutions of a Volterra integral equation. It is not enough to assume that $I^{1-\alpha}u$ is differentiable for a.e. t. This has been noted long ago in the monograph [22], see [22, Definition 2.4] and the related comments in [22, Notes to §2.6].

As in the books Diethelm [7, Definition 3.2] and Kilbas, Srivastava and Trujillo [18, (2.4.1)], the Caputo fractional differential operator (or Caputo derivative) is defined via the R–L derivative as follows.

Definition 2.6. The Caputo fractional derivative $D^\alpha_* u$ of order $\alpha \in (0,1)$ is defined when $u(0)$ exists and $D^\alpha(u - u(0))$ exists at a point $t \in [0,T]$ by $D^\alpha_* u(t) := D^\alpha(u - u(0))(t)$.

Recall that we always use the fixed initial point 0, changing the initial point changes the fractional derivative. We interpret the statement $u(0)$ exists to mean that $u(0) = \lim_{t\to 0+} u(t)$, that is, u is continuous at 0, for otherwise $u(0)$ could have an arbitrary value not related to the behaviour of $u(t)$ for $t \in (0,T]$. This is often implicit but not usually explicit. The Caputo derivative is usually considered for continuous functions, then $u(0)$ does exist. If we want to relate Caputo fractional differential equations and integral equations then it is necessary to suppose that $I^{1-\alpha}(u - u(0)) \in AC[0,T]$ (equivalently $I^{1-\alpha}(u) \in AC[0,T]$) then $D^\alpha_*(t)$ is defined for a.e. t. This AC condition is weaker than having $u \in AC$ and, if it is imposed, then there is an equivalence between the fractional differential equation and an integral equation as follows.

Lemma 2.7. *Let $0 < \alpha < 1$ and suppose that $f \in L^p$ for some $p > 1/\alpha$. Then the following assertions are equivalent.*

(i) $u \in C[0,T]$, $I^{1-\alpha}(u) \in AC$, and u satisfies $D^\alpha_* u(t) = f(t)$ for a.e. $t \in [0,T]$, and $u(0) = u_0$.

(ii) $u \in C[0,T]$ satisfies $u(t) = u_0 + I^\alpha f(t)$ for every $t \in [0,T]$.

This is proved in [19, Corollary 4.2] and also in [20, Lemma 4]. When f is continuous the equivalence is well-known and can be found in Diethelm's book [7, Lemma 6.2].

We will make any necessary AC conditions explicit in this paper.

There is another frequently used definition of Caputo derivative, which uses the ordering $I^{1-\alpha}(Du)$, and also depends on the initial point. To distinguish between the definitions we will refer to this as the Caputo-C derivative.

Definition 2.8. For $\alpha \in (0,1)$, if u is differentiable a.e. on $[0,T]$ and $u' = Du \in L^1[0,T]$ the Caputo-C fractional derivative, denoted $D_C^\alpha u(t)$, is defined for a.e. t by

$$D_C^\alpha u(t) := I^{1-\alpha} Du(t).$$

This defines $D_C^\alpha u$ as an L^1 function. However, in order to prove any result about a Caputo fractional equation with the definition D_C^α, it is not sufficient to suppose that $u' \in L^1$ but it is necessary to have $u \in AC$. Consider the Cantor function φ (aka Lebesgue's singular function) which is continuous with derivative $\varphi'(t) = 0$ for a.e. t, so if $D_C^\alpha u(t) = f(t)$ a.e. then also $D_C^\alpha(u + c\varphi)(t) = f(t)$ a.e. for every constant c and nothing can be deduced. Moreover, there is not an equivalence such as in Lemma 2.7, since when f is continuous it does not follow that $I^\alpha f \in AC$ (Proposition 2.4 (5)). A comprehensive discussion of this fact is given in [20]. When $u \in AC$, as shown in Diethelm [7, Theorem 3.1], also in [29, Proposition 4.4], for $0 < \alpha < 1$ the two definitions $D_*^\alpha u$ and $D_C^\alpha u$ coincide, so usually there is no reason to consider $D_C^\alpha u$.

3 Fractional derivative equation with impulses

For an interval $[0,T]$ with subintervals defined by $0 = t_0 < t_1 < t_2 \cdots < t_m = T$, the space of piecewise continuous functions is defined by

$$\begin{aligned} PC[0,T] = \{u : [0,T] \to \mathbb{R} : u \in C(t_k, t_{k+1}),\ k = 0, \dots, m-1, \\ \text{and there exist } u(t_k^-) \text{ and } u(t_k^+) \text{ with } u(t_k^-) = u(t_k)\}, \end{aligned}$$

where $u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t = t_k$. Also the space $PC^1[0,T]$ is defined by

$$PC^1[0,T] := \{u \in PC[0,T] : u' \in PC[0,T]\}.$$

For the case of two subintervals, say $t_0 = 0, t_1 = 1, t_2 = 2$, this means that for $u \in PC[0,2]$, if we define

$$\begin{aligned} u_1(t) &= u(t), \quad t \in (0,1), \quad u_1(0) = \lim_{t \to 0^+} u(t), \quad u_1(1) = \lim_{t \to 1^-} u(t), \\ u_2(t) &= u(t), \quad t \in (1,2), \quad u_2(1) = \lim_{t \to 1^+} u(t), \quad u_2(2) = \lim_{t \to 2^-} u(t) \end{aligned}$$

then $u_1 \in C[0,1]$ and $u_2 \in C[1,2]$.

We will use the Heaviside function which is defined by

$$H(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{3.1}$$

Clearly H is piecewise continuous, $H'(t) = 0$ except at $t = 0$, but obviously $H \notin AC$.

There are two ways the notion of solution has been interpreted for the problem

$$\begin{aligned} D_*^\alpha u(t) &= f(t, u(t)), \quad t \in (0,T) \setminus \{t_1, \dots, t_{m-1}\} \\ u(0) &= u_0, \quad u(t_k^+) = u(t_k^-) + y_k, \quad k = 1, 2 \dots m-1. \end{aligned} \tag{3.2}$$

Here y_k are constants. The first way is to change the initial point (lower limit) of the FD to t_k on each subinterval and for a piecewise continuous function u consider a sequence of problems of the form

$$\begin{aligned} D_*^\alpha u(t) &= f(t, u(t)), \quad t \in (0, t_1), \text{ with initial condition } u(0) = u_0, \\ D_{*,t_k}^\alpha u(t) &= f(t, u(t)), \quad t \in (t_k, t_{k+1}), \text{ with initial condition } u(t_k) = u(t_k^-) + y_k, \end{aligned} \tag{3.3}$$

where
$$D^{\alpha}_{*,t_k}u(t) := D\left(\frac{1}{\Gamma(1-\alpha)}\int_{t_k}^{t}(t-s)^{1-\alpha}(u(s)-u(t_k))\,ds\right),$$
This first method is used in many papers (see for example, [1–5]). In all of these papers the formulas for the integral equations seem to be correct but the quoted ones use the Caputo-C derivative and hence any claimed equivalence with the integral equation is not valid even on the first interval $[0, t_1]$.

The second way, which is implied by how the problem is written, is to consider the FD with a fixed initial point at 0 as written in (3.2).

The formula of solutions for Eq. (3.2) is given by Fečkan, Zhou, Wang [10,24,27], to be
$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t f(s,u(s))\,ds \ \text{ for } t \in [0,t_1), \\ u_0 + y_1 + \frac{1}{\Gamma(\alpha)}\int_0^t f(s,u(s))\,ds \ \text{ for } t \in (t_1,t_2], \\ \cdots \\ u_0 + y_1 + \cdots + y_m + \frac{1}{\Gamma(\alpha)}\int_0^t f(s,u(s))\,ds \ \text{ for } t \in (t_{m-1},T]. \end{cases} \tag{3.4}$$
We consider a simple special case, namely the problem
$$D^{\alpha}_{*}u(t) = g(t), \quad t \in (0,2)\setminus\{1\}, \quad u(0) = 0, \quad u(1^+) = u(1^-) + y, \tag{3.5}$$
where g is continuous on [0,2] and y is a constant.

The claim from [10, Lemma 2.7] for $g \in C[0,T]$, is that, for y a constant, the solution as given by (3.4) is
$$u(t) = \begin{cases} u_1(t) & \text{when } t < 1, \\ y + u_1(t) & \text{when } t \ge 1, \end{cases} \tag{3.6}$$
where $u_1(t) = I^{\alpha}g(t),\ t \in [0,2]$ satisfies $D^{\alpha}_{*}u_1(t) = g(t),\ a.e.\ t \in [0,2],\ u(0) = 0$. One implication uses [10, Lemma 2.6] which is essentially a correct result but it is stated without giving the necessary hypotheses and is then used incorrectly when these hypotheses are not satisfied. A possible reason for the claim that the function defined in (3.6) satisfies (3.5) is that the Caputo derivative of a constant is 0. This is a misunderstanding, the Caputo derivative of a constant on the whole interval is 0, but the claimed solution is actually $u(t) = u_1 + yH(t-1)$ for $t \in [0,2]$ and the Caputo derivative of the Heaviside function is not zero, as we will show below. u_1 is certainly the solution of the problem (3.5) on the interval $[0,1)$, and is the solution without an impulse on all of $[0,2]$.

We give the simple example of the Heaviside function to show that the result claimed is not correct.

Example 3.1. Consider the problem
$$D^{\alpha}_{*}u(t) = 0, \quad t \in (0,2)\setminus\{1\}, \quad u(0) = 0, \quad u(1^+) = u(1^-) + 1. \tag{3.7}$$
The solution claimed in [10] would be $u(t) = 0,\ t \le 1,\ u(t) = 1,\ t > 1$, that is $u(t) = H(t-1)$. Here $u(0) = 0$, and $I^{1-\alpha}u(t) = 0$ for $t \le 1$ while, for $t > 1$, we have
$$\begin{aligned} I^{1-\alpha}u(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(s)\,ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_1^t (t-s)^{-\alpha}\,ds = \frac{1}{\Gamma(2-\alpha)}(t-1)^{1-\alpha}. \end{aligned}$$

Thus $I^{1-\alpha}u$ is differentiable on $(1,2]$, and $I^{1-\alpha}u \in AC[0,2]$, however the Caputo fractional derivative is not 0 but is

$$D_*^\alpha u(t) = D(I^{1-\alpha}(u-u(0))(t) = D(I^{1-\alpha}u)(t) = \begin{cases} 0, & \text{if } t<1, \\ \frac{1}{\Gamma(1-\alpha)}(t-1)^{-\alpha}, & \text{if } t>1. \end{cases}$$

Thus u is not a solution of the given FD equation (3.7).

The same argument applies to the more general case of equation (3.5) with g continuous when the claimed solution would be $u = u_1 + yH(t-1)$ where $D_*^\alpha u_1(t) = g(t)$ for a.e. $t \in [0,2]$.

Remark 3.2. The function $u(t) = H(t-1)$ gives a very simple example of a function where $D_C^\alpha u(t) = 0$ for the Caputo-C derivative whereas $D_*^\alpha u(t) \neq 0$ for $t > 1$. Using the Caputo-C definition leads to an error, the reason is that H is not absolutely continuous.

Remark 3.3. There are other difficulties with the set-up in [10].

1. The concept of solution is that $u \in PC^1$, but for g continuous $I^\alpha g$ is not C^1 in general, and indeed might not be AC. Therefore on the first interval it is usually not the case that $u \in C^1[0,t_1]$; the class of solutions is too small.

2. The result for g continuous is then applied in [10] to $f(t,u(t))$ where f is continuous. But, since $u(t)$ is not continuous at the impulse points, $g(t) = f(t,u(t))$ is unlikely to be continuous. However, the result is quoted in [24] for $g \in PC$.

The real problem is how to define the concept of solution for such problems. We show that the suggested concept of solution given in [10] is never correct, in fact also a weaker notion of solution which allows g to have countably many pointwise discontinuities is not adequate.

We recall some facts from real analysis.

Lemma 3.4. *Suppose that*

1) *h is continuous on $[a,b]$,*

2) *h' exists and is finite for all but a countable subset of (a,b), and*

3) *h' is Lebesgue integrable.*

Then $h \in AC[a,b]$ and $h(t) - h(a) = \int_a^t h'(s)ds$ for all $a \le t \le b$.

This is given, for example, in Hewitt and Stromberg's book [17, Exercise 18.41]; see also 18.17 in that book. A complete proof, using the Banach–Zaretsky theorem, is given by Heil in [16].

Corollary 3.5. *If h is continuous on $[a,b]$ and $h'(t) = 0$ except on a countable subset of (a,b), then $h(t) = c$, a constant.*

Proof. By Lemma 3.4, $h \in AC$ and $h(t) - h(a) = \int_a^t 0\,ds = 0$, so $h(t) \equiv h(a) = c$. □

For simplicity we consider the problem with $u(0) = 0$ and a single impulse. We will look for solutions that are piecewise continuous on $[0,2]$ for the problem

$$D_*^\alpha u(t) = g(t), \quad t \in (0,2) \setminus \{1\}, \quad u(0) = 0, \quad u(1^+) = u(1^-) + y, \tag{3.8}$$

where y is a constant. It is implicit that $I^{1-\alpha}u$ should be suitably differentiable. We will show that a solution of the type proposed in [10] cannot exist. We suppose that

$$u(t) = \begin{cases} u_1(t), & t \in [0,1], \\ u_2(t), & t \in [1,2], \end{cases} \tag{3.9}$$

where $u_1(t) = I^\alpha g(t)$ will be the solution of the problem $D_*^\alpha u(t) = g(t)$, $u(0) = 0$, without an impulse, and $u_2 \in C[1,2]$ with $u_2(1) = u_1(1) + y$.

Theorem 3.6. *Suppose that $g \in L^p[0,2]$ for some $p > 1/\alpha$ and let Z be a countable subset of $(0,2)$. Then for $y \neq 0$ there does not exist a piecewise continuous function u as in (3.9) with $I^{1-\alpha}u$ differentiable on $(0,2) \setminus Z$, that satisfies*

$$D_*^\alpha u(t) = D(I^{1-\alpha}u)(t) = g(t), \quad \textit{for } t \in (0,2) \setminus Z, \quad u(0) = 0, \quad u(1^+) = u(1^-) + y.$$

Proof. Suppose that a piecewise continuous solution exists. We have $u_1(t) = I^\alpha g(t)$ for $t \in [0,2]$ by Lemma 2.7, where $I^\alpha g$ is (Hölder) continuous on $[0,2]$ by Proposition 2.4 (2). Let $w(t) = u_2(t) - u_1(t), t \in [1,2]$. Extend w to $[0,1]$ by defining $w(t) = w(1)$ for $t \in [0,1]$, then $w \in C[0,2]$. We can then write $u(t) = u_1(t) + w(t)H(t-1)$ for $t \in [0,2]$. For $t < 1$ we have $I^{1-\alpha}u(t) = I^{1-\alpha}u_1(t)$ and $D(I^{1-\alpha}u)(t) = g(t)$ for $t \in (0,1) \setminus Z$ by the assumptions, that is, u_1 is a solution of the problem on $[0,1)$. For $t > 1$,

$$\begin{aligned} I^{1-\alpha}u(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_1(s)\, ds + \frac{1}{\Gamma(1-\alpha)} \int_1^t (t-s)^{-\alpha} w(s)\, ds \\ &= I^{1-\alpha}u_1(t) + \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau-\sigma)^{-\alpha} \widehat{w}(\sigma)\, ds, \end{aligned}$$

where $s = 1+\sigma$, $t = 1+\tau$, $\widehat{w}(\sigma) = w(\sigma+1)$. The derivative of the first term is $g(t)$ for $t \in (1,2) \setminus Z$. The second integral is $I^{1-\alpha}\widehat{w}(\tau)$, which is continuous for w continuous, and its derivative must be 0 except on Z in order for u to be a solution. Therefore, by Corollary 3.5, we must have $I^{1-\alpha}\widehat{w}(\tau) = c$ for $\tau \in [0,1]$. By the semigroup property of fractional integrals, $I\widehat{w}(\tau) = I^\alpha I^{1-\alpha}\widehat{w}(\tau) = c\frac{\tau^\alpha}{\Gamma(1+\alpha)}$ for all τ. Both sides of this equation are AC functions, thus have equal derivatives almost everywhere, that is, $\widehat{w}(\tau) = c\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}$ a.e.. This gives $w(t) = c\frac{(t-1)^{\alpha-1}}{\Gamma(\alpha)}$ for a.e. $t > 1$. The function w is continuous on $(1,2]$, but does not have a finite limit as $t \to 1^+$, and the impulse condition cannot be satisfied. This proves that no such solution can exist. □

It is a problem of how to define a solution of the impulsive fractional differential equation for the Caputo FD, $D_*^\alpha u$, with a fixed initial point (lower limit), so as to have an equivalence with some, as yet unknown, integral equation. Our calculations suggest that the integral equation must be considered in a space which allows singularities at the jumps.

4 Non-instantaneous impulses

A simplified problem with non-instantaneous impulses is given by

$$\begin{aligned} D_*^\alpha u(t) = f(t), &\quad t \in (t_k, s_k],\ k = 0,1,\ldots,m, \ \ u(0) = u_0 \\ u(t) = g_k(t), &\quad t \in (s_k, t_{k+1}],\ k = 0,1,\ldots,m-1. \end{aligned} \tag{4.1}$$

where $\alpha \in (0,1)$, $0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_m < s_m = T$, $f : [0,T] \to \mathbb{R}$ is piecewise continuous, and $g_k : [s_k, t_{k+1}] \to \mathbb{R}$ are continuous for all $k = 0,1,\ldots,m-1$.

Again there are two approaches, the first one is to change the initial point at each t_k and study a sequence of problems, this corresponds to defining a new initial value on each new subinterval without taking account of previous behaviour, and using new definitions of fractional derivatives. That is, the following problem is studied.

$$\begin{aligned}
D^{\alpha}_{*,t_0}u(t) &= f(t), \quad t \in (t_0, s_0],\ u(t_0) = u_0,\\
D^{\alpha}_{*,t_k}u(t) &= f(t), \quad t \in (t_k, s_k],\ k = 1,\ldots,m,\ \ u(t_k) = g_k(t_k),\\
u(t) &= g_k(t), \quad t \in (s_k, t_{k+1}],\ k = 0,1,\ldots,m-1,
\end{aligned}$$

where $D^{\alpha}_{*,t_k}u(t)$ is defined as in (3.3).

This approach gives correct mathematics, connections to applications are unclear. The second approach, as proposed by Wang, Zhou, Lin [28], and implied by the equation (4.1) as written, is to keep the initial point fixed at t_0. It was claimed in [28, Lemma 2.7] that the solution $u \in PC$ of (4.1) is given by

$$\begin{aligned}
&u(t) = u_0 + (I^{\alpha}f)(t), \quad t \in (t_0, s_0],\\
&u(t) = g_k(t), \quad t \in (s_{k-1}, t_k],\ k = 1,2,\ldots,m,\\
&u(t) = (I^{\alpha}f)(t) + g_k(t_k) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_k}(t_k - s)^{\alpha-1}f(s)\,ds, \quad t \in (t_k, s_k],\ k = 1,2,\ldots,m,
\end{aligned} \tag{4.2}$$

A similar claim is made in [25, Lemma 2.5]. This has been used by several authors, for example by Wang, Fečkan, Zhou [24]. Agarwal, Hristova, O'Regan [1, 2], discuss both forms. They point out that the proposed solutions given by the two approaches differ, but they concentrate on the first approach.

It was claimed by Fernandez, Ali, Zada [12] that the 'solution' in (4.2) does not take proper account of the non-instantaneous impulse so is not correct. Since the change of initial point always 'forgets' what has gone before this is not entirely clear. Also, the paper [12] uses the definition D^{α}_{C} which does not give an equivalence with the integral operator version so precision is lost. Fernandez, Ali, Zada write that the problems only arise in the case of non-instantaneous impulsive fractional differential equations, and that for instantaneous impulsive fractional differential equations the method of [10] is correct, but we showed above that this is false.

We now give a simple example to show that the solution claimed in (4.2) is not correct. We use as before the initial point 0.

Example 4.1. We take $f \equiv 0$ and consider the following problem.

$$\begin{aligned}
D^{\alpha}_{*}u(t) &= 0, \quad t \in (0,1] \cup (2,3],\ u(0) = 0,\\
u(t) &= 1, \quad t \in (1,2].
\end{aligned} \tag{4.3}$$

According to (4.2) the solution should be

$$u(t) = \begin{cases} 0, & t \in [0,1],\\ 1, & t \in (1,2],\\ 1, & t \in (2,3]. \end{cases} \tag{4.4}$$

For this function we have

$$\Gamma(1-\alpha)(I^{1-\alpha}u)(t) = \int_0^t (t-s)^{-\alpha}u(s)\,ds = \begin{cases} 0, & \text{for } 0 < t \le 1, \\ \frac{1}{1-\alpha}(t-1)^{1-\alpha}, & \text{for } 1 < t \le 3. \end{cases}$$

For $0 < t \le 1$ we have $D(I^{1-\alpha})u(t) = 0$ and for $t \in (1,2]$ we have $u(t) = 1$ as required, but for $t \in (2,3]$, $D(I^{1-\alpha})u(t) = \frac{1}{\Gamma(1-\alpha)}(t-1)^{-\alpha} \neq 0$, so u is not a solution of the problem (4.3).

5 Caputo fractional derivative of order between 1 and 2

The paper Wang–Zhou–Fečkan [26] discusses a problem of order $\beta \in (1,2)$. We write $\beta = 1+\alpha$ and will consider the corresponding initial value problem with impulses

$$\begin{aligned} D_*^{1+\alpha}u(t) &= h(t), \quad t \in [0,T] \setminus \{t_1,\dots,t_m\},\ u(0) = u_0,\ u'(0) = u_1, \\ u(t_k^+) &= u(t_k^-) + y_k, \quad u'(t_k^+) = u'(t_k^-) + z_k, \quad k = 1,2,\dots,m. \end{aligned} \tag{5.1}$$

The paper [26] discusses and gives a formula for the solution of the boundary value problem with $u(0) = 0$, $u'(1) = 0$ and also gives an example with other boundary conditions (BCs). Fu–Liu [13] and Liu–Jia [21] consider other boundary value problems and use the same method of finding a formula for solutions as [26]. In all these papers the Caputo-C definition is used, but that is not the main difficulty. We will show that the method given to obtain the formula is not correct.

It is stated that the general solution u of the equation on each interval (t_k, t_{k+1}), $k = 1,\dots,m)$ is given by

$$u(t) = I^{1+\alpha}h(t) + a_k + b_k t, \quad \text{for } t \in (t_k, t_{k+1}) \tag{5.2}$$

where a_k, b_k are constants. This is correct on the first interval $(0, t_1)$ but is not correct thereafter. It seems to be using the fact that $D_*^{1+\alpha}(a+bt) = 0$ when a, b are constants, which is true when a, b are constant on the whole interval $[0,T]$ but, as we saw previously, this is not true for different constants on subintervals.

We give an example of a simpler problem to show that this procedure does not give a solution of the impulsive FD equation.

Example 5.1. Consider the problem for h continuous with the simplest initial conditions

$$\begin{aligned} D_*^{1+\alpha}u(t) &= h(t), \quad t \in (0,2) \setminus \{1\}, \quad u(0) = 0, \quad u'(0) = 0, \\ u(1^+) &= u(1^-) + y, \quad u'(1^+) = u'(1^-) + z, \end{aligned} \tag{5.3}$$

where y, z are constants. On the interval $(0,1)$ the solution is $u(t) = I^{1+\alpha}h(t)$ as given in [26]. On the interval $(1,2]$ the solution claimed in [26] would be of the form $u(t) = a + bt + I^{1+\alpha}h(t)$. Then on the interval $[0,2]$, $u(t) = I^{1+\alpha}h(t) + aH(t-1) + btH(t-1)$. Since we take $u(0) = 0$, $u'(0) = 0$, we have

$$D_*^{1+\alpha}u(t) = D^{1+\alpha}(u - u(0) - tu'(0))(t) = D^{1+\alpha}u(t) = D^2(I^{1-\alpha}u)(t),$$

where $(I^{1-\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(s)\,ds$. For $t \in (0,1)$, $D^2(I^{1-\alpha}u)(t) = D^2(I^2h)(t) = h(t)$ (since h is continuous) as required. For $t \in (1,2)$,

$$I^{1-\alpha}u(t) = I^2h(t) + \frac{1}{\Gamma(1-\alpha)}\int_1^t (t-s)^{-\alpha}(a+bs)\,ds.$$

We have $\int_1^t (t-s)^{-\alpha}\,ds = \frac{1}{1-\alpha}(t-1)^{1-\alpha}$, and, integrating by parts gives,

$$\int_1^t (t-s)^{-\alpha} s\,ds = \frac{1}{1-\alpha}(t-1)^{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)}(t-1)^{2-\alpha}.$$

Therefore we have

$$I^{1-\alpha}u(t) = I^2 h(t) + \frac{a+b}{(1-\alpha)\Gamma(1-\alpha)}(t-1)^{1-\alpha} + \frac{b}{(2-\alpha)\Gamma(2-\alpha)}(t-1)^{2-\alpha},$$

which gives

$$D_*^{1+\alpha}u(t) = D^2 I^{1-\alpha}u(t) = h(t) - \alpha\frac{a+b}{\Gamma(1-\alpha)}(t-1)^{-\alpha-1} + \frac{b}{\Gamma(1-\alpha)}(t-1)^{-\alpha}.$$

Thus, for $t \in (1,2)$ the fractional derivative is not $h(t)$, except when $a = b = 0$. This would imply that $y = z = 0$ so it would not be an impulsive problem.

6 A Riemann–Liouville problem

In a recent paper, Zhang [32], an impulsive problem with the R–L derivative was studied. Some definitions lack proper conditions for their validity. A claimed equivalence ([32, Proposition 2]) between the fractional differential equation without an impulse and an integral equation is not given hypotheses.

A simple version of the problem, where h is a constant, is as follows.

$$\begin{aligned} &D^\alpha u(t) = f(t), \quad t \in (0,2)\setminus\{1\},\\ &I^{1-\alpha}u(0) = u^{(0)},\\ &I^{1-\alpha}u(1^+) - I^{1-\alpha}u(1^-) = h. \end{aligned} \tag{6.1}$$

It is claimed that there are two piecewise defined functions that satisfy this problem, which then gives a one-parameter family of solutions. 'Solution' is not defined in the paper. The first solution given is correct. We now prove this since it is relevant to the further discussion. During the proof we will see what are the implicit assumptions needed to have the solution.

Lemma 6.1. *The problem* (6.1) *has the solution*

$$u(t) = \begin{cases} \frac{u^{(0)}t^{\alpha-1}}{\Gamma(\alpha+1)} + I^\alpha f(t), & t \in (0,1),\\ \frac{u^{(0)}t^{\alpha-1}}{\Gamma(\alpha+1)} + I^\alpha f(t) + h\frac{(t-1)^{\alpha-1}}{\Gamma(\alpha)}, & t \in (1,2). \end{cases} \tag{6.2}$$

Proof. Writing $w(t) = I^{1-\alpha}u(t)$ we see that (6.1) is the impulsive problem for on ODE namely

$$w'(t) = f(t), \quad t \in (0,2)\setminus\{1\}, \quad w(0) = u^{(0)}, \quad w(1^+) - w(1^-) = h. \tag{6.3}$$

Therefore the equation (6.1) as written implies that w should be differentiable on $(0,1)$ and $(1,2)$ and the one-sided limits of w should exist at 0 and 1. Thus w is piecewise continuous and it must be supposed that f is at least integrable. The ODE problem then has solution $w(t) = u^{(0)} + \int_0^t f(s)\,ds$, for $t \in [0,1)$ and $w(1^-) = u^{(0)} + \int_0^1 f(s)\,ds$. Then on the interval $(1,2)$ we have

$$w(t) = \big(u^{(0)} + \int_0^1 f(s)\,ds + h\big) + \int_1^t f(s)\,ds = u^{(0)} + h + \int_0^t f(s)\,ds.$$

This can be written, with H denoting the Heaviside function,

$$w(t) = u^{(0)} + If(t) + hH(t-1). \tag{6.4}$$

Since $I^{1-\alpha}u(t) = w(t)$, applying the operator I^α and using the semigroup property we have

$$Iu(t) = I^\alpha w(t) = \frac{u^{(0)}t^\alpha}{\Gamma(\alpha+1)} + I(I^\alpha f(t)) + \frac{h(t-1)^\alpha H(t-1)}{\Gamma(\alpha+1)}. \tag{6.5}$$

We suppose that $f \in L^p$ for some $p > 1/\alpha$ so that $I^\alpha f$ is continuous and $I(I^\alpha f)$ has derivative $I^\alpha f$. If u is continuous on each open subinterval then differentiating (6.5) gives

$$u(t) = D(I^\alpha w)(t) = \frac{u^{(0)}t^{\alpha-1}}{\Gamma(\alpha)} + I^\alpha f(t) + \frac{h(t-1)^{\alpha-1}H(t-1)}{\Gamma(\alpha)}, \qquad t \neq 0, 1. \tag{6.6}$$

This is (6.2). We now show that if u satisfies (6.6) then $w = I^{1-\alpha}u$ satisfies (6.4). From (6.6) we obtain

$$I^{1-\alpha}u(t) = u^{(0)} + If(t) + \frac{h}{\Gamma(1-\alpha)} \int_1^t (t-s)^{-\alpha} \frac{(s-1)^{\alpha-1}}{\Gamma(\alpha)}\, ds$$

Changing the variable to $s = 1 + \sigma$, and writing $\tau = 1 + t$ in the last integral we see that

$$\int_1^t (t-s)^{-\alpha}(s-1)^{\alpha-1}\, ds = \int_0^\tau (\tau-\sigma)^{-\alpha}\sigma^{\alpha-1}\, d\sigma = B(1-\alpha,\alpha) = \Gamma(1-\alpha)\Gamma(\alpha).$$

This gives (6.4) which then shows that (6.3) holds. □

Remark 6.2. We obtain exactly one solution so the claims in [32] and in [33] that there are two distinct ones seems to be wrong. The calculations in these papers are not fully given and exactly what 'solution' is intended to be is not clear. However Lemma 6.1 uses what seem to be necessary hypotheses, so any explanation is far from clear.

7 Conclusion

It has been shown with simple counter-examples that the method proposed in [10, 26] for dealing with instantaneous impulses, and in [28] for non-instantaneous impulses, for a Caputo fractional differential equation with the fractional derivative having a fixed initial point, do not give solutions of the problem. Moreover, it was shown that no piecewise continuous solution of the type proposed is possible. It is an open problem of how to define a solution in order to have an equivalence with an integral equation. Some faults in a Riemann–Liouville impulsive problem studied in [32] were also pointed out.

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