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Infinity many solutions of a quasilinear periodic boundary value problem

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Abstract. Quasilinear boundary value problems with periodic conditions have been extensively investigated because of their importance in nonlinear analysis, mechanics, and dynamical systems. In this paper, we examine a class of second-order quasilinear differential equations with derivative-dependent principal part and periodic boundary conditions. Under a framework of structural and growth assumptions, we prove the existence of infinitely many weak solutions. Our approach is variational and based on Ricceri's infinite critical points theorem, which ensures the existence of an unbounded sequence of critical points of the associated energy functional. The obtained results generalize and complement recent contributions in the literature concerning one, two, and three-solution problems. Several corollaries and examples are also presented to illustrate the applicability of the main theorem.

Keywords: critical point theory, multiple solutions, quasilinear periodic, variational methods.

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1 Introduction

The study of quasilinear differential equations and systems has been a fertile area of research due to their profound applications in mathematical physics and engineering, describing phenomena such as non-Newtonian fluids, quantum field theory, and nonlinear elasticity. A central and challenging problem in this field is to determine the number of solutions that boundary value problems for such equations can possess.

In this study, we investigate the boundary value problem

$$\begin{cases} -p(u'(x))u''(x) + \zeta(x)u(x) = \lambda f(x,\ u(x)) + \mu g(x,\ u(x)), & x \in (0,1), \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{1.1}$$

where the real parameters satisfy $\lambda > 0$, $\mu \geq 0$, and the nonlinearities $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the L^1-Carathéodory conditions. Our analysis is carried out under the following framework

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(H_1) The function $p : \mathbb{R} \to (0, \infty)$ is continuous, monotone nondecreasing on $(0, \infty)$, and there exist constants $M \geq m > 0$ such that

$$m \leq p(t) \leq M, \qquad \forall t \in \mathbb{R}. \tag{1.2}$$

(H_2) The function ζ belongs to $C([0, 1])$, and there exist constants $0 < \zeta_0 \leq \zeta_1$ satisfying

$$\zeta_0 \leq \zeta(x) \leq \zeta_1, \qquad \forall x \in [0, 1]. \tag{1.3}$$

In the last two decades, variational methods and critical point theory have emerged as powerful tools for tackling the problem of determining the number of solutions to boundary value problems. A significant direction has been the search for multiple solutions, with a particular focus on obtaining three or even infinitely many solutions under various growth conditions on the nonlinearity. Seminal work in this direction includes the three critical points theorem of Ricceri [12]. In [4] Bonanno and Livrea investigate a second-order Hamiltonian system and by applying critical point theorems due to Averna–Bonanno and Bonanno, they establish the existence of an explicit interval for which the system admits at least three distinct periodic solutions. Afrouzi and Heidarkhani in [1] investigate a quasilinear boundary value problem and extend previous work by Livrea [9] and provide new multiplicity theorems under explicit conditions. In [5] Graef, Heidarkhani, and Kong investigate a quasilinear elliptic system. Shen and Liu in [13] investigate the existence of infinitely many solutions for a second-order quasilinear periodic boundary value problem with impulsive effects. In [14] Wang, Liu and Song studied the existence of at least three periodic solutions for a quasilinear periodic boundary value problem. See also [6, 10, 11, 15]. Recently, Amoroso, Bonanno and Perera in [2] investigated a nonlinear elliptic p-Laplacian equation in the whole space $\mathbb{R}^N$, applying a critical points theorem to obtain infinitely many weak solutions for a parameter-dependent problem, and also provided conditions for the solutions to be nonnegative. In 2024, Heidarkhani and Moradi in [7] studied a quasilinear periodic boundary value problem, and by utilizing a three critical points theorem, they established the existence of at least three weak solutions for specific intervals of the parameters. In the same year, Heidarkhani, Moradi, Caristi and Ferrara in [8] established the existence of one, two, and three solutions to problem (1.1) by employing critical point theorems. In this paper, we use Ricceri's infinitely many solutions theorem to establish the existence of solutions to this problem.

This paper is organized into four sections. In Section 2, we present Ricceri's infinity many solutions theorem, the appropriate functional space for the problem, and the required functionals. In Section 3, we prove Theorem 3.1 by applying the necessary hypotheses and drawing inspiration from Theorem 2.1, which ensures the existence of infinity many solutions for problem (1.1). In the final section, by setting $\lambda = 1$ and $\mu = 0$, we present Corollary 4.2 as an application of Theorem 3.1. In Corollary 4.3, under certain conditions, we define λ in such a way that, by applying the separability of the function f, where one part belongs to L^1-function and the other part to continuous function, the problem continues to have infinity many solutions. In Theorem 4.4, by modifying the conditions, we define the function f in such a manner that it no longer depends on the variable x. In Corollary 4.5, we analyze a problem in which the second part is continuous. An example which is related to this corollary is presented. In Theorem 4.6, we have examined part (c) of Theorem 2.1. Finally, another illustrative example is provided.

2 Preliminaries

Our main abstract tool is the following theorem.

Theorem 2.1 ([3, Theorem 2.1]). *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\begin{aligned} \varphi(r) &:= \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(u)}{r-\Phi(u)}, \\ \gamma &:= \liminf_{r\to+\infty} \varphi(r), \\ \delta &:= \liminf_{r\to(\inf_X \Phi)^+} \varphi(r). \end{aligned}$$

Then the following hold:

(*a*) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$J_\lambda = \Phi - \lambda\Psi,$$

to $\Phi^{-1}((-\infty,\ r))$ admits a global minimum, which is a critical point (local minimum) of J_λ in X.

(*b*) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(1) J_λ possesses a global minimum, or

(2) there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(*c*) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(1) there is a global minimum of Φ which is a local minimum of J_λ, or

(2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of J_λ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.

Let

$$X := \{u : [0,1] \to \mathbb{R} \mid u \text{ is absolutely continuous, } u(0) = u(1),\ u' \in L^2(0,1)\}.$$

Equivalently, X may be denoted by $H^1_{\text{per}}(0,1)$ or $W^{1,2}_{\text{per}}(0,1)$, the Sobolev space of 1-periodic functions with square-integrable weak derivative. Note that X is infinite-dimensional.

We endow X with the inner product

$$\langle u, v\rangle := \int_0^1 \left(u'(x)v'(x) + u(x)v(x)\right) dx,$$

and the induced norm

$$\|u\| := \left(\int_0^1 \left(|u'(x)|^2 + |u(x)|^2\right) dx\right)^{1/2}. \tag{2.1}$$

With this structure X is a Hilbert space and we denote its dual by X^*. (Any other equivalent Hilbert norm on $H^1_{\text{per}}(0,1)$ may be used, but (2.1) is convenient for variational formulations.)

For the function p, define

$$h(y) := \int_0^y \left(\int_0^\tau p(\xi)\, d\xi \right) d\tau \quad \text{for } y \in \mathbb{R}.$$

Then, by differentiation with respect to y,

$$h'(y) = \int_0^y p(\xi)\, d\xi, \qquad h''(y) = p(y) \quad \text{for all } y \in \mathbb{R},$$

Define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) := \int_0^1 h\big(u'(x)\big)dx + \frac{1}{2}\int_0^1 \zeta(x)|u(x)|^2\, dx, \tag{2.2}$$

$$\Psi(u) := \int_0^1 F\big(x, u(x)\big)dx + \frac{\mu}{\lambda}\int_0^1 G\big(x, u(x)\big)dx, \tag{2.3}$$

where

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\, d\xi \qquad \big((x,t) \in [0,1] \times \mathbb{R}\big).$$

The energy functional associated with (1.1) is then

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u) \qquad (u \in X),$$

which is readily seen to be Gâteaux differentiable.

Definition 2.2. A function $u \in X$ is called a weak solution of problem (1.1) if it satisfies

$$\int_0^1 h'(u'(x))y'(x)dx + \int_0^1 \zeta(x)u(x)y(x)dx \\ - \lambda \int_0^1 f(x,\ u(x))y(x)dx - \mu \int_0^1 g(x,\ u(x))y(x)dx = 0,$$

for every test function $y \in X$.

The following lemma, whose proof can be found in [8, Lemma 2.6], connects critical points of J_λ with weak solutions of (1.1).

Lemma 2.3. *If $u \in X$ is a critical point of J_λ, then $u \in X$ is a solution of (1.1).*

3 Main results

For convenience we introduce the quantities

$$f_\infty := \liminf_{\xi \to +\infty} \frac{\int_0^1 \sup_{|v| < \xi} F(x, v(x))dx}{\xi^2},$$

$$g_\infty := \lim_{\xi \to +\infty} \frac{\int_0^1 \sup_{|v| < \xi} G(x, v(x))dx}{\xi^2}$$

and

$$B := \limsup_{\xi \to +\infty} \frac{\int_0^1 F(x,\xi)dx}{\xi^2}.$$

Our principal result reads as follows.

Theorem 3.1. *Assume that the nonlinearity f satisfies*

(A_1) $f(x,t) \geq 0$ *for every* $(x,t) \in [0,1] \times ([0,\frac{1}{4}] \cup [\frac{3}{4},1])$;

(A_2) $\frac{8(2M+\zeta_1)}{\min\{\zeta_0,m\}} f_\infty < B$.

Then, for each

$$\lambda \in \Lambda_1 := \left(\frac{2M+\zeta_1}{B}, \frac{\min\{\zeta_0,m\}}{8f_\infty} \right)$$

and for every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ whose potential $G(x,\ t) := \int_0^t g(x,\xi)d\xi$ is non-negative on $[0,1] \times [0,+\infty)$ and satisfies

$$g_\infty < +\infty, \tag{3.1}$$

if we set

$$\mu_{g,\lambda} := \frac{\min\{\zeta_0,m\} - 8\lambda f_\infty}{8g_\infty} \qquad (\textit{with } \mu_{g,\lambda} = +\infty \textit{ when } g_\infty = 0),$$

problem (1.1) *admits an unbounded sequence of weak solutions for every $\mu \in [0, \mu_{g,\lambda})$.*

Proof. Our aim is to apply Theorem 2.1(b) to problem (1.1). To this end, fix $\lambda \in \Lambda_1 = (\lambda_1, \lambda_2)$ and a function g satisfying the hypotheses. Because $\lambda < \lambda_2$, we have

$$\mu_{g,\lambda} := \frac{\min\{\zeta_0,m\} - 8\lambda f_\infty}{8g_\infty} > 0.$$

Choose any $\mu \in (0,\ \mu_{g,\lambda})$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined in (2.2) and (2.3). Standard arguments show that Φ is Gâteaux differentiable, sequentially weakly lower semicontinuous, coercive, and its derivative is given by

$$\Phi'(u)(v) = \int_0^1 h'(u'(x))v'(x)\,dx + \int_0^1 \zeta(x)u(x)v(x)\,dx \qquad (v \in X).$$

The functional Ψ is well defined, sequentially weakly upper semicontinuous, and its Gâteaux derivative equals

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)\,dx + \frac{\mu}{\lambda}\int_0^1 g(x,u(x))v(x)\,dx \qquad (v \in X).$$

Consequently the energy functional $J_\lambda = \Phi - \lambda\Psi$ satisfies the regularity requirements of Theorem 2.1.

Let $\{c_n\}$ be a sequence of positive numbers with $c_n \to +\infty$ and set $r_n := \frac{\min\{\zeta_0,m\}}{8}c_n^2$. For any $u \in X$ with $\Phi(u) \leq r_n$ one has the estimate $\max_{x\in[0,1]}|u(x)| \leq c_n$ (see the embedding inequality derived in [8, Theorem 3.1]). Hence, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)}{r_n} \\
&= \frac{8}{\min\{\zeta_0,m\}}\left(\frac{\int_0^1 \sup_{|v|<c_n} F(x,v)dx}{c_n^2} + \frac{\mu}{\lambda}\frac{\int_0^1 \sup_{|v|<c_n} G(x,v)dx}{c_n^2} \right).
\end{aligned}$$

Moreover, from the assumption (A_2) and (3.1):

$$\begin{aligned}\gamma &= \liminf_{r\to+\infty} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n)\\ &\le \frac{8}{\min\{\zeta_0, m\}}(f_\infty + \frac{\mu}{\lambda} g_\infty) < +\infty.\end{aligned}$$

Because $\mu < \mu_{g,\lambda}$, the inequality $\frac{8}{\min\{\zeta_0,m\}}(f_\infty + \frac{\mu}{\lambda} g_\infty) < \frac{1}{\lambda}$ holds, whence $\gamma < 1/\lambda$.

Next we show that J_λ is unbounded from below. From $\lambda > (2M + \zeta_1)/B$ we can select a constant τ and a sequence $\{\eta_n\}$ with $\eta_n \to +\infty$ such that

$$\frac{1}{\lambda} < \tau < \frac{\int_{1/4}^{3/4} F(x, \eta_n)\, dx}{(2M + \zeta_1)\eta_n^2} \quad \text{for all large } n.$$

Define the family of test functions

$$w_n(x) = \begin{cases} \eta_n & x \in [0,\ \frac{1}{4}],\\ 2\eta_n x + \frac{\eta_n}{2} & x \in [\frac{1}{4},\ \frac{1}{2}],\\ -2\eta_n x + \frac{5\eta_n}{2} & x \in [\frac{1}{2},\ \frac{3}{4}],\\ \eta_n & x \in [\frac{3}{4}, 1]. \end{cases} \tag{3.2}$$

Clearly $w_n \in X$, for each $n \in \mathbb{N}$. A direct computation gives

$$\Phi(w_n) < \frac{1}{4} h(2\eta_n) + \frac{1}{4} h(-2\eta_n) + \zeta_1 \eta_n^2.$$

Using (A_1) and the non-negativity of G we obtain

$$\Psi(w_n) \ge \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \eta_n) dx. \tag{3.3}$$

Consequently,

$$J_\lambda(w_n) = \Phi(w_n) - \lambda \Psi(w_n) \le (2M + \zeta_1)(1 - \lambda\tau)\eta_n^2, \tag{3.4}$$

for all sufficiently large n. Since $\lambda\tau > 1$ and $\eta_n \to +\infty$, it follows that

$$\lim_{n\to+\infty} J_\lambda(w_n) = -\infty.$$

Hence J_λ is unbounded from below and possesses no global minimum.

All assumptions of Theorem 2.1(b) are now fulfilled. Therefore there exists a sequence $\{u_n\} \subset X$ of critical points of J_λ with $\lim_{n\to+\infty} \Phi(u_n) = +\infty$. By Lemma 2.3 each u_n is a weak solution of (1.1), and the sequence is unbounded because Φ is coercive. This completes the proof. $\square$

4 Applications and examples

In this section, we discuss some consequences and applications of the result previously obtained. First, we record an immediate consequence of Theorem 3.1 when the perturbation g is absent.

Corollary 4.1. *Under the assumptions (A_1) and (A_2) of Theorem 3.1, for every $\lambda \in \Lambda_1$ the problem*

$$\begin{cases} -p(u'(x))u''(x) + \zeta(x)u(x) = \lambda f(x,\ u(x)), \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{4.1}$$

admits an unbounded sequence of weak solutions in X.

Taking $\lambda = 1$ we obtain a result for the "non-parametric" problem.

Corollary 4.2. *Assume (A_1) holds and that*

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} f_\infty < 2M + \zeta_1 < B.$$

Then, the problem

$$\begin{cases} -p(u'(x))u''(x) + \zeta(x)u(x) = f(x,\ u(x))\,, \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{4.2}$$

has an unbounded sequence of weak solutions in X.

Proof. By assumption, we have $\lambda_1 < 1$ and $\lambda_2 > 1$. Hence $1 \in \Lambda_1$ and consequently, problem (4.2) admits a solution by Corollary 4.1. □

The statement below represents a particular manifestation of Theorem 3.1. When the nonlinearity splits into an x-dependent factor and a u-dependent factor we have the following specialisation.

Corollary 4.3. *Let $f_1 \in L^1([0,1])$ with $f_1(x) > 0$ a.e., and let $f_2 \in C(\mathbb{R})$ be non-negative. Set $\widetilde{F}(t) := \int_0^t f_2(s)\,ds$ and suppose that*

$$\liminf_{\xi \to +\infty} \frac{\widetilde{F}(\xi)}{\xi^2} < +\infty, \qquad \limsup_{\xi \to +\infty} \frac{\widetilde{F}(\xi)}{\xi^2} = +\infty.$$

Then, for every

$$\lambda \in \left(0, \frac{\min\{\zeta_0, m\}}{8\|f_1\|_{L^1} \liminf_{\xi \to +\infty} \widetilde{F}(\xi)/\xi^2}\right)$$

the problem

$$\begin{cases} l - p(u'(x))u''(x) + \zeta(x)u(x) = \lambda f_1(x) f_2(u(x))\,, \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{4.3}$$

admits an unbounded sequence of weak solution in X.

Proof. Define $f(x,u) := f_1(x)f_2(u)$. Then $F(x,u) = f_1(x)\widetilde{F}(u)$. The positivity of f_1 together with the non-negativity of f_2 implies (A_1). Moreover,

$$f_\infty = \|f_1\|_{L^1} \liminf_{\xi \to +\infty} \frac{\widetilde{F}(\xi)}{\xi^2}, \qquad B = \|f_1\|_{L^1} \limsup_{\xi \to +\infty} \frac{\widetilde{F}(\xi)}{\xi^2} = +\infty,$$

so (A_2) is automatically satisfied. Theorem 3.1 (with $g \equiv 0$) yields the conclusion. □

The next theorem treats the case in which the functions $f(x,u)$ and $g(x,u)$ do not explicitly depend on x.

Theorem 4.4. *Let $k, q \in C(\mathbb{R})$ be such that their primitives $K(t) := \int_0^t k(s)\,ds$ and $Q(t) := \int_0^t q(s)\,ds$ are non-negative for $t \geq 0$. Define*

$$f'_\infty := \liminf_{t\to+\infty} \frac{\max_{|\xi|\leq t} K(\xi)}{t^2}, \qquad B' := \limsup_{t\to+\infty} \frac{K(t)}{t^2},$$

and assume that

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} f'_\infty < B'.$$

Set

$$Q_\infty := \limsup_{t\to+\infty} \frac{\max_{|\xi|\leq t} Q(\xi)}{t^2} \ (< \infty).$$

Then, for each $\lambda \in \left(\frac{2M+\zeta_1}{B'}, \frac{\min\{\zeta_0,m\}}{8f'_\infty}\right)$ and for every $\mu \in [0, \mu^*)$ where $\mu^* := \frac{\min\{\zeta_0,m\}-8\lambda f'_\infty}{8Q_\infty}$ ($\mu^* = +\infty$ if $Q_\infty = 0$), the autonomous problem

$$\begin{cases} -p\big(u'(x)\big)u''(x) + \zeta(x)u(x) = \lambda k\big(u(x)\big) + \mu q\big(u(x)\big), \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{4.4}$$

possesses an unbounded sequence of weak solutions.

Proof. We aim to apply Theorem 3.1 with the choices $f(x,t) = k(t)$ and $g(x,t) = q(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. Both k and q are continuous, hence they are L^1-Carathéodory functions. We must verify that the assumptions (A_1) and (A_2) of Theorem 3.1 are satisfied.

For the autonomous case, condition (A_1) requires $f(x,t) \geq 0$ for every $(x,t) \in [0,1] \times \big([0,\frac{1}{4}] \cup [\frac{3}{4},1]\big)$, i.e. $k(t) \geq 0$ for $t \in [0,\frac{1}{4}] \cup [\frac{3}{4},1]$. Here, we only assume $K(t) = \int_0^t k(s)\,ds \geq 0$ for all $t \geq 0$. This does not, in general, imply pointwise non-negativity of k on the above intervals. However, in the proof of Theorem 3.1 the sole purpose of (A_1) is to guarantee that for the test functions w_n defined in (3.2) – which take the constant value $\eta_n > 0$ on $[0,\frac{1}{4}]$ and $[\frac{3}{4},1]$ – one has

$$\int_0^{1/4} F(x,\eta_n)\,dx \geq 0 \quad \text{and} \quad \int_{3/4}^1 F(x,\eta_n)\,dx \geq 0.$$

For our choice $f(x,t) = k(t)$ we have $F(x,t) = K(t)$, independent of x. Thus the two integrals become $\frac{1}{4}K(\eta_n)$ and $\frac{1}{4}K(\eta_n)$, respectively. Since $\eta_n > 0$, we have $K(\eta_n) \geq 0$, so the required non-negativity holds. Consequently, the key estimate $\Psi(w_n) \geq \int_{1/4}^{3/4} F(x,\eta_n)\,dx$ (equation (3.3) in the proof of Theorem 3.1) remains valid.

Now, we compute the quantities f_∞ and B appearing in Theorem 3.1 for our special nonlinearities. Because $F(x,t) = K(t)$ does not depend on x,

$$\begin{aligned} f_\infty &:= \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|v|<\xi} F(x,v)\,dx}{\xi^2} \\ &= \liminf_{\xi\to+\infty} \frac{\sup_{|v|<\xi} K(v)}{\xi^2} = f'_\infty, \\ B &:= \limsup_{\xi\to+\infty} \frac{\int_0^1 F(x,\xi)\,dx}{\xi^2} \\ &= \limsup_{\xi\to+\infty} \frac{K(\xi)}{\xi^2} = B'. \end{aligned}$$

Thus the inequality required in (A_2) of Theorem 3.1 becomes exactly

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} f'_\infty < B',$$

For the perturbation g we have $G(x,t) = Q(t)$. The non-negativity condition on G demanded in Theorem 3.1 is that $G(x,t) \geq 0$ for all $(x,t) \in [0,1] \times [0,+\infty)$; this is guaranteed by $Q(t) \geq 0$. Moreover,

$$g_\infty = \lim_{\xi \to +\infty} \frac{\int_0^1 \sup_{|v|<\xi} G(x,v)\, dx}{\xi^2} = \lim_{\xi \to +\infty} \frac{\sup_{|v|<\xi} Q(v)}{\xi^2} = Q_\infty,$$

so the requirement $g_\infty < +\infty$ coincides with $Q_\infty < +\infty$.

All hypotheses of Theorem 3.1 are therefore satisfied. Applying it with the above identifications, we obtain that for every $\lambda \in \left(\frac{2M+\zeta_1}{B'}, \frac{\min\{\zeta_0,m\}}{8f'_\infty}\right)$ and for every $\mu \in [0,\mu^*)$ where $\mu^* = \frac{\min\{\zeta_0,m\} - 8\lambda f'_\infty}{8Q_\infty}$ (interpreted as $+\infty$ when $Q_\infty = 0$), problem (4.4) admits an unbounded sequence of weak solutions. □

Corollary 4.5. *Let $k : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function such that*

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} \liminf_{t \to +\infty} \frac{K(t)}{t^2} < \limsup_{t \to +\infty} \frac{K(t)}{t^2}.$$

Take $\lambda \in \left(\frac{2M+\zeta_1}{\limsup_{t\to+\infty} K(t)/t^2}, \frac{\min\{\zeta_0,m\}}{8 \liminf_{t\to+\infty} K(t)/t^2}\right)$ and let $q \in C(\mathbb{R})$ be such that $t\,q(t) \geq 0$ for all t and $\lim_{|t|\to+\infty} \frac{q(t)}{|t|} = 0$. Then, for every $\mu \geq 0$, problem (4.4) admits an unbounded sequence of weak solutions.

Proof. We check that the assumptions of Theorem 4.4 hold, and hence its conclusion provides the desired sequence of solutions.

Because k is non-negative, its primitive K is non-negative and non-decreasing on $[0,\infty)$. Consequently, for any $t \geq 0$,

$$\max_{|\xi| \leq t} K(\xi) = K(t).$$

Hence the quantities f'_∞ and B' defined in Theorem 4.4 become

$$f'_\infty = \liminf_{t \to +\infty} \frac{K(t)}{t^2}, \qquad B' = \limsup_{t \to +\infty} \frac{K(t)}{t^2}.$$

The assumed inequality

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} \liminf_{t \to +\infty} \frac{K(t)}{t^2} < \limsup_{t \to +\infty} \frac{K(t)}{t^2}$$

is therefore exactly the assumption of Theorem 4.4.

Now, let $q \in C(\mathbb{R})$ satisfy $t\,q(t) \geq 0$ for all t and $\lim_{|t|\to+\infty} \frac{q(t)}{|t|} = 0$. Define $Q(t) = \int_0^t q(s)\, ds$. For $t \geq 0$ the condition $s\,q(s) \geq 0$ implies $q(s) \geq 0$ for $s \geq 0$; hence $Q(t) \geq 0$. For $t \leq 0$ we write $Q(t) = -\int_t^0 q(s)\, ds$. Since $s \leq 0$, the inequality $s\,q(s) \geq 0$ gives $q(s) \leq 0$, so again $Q(t) \geq 0$. Thus $Q(t) \geq 0$ for all $t \in \mathbb{R}$.

In the following, we claim that $Q_\infty = 0$, where

$$Q_\infty := \limsup_{t\to+\infty} \frac{\max_{|\xi|\le t} Q(\xi)}{t^2}.$$

Because Q is even (or at least symmetric in the sense that $Q(-t)$ behaves similarly), it suffices to estimate $Q(t)$ for large positive t. Given $\varepsilon > 0$, the hypothesis $\lim_{|t|\to+\infty} q(t)/|t| = 0$ yields a constant $T_\varepsilon > 0$ such that

$$|q(s)| \le \varepsilon s \quad \text{for all } s \ge T_\varepsilon.$$

For $t > T_\varepsilon$ we split the integral:

$$Q(t) = \int_0^{T_\varepsilon} q(s)\,ds + \int_{T_\varepsilon}^t q(s)\,ds =: C_\varepsilon + \int_{T_\varepsilon}^t q(s)\,ds,$$

where C_ε is a constant depending on ε. Using the bound on $|q(s)|$,

$$\Big|\int_{T_\varepsilon}^t q(s)\,ds\Big| \le \int_{T_\varepsilon}^t |q(s)|\,ds \le \varepsilon \int_{T_\varepsilon}^t s\,ds = \frac{\varepsilon}{2}\big(t^2 - T_\varepsilon^2\big).$$

Hence for $t \ge T_\varepsilon$,

$$Q(t) \le |C_\varepsilon| + \frac{\varepsilon}{2} t^2.$$

The same estimate holds for $Q(-t)$ because $|q(s)|$ satisfies the same growth condition. Consequently,

$$\max_{|\xi|\le t} Q(\xi) \le |C_\varepsilon| + \frac{\varepsilon}{2} t^2 \quad \text{for all } t \ge T_\varepsilon,$$

and therefore

$$\frac{\max_{|\xi|\le t} Q(\xi)}{t^2} \le \frac{|C_\varepsilon|}{t^2} + \frac{\varepsilon}{2}.$$

Passing to the limit superior as $t \to +\infty$ gives

$$Q_\infty = \limsup_{t\to+\infty} \frac{\max_{|\xi|\le t} Q(\xi)}{t^2} \le \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude $Q_\infty = 0$.

Take

$$\lambda \in \Big(\frac{2M + \zeta_1}{\limsup_{t\to+\infty} K(t)/t^2}, \frac{\min\{\zeta_0, m\}}{8 \liminf_{t\to+\infty} K(t)/t^2} \Big).$$

For the perturbation q, we have shown $Q(t) \ge 0$ and $Q_\infty = 0$. Theorem 4.4 then guarantees the existence of an unbounded sequence of weak solutions for every μ in the interval $[0, \mu^*)$ with

$$\mu^* = \frac{\min\{\zeta_0, m\} - 8\lambda f'_\infty}{8 Q_\infty}.$$

Because $Q_\infty = 0$, the denominator is zero and, by convention, $\mu^* = +\infty$. Thus the conclusion holds for every $\mu \ge 0$.

This completes the proof. □

Using part (c) of Theorem 2.1 we can also obtain solutions that accumulate at zero. Define

$$f_0 := \liminf_{\xi \to 0^+} \frac{\int_0^1 \sup_{|v|<\xi} F(x,v)\,dx}{\xi^2},$$

$$B_0 := \limsup_{\xi \to 0^+} \frac{\int_0^1 F(x,\xi)\,dx}{\xi^2}.$$

Theorem 4.6. *Assume (A_1) holds and that*

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} f_0 < B_0.$$

Let g be an L^1-Carathéodory function with $G(x,t) \geq 0$ on $[0,1] \times [0,+\infty)$ and set

$$g_0 := \lim_{t \to 0^+} \frac{\int_0^1 \sup_{|\xi| \leq t} G(x,\xi)\,dx}{t^2} \quad (< \infty).$$

Then, for every $\lambda \in \Lambda_2 := \left(\frac{2M+\zeta_1}{B_0}, \frac{\min\{\zeta_0,m\}}{8f_0}\right)$ and for every $\mu \in [0, \mu'_{g,\lambda})$ where $\mu'_{g,\lambda} := \frac{\min\{\zeta_0,m\}-8\lambda f_0}{8g_0}$ ($\mu'_{g,\lambda} = +\infty$ if $g_0 = 0$), problem (1.1) possesses a sequence of weak solutions that converges strongly to zero in X.

Proof. Fix $\lambda \in \Lambda_2 = (\lambda_3, \lambda_4)$ and assume that g satisfies the assumptions of the theorem. Since $\lambda < \lambda_4$, we have

$$\mu'_{g,\lambda} := \frac{\min\{\zeta_0, m\} - 8\lambda f_0}{8g_0} > 0.$$

Now fix $\mu \in (0, \mu'_{g,\lambda})$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined in (2.2) and (2.3), and the energy functional

$$J_\lambda(u) := \Phi(u) - \lambda \Psi(u), \qquad u \in X.$$

As shown in the proof of Theorem 3.1, the functional Φ is coercive, sequentially weakly lower semicontinuous, and Gâteaux differentiable. Moreover, Ψ is well defined, sequentially weakly upper semicontinuous, and Gâteaux differentiable. It follows that $J_\lambda = \Phi - \lambda\Psi$ fulfills the regularity assumptions of Theorem 2.1.

Recall that $\delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r)$. Since $\Phi(u) \geq 0$ for all $u \in X$ and $\Phi(0) = 0$, we have $\inf_X \Phi = 0$. Hence $\delta = \liminf_{r \to 0^+} \varphi(r)$.

Let $\{c_n\}$ be a sequence of positive numbers such that $\lim_{n \to +\infty} c_n = 0$ and

$$\lim_{n \to +\infty} \frac{\int_0^1 \sup_{|t|<c_n} F(x,t)dx}{c_n^2} = f_0.$$

Set $r_n := \frac{\min\{\zeta_0,m\}}{8} c_n^2$ for each $n \in \mathbb{N}$. For any $u \in X$ with $\Phi(u) < r_n$, we have the estimate

$$\max_{x \in [0,1]} |u(x)| \leq \sqrt{\frac{8r_n}{\min\{\zeta_0, m\}}} = c_n.$$

Consequently,

$$\sup_{\Phi(u)<r_n} \Psi(u) \leq \int_0^1 \sup_{|t| \leq c_n} F(x,t)dx + \frac{\mu}{\lambda} \int_0^1 \sup_{|t| \leq c_n} G(x,t)dx.$$

Therefore, as in the proof of Theorem 3.1, we obtain

$$\varphi(r_n) \leq \frac{8}{\min\{\zeta_0, m\}} \left(\frac{\int_0^1 \sup_{|t|<c_n} F(x,t)dx}{c_n^2} + \frac{\mu}{\lambda} \frac{\int_0^1 \sup_{|t|<c_n} G(x,t)dx}{c_n^2} \right).$$

Taking the limit inferior and using the definitions of f_0 and g_0 yields

$$\delta \leq \frac{8}{\min\{\zeta_0, m\}} \left(f_0 + \frac{\mu}{\lambda} g_0 \right) < +\infty.$$

Since $\mu < \mu'_{g,\lambda}$, we have

$$\frac{8}{\min\{\zeta_0, m\}} \left(f_0 + \frac{\mu}{\lambda} g_0 \right) < \frac{8}{\min\{\zeta_0, m\}} \left(f_0 + \frac{\min\{\zeta_0, m\} - 8\lambda f_0}{8\lambda g_0} g_0 \right) = \frac{1}{\lambda},$$

hence $\delta < 1/\lambda$, i.e., $\lambda < 1/\delta$.

From $\lambda > \lambda_3$, we have

$$\frac{1}{\lambda} < \frac{B_0}{2M + \zeta_1}.$$

Hence there exist a sequence $\{\eta_n\}$ of positive numbers with $\eta_n \to 0^+$ and a constant $\sigma > 0$ such that for all sufficiently large n,

$$\frac{1}{\lambda} < \sigma < \frac{1}{2M + \zeta_1} \frac{\int_0^1 F(x, \eta_n)dx}{\eta_n^2}.$$

Define the test functions w_n as in (3.2). Then, as computed in the proof of Theorem 3.1,

$$\Phi(w_n) < \frac{1}{4}h(2\eta_n) + \frac{1}{4}h(-2\eta_n) + \zeta_1\eta_n^2 \leq (2M + \zeta_1)\eta_n^2,$$

and by condition (A_1) together with the non-negativity of G,

$$\Psi(w_n) \geq \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \eta_n)dx \geq \frac{1}{2} \int_0^1 F(x, \eta_n)dx.$$

Consequently,

$$J_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) \leq (2M + \zeta_1)\eta_n^2 - \frac{\lambda}{2} \int_0^1 F(x, \eta_n)dx < (2M + \zeta_1)(1 - \lambda\sigma)\eta_n^2.$$

Since $\lambda\sigma > 1$ and $\eta_n \to 0^+$, it follows that $J_\lambda(w_n) < 0 = J_\lambda(0)$ for all large n. Therefore, 0 is not a local minimum of J_λ.

Thus alternative (2) of Theorem 2.1(c) must hold. Hence there exists a sequence $\{u_n\}$ of pairwise distinct critical points of J_λ such that

$$\lim_{n \to +\infty} \Phi(u_n) = 0 \quad \text{and} \quad u_n \rightharpoonup 0 \text{ weakly in } X.$$

Since Φ is coercive and u_n satisfies $\Phi(u_n) \to 0$, the sequence $\{u_n\}$ is bounded in X. Moreover, from the inequality

$$\frac{\min\{\zeta_0, m\}}{2} \|u_n\|^2 \leq \Phi(u_n),$$

we obtain $\|u_n\| \to 0$, i.e., u_n converges strongly to 0 in X.

By Lemma 2.3, each u_n is a weak solution of problem (1.1) and the proof is complete. □

Remark 4.7. If $f_0 = 0$ and $B_0 = +\infty$, then $\Lambda_2 = (0, +\infty)$. Hence, under the remaining assumptions, for every $\lambda > 0$ and every $\mu \in \left[0, \frac{\min\{\zeta_0, m\}}{8g_0}\right)$ (or any $\mu \geq 0$ when $g_0 = 0$) problem (1.1) admits a sequence of weak solutions converging strongly to 0 in X.

Now, we illustrate Corollary 4.5 through the following example.

Example 4.8. We consider the following problem

$$\begin{cases} -p(u'(x))u''(x) + u(x) = \lambda k(u(x)), & x \in (0,1), \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases} \tag{4.5}$$

where $p(t) = \frac{2}{\pi} \arctan t + 2$ ($m = 1$, $M = 3$), $\zeta(x) \equiv 1$ ($\zeta_0 = \zeta_1 = 1$), and $k(s) = 2s \sin s + s^2 \cos s + 2s$. Then $K(t) = t^2 \sin t + t^2$, whence

$$\limsup_{t \to +\infty} \frac{K(t)}{t^2} = 2 \quad \text{and} \quad \liminf_{t \to +\infty} \frac{K(t)}{t^2} = 0$$

The condition of Corollary 4.5 reads $\frac{8(2\cdot 3+1)}{1} \cdot 0 < 2$, which is trivially true. Therefore, for every $\lambda > \frac{7}{2}$ problem (4.4) admits an unbounded sequence of solutions.

In the next example, we construct a family of data satisfying all assumptions of Theorem 3.1 to explicitly demonstrate the existence of infinitely many weak solutions for a class of quasilinear periodic boundary value problems.

Example 4.9. Consider the periodic problem (1.1) where

$$p(t) = 1 + \frac{t^2}{1 + t^2}, \qquad \zeta(x) \equiv 1, \qquad t \in \mathbb{R}, x \in [0,1].$$

Clearly, $p \in C(\mathbb{R})$, p is nondecreasing on $(0, \infty)$, and $1 \leq p(t) \leq 2$. Thus, (D_1) is satisfied with $m = 1$ and $M = 2$, and (D_2) holds with $\zeta_0 = \zeta_1 = 1$.

Let

$$f_1(x) = 1 + \tfrac{1}{2} \sin(2\pi x), \qquad x \in [0,1],$$

so that $f_1 \in L^1([0,1])$ and $f_1(x) \geq \frac{1}{2} > 0$. Define $f_2 : \mathbb{R} \to [0, \infty)$ by

$$f_2(\xi) = r(\xi) + \sum_{k=1}^{\infty} A_k \, \varphi_k(\xi), \qquad \xi \in \mathbb{R},$$

where $r(\xi) = \xi$ for $\xi \geq 1$, $A_k = k^6$, $\varepsilon_k = k^{-2}$, and $\varphi_k(\xi) = \varphi\big((\xi - k)/\varepsilon_k\big)$ for a nonnegative function $\varphi \in C_c^\infty([-1,1])$. $f_2 \in C(\mathbb{R})$ and $f_2(\xi) \geq 0$ for all ξ. Set

$$f(x,u) = f_1(x) f_2(u), \qquad g(x,u) \equiv 0.$$

Then f is an L^1–Carathéodory function, and the potential of f_2 is

$$\widetilde{F}(\xi) = \int_0^\xi f_2(s) \, ds.$$

Since $f_1(x) \geq \frac{1}{2} > 0$ and $f_2(\xi) \geq 0$ for all (x, ξ), we have $F(x,t) = f_1(x) \widetilde{F}(t) \geq 0$ for $t \geq 0$. Thus, there exist intervals of t (e.g., $t \in [0, 1/4] \cup [3/4, 1]$) where $F(x,t) \geq 0$ uniformly in x, satisfying (A_1).

The base function $r(\xi) = \xi$ ensures that $\int_0^\xi r(s)\,ds = \frac{1}{2}\xi^2$, thus

$$f_\infty = \liminf_{\xi\to\infty} \frac{\int_0^1 f_1(x)\widetilde{F}(\xi)\,dx}{\xi^2} = \|f_1\|_{L^1}\cdot\tfrac{1}{2} < +\infty.$$

constant C. Consequently,

$$\frac{\widetilde{F}(k)}{k^2} \gtrsim Ck^3 \to +\infty \quad (k\to\infty),$$

so

$$B = \limsup_{\xi\to\infty} \frac{\int_0^1 F(x,\xi)\,dx}{\xi^2} = \|f_1\|_1 \cdot \limsup_{\xi\to\infty} \frac{\widetilde{F}(\xi)}{\xi^2} = +\infty.$$

Hence, the inequality

$$\frac{8(2M+\zeta_1)}{\min\{\zeta_0, m\}} f_\infty < B$$

trivially holds, fulfilling (A_2). Now we compute the interval Λ_1 from Theorem 3.1,

$$\Lambda_1 = \left(\frac{2M+\zeta_1}{B}, \frac{\min\{\zeta_0, m\}}{8f_\infty}\right) = (0, \tfrac{1}{4}).$$

Thus $\Lambda_1 = (0, 1/4)$. The function f is measurable in x and continuous in u, so it is an L^1–Carathéodory function. Taking $g \equiv 0$ yields $g_\infty = 0$. According to Theorem 3.1, when $g_\infty = 0$ we set $\mu_{g,\lambda} = +\infty$. Hence every $\mu \geq 0$ is admissible.

All hypotheses of Theorem 3.1 (particularly (A_1) and (A_2)) are satisfied. Therefore, there exists an open interval of parameters $\Lambda_1 = (0, \frac{1}{4})$ such that for every $\lambda \in \Lambda_1$ and admissible $\mu \geq 0$, the quasilinear periodic problem (1.1) admits an unbounded sequence of weak solutions.

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