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A scalar differential delay equation with level-dependent response time

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Abstract. We examine the dynamics of some model scalar differential equations with negative feedback and a particular kind of state-dependent delay. The equations are chosen as maximally simple models representing the assumptions that larger deviations from equilibrium either strengthen or weaken the feedback response, and either increase or decrease the delay time. We establish several dynamical behaviors for various of our examples, including uniqueness/non-uniqueness of "slowly oscillating" periodic solutions and chaos.

Keywords: delay equations, state-dependent delay, periodic solutions, chaotic solutions.

2020 Mathematics Subject Classification: 34K13, 34K23.

1 Introduction

We choose and fix $d \in (0,1)$ and write $I = [-1,1]$. Our goal in this paper is to study the dynamics of the following two scalar-valued differential delay equations:

$$x'(t) = -a\,\mathrm{sign}(x(t-1))\chi_I(x(t-1)) - b\,\mathrm{sign}(x(t-d))\chi_{\mathbb{R}\setminus I}(x(t-d)), \quad a,b>0; \tag{EF}$$

$$x'(t) = -a\,\mathrm{sign}(x(t-1))\chi_{\mathbb{R}\setminus I}(x(t-1)) - b\,\mathrm{sign}(x(t-d))\chi_I(x(t-d)), \quad a,b>0. \tag{ES}$$

In the customary manner we write $C = C([-1,0],\mathbb{R})$ for the Banach space of continuous real-valued functions on $[-1,0]$, equipped with the sup norm. Given any continuous real-valued function x whose domain includes $[t-1,t]$ we write x_t for the member of C given by the formula $x_t(s) = x(t+s)$. We shall regard the state spaces for Equations (EF) and (ES) as situated in C.

In Equations (EF) and (ES), the current level $x(s)$ of a solution induces, depending on its size, a response exactly $d \in (0,1)$ time units or exactly 1 time unit in the future. For this reason, we describe these equations as having *level-dependent response time*. Figure 1.1 illustrates the basic feedback mechanism for Equation (EF). The intersections of the current state x_t with

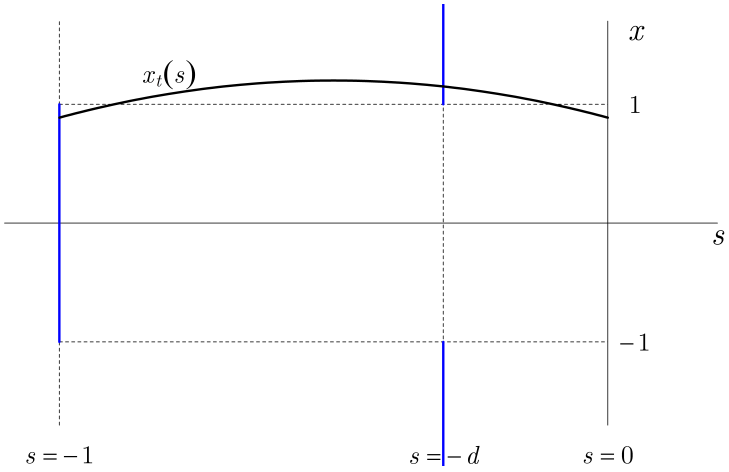


Figure 1.1: A state x_t for Equation (EF).

the blue lines represent the points of the state that contribute to $x'(t)$. In the particular case illustrated in the figure, $x'(t) = -a - b$.

Similarly, Figure 1.2 illustrates the basic feedback mechanism for Equation (ES). The intersections of the current state x_t with the blue lines represent the points of the state that contribute to $x'(t)$. In the particular case illustrated in the figure, $x'(t) = b$.

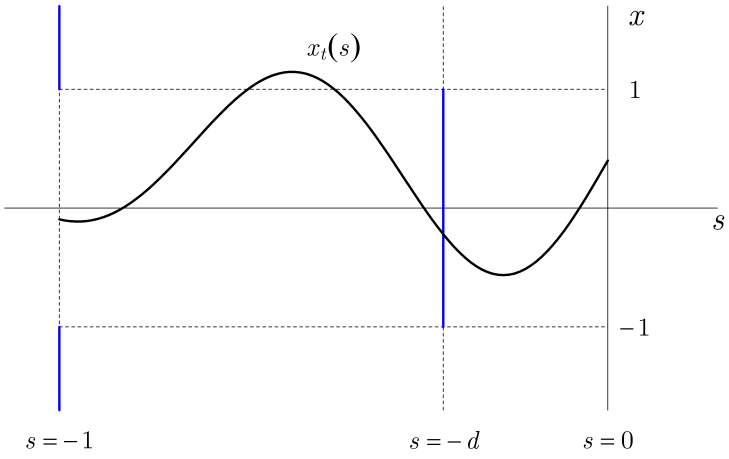


Figure 1.2: A state x_t for Equation (ES).

Our study of Equations (EF)–(ES) is part of an attempt to understand equations that fit (or that have approximations that fit) into the general form

$$x'(t) = \int_0^1 g(u, x(t-u))\, du. \tag{1.1}$$

Roughly speaking, in Equation (1.1), the current level $x(s)$ of a solution x contributes $g(u, x(s))$ to the derivative $x'(s+u)$ of x exactly u time units in the future. Equation (1.1) provides a flexible framework to express various heuristic modeling assumptions regarding state-dependent delays. If we impose the *negative feedback condition* that $xg(u, x) \leq 0$ for all u and all x, then casual numerical exploration of various instances of Equation (1.1) yields, in many cases, the

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"stereotypical" negative-feedback picture for scalar-valued delay equations – in particular, a strongly attractive domain of solutions that are in some sense "slowly oscillating," and (often) within this set an attractive periodic solution. However, it is not obvious (to these authors) how to carefully define these slowly oscillating solutions (let alone, for example, a useful notion of "oscillation speed" or a discrete Lyapunov function) for anything close to the breadth of situations in Equation (1.1) for which this behavior seems to occur. More generally, much about Equation (1.1) in the negative feedback case does not seem well-understood.

Hence the need for some case studies. Equations (EF)–(ES) are intended to be maximally simple models invoking Equation (1.1) and expressing four broad heuristic assumptions, described as follows. If we regard a response time of d as "fast" and a response time of 1 as "slow," and a deviation of more than 1 unit from the equilibrium at 0 as "large," then:

- in Equation (EF) with $a \leq b$, large deviations from the equilibrium at 0 induce a *faster* and *stronger* response;
- in Equation (EF) with $b \leq a$, large deviations from the equilibrium at 0 induce a *faster* and *weaker* response;
- in Equation (ES) with $b \leq a$, large deviations from the equilibrium at 0 induce a *slower* and *stronger* response;
- in Equation (ES) with $a \leq b$, large deviations from the equilibrium at 0 induce a *slower* and *weaker* response.

Either Equation (EF) or (ES) can, of course, be rescaled so that the longest delay is different from 1, or so that the "cutoff" between small and large deviations from equilibrium is different from 1.

Each of these equations exhibits negative feedback in the sense that deviations from 0 in a given direction induce movement in the opposite direction. As we shall show, under appropriate restrictions on d, Equations (EF) and (ES) admit forward-invariant subsets of "slowly oscillating" solutions where explicit and elementary – though nontrivial – calculations are possible. We hope that our analyses of Equations (EF)–(ES) yield some insight into broader families of equations.

We shall, in particular, study each of Equations (EF)–(ES) with $a \neq b$, $a, b \in \{1, 2\}$, and (for reasons that we shall explain) $d \in [2/3, 1)$. A brief sketch of our findings is as follows.

- For Equation (EF) with $a = 1$ and $b = 2$, all slowly oscillating solutions tend toward a single slowly oscillating periodic solution.
- For Equation (EF) with $a = 2$ and $b = 1$, there are multiple slowly oscillating periodic solutions. For some values of d one of these periodic solutions attracts most slowly oscillating solutions; but for other values of d the solution semiflow is, in an appropriate sense, chaotic on the set of slowly oscillating solutions.
- For Equation (ES) with $a = 2$ and $b = 1$, there are exactly two stable slowly oscillating periodic solutions.
- For Equation (ES) with $a = 1$ and $b = 2$, all slowly oscillating solutions tend toward a single slowly oscillating periodic solution.

A conceptual relative of Equation (1.1) is the equation

$$x'(t+\eta(x(t))) = h(x(t)) \tag{1.2}$$

introduced by Walther in [13], where the current level $x(s)$ of x induces some feedback response $\eta(x(s))$ time units in the future. Walther recast this equation as a state-dependent delay equation of the form

$$x'(t) = f(x(t-d(x_t))), \tag{1.3}$$

having imposed assumptions on h to guarantee the injectivity of the map $t \mapsto t+\eta(x(t))$. Most of the best-understood nonlinear scalar-valued delay equations (including those with constant delay) are of the form (1.3); the monographs [2] and [3] are standard references for (1.3) in the constant- and state-dependent-delay cases, respectively. Work is ongoing to understand various specific subfamilies of (1.3) where the "delay functional" $d(x_t)$ has various features (see, for example, [6] and [7], and [5]). Scalar-valued delay equations that do not fit into the framework (1.3) include cases where the specification of the delay d requires an auxiliary quantity (e.g. [1]), and cases where multiple feedback mechanisms are operating simultaneously (e.g. [4,9]). Our initial impetus in studying Equations (EF)–(ES) was to explore equations with a feedback structure akin to that in Equation (1.2), but where multiple past time points can exert a simultaneous influence.

The paper is organized as follows. In Section 2 we briefly sketch an existence-and-uniqueness result sufficient for our needs. In Section 3 we describe the forward-invariant subsets of "slowly oscillating" solutions on which we focus. We analyze the dynamics of solutions in these sets with a standard approach: we identify subsets $\Sigma \subseteq C$ to which solutions return infinitely often, and define return maps $R : \Sigma \to \Sigma$ that track solutions from one intersection with Σ to the next. These maps R are semiconjugate to interval maps $F : [0,1] \to [0,1]$, which are the maps that we actually analyze. We define Σ, R and F in Section 4. In Section 5 we present explicit formulas for F for each of our equations of interest, and note the implications for the dynamics of solutions. Since the main ideas here are standard, our proofs in Section 5 are brief and in some cases only sketched. By contrast, the actual derivations of the formulas for F, while entirely elementary, are neither brief nor routine, and we present these derivations in full in Appendix A.

2 Existence and uniqueness

We recall that $I = [-1,1]$, that $d \in (0,1)$, and that a and b are strictly positive. For either of Equations (EF)–(ES), let us write $f(x_t)$ for the expression on the right-hand side; so for example, with Equation (EF) we have

$$f(x_t) = -a\,\mathrm{sign}(x_t(-1))\chi_I(x_t(-1)) - b\,\mathrm{sign}(x_t(-d))\chi_{\mathbb{R}\setminus I}(x_t(-d)).$$

The possible values for f are

$$0,\ \pm a,\ \pm b,\ \pm a \pm b$$

(the values on this list might overlap). By a *solution* of either of Equations (EF)–(ES) we mean a continuous function $x : [-1,\infty) \to \mathbb{R}$ for which

$$x(t) = x(0) + \int_0^t f(x_s)\,ds \quad \text{for all} \quad t \geq 0.$$

At points t for which neither $x(t-d)$ nor $x(t-1)$ lie in $\{-1,0,1\}$, x will be differentiable, with a locally constant derivative of $x'(t) = f(x_t)$. We call x a *continuation of* x_0 *as a solution* (of whichever equation we are considering).

Given $\varphi \in C$, we write $Lip(\varphi)$ for the smallest possible Lipschitz constant for φ (if it exists): namely,

$$Lip(\varphi) = \sup_{s \neq t} \frac{|\varphi(s) - \varphi(t)|}{|s-t|}.$$

We write $\partial\varphi^{-1}\{-1,0,1\}$ for the boundary of the set $\{t \in [-1,0] \ : \ \varphi(t) \in \{-1,0,1\}\}$, and define

$$X = \{\varphi \in C \ : \ \partial\varphi^{-1}\{-1,0,1\} \text{ is finite and } \mathrm{Lip}(\varphi) \leq a+b\}.$$

We shall make no particular effort here to identify the strongest possible existence and uniqueness theorem for our equations, but content ourselves with stating the following elementary result.

Proposition 2.1. *Assume that $d \in (0,1)$ and that X is as above, and suppose that $\varphi \in X$. Then for either of Equations (EF)–(ES), there is a unique continuation $x : [-1,\infty) \to \mathbb{R}$ of φ as a solution. Moreover, $x_t \in X$ for all $t \geq 0$ and x is differentiable, with $x'(t) = f(x_t)$, except at isolated points.*

3 Slowly oscillating solutions

We shall henceforth assume that solutions of Equations (EF)–(ES) have segments that lie in X, where X is as in Proposition 2.1. In addition to this relatively innocuous condition, in this section we introduce another more substantial restriction that will make our analysis much easier. This restriction is

$$a+b \leq \frac{1}{1-d}. \tag{OS}$$

Since $a+b$ is a Lipschitz constant for solutions x, the implications of (OS) are that

(OSi) if $x(t-1)$ and $x(t-d)$ both contribute nontrivially to $x'(t)$, they must have the same sign;

(OSii) for Equation (EF), if $x'(t) \neq 0$, then $x'(t)x(t-1)$ is strictly negative;

(OSiii) for Equation (ES), if $x'(t) \neq 0$, then $x'(t)x(t-d)$ is strictly negative.

Figures 3.1 and 3.2 illustrate the situation for Equations (EF) and (ES) respectively. The diagonal dashed lines have slope with absolute value $(1-d)^{-1} \geq a+b$, and so no solution segment can cross these dashed lines with a "steeper" slope.

Condition (OS) allows us to identify forward-invariant sets of "slowly oscillating" solutions. The ideas are entirely familiar. To begin with, we define the number

$$h = \begin{cases} 1, & \text{for Equation (EF);} \\ d, & \text{for Equation (ES).} \end{cases}$$

The point is that, for either of our equations, if $x'(t) \neq 0$, then $x(t-h)x'(t)$ is strictly negative.

Let us define the sets

$$S_1^+ = \{\varphi \in X \ : \ \varphi(s) > 0 \text{ for all } s \in (-1,0]\}; \quad S_d^+ = \{\varphi \in X \ : \ \varphi(s) > 0 \text{ for all } s \in (-d,0]\}.$$

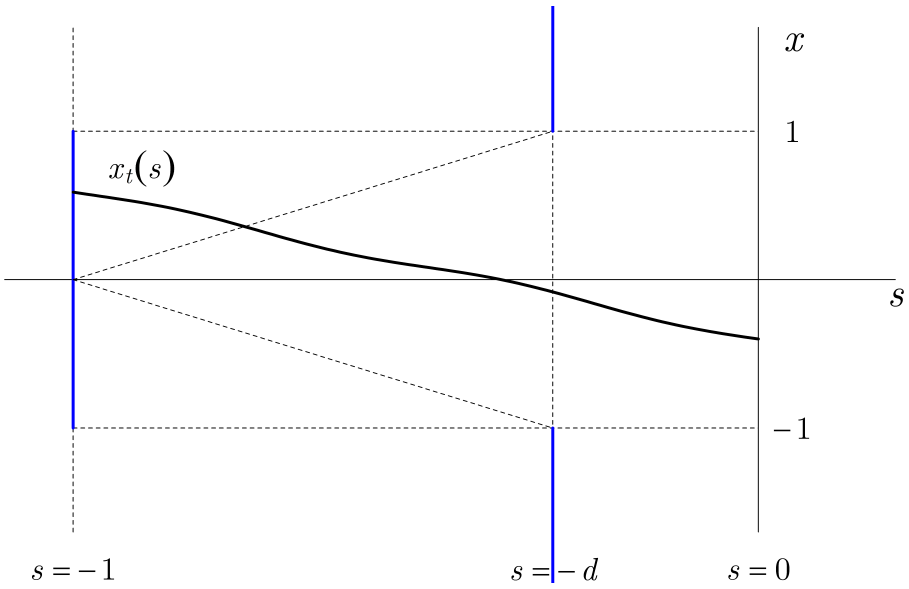


Figure 3.1: Condition (OS) for Equation (EF).

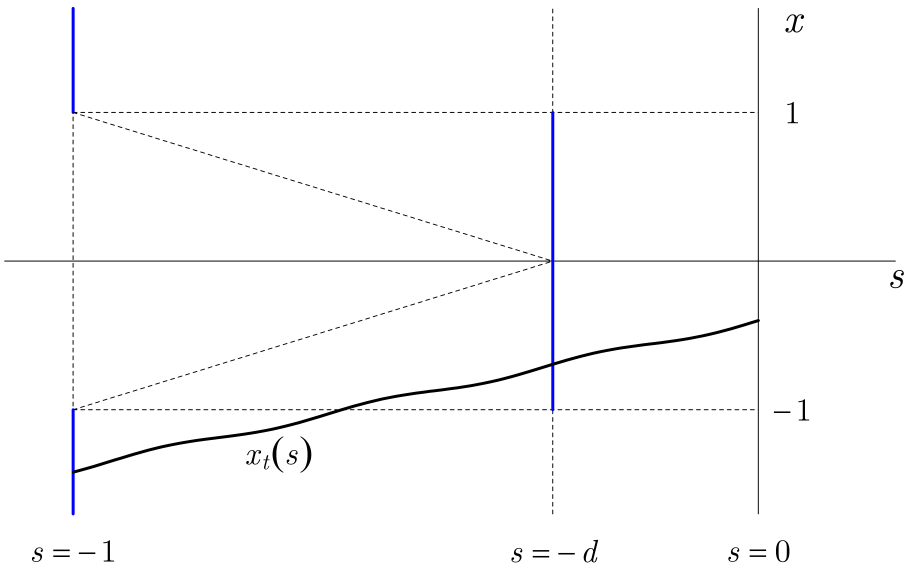


Figure 3.2: Condition (OS) for Equation (ES).

Suppose that $\varphi = x_0 \in S_1^+$ has continuation x as a solution of Equation (EF). Then, by observation (OSii) above, we know that $x'(t)$ will be nonpositive for all $t \in [0, z+1]$, where z is the right-hand endpoint of the first positive interval (usually a one-point interval) of zeros of x. (z must exist because otherwise x would be nonincreasing on the positive half-line toward a nonnegative equilibrium, which is not possible since both a and b are strictly positive.) Observe that x_{z+1} lies in $-S_1^+$. Since our feedback function is odd, we conclude that solutions of Equation (EF) beginning in (or entering) S_1^+ will be (eventually) "slowly oscillating" in the sense that all of their positive zeros (more correctly, right endpoints of intervals of positive zeros) will be more than one unit apart.

An almost identical argument shows that solutions of Equation (ES) beginning in (or entering) S_d^+ will be (eventually) "slowly oscillating" in the sense that all of their positive zeros (more correctly, right endpoints of intervals of positive zeros) will be more than d units apart.

Remark 3.1. Under condition (OS), we can use familiar ideas along with the kind of reasoning above to define "oscillation speeds" for our equations – roughly speaking, the number of sign changes per 1 unit time for Equation (EF) and the number of sign changes per d units time for Equation (ES). Since we are concerned only with slowly oscillating solutions in this paper, we won't pursue this here.

We now turn to a standard idea to study the dynamics of slowly oscillating solutions. We shall identify subsets $\Sigma_1 \subseteq S_1^+$ and $\Sigma_d \subseteq S_d^+$ with the features that, if $x_0 \in \Sigma_h$ ($h \in \{1, d\}$, as appropriate), then there is a minimum positive ζ for which $x_\zeta \in -\Sigma_h$. We then define the *return map* $R_h : \Sigma_h \to \Sigma_h$ by the formula

$$R_h(x_0) = -x_\zeta.$$

Since Equations (EF) and (ES) have odd feedback, the dynamics of R_h capture the dynamics of the solution semiflow on the set of solutions passing through Σ_h (and indeed on the set of all slowly oscillating solutions, as we'll see shortly). Moreover, we will be able to define a map $Z_h : \Sigma_h \to [0,1] \subseteq \mathbb{R}$ and a map $F_h : [0,1] \to [0,1]$ such that the following semiconjugacy holds:

$$Z_h \circ R_h = F_h \circ Z_h.$$

It is the maps $F_h : [0,1] \to [0,1]$ that we will study explicitly; and results about the dynamics of these maps (periodic points, chaos, etc.) will translate to corresponding results for R_h, and hence for the solution semiflow.

4 The return maps for Equations (EF) and (ES)

In this section we define the sets Σ_h, and the maps R_h and Z_h as described above. Given $\varphi \in C$, we write $|\varphi|$ for its sup norm.

4.1 The Equation (EF) (i.e. $h = 1$)

Our set Σ_1 is defined as follows:

$$\Sigma_1 = \left\{ \varphi \in S_1^+ : \ \varphi^{-1}(1,\infty) \cap (-1,0] = (-v, 0) \text{ for some } v \in [0,1] \right\}.$$

In particular, if $\varphi \in S_1^+$ with $|\varphi| \leq 1$ then $\varphi \in \Sigma_1$ (with $v = 0$). If $\varphi \in S_1^+$ and $|\varphi| > 1$, then for φ to belong to Σ_1 it must be the case that $\varphi(0) = 1$ and that $\varphi(t) > 1$ precisely on some

interval $(-v, 0)$ (or on $[-1, 0)$). Figure 4.1 shows two elements of Σ_1, one with $v = 0$ and one with $v = 1/2$.

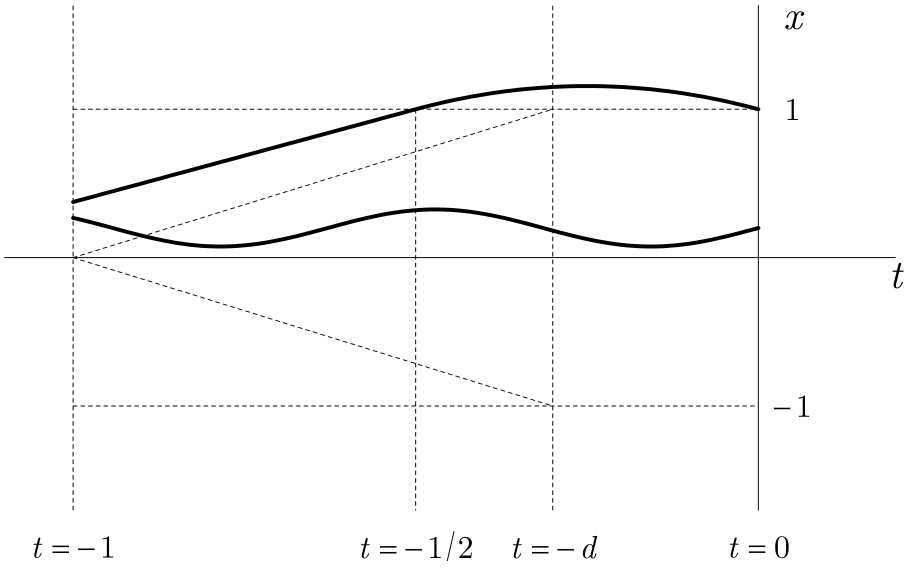


Figure 4.1: Two elements of Σ_1.

Let us define the map $Z_1 : \Sigma_1 \to [0, 1]$ by the formula $Z_1(\varphi) = v$, where v is as above. We also continue to write x for the continuation of φ as a solution of (EF), and z for the right endpoint of the first positive interval of zeros of x. We claim that

(i) there is a first positive time $\zeta > z$ for which $x_\zeta =: -R_1(\varphi) \in -\Sigma_1$; and

(ii) $Z_1(\varphi) = v$ completely determines $R_1(\varphi)$.

We prove this claim in two cases.

$v > 0$ **case.** By (OSii), x will be nonincreasing on $[z, z + 1]$, and since $x(t - 1) \in (0, 1]$ for at least some $t \in (z, z + 1)$, (OSi) assures us that $x(z + 1)$ is strictly negative. Again by (OSii), x will be nondecreasing on $[z + 1, z' + 1]$, where z' is right endpoint of the first interval of zeros of x after $z + 1$. If $x(z + 1) \geq -1$, then $\zeta = z + 1$. If $x(z + 1) < -1$ then $x'(t)$ will be nonnegative until the right-hand endpoint t_2 of the first interval after $z + 1$ where $x(t) = -1$; in this case, $\zeta = t_2$. Finally, if $Z_1(\varphi) = Z_1(\psi) > 0$, then φ and ψ continue in exactly the same way as solutions of (EF), and so in particular $R_1(\varphi) = R_1(\psi)$.

$v = 0$ **case.** If $Z_1(\varphi) = 0$, then $x'(t) = -a$ for all $t \in (0, z + 1)$, and so x_{z+1} is completely determined by the fact that $v = 0$. As before, ζ will either be equal to or greater than $z + 1$, depending on whether or not $x(z + 1) \geq -1$.

Our claim shows that the map $F_1 : [0, 1] \to [0, 1]$ satisfying $F_1 \circ Z_1 = Z_1 \circ R_1$ exists; in the following section we make explicit calculations for F_1. Furthermore, similar reasoning as in the above claim shows that any solution beginning in S_1^+ enters $-\Sigma_1$; thus the dynamics of R_1 capture the dynamics of all "slowly oscillating" solutions.

4.2 The Equation (ES) (i.e. $h = d$)

Our set Σ_d is defined very similarly as Σ_1.

$$\Sigma_d = \left\{ \varphi \in S_d^+ : \ \varphi^{-1}(1, \infty) \cap (-1, 0] = (-v, 0) \text{ for some } v \in [0, 1] \right\}.$$

In particular, if $\varphi \in S_d^+$ with $|\varphi| \leq 1$ then $\varphi \in \Sigma_d$ (with $v = 0$). If $\varphi \in S_d^+$ and $|\varphi| > 1$, then $\varphi(0) = 1$ and $\varphi(t) > 1$ precisely on some interval $(-v, 0)$ (or on $[-1, 0)$). Figure 4.2 shows two elements of Σ_d, one with $v = 0$ and one with $v = 3/5$.

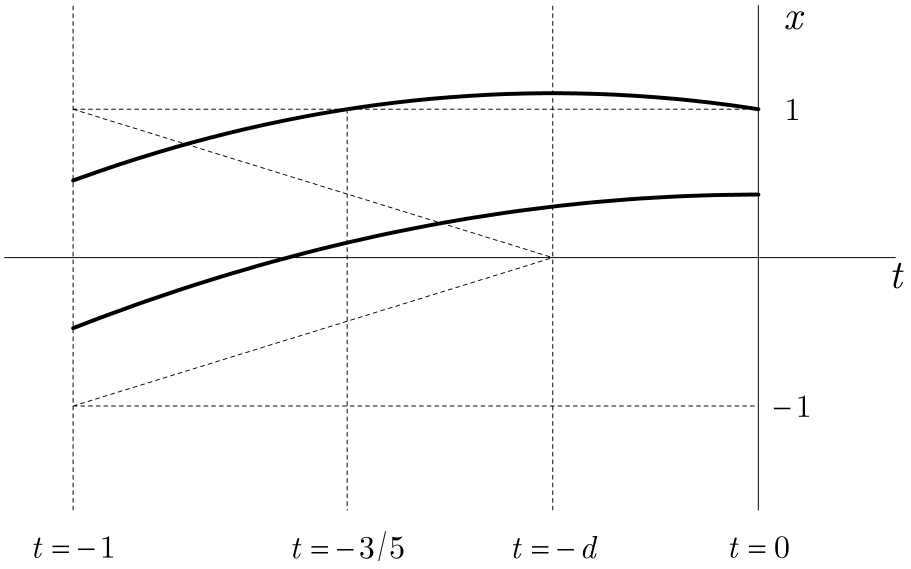


Figure 4.2: Two elements of Σ_d.

We define the map $Z_d : \Sigma_d \to [0, 1]$ by the formula $Z_d(\varphi) = v$, where v is as above. We also continue to write x for the continuation of φ as a solution of (ES), and z for the right endpoint of the first positive interval of zeros of x. As above, we claim that

(i) there is a first positive time $\zeta > z$ for which $x_\zeta =: -R_d(\varphi) \in -\Sigma_d$; and

(ii) $Z_d(\varphi) = v$ completely determines $R_d(\varphi)$.

The proof is virtually the same as in the $h = 1$ case (using (OSiii) instead of (OSii) where appropriate) and we omit it. Our claim shows that the map $F_d : [0, 1] \to [0, 1]$ satisfying $F_d \circ Z_d = Z_d \circ R_d$ exists; in the following section we make explicit calculations for F_d. Again, we observe that all solutions beginning in S_d^+ enter $-\Sigma_d$ and so the dynamics of R_d capture the dynamics of all "slowly oscillating" solutions.

5 Main results

In this section, we consider Equations (EF) and (ES) with $a \neq b$ and $a, b \in \{1, 2\}$. For condition (OS) to apply we must have $d \geq 2/3$, so we will make this assumption as well. We write both F_1 and F_d simply as F. For all four of our equations, we state the formulas for F and comment on some implications for the dynamics of our delay equations. The derivations of the formulas for F are deferred to Appendix A.

5.1 Equation (EF) with $a = 1$ and $b = 2$

We remind the reader that the equation under consideration here is

$$x'(t) = -\operatorname{sign}(x(t-1))\chi_{[-1,1]}(x(t-1)) - 2\operatorname{sign}(x(t-d))\chi_{(-\infty,-1)\cup(1,\infty)}(x(t-d)), \qquad \text{(I)}$$

where $d \in [2/3, 1)$.

Proposition 5.1. *For Equation (I) and where $F : [0,1] \to [0,1]$ is as defined in Section 4, we have the following formulas.*

For $d \in [2/3, 3/4)$:

$$F(v) = \begin{cases} 0, & v \in [0, -2d+2); \\ v + 2d - 2, & v \in [-2d+2, d); \\ -v + 4d - 2, & v \in [d, 4d-2); \\ 0, & v \in [4d-2, 1]. \end{cases}$$

For $d \in [3/4, 1)$:

$$F(v) = \begin{cases} 0, & v \in [0, -2d+2); \\ v + 2d - 2, & v \in [-2d+2, d); \\ -v + 4d - 2, & v \in [d, 1]. \end{cases}$$

For any value of $d \in [2/3, 1)$, the above formulas show that $F : [0,1] \to [0,1]$ is continuous, with $F(0) = 0$ and $F(v) < v$ for all $v \in (0,1]$. Figure 5.1, for example, shows the graph of F in the case that $d = 0.8$.

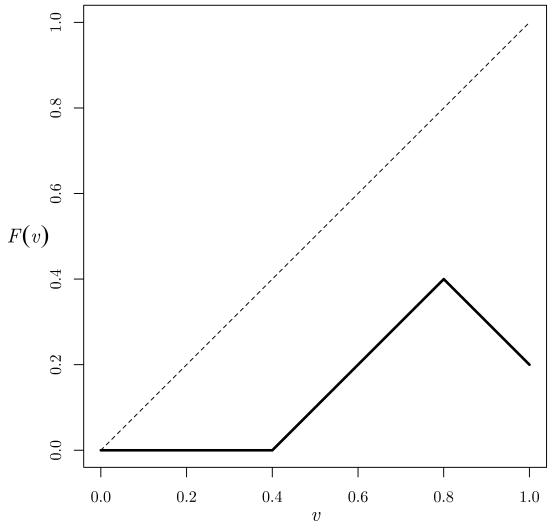


Figure 5.1: The graph of F for Equation (I), with $d = 0.8$.

Writing F^k for the k-fold composition of F with itself, we see that $F^k(v) \to 0$ for all $v \in [0,1]$; in fact, $F^k(v) = 0$ for all sufficiently large k. We therefore have the following corollary.

Corollary 5.2. *Any solution of x of Equation (I) with $x_0 \in \Sigma_1$ eventually coincides with the unique periodic solution p of Equation (I) for which $p_0 \in \Sigma_1$ and $Z_1(p_0) = 0$.*

Figure 5.2 shows this periodic solution p (black) as well as another solution x (red) with $x_0 \in \Sigma_1$.

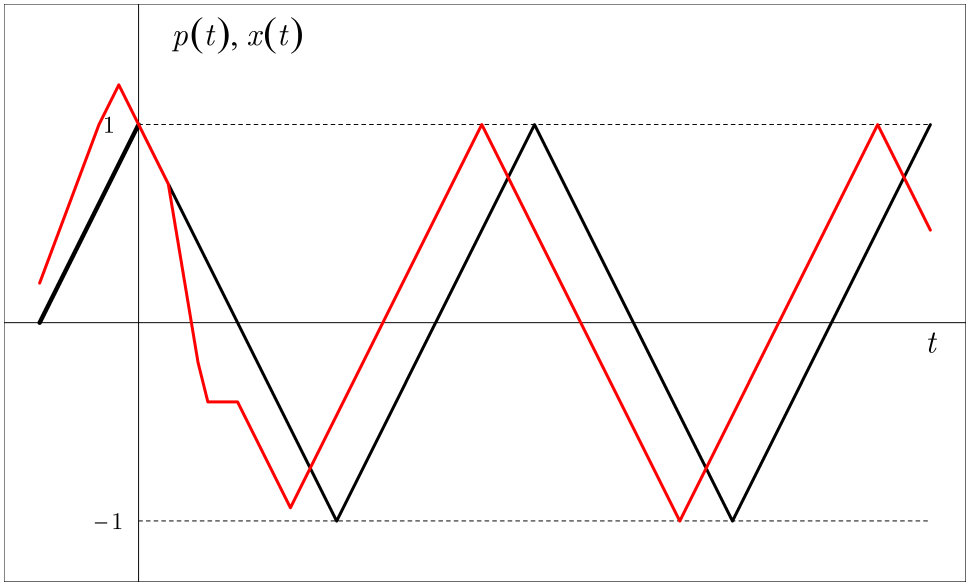


Figure 5.2: The periodic solution p, and another solution x.

5.2 Equation (EF) with $a = 2$ and $b = 1$

We remind the reader that the equation under consideration here is

$$x'(t) = -2\,\mathrm{sign}(x(t-1))\chi_{[-1,1]}(x(t-1)) - \mathrm{sign}(x(t-d))\chi_{(-\infty,-1)\cup(1,\infty)}(x(t-d)), \qquad \text{(II)}$$

where $d \in [2/3, 1)$.

Proposition 5.3. *For Equation (II) and where $F : [0,1] \to [0,1]$ is as defined above, we have the following formulas.*

For $d \in \left[\frac{2}{3}, \frac{3}{4}\right)$:

$$F(v) = \begin{cases} -\frac{2}{3}v + \frac{1}{3}d + \frac{2}{3}, & v \in \left[0, d - \frac{1}{2}\right); \\ -\frac{10}{9}v + \frac{7}{9}d + \frac{4}{9}, & v \in \left[d - \frac{1}{2}, -\frac{1}{5}d + \frac{2}{5}\right); \\ -\frac{5}{3}v + \frac{2}{3}d + \frac{2}{3}, & v \in \left[-\frac{1}{5}d + \frac{2}{5}, -\frac{1}{2}d + 1\right); \\ v + 2d - 2, & v \in \left[-\frac{1}{2}d + 1, d\right); \\ 2v + d - 2, & v \in \left[d, \frac{1}{2}d + \frac{1}{2}\right); \\ \frac{1}{3}d + \frac{2}{3}, & v \in \left[\frac{1}{2}d + \frac{1}{2}, 1\right]. \end{cases}$$

For $d \in \left[\frac{3}{4}, 1\right)$:

$$F(v) = \begin{cases} -\frac{2}{3}v + \frac{1}{3}d + \frac{2}{3}, & v \in [0, -d + 1); \\ -v + 1, & v \in \left[-d + 1, d - \frac{1}{2}\right); \\ -\frac{5}{3}v + \frac{2}{3}d + \frac{2}{3}, & v \in \left[d - \frac{1}{2}, -\frac{1}{2}d + 1\right); \\ v + 2d - 2, & v \in \left[-\frac{1}{2}d + 1, d\right); \\ 2v + d - 2, & v \in \left[d, \frac{1}{2}d + \frac{1}{2}\right); \\ \frac{1}{3}d + \frac{2}{3}, & v \in \left[\frac{1}{2}d + \frac{1}{2}, 1\right]. \end{cases}$$

Figure 5.3 shows the graph of F in the case that $d = 0.8$.

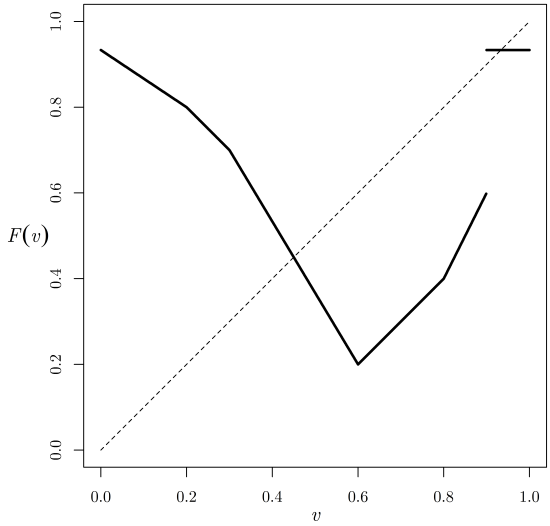


Figure 5.3: The graph of F for Equation (II), with $d = 0.8$.

Of the four equations that we are studying in detail, (II) presents (by far) the most complicated dynamics for F, and is the only one that exhibits bifurcations with respect to d. Here we describe only partial results; and we only sketch the proofs since the ideas are standard. To begin with, we define the following numbers:

$$v_a = \frac{1}{4} - \frac{d}{4}; \quad v_p = \frac{1}{4} + \frac{d}{4}; \quad v_b = \frac{1}{2} + \frac{d}{2}; \quad v_q = \frac{2}{3} + \frac{d}{3}.$$

The following observations hold for all $d \in [2/3, 1)$.

- $v_a < v_p < v_b < v_q$.
- v_b is the single point of discontinuity of F, and $F(v) = v_q$ for all $v \in [v_b, 1]$; v_q is accordingly a stable fixed point of F.
- v_p is an unstable fixed point of F.
- $F(v_a) = v_b$, and $F^{-1}[v_b, 1] = [0, v_a] \cup [v_b, 1]$. It follows that F^2 is continuous on (v_a, v_b), and that $F^2([0, v_a] \cup [v_b, 1]) = \{v_q\}$.

In addition we see that, around v_p, F^2 has slope $25/9$ and so $F^2(v) < v$ for v slightly less than v_p and $F^2(v) > v$ for v slightly greater than v_p. On the other hand,

$$\lim_{v \to v_a^+} F^2(v) = \lim_{v \to v_b^-} F(v) = 2d - 1 > v_a$$

and

$$\lim_{v \to v_b^-} F^2(v) = F(2d - 1) < v_b.$$

Thus F^2 has a fixed point strictly between v_a and v_p and another strictly between v_p and v_b. In other words, F has an orbit of minimal period 2.

Observe also that the minimum value attained by F is $\frac{3}{2}d - 1$. We see that

$$\frac{3}{2}d - 1 \leq v_a \text{ if and only if } d \leq \frac{5}{7}.$$

We therefore see that F maps the interval (v_a, v_b) to itself if and only if $d \geq 5/7$.

The following three figures show the graphs of F (black) and F^2 (red) for $d = 2/3$, $d = 5/7$, and $d = 4/5$ respectively. The values of v_a, v_p, v_b, and v_q are shown.

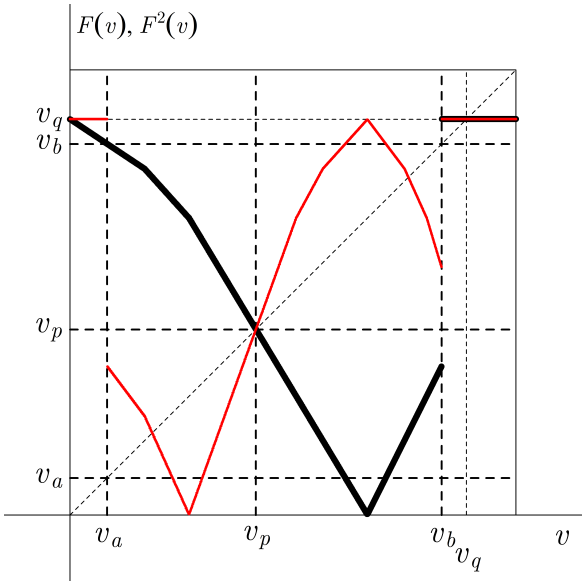


Figure 5.4: The graphs of F and F^2 for Equation (II), with $d = 2/3$.

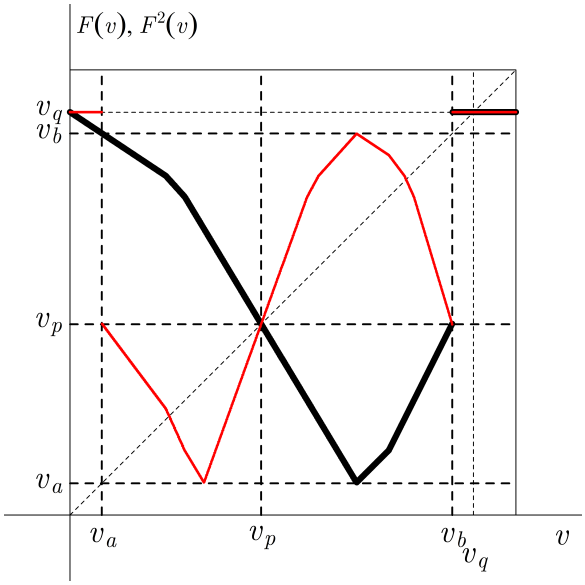


Figure 5.5: The graphs of F and F^2 for Equation (II), with $d = 5/7$.

The following proposition, while far from complete, offers some more information.

Proposition 5.4. *With notation as above, the following hold.*

(i) If $d \in [2/3, 9/13)$, $F^k(v)$ is eventually equal to v_q for all v in a dense subset of $[0, 1]$.

(ii) If $d \in (5/7, 3/4)$, F is chaotic (in the sense of Li–Yorke).

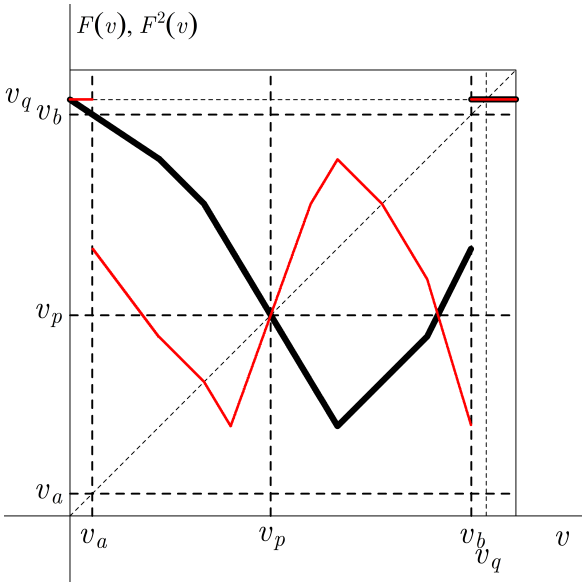


Figure 5.6: The graphs of F and F^2 for Equation (II), with $d = 4/5$.

Sketch of proof. With the formula for F in hand, tedious but elementary calculations allow us to calculate the formula for F^2 for $d \in [2/3, 3/4)$. We present this formula Appendix B. All the assertions that we make for the remainder of this proof follow from this formula.

When $d \in [2/3, 9/13)$, $F^{-2}(v_a, v_b)$ consists of four pairwise disjoint open intervals I_1, I_2, I_3, I_4 on which F^2 is expanding: that is, there is some $\sigma > 1$ such that if v, w belong to any single one of the intervals I_i, then $|F^2(v) - F^2(w)| \geq \sigma|v - w|$. It follows that, given two distinct v, w in any single one of the intervals I_i, there is some k for which $F^{2k}(v)$ and $F^{2k}(w)$ do *not* lie in a single subinterval I_j; accordingly, by the intermediate value theorem there is some u (semi-strictly) between v and w that is eventually mapped by F^2 outside of (v_a, v_b) and hence to v_q. This proves point (i).

When $d \in (5/7, 3/4)$, if v is slightly larger than v_p then $(F^2)^3(v) > v$. On the other hand, if $v = 1 - d/2$ (this is the local maximum point of F^2 inside (v_p, v_b)) then $(F^2)^3(v) < v$. It follows by the intermediate value theorem that $(F^2)^3$ has a fixed point strictly between v_p and $1 - d/2$; this fixed point is not a fixed point of F^2 and hence is a 3-periodic point of F^2. Then classical results of Li and Yorke [8] and Sharkovskii [12] show that F^2 has periodic points of all periods, and is moreover chaotic in a certain sense. □

Figure 5.7 shows an "irregular" slowly oscillating solution of Equation (II) with $d = 0.72$.

Remark 5.5. Neither statement in Proposition 5.5.4 seems best possible; in particular, we are convinced that F^2 is chaotic for a larger range of d than we have described here. In the first place, the argument we gave above appears to extend to values of d up to $434/559 \approx 0.776$. In the second place, even for larger values of d, the shape of F^2 provides compelling graphical evidence of chaotic behavior. For example, Figure 5.8 shows a close-up of the graph of F^2 (red) near the larger of the 2-periodic points of F in the case that $d = 4/5$. A particular backward orbit of this periodic point is illustrated as a cobweb diagram, and shows that this periodic point is a so-called *snap-back repeller*. Although we have not carried out all the details, it seems that classical results of Marotto ([10] and [11]) extend in a straightforward way to this situation (where F^2 is only piecewise smooth and only strictly expanding on "one side" of the



Figure 5.7: A slowly oscillating solution x of Equation (II).

fixed point) to establish that F^2 has periodic solutions of infinitely many distinct periods and a "scrambled set" in the sense of Li and Yorke [8].

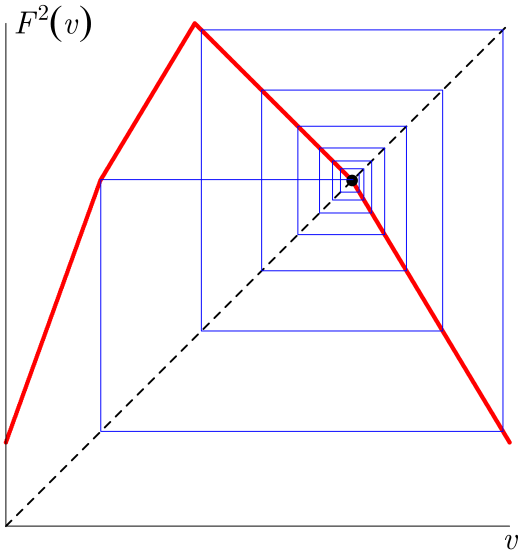


Figure 5.8: A snap-back repeller for F^2 with $d = 4/5$.

5.3 Equation (ES) with $a = 2$ and $b = 1$

We remind the reader that the equation under consideration here is

$$x'(t) = -2\,\mathrm{sign}(x(t-1))\chi_{(-\infty,-1)\cup(1,\infty)}(x(t-1)) - \mathrm{sign}(x(t-d))\chi_{[-1,1]}(x(t-d)), \qquad \text{(III)}$$

where $d \in [2/3, 1)$.

Proposition 5.6. *For Equation (III) and where $F : [0,1] \to [0,1]$ is as defined above, we have the following formulas.*

For $d \in [2/3, 3/4)$:

$$F(v) = \begin{cases} 0, & v \in \left[0, -\frac{3}{2}d + \frac{3}{2}\right); \\ \frac{4}{3}v + 2d - 2, & v \in \left[-\frac{3}{2}d + \frac{3}{2}, d\right); \\ \frac{8}{3}v + \frac{2}{3}d - 2, & v \in \left[d, -d + \frac{3}{2}\right); \\ 2v - 1, & v \in \left[-d + \frac{3}{2}, \frac{1}{2}d + \frac{1}{2}\right]; \\ 1, & v \in \left(\frac{1}{2}d + \frac{1}{2}, 1\right]. \end{cases}$$

For $d \in [3/4, 1)$:

$$F(v) = \begin{cases} 0, & v \in \left[0, -\frac{3}{2}d + \frac{3}{2}\right); \\ \frac{4}{3}v + 2d - 2, & v \in \left[-\frac{3}{2}d + \frac{3}{2}, -3d + 3\right); \\ v + d - 1, & v \in [-3d + 3, d); \\ 2v - 1, & v \in \left[d, \frac{1}{2}d + \frac{1}{2}\right]; \\ 1, & v \in \left(\frac{1}{2}d + \frac{1}{2}, 1\right]. \end{cases}$$

For any value of $d \in [2/3, 1)$, the above formulas show that $F : [0,1] \to [0,1]$ satisfies $F(v) = 0$ for $v \in [0, (3-3d)/2)$, $F(v) < v$ for all $v \in [(3-3d)/2, (d+1)/2)$, and $F(v) = 1$ for all $v \in [(d+1)/2, 1]$. Figure 5.9 shows the graph of F in the case that $d = 0.8$.

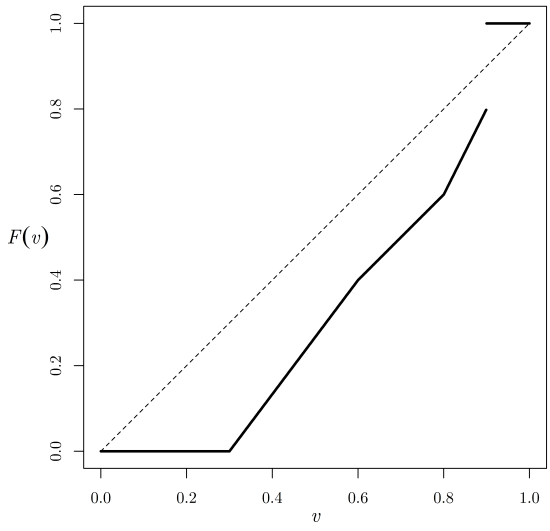


Figure 5.9: The graph of F for Equation (III), with $d = 0.8$.

We therefore see that $F^k(v)$ is eventually 0 for all $v \in [0, (d+1)/2)$; otherwise, $F^k(v) = 1$ for all $k \in \mathbb{N}$.

Corollary 5.7. *Any solution of x of Equation (III) with $x_0 \in \Sigma_d$ eventually coincides with the unique periodic solution p of Equation (III) for which $p_0 \in \Sigma_d$ and $Z(p_0) = 0$, or the unique periodic solution q of Equation (III) for which $q_0 \in \Sigma_d$ and $Z(q_0) = 1$.*

5.4 Equation (ES) with $a = 1$ and $b = 2$

We remind the reader that the equation under consideration here is

$$x'(t) = -\operatorname{sign}(x(t-1))\chi_{(-\infty,-1)\cup(1,\infty)}(x(t-1)) - 2\operatorname{sign}(x(t-d))\chi_{[-1,1]}(x(t-d)), \qquad \text{(IV)}$$

where $d \in [2/3, 1)$.

Proposition 5.8. *For Equation (IV) and where $F : [0,1] \to [0,1]$ is as defined above, we have the following formulas.*

For $d \in [\frac{2}{3}, \frac{3}{4})$:

$$F(v) = \begin{cases} -v + 2d - 1, & v \in \left[0, d - \frac{1}{2}\right); \\ v + 2d - 1, & v \in \left[d - \frac{1}{2}, \frac{1}{4}\right); \\ -\frac{1}{3}v + 2d - \frac{2}{3}, & v \in \left[\frac{1}{4}, d\right); \\ \frac{1}{3}v + \frac{4}{3}d - \frac{2}{3}, & v \in [d, 1]. \end{cases}$$

For $d \in [\frac{3}{4}, 1)$:

$$F(v) = \begin{cases} -v + 2d - 1, & v \in \left[0, d - \frac{1}{2}\right); \\ v - 2d + 2, & v \in \left[d - \frac{1}{2}, 3d - 2\right); \\ -\frac{1}{3}v + 2d - \frac{2}{3}, & v \in [3d - 2, d); \\ \frac{1}{3}v + \frac{4}{3}d - \frac{2}{3}, & v \in [d, 1]. \end{cases}$$

Figure 5.10 shows the graph of F in the case that $d = 0.8$.

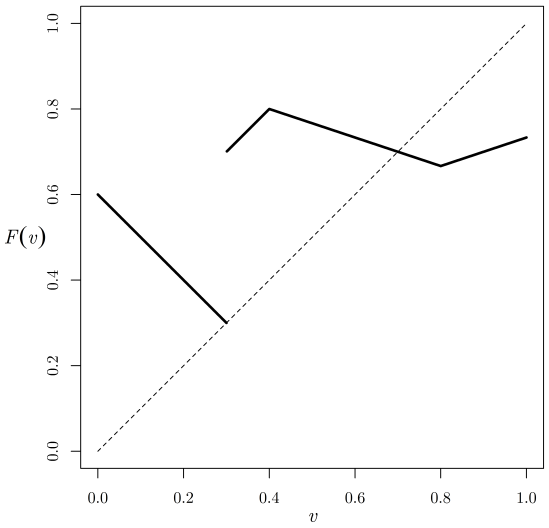


Figure 5.10: The graph of F for Equation (IV), with $d = 0.8$.

Elementary calculations show the following: if $d \in [2/3, 3/4)$, $F^k(v)$ eventually enters $[1/4, 1]$, where F is a contraction mapping with unique fixed point $3d/2 + 1/2$; if $d \in [3/4, 1)$, $F^k(v)$ eventually enters $[3d - 2, 1]$, where F is likewise a contraction mapping with unique fixed point $3d/2 + 1/2$. Accordingly, we have

Proposition 5.9. *For Equation (IV), all solutions beginning in Σ_d converge to the unique periodic solution p for which $Z(p_0) = 3d/2 + 1/2$.*

A Proof of the formulas for F

A.1 Equation (I)

We take $\varphi \in \Sigma_1$ with continuation x. We write z for the right endpoint of the first positive interval (usually a one-point interval) of zeros of x. If $x(z+1) \geq -1$, then $F(v) = 0$. Otherwise, we write t_1 for the first time after z that $x(t) = -1$, and t_2 for the first time after $z+1$ that $x(t) = -1$; in this case $t_1 < z+1 < t_2$ and $F(v) = t_2 - t_1$.

The positive times at which the slope of x can change and that might occur before t_2 are

- $d - v$ (if $v < d$) and $1 - v$;
- d and 1;
- $z + 1$;
- $t_1 + d$ and $t_1 + 1$ (if t_1 is defined).

The different cases in the formula for F reflect variations in the orders of these critical points. Note, though, that condition (OSii) guarantees that $z + 1 \leq t_1 + d$.

The reader may find it useful to refer to Figure A.1.

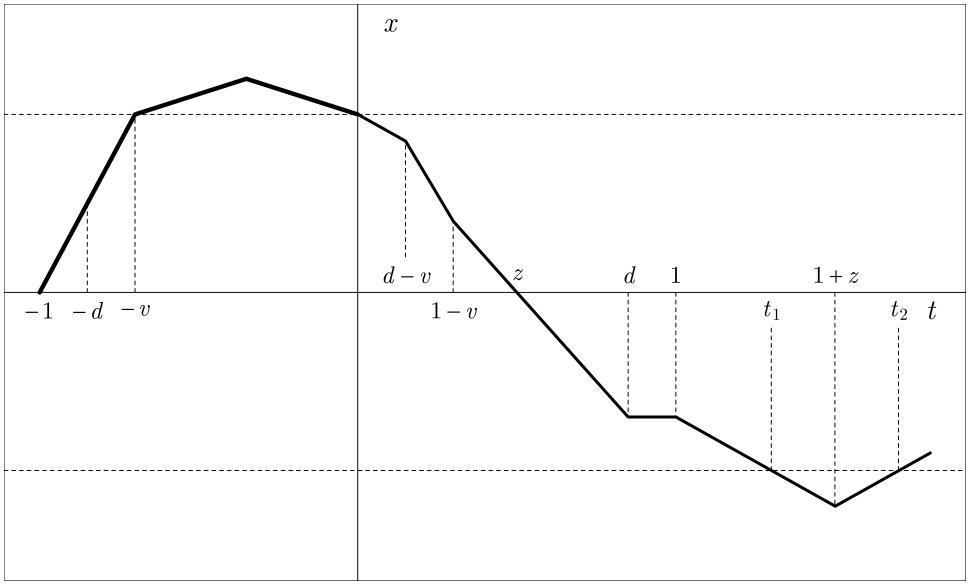


Figure A.1: A solution of Equation (I) with initial condition $\varphi \in \Sigma_1$.

As we established in Section 4, $F(0)$ is well-defined; for the purposes of our explicit calculations, if $Z(\varphi) = 0$ we shall assume that $\varphi(0) = 1$.

Case 1: $v \in [0, -d+1)$. In this case $v < d$ also (since $d \geq 2/3$) and so the order of the first few positive critical points of x is

$$d - v \leq d < 1 - v \leq 1 < z + 1.$$

We have

- $x'(t) = -1$ for $t \in (0, d - v)$, and $x(d - v) = 1 + v - d > 0$.
- $x'(t) = -3$ for $t \in (d - v, d)$, and $x(d) = 1 - 2v - d$.

- $x'(t) = -1$ for $t \in (d, 1-v)$, and $x(1-v) = -v \leq 0$.
- $x'(t) = 0$ for $t \in (1-v, 1)$, and $x(1) = -v$.
- $x'(t) = -1$ for $t \in (1, z+1)$, and $x(z+1) = -v - z$.

Case 1a: $v \in \left[0, -\frac{1}{2}d + \frac{1}{2}\right)$. In this case $x(d) > 0$, $z \in (d, 1-v]$, and

$$z = d + x(d) = d + (1 - 2v - d) = 1 - 2v.$$

It follows that

$$x(z+1) = -v - z = -v - (1-2v) = -1 + v \geq -1.$$

We conclude that $F(v) = 0$.

Case 1b: $v \in \left[-\frac{1}{2}d + \frac{1}{2}, -d + 1\right)$. In this case $x(d) \leq 0$, $z \in (d - v, d]$, and

$$z = d - v + \frac{x(d-v)}{3} = d - v + \frac{1 + v - d}{3} = \frac{1 + 2d - 2v}{3}.$$

It follows that

$$x(z+1) = -v - z = -(v+z) = -\frac{1 + 2d + v}{3} \geq -\frac{1 + 2d + (1-d)}{3} = -\frac{2+d}{3} \geq -1.$$

We conclude that $F(v) = 0$.

Case 2: $v \in [-d + 1, d)$. In this case the order of the first few positive critical points of x is

$$d - v < 1 - v \leq d < 1 < z + 1.$$

We have

- $x'(t) = -1$ for $t \in (0, d-v)$, and $x(d-v) = 1 + v - d > 0$.
- $x'(t) = -3$ for $t \in (d-v, 1-v)$, and $x(1-v) = 2d + v - 2$.
- $x'(t) = -2$ for $t \in (1-v, d)$, and $x(d) = -v < 0$.
- $x'(t) = 0$ for $t \in (d, 1)$, and $x(1) = -v$.
- $x'(t) = -1$ for $t \in (1, z+1)$, and $x(z+1) = -v - z$.

Case 2a: $v \in [-d + 1, -2d + 2)$ (observe that $-2d + 2 \leq d$ since $d \geq 2/3$). In this case $x(1-v) < 0$, $z \in (d - v, 1 - v)$, and

$$z = d - v + \frac{x(d-v)}{3} = d - v + \frac{1 + v - d}{3} = \frac{1 + 2d - 2v}{3}.$$

Thus we have

$$x(z+1) = -v - z = -\frac{1 + 2d + v}{3} > -\frac{1 + 2d + 2 - 2d}{3} = -1.$$

We conclude that $F(v) = 0$.

Case 2b: $v \in [-2d + 2, d)$. In this case $x(1-v) \geq 0$, $z \in [1 - v, d)$, and

$$z = 1 - v + \frac{x(1-v)}{2} = 1 - v + \frac{2d + v - 2}{2} = d - \frac{v}{2}.$$

Thus we have

$$x(z+1) = -v - z = -d - \frac{v}{2}.$$

Therefore $F(-2d+2) = 0$ and if $v \in (-2d+2, d)$ we have

$$t_1 = 1 + [x(1) - (-1)] = 2 - v \in (1, z+1).$$

We know that $x'(t) = 1$ for $t \in (z+1, t_1 + d)$, and so we have

$$x(t_1 + d) = x(z+1) + (t_1 + d - (z+1)) = -v - d + 1 > -1.$$

Therefore we see that $t_2 \in (z+1, t_1 + d)$, and in fact

$$F(v) = t_2 - t_1 = 2(z + 1 - t_1) = v + 2d - 2.$$

Case 3: $v \in [d, 1]$. Observe that in this case $1 - v < d$ (since $d \geq 2/3$). The order of the first few positive critical points of x is

$$1 - v < d < 1 < z + 1.$$

We have

- $x'(t) = -3$ for $t \in (0, 1-v)$, and $x(1-v) = 3v - 2 \geq 0$.
- $x'(t) = -2$ for $t \in (1-v, d)$, and $x(d) = v - 2d < 0$.
- $x'(t) = 0$ for $t \in (d, 1)$, and $x(1) = v - 2d$.
- $x'(t) = -1$ for $t \in (1, z+1)$, and $x(z+1) = v - 2d - z$.

Observe that z must fall in the interval $[1-v, d)$, and that

$$z = 1 - v + \frac{x(1-v)}{2} = 1 - v + \frac{3v-2}{2} = \frac{v}{2}.$$

Therefore

$$x(z+1) = \frac{v}{2} - 2d.$$

Case 3a: $v \in [d, 4d-2)$. In this case $x(z+1) < -1$, and we have

$$t_1 = 1 + [x(1) - (-1)] = 2 + v - 2d.$$

As in case 2b, we know that $x'(t) = 1$ for $t \in (z+1, t_1 + d)$, and so we have

$$x(t_1 + d) = x(z+1) + (t_1 + d - (z+1)) = 1 + v - 3d > -1.$$

Therefore we see that $t_2 \in (z+1, t_1 + d)$, and in fact

$$F(v) = t_2 - t_1 = 2(z + 1 - t_1) = -v + 4d - 2.$$

Case 3b: $v \in [4d-2, 1]$. *This case is nontrivial only for $d < \frac{3}{4}$. When $d = \frac{3}{4}$ this case occurs precisely when $v = 1$; when $d > \frac{3}{4}$, $4d - 2 > 1$ and this case cannot occur.* In this case $x(z+1) \geq -1$ and $F(v) = 0$.

As just explained, not all of the cases we have enumerated above are actually possible for all values of d. In particular, if $d \geq 3/4$ then $4d - 2 \geq 1$ and so case 3a (with v going all the way up to 1) covers all the possibilities for case 3. (We shall encounter some analogous situations as we examine our other equations, and will remark on them in the same way, with italic type.) This completes the proof. □

A.2 Equation (II)

We take $\varphi \in \Sigma_1$ with continuation x. We write z for the right endpoint of the first positive interval of zeros of x. If $x(z+1) \geq -1$, then $F(v) = 0$. Otherwise, we write t_1 for the first time after z that $x(t) = -1$, and t_2 for the first time after $z+1$ that $x(t) = -1$; in this case $t_1 < z+1 < t_2$ and $F(v) = t_2 - t_1$.

The positive times at which the slope of x can change and that might occur before t_2 are

- $d - v$ (if $v < d$) and $1 - v$;
- d and 1;
- $z + 1$;
- $t_1 + d$ and $t_1 + 1$ (if t_1 is defined).

As for Equation (I), the different cases in the formula for F reflect variations in the orders of these critical points. We repeat that condition (OSii) guarantees that $z + 1 \leq t_1 + d$. We repeat that $F(0)$ is well-defined, and so if $Z(\varphi) = 0$ we may and shall assume that $\varphi(0) = 1$.

Case 1: $v \in [0, -d+1)$. In this case $v < d$ also (since $d \geq 2/3$) and so the order of the first few positive critical points of x is

$$d - v \leq d < 1 - v \leq 1 < z + 1.$$

We have

- $x'(t) = -2$ for $t \in (0, d - v)$, and $x(d - v) = 1 + 2v - 2d$.
- $x'(t) = -3$ for $t \in (d - v, d)$, and $x(d) = 1 - v - 2d < 0$.
- $x'(t) = -2$ for $t \in (d, 1 - v)$, and $x(1 - v) = v - 1 \geq -1$.
- $x'(t) = 0$ for $t \in (1 - v, 1)$, and $x(1) = v - 1$.
- $x'(t) = -2$ for $t \in (1, z + 1)$, and $x(z + 1) = v - 1 - 2z$.

Case 1a: $v \in \left[0, d - \frac{1}{2}\right)$. *This is only a proper subcase of case 1 if $d < \frac{3}{4}$. If $d \geq \frac{3}{4}$, this condition holds for all of case 1.* In this case $x(d - v) < 0$, $z \in (0, d - v)$, and

$$z = \frac{1}{2}.$$

It follows that

$$x(z + 1) = v - 1 - 2z = v - 2 < -1.$$

Therefore we have

$$t_1 = 1 + \frac{x(1) - (-1)}{2} = 1 + \frac{v}{2}.$$

Now, for $t \in (z + 1, t_1 + d)$ we have $x'(t) = 2$; so

$$x(t_1 + d) = x(z + 1) + 2(t_1 + d - z - 1) = 2v + 2d - 3 < -1.$$

Therefore $t_2 > t_1 + d$. Now, we will have $x'(t) = 3$ for $t \in (t_1 + d, \min(t_1 + 1, t_2 + d))$; but since

$$t_1 + 1 < t_1 + 2d < t_2 + d,$$

the above minimum is actually $t_1 + 1$ and we have

$$x(t_1 + 1) = x(t_1 + d) + 3(t_1 + 1 - t_1 - d) = 2v - d > -1.$$

We conclude that t_2 lies in $(t_1 + d, t_1 + 1)$, and that

$$t_2 = t_1 + d + \frac{-1 - x(t_1 + d)}{3} = \frac{5}{3} + \frac{d}{3} - \frac{v}{6}.$$

Thus, in this case,

$$F(v) = t_2 - t_1 = -\frac{2}{3}v + \frac{1}{3}d + \frac{2}{3}.$$

Case 1b: $v \in [d - 1/2, 1 - d)$. *This is only a possible subcase of case 1 if* $d < \frac{3}{4}$. *If* $d \geq \frac{3}{4}$, *this subcase does not occur.* In this case $x(d - v) \geq 0$, $z \in [d - v, d)$, and

$$z = d - v + \frac{x(d - v)}{3} = d - v + \frac{1 + 2v - 2d}{3} = \frac{1 + d - v}{3}.$$

It follows that

$$x(z + 1) = v - 1 - 2z = \frac{5v - 5 - 2d}{3} < -\frac{7d}{3} < -1.$$

Thus $t_1 \in (1, z + 1)$, and

$$t_1 = 1 + \frac{x(1) - (-1)}{2} = 1 + \frac{v}{2}.$$

As above, for $t \in (z + 1, t_1 + d)$, we have $x'(t) = 2$, and so

$$x(t_1 + d) = x(z + 1) + 2(t_1 + d - z - 1) = \frac{-7 + 10v + 2d}{3}.$$

This quantity is less than -1 for $v < -\frac{1}{5}d + \frac{2}{5}$.

Case 1bi: $v \in \left[d - \frac{1}{2}, -\frac{1}{5}d + \frac{2}{5}\right)$. As just remarked, in this case we know that $x(t_1 + d) < -1$. Thus $t_2 > t_1 + d$. Now, we will have $x'(t) = 3$ for $t \in (t_1 + d, \min(t_1 + 1, t_2 + d))$; but since

$$t_1 + 1 < t_1 + 2d < t_2 + d,$$

the above minimum is actually $t_1 + 1$ and we have

$$x(t_1 + 1) = x(t_1 + d) + 3(t_1 + 1 - t_1 - d) = \frac{2 + 10v - 7d}{3} \geq -1 + d > -1.$$

We conclude that $t_2 \in (t_1 + d, t_1 + 1)$, and

$$t_2 = t_1 + d + \frac{-1 - x(t_1 + d)}{3} = \frac{13 - 11v/2 + 7d}{9}.$$

Thus in this case we have

$$F(v) = t_2 - t_1 = -\frac{10}{9}v + \frac{7}{9}d + \frac{4}{9}.$$

Case 1bii: $v \in \left[-\frac{1}{5}d + \frac{2}{5}, 1 - d\right)$. In this case $x(t_1 + d) \geq -1$, and so $t_2 \in (z + 1, t_1 + d]$ and

$$t_2 = z + 1 + \frac{-1 - x(z + 1)}{2} = \frac{10 - 7v + 4d}{6}.$$

Thus in this case we have

$$F(v) = t_2 - t_1 = -\frac{5}{3}v + \frac{2}{3}d + \frac{2}{3}.$$

Case 2: $v \in [-d + 1, d)$. In this case the order of the first few positive critical points of x is

$$d - v < 1 - v \leq d \leq 1 < z + 1.$$

We have

- $x'(t) = -2$ for $t \in (0, d-v)$, and $x(d-v) = 1 + 2v - 2d$.
- $x'(t) = -3$ for $t \in (d-v, 1-v)$, and $x(1-v) = -2 + 2v + d$.
- $x'(t) = -1$ for $t \in (1-v, d)$, and $x(d) = -1 + v$.
- $x'(t) = 0$ for $t \in (d, 1)$, and $x(1) = -1 + v > -1$.
- $x'(t) = -2$ for $t \in (1, z+1)$, and $x(z+1) = -1 + v - 2z$.

Case 2a: $v \in \left[-d+1, d-\frac{1}{2}\right)$. *This is only a subcase of case 2 for $d > \frac{3}{4}$; it does not occur for $d < \frac{3}{4}$. When $d = \frac{3}{4}$, we include this case in our formula for F but it does not actually occur.* In this case $x(d-v) < 0$, $z = 1/2$, and

$$x(z+1) = -2 + v < -1.$$

It follows that

$$t_1 = 1 + \frac{x(1) - (-1)}{2} = 1 + \frac{v}{2}.$$

We know that $x'(t) = 2$ for $t \in (z+1, t_1 + d)$, and so

$$x(t_1 + d) = x(z+1) + 2(t_1 + d - z - 1) = -3 + 2v + 2d \geq -1.$$

Thus $t_2 \in (1+z, t_1 + d]$ and

$$t_2 = z + 1 + \frac{-1 - x(z+1)}{2} = 2 - \frac{v}{2}.$$

Thus in case 2*a* we have

$$F(v) = t_2 - t_1 = -v + 1.$$

Case 2b: $v \in \left[d - \frac{1}{2}, -\frac{1}{2}d + 1\right)$. *When $d < \frac{3}{4}$ this case, viewed as a subcase of case 2, can be written $v \in \left[-d+1, -\frac{1}{2}d + 1\right)$.* In this case $x(d-v) \geq 0$ but $x(1-v) < 0$, so

$$z = d - v + \frac{x(d-v)}{3} = \frac{1 - v + d}{3}$$

and

$$x(z+1) = \frac{-5 + 5v - 2d}{3} < -1.$$

Thus

$$t_1 = 1 + \frac{x(1) + 1}{2} = 1 + \frac{v}{2}.$$

We know that $x'(t) = 2$ for $t \in (z+1, t_1 + d)$ and so

$$x(t_1 + d) = x(z+1) + 2(t_1 + d - z - 1) = \frac{-7 + 10v + 2d}{3}.$$

If $d < \frac{3}{4}$ we know that v is greater than or equal to $-d + 1$; if $d \geq \frac{3}{4}$, we know that v is actually greater than or equal to $d - \frac{1}{2}$. Either way, the above expression for $x(t_1 + d)$ is greater than or equal to -1. We are therefore assured that t_2 occurs on $(z+1, t_1 + d)$, and by symmetry

$$F(v) = t_2 - t_1 = 2(z + 1 - t_1) = -\frac{5}{3}v + \frac{2}{3}d + \frac{2}{3}.$$

Case 2c: $v \in \left[-\frac{1}{2}d+1, d\right)$. *When $d = \frac{2}{3}$ this case does not actually occur.* In this case $x(1-v) \geq 0$ but $x(d) < 0$, so

$$z = 1 - v + x(1-v) = -1 + v + d$$

and

$$x(z+1) = 1 - v - 2d \leq -1.$$

As above, we get

$$t_1 = 1 + \frac{x(1)+1}{2} = 1 + \frac{v}{2}$$

and compute

$$x(t_1 + d) = x(z+1) + 2(t_1 + d - z - 1) = 3 - 2v - 2d > -1.$$

We conclude that, as above, $t_2 \in (z+1, t_1+d)$ and

$$F(v) = t_2 - t_1 = 2(z + 1 - t_1) = v + 2d - 2.$$

This concludes case 2.

Case 3: $v \in [d, 1]$. In this case the order of the first few positive critical points of x is

$$1 - v < d < 1 < z + 1.$$

We have

- $x'(t) = -3$ for $t \in (0, 1-v)$, and $x(d-v) = -2 + 3v \geq 0$.
- $x'(t) = -1$ for $t \in (1-v, d)$, and $x(d) = -1 + 2v - d$.
- $x'(t) = 0$ for $t \in (d, 1)$, and $x(1) = -1 + 2v - d > -1$.
- $x'(t) = -2$ for $t \in (1, z+1)$, and $x(z+1) = -1 + 2v - d - 2z$.

Case 3a: $v \in \left[d, \frac{1}{2}d + \frac{1}{2}\right)$. In this case, $x(d) < 0$ and

$$z = 1 - v + x(d-v) = -1 + 2v$$

and

$$x(z+1) = 1 - 2v - d \leq -1.$$

Accordingly, we have

$$t_1 = 1 + \frac{x(1)+1}{2} = 1 + v - \frac{d}{2}.$$

As we have computed several times by now, $x'(t) = 2$ for all $t \in (z+1, t_1+d)$, and accordingly we have

$$x(t_1 + d) = x(z+1) + 2(t_1 + d - z - 1) = 3 - 4v \geq -1.$$

We therefore see that t_2 lies in $(z+1, t_1+d)$ and that

$$F(v) = t_2 - t_1 = 2(z + 1 - t_1) = 2v + d - 2.$$

Case 3b: $v \in \left[\frac{1}{2}d + \frac{1}{2}, 1\right]$. In this case $x(1) \geq 0$ and

$$z = 1 + \frac{x(1)}{2} = \frac{1 + 2v - d}{2}$$

and $x(z+1) = -2$. In this case

$$t_1 = z + \frac{1}{2} = \frac{2+2v-d}{2},$$

and

$$x(t_1+d) = x(z+1) + 2(t_1+d-z-1) = -3+2d \leq -1.$$

We then have $x'(t) = 3$ for $t \in (t_1+d, \min(t_1+1, t_2+d))$. But since $t_1+d \leq t_2$, we have $t_2+d \geq t_1+2d > t_1+1$, and we are assured that the above minimum is t_1+1. Accordingly, we get

$$x(t_1+1) = x(t_1+d) + 3(1-d) = -d \geq -1.$$

We conclude that

$$t_2 = t_1 + d + \frac{-1-x(t_1+d)}{3} = \frac{10+6v-d}{6}.$$

Therefore in this case we have

$$F(v) = t_2 - t_1 = \frac{1}{3}d + \frac{2}{3}.$$

Putting together the situations where each case can occur yields our stated formulas for F in the $d < \frac{3}{4}$ and the $d \geq \frac{3}{4}$ cases. □

A.3 Equation (III)

As above, we take $\varphi \in \Sigma_d$ with continuation x. This means that $\varphi \geq 0$ on $[-d, 0]$, which (by (OSi)) forces $\varphi \geq -1$ overall. We will again write z for the right endpoint of the first positive interval of zeros of x. As above, we will assume that $\varphi(0) = 1$ if $Z_d(\varphi) = 0$.

If $x(z+d) \geq -1$, then $F(v) = 0$. Otherwise, we write t_1 for the first time after z that $x(t) = -1$, and t_2 for the first time after $z+d$ that $x(t) = -1$; in this case $t_1 < z+d < t_2$ and $F(v) = t_2 - t_1$.

The positive times at which the slope of x can change and that might occur before t_2 are

- $d - v$ (if $v < d$) and $1 - v$;
- d and 1;
- $z + d$;
- $t_1 + d$ and $t_1 + 1$.

The different cases in the formula for F reflect variations in the orders of these critical points. Note, though, that condition (OSiii) guarantees that $1 < z + d$.

The reader may find it useful to refer to Figure A.2.

Case 1: $v \in [0, -d+1)$. In this case the order of the first several critical points is

$$0 < d - v \leq d < 1 - v \leq 1 < z + d.$$

We have

- $x'(t) = -1$ for $t \in (0, d-v)$ and $x(d-v) = 1+v-d$.
- $x'(t) = 0$ for $t \in (d-v, d)$ and $x(d) = 1+v-d$.

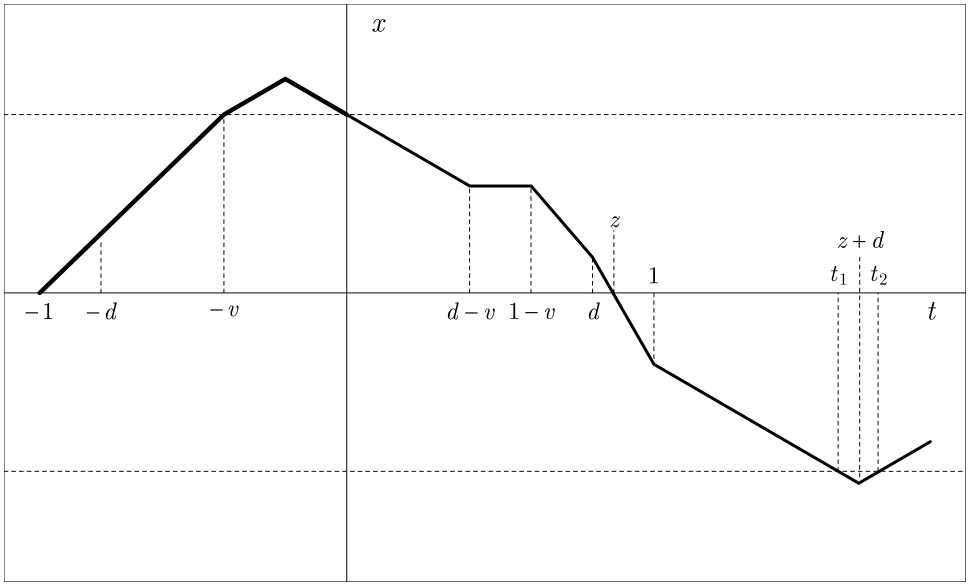


Figure A.2: A solution of Equation (III) with initial condition $\varphi \in \Sigma_d$.

- $x'(t) = -1$ for $t \in (d, 1-v)$ and $x(1-v) = 2v \geq 0$.
- $x'(t) = -3$ for $t \in (1-v, 1)$ and $x(1) = -v \leq 0$.
- $x'(t) = -1$ for $t \in (1, z+d)$, and $x(z+d) = 1-v-z-d$.

The above calculations show that

$$z = 1 - v + \frac{x(1-v)}{3} = 1 - v + \frac{2v}{3} = 1 - \frac{v}{3}$$

and accordingly that

$$x(z+d) = -\frac{2v}{3} - d > -1.$$

We conclude that $F(v) = 0$ in this case.

Case 2: $v \in [-d+1, d)$. In this case the order of the first several critical points is

$$0 < d - v < 1 - v \leq d < 1 < z + d.$$

We have

- $x'(t) = -1$ for $t \in (0, d-v)$, and $x(d-v) = 1+v-d$.
- $x'(t) = 0$ for $t \in (d-v, 1-v)$, and $x(1-v) = 1+v-d > 0$.
- $x'(t) = -2$ for $t \in (1-v, d)$, and $x(d) = 3-v-3d$.
- $x'(t) = -3$ for $t \in (d, 1)$, and $x(1) = -v \leq 0$.
- $x'(t) = -1$ for $t \in (1, z+d)$, and $x(z+d) = 1-v-z-d$.

Case 2a: $v \in [-d+1, -3d+3)$. *If $d < \frac{3}{4}$, this subcase covers all of case 2. If $d > \frac{3}{4}$, this is a proper subcase of case 2. We include this case in the formula for F in the $d = \frac{3}{4}$ case even though it does not occur.* In this case $x(d) > 0$ and so $z \in (d, 1)$; we have

$$z = d + \frac{x(d)}{3} = 1 - \frac{v}{3} \quad \text{and} \quad x(z+d) = -\frac{2v}{3} - d.$$

Case 2ai: $v \in \left[-d+1, -\frac{3}{2}d+\frac{3}{2}\right)$. In this case $x(z+d) > -1$ and $F(v) = 0$.

Case 2aii: $v \in \left[-\frac{3}{2}d+\frac{3}{2}, \min(-3d+3, d)\right)$. If v is actually equal to $-\frac{3}{2}d+\frac{3}{2}$, we have $F(v) = 0$. Otherwise, $t_1 \in (1, z+d)$ and

$$t_1 = 1 + \frac{x(1)+1}{1} = 2 - v.$$

We know that $x'(t) = 1$ for all $t \in (z+d, t_1+d)$, and so

$$x(t_1+d) = 1 - \frac{4v}{3} - d > -1.$$

We conclude that $t_2 \in (z+d, t_1+d)$ and

$$F(v) = 2(z+d-t_1) = \frac{4}{3}v + 2d - 2.$$

(Observe that this formula is consistent with the conclusion that $F(v) = 0$ for $v = -\frac{3}{2}d+\frac{3}{2}$.)

Case 2b: $v \in [-3d+3, d)$. *This case only occurs for $d \geq \frac{3}{4}$.* In this case $x(d) \leq 0$ and so we have

$$z = 1 - v + \frac{x(1-v)}{2} = \frac{3-v-d}{2} \quad \text{and} \quad x(z+d) = \frac{-1-v-d}{2} \leq -1.$$

Therefore

$$t_1 = 1 + \frac{x(1)+1}{1} = 2 - v$$

and, similarly to above,

$$x(t_1+d) = x(z+d) + (t_1+d-z-d) = -v > -1.$$

Therefore in this case we again have

$$F(v) = 2(z+d-t_1) = v + d - 1.$$

Case 3: $v \in [d, 1]$. In this case the order of the first several critical points is

$$0 \leq 1 - v < d < 1 < z + d.$$

We have

- $x'(t) = 0$ for $t \in (0, 1-v)$, and $x(1-v) = 1$.
- $x'(t) = -2$ for $t \in (1-v, d)$, and $x(d) = 3 - 2d - 2v$.
- $x'(t) = -3$ for $t \in (d, 1)$, and $x(1) = -2v + d < 0$.
- $x'(t) = -1$ for $t \in (1, z+d)$, and $x(z+d) = 1 - 2v - z$.

Case 3a: $v \in \left[d, -d+\frac{3}{2}\right)$. *This case is only possible for $d < \frac{3}{4}$; when $d \geq \frac{3}{4}$, this case does not occur.* In this case $x(d) > 0$ and $z \in (d, 1)$, with

$$z = d + \frac{x(d)}{3} = \frac{3-2v+d}{3} \quad \text{and} \quad x(z+d) = \frac{-4v-d}{3} < -1;$$

in this case we also have $x(1) \geq -1$. Thus $t_1 \in (1, z+d)$ and

$$t_1 = 1 + \frac{x(1)+1}{1} = 2 - 2v + d.$$

We will have $x'(t) = 1$ for all $t \in (z+d, t_1+d)$ and accordingly

$$x(t_1+d) = x(z+d) + (t_1+d-z-d) = \frac{3-8v+d}{3} > -1.$$

Thus we see that $t_1 \in (z+d, t_1+d)$, and, as before,

$$F(v) = 2(z+d-t_1) = \frac{8}{3}v + \frac{2}{3}d - 2.$$

Case 3b: $v \in \left[-d+\frac{3}{2}, 1\right]$. *This subcase actually encompasses all of case 3 if* $d \geq \frac{3}{4}$. In this case $x(d) \leq 0$ and $z \in (1-v, d]$, with

$$z = 1 - v + \frac{x(1-v)}{2} = \frac{3-2v}{2} \quad \text{and} \quad x(z+d) = \frac{-1-2v}{2} < -1.$$

Observe also that $x(d) \geq -1$.

Case 3bi: $v \in \left[-d+\frac{3}{2}, \frac{1}{2}d+\frac{1}{2}\right)$. In this case $x(1) > -1$, and so $t_1 \in (1, z+d)$ and

$$t_1 = 1 + \frac{x(1)+1}{1} = 2 - 2v + d.$$

$x'(t) = 1$ for $t \in (z+d, t_1+d)$, so

$$x(t_1+d) = x(z+d) + (t_1+d-z-d) = -2v + d > -1.$$

Thus, similarly as before,

$$F(v) = 2(z+d-t_1) = 2v - 1.$$

Case 3bii: $v \in \left[\frac{1}{2}d+\frac{1}{2}, 1\right]$. In this case $x(1) \leq -1$, and so $t_1 \in (d, 1]$ and

$$t_1 = d + \frac{x(d)+1}{3} = \frac{4-2v+d}{3}.$$

As before, $x'(t) = 1$ for $t \in (z+d, t_1+d)$ and

$$x(t_1+d) = x(z+d) + (t_1 - z) = \frac{-2-2v+d}{3}.$$

This quantity is equal to -1 for $v = \frac{1}{2}d+\frac{1}{2}$ and less than -1 for $v > \frac{1}{2}d+\frac{1}{2}$. So for $v = \frac{1}{2}d+\frac{1}{2}$, we have $F(v) = d$, and for $v \in \left(\frac{1}{2}d+\frac{1}{2}, 1\right]$ we have that $x'(t) = 0$ for $t \in (t_1+d, t_1+1)$, and so $F(v) = 1$.

This completes the proof. □

A.4 Equation (IV)

We once more take $\varphi \in \Sigma_d$ with continuation x, and once more take z to be the right endpoint of the first positive interval of zeros of x. We again assume that $\varphi(0) = 1$ if $Z_d(\varphi) = 0$.

If $x(z+d) \geq -1$, then $F(v) = 0$. Otherwise, we write t_1 for the first time after z that $x(t) = -1$, and t_2 for the first time after $z+d$ that $x(t) = -1$; in this case $t_1 < z+d < t_2$ and $F(v) = t_2 - t_1$.

The positive times at which the slope of x can change and that might occur before t_2 are

- $d - v$ (if $v < d$) and $1 - v$;

- d and 1;
- $z + d$;
- $t_1 + d$ and $t_1 + 1$.

The different cases in the formula for F reflect variations in the orders of these critical points. Note, though, that condition (OSiii) guarantees that $1 < z + d$.

Case 1: $v \in [0, -d + 1)$. In this case the order of the first several critical points is

$$0 < d - v \leq d < 1 - v \leq 1 < z + d.$$

We have

- $x'(t) = -2$ for $t \in (0, d - v)$, and $x(d - v) = 1 + 2v - 2d$.
- $x'(t) = 0$ for $t \in (d - v, d)$, and $x(d) = 1 + 2v - 2d$.
- $x'(t) = -2$ for $t \in (d, 1 - v)$, and $x(1 - v) = -1 + 4v$.
- $x'(t) = -3$ for $t \in (1 - v, 1)$, and $x(1) = -1 + v \leq 0$.
- $x'(t) = -2$ for $t \in (1, z + d)$, and $x(z + d) = 1 + v - 2d - 2z$.

Case 1a: $v \in \left[0, d - \frac{1}{2}\right)$. *This case is only a proper subcase of case 1 when $d < \frac{3}{4}$; when $d \geq \frac{3}{4}$, this subcase covers the entirety of case 1.* In this case $x(d - v) < 0$, and so $z = 1/2$ and

$$x(z + d) = v - 2d < -1.$$

Since $x(1) \geq -1$, in this case we have

$$t_1 = 1 + \frac{x(1) + 1}{2} = \frac{2 + v}{2}.$$

We have $x'(t) = 2$ for $t \in (z + d, t_1 + d)$ and so

$$x(t_1 + d) = x(z + d) + 2(t_1 + d - z - d) = 1 + 2v - 2d = x(d) > -1.$$

Thus in this case we have

$$F(v) = 2(z + d - t_1) = -v + 2d - 1.$$

Observe that, when $v = d - \frac{1}{2}$, $x(t) = 0$ for $t \in [d - v, d]$. This introduces a discontinuity in F as v crosses $d - \frac{1}{2}$, as we are about to calculate.

Case 1b: $v \in \left[d - \frac{1}{2}, \frac{1}{4}\right)$. *This subcase only occurs for $d < \frac{3}{4}$.* In this case $x(d) \geq 0$ and $x(1 - v) < 0$, so

$$z = d + \frac{x(d)}{2} = \frac{1 + 2v}{2} \text{ and } x(z + d) = -v - 2d < -1.$$

Since $x(1) > -1$, we again have in this case

$$t_1 = 1 + \frac{x(1) + 1}{2} = \frac{2 + v}{2}.$$

We again have $x'(t) = 2$ for $t \in (z+d, t_1+d)$, and

$x(t_1+d) = x(z+d)+2(t_1+d-z-d) = 1-2v-2d \geq -1$ (recall that $d<3/4$ in this subcase).

Thus

$$F(v) = 2(z+d-t_1) = v+2d-1.$$

Case 1c: $v \in \left[\frac{1}{4}, -d+1\right)$. *This subcase only occurs for* $d < \frac{3}{4}$. In this case $z \in [1-v, 1)$ and

$$z = 1-v+\frac{x(1-v)}{3} = \frac{2+v}{3} \text{ and } x(z+d) = \frac{-1+v-6d}{3}.$$

As in the other case 1 subcases we have $t_1 = \dfrac{2+v}{2}$, and

$$x(t_1+d) = x(z+d)+2(t_1+d-z-d) = \frac{1+2v-6d}{3} > -1.$$

Since $d < 3/4$ and $d > 1/4$, this quantity is at least -1. We conclude again that

$$F(v) = 2(z+d-t_1) = -\frac{1}{3}v+2d-\frac{2}{3}.$$

Case 2: $v \in [-d+1, d)$. In this case the order of the first several critical points is

$$0 < d-v < 1-v \leq d < 1 < z+d.$$

We have

- $x'(t) = -2$ for $t \in (0, d-v)$, and $x(d-v) = 1+2v-2d$.
- $x'(t) = 0$ for $t \in (d-v, 1-v)$, and $x(1-v) = 1+2v-2d$.
- $x'(t) = -1$ for $t \in (1-v, d)$, and $x(d) = 2+v-3d$.
- $x'(t) = -3$ for $t \in (d, 1)$, and $x(1) = -1+v < 0$.
- $x'(t) = -2$ for $t \in (1, z+d)$, and $x(z+d) = 1+v-2d-2z$.

Case 2a: $v \in \left[-d+1, d-\frac{1}{2}\right)$. *This case does not occur when* $d < \frac{3}{4}$. In this case $x(d-v) \leq 0$ so $z = 1/2$ and

$$x(z+d) = v-2d < -1.$$

Since $x(1) > -1$, the same calculation as in case 1 shows that $t_1 = \dfrac{2+v}{2}$ and

$$x(t_1+d) = 1+2v-2d > -1$$

and so

$$F(v) = 2(z+d-t_1) = -v+2d-1.$$

Case 2b: $v \in \left[d-\frac{1}{2}, 3d-2\right)$. *This case does not occur when* $d < \frac{3}{4}$. In this case $z \in (1-v, d]$ and

$$z = 2+v-2d.$$

The same calculations as in case 1 yield $t_1 = \dfrac{2+v}{2}$ and

$$x(t_1+d) = -5-2v+6d > -1,$$

and so

$$F(v) = 2(z + d - t_1) = v - 2d + 2.$$

Case 2c: $v \in [\max(-d+1, 3d-2), d)$. In this case $x(d) \geq 0$ and so

$$z = d + \frac{x(d)}{3} = \frac{2+v}{3}, \text{ and } x(z+d) = \frac{-1+v-6d}{3}.$$

Now calculations like those in case 1c show that

$$t_1 = \frac{2+v}{2} \text{ and } x(t_1+d) = \frac{1+2v-6d}{3} > -1,$$

and so again we conclude that

$$F(v) = 2(z + d - t_1) = -\frac{1}{3}v + 2d - \frac{2}{3}.$$

Case 3: $v \in [d, 1]$. In this case the order of the first several critical points is

$$0 \leq 1 - v < d < 1 < z + d.$$

We have

- $x'(t) = 0$ for $t \in (0, 1-v)$, and $x(1-v) = 1$.
- $x'(t) = -1$ for $t \in (1-v, d)$, and $x(d) = 2 - v - d \geq 0$.
- $x'(t) = -3$ for $t \in (d, 1)$, and $x(1) = -1 - v + 2d \leq 0$.
- $x'(t) = -2$ for $t \in (1, z+d)$, and $x(z+d) = 1 - v - 2z$.

We see that $z \in [d, 1]$ and

$$z = d + \frac{x(d)}{3} = \frac{2-v+2d}{3} \text{ and } x(z+d) = \frac{-1-v-4d}{3} \leq -1.$$

Since $x(1) \geq -1$ we see that

$$t_1 = 1 + \frac{x(1)+1}{2} = \frac{2-v+2d}{2}.$$

We again have $x'(t) = 2$ for $t \in (z+d, t_1+d)$ and so

$$x(t_1+d) = x(z+d) + 2(t_1 - z) = \frac{1-2v-2d}{3} \geq -1.$$

We conclude once more that

$$F(v) = 2(z + d - t_1) = \frac{1}{3}v + \frac{4}{3}d - \frac{2}{3}.$$

This completes the proof. □

B Formulas for F^2 in Equation (II)

With the formulas for F in hand, the formulas for F^2 can be obtained through tedious but elementary calculations. As d runs from $2/3$ to 1 the cases proliferate, so we shall content ourselves with presenting the formula for the case that $d \in [2/3, 3/4)$. This formula is as follows.

$$F^2(v) = \begin{cases} \frac{1}{3}d + \frac{2}{3}, & 0 \le v \le -\frac{1}{4}d + \frac{1}{4}; \\ -\frac{4}{3}v + \frac{5}{3}d - \frac{2}{3}, & -\frac{1}{4}d + \frac{1}{4} < v \le d - \frac{1}{2}; \\ -\frac{20}{9}v + \frac{23}{9}d - \frac{10}{9}, & d - \frac{1}{2} \le v \le -\frac{1}{5}d + \frac{2}{5}; \\ -\frac{5}{3}v + \frac{8}{3}d - \frac{4}{3}, & -\frac{1}{5}d + \frac{2}{5} \le v \le \frac{7}{10}d - \frac{1}{5}; \\ \frac{25}{9}v - \frac{4}{9}d - \frac{4}{9}, & \frac{7}{10}d - \frac{1}{5} \le v \le \frac{13}{25}d + \frac{4}{25}; \\ \frac{50}{27}v + \frac{1}{27}d - \frac{8}{27}, & \frac{13}{25}d + \frac{4}{25} \le v \le -\frac{1}{5}d + \frac{7}{10}; \\ \frac{10}{9}v - \frac{1}{9}d + \frac{2}{9}, & -\frac{1}{5}d + \frac{7}{10} \le v \le -\frac{1}{2}d + 1; \\ -\frac{2}{3}v - d + 2, & -\frac{1}{2}d + 1 \le v \le d; \\ -\frac{4}{3}v - \frac{1}{3}d + 2, & d \le v \le \frac{3}{4}; \\ -\frac{20}{9}v - \frac{1}{3}d + \frac{24}{9}, & \frac{3}{4} \le v \le -\frac{3}{5}d + \frac{6}{5}; \\ -\frac{10}{3}v - d + 4, & -\frac{3}{5}d + \frac{6}{5} \le v < \frac{1}{2}d + \frac{1}{2}; \\ \frac{1}{3}d + \frac{2}{3}, & \frac{1}{2}d + \frac{1}{2} \le v \le 1. \end{cases}$$

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