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Solutions of Klein–Gordon–Maxwell systems with the term of Choquard and critical or supercritical growth

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Abstract. This article concerns the following Klein–Gordon–Maxwell systems with the term of Choquard

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u) + (I_\alpha * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

and

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = \lambda|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, $\lambda > 0$ is a parameter, $q \geq 6$, I_α is a Riesz potential whose order is α. Under some suitable assumptions on α, p, V and f, we establish the existence and multiplicity of solutions for the systems by variational methods and Moser iteration. Our results extend several recent findings on solutions to these systems.

Keywords: Klein–Gordon–Maxwell system, Choquard nonlinearities, variational methods, Moser iteration.

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1 Introduction and main results

In this paper, we study the following Klein–Gordon–Maxwell systems with the term of Choquard

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u) + (I_\alpha * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = \lambda |u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\omega > 0$ is a constant, $\lambda > 0$ is a parameter, $0 < \alpha < 3$, $\frac{3+\alpha}{3} < p < 3+\alpha$, I_α is a Riesz potential whose order is α. Here, I_α is given by

$$I_\alpha = \frac{\Gamma(\frac{3-\alpha}{2})}{2^\alpha \pi^{\frac{3}{2}} \Gamma(\frac{\alpha}{2})|x|^{3-\alpha}},$$

where Γ denotes the Gamma function. Moreover, V, f satisfy the following assumptions:

(V) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) = V_0 > 0$, and for each $M > 0$,

$$\text{meas}\left\{x \in \mathbb{R}^3 : V(x) \le M\right\} < \infty;$$

(F_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist constants $s \in (2,6)$ and $c_1 > 0$ such that

$$|f(x,t)| \le c_1(1 + |t|^{s-1}),$$

for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^3$;

(F_2) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

(F_3) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = +\infty$ uniformly in $x \in \mathbb{R}^3$, and there exists $r_0 \ge 0$ such that

$$F(x,t) \ge 0, \quad \forall x \in \mathbb{R}^3, \ |t| \ge r_0,$$

where $F(x,t) := \int_0^t f(x,s)ds$;

(F_4) there exist $\mu \in (2, 2p], c_2 \ge 0$ such that

$$f(x,t)t - \mu F(x,t) \ge -c_2|t|^2, \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

(F_5) $f(x,t) = -f(x,-t), \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

If we replace $f(x,u) + (I_\alpha * |u|^p)|u|^{p-2}u$ by a new nonlinearity $g(x,u)$ in (1.1), then we obtain the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = g(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

This type of system which was firstly introduced by Benci and Fortunato in [2] as a model describing solitary waves for the nonlinear stationary Klein–Gordon equation in the three-dimensional space interacting with the electrostatic field. Because of this, many scholars have studied the system (1.3) with constant potential $V(x) = m^2 - \omega^2$ and the pure power nonlinearity $g(x,u) = |u|^{p-2}u$, some early studies can be found in [1,3,12,13,25,35] and the references therein.

Recently, there has been significant attention on interesting conditions for g in (1.3). The existence and multiplicity of solutions under these conditions have been a focus of study for many researchers. The nonlinearity g to be sublinear (see [15, 18, 19]), asymptotically linear(see [24,38]), superlinear and subcritical (see [7–9,17,23,30,31,34,39–41,44]), critical (see [4–6,10,33,36,37,43]), critical or supercritical (see [11,32]).

On the other hand, when $\lambda = 0, \phi \to 0$, system (1.2) reduces to

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u.$$

It is called nonlinear Choquard type equation. For the physical background, we refer to [21,26] and references therein. For decades, researchers have employed variational techniques to prove the existence and analyze the qualitative behavior of solutions to Choquard type equations, see [14,16,27,29,42] and references therein.

As we can observe, the previous results in this field need assume that the nonlinearity satisfies subcritical or critical growth condition. To the best of our knowledge, there is no literature considering Klein–Gordon–Maxwell system with the term of Choquard, not to mention this system with the term of Choquard and critical or supercritical growth. Motivated by the aforementioned works, especially by [20,32], in this paper, we will study system (1.3) with Choquard nonlinearities, and the nonlinearity may be critical or supercritical. As we all known, there are very few results to system (1.3) with Choquard nonlinearities and critical or supercritical growth. So, our results are novelty. We can now state our main results.

Theorem 1.1. *Suppose that (V) and (F_1)–(F_4) hold, $0 < \alpha < 3$, $\frac{3+\alpha}{3} < p < 3+\alpha$. Then the system* (1.1) *has a nontrivial solution.*

Theorem 1.2. *Suppose that (V) and (F_1)–(F_5) hold, $0 < \alpha < 3$, $\frac{3+\alpha}{3} < p < 3+\alpha$. Then the system* (1.1) *has a sequence of solutions $\{(u_n, \phi_{u_n})\}$ satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 - \omega\phi_{u_n}u_n^2)dx - \frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx - \int_{\mathbb{R}^3} F(x,u_n)dx \to +\infty$$

as $n \to +\infty$.

Theorem 1.3. *Suppose that (V) holds, $1 < \alpha < 2$, $\alpha + 1 \leq p < 3$, $q \geq 6$. Then there exists $\lambda_* > 0$ such that for $\lambda \in (0, \lambda_*)$, the system* (1.2) *has a nontrivial solution.*

Remark 1.4. Comparing with above mentioned articles, we prove that the existence and multiplicity of solutions for system (1.1)–(1.2). As far as we know, it seems that Theorem 1.1–Theorem 1.3 are the first results about the existence of solutions for the Klein–Gordon–Maxwell system with Choquard nonlinearities.

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem and give some preliminary lemmas. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3.

2 Preliminaries

In the following, we give notations and introduce the variational setting for system (1.1)–(1.2).

- $\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ is the Sobolev space endowed with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

- $H := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\}$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \right)^{\frac{1}{2}}.$$

- $L^s(\mathbb{R}^3)$ is the Lebesgue space endowed with the norm

$$\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx \right)^{\frac{1}{s}}, \qquad \forall s \in [1, +\infty)$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^3} |u(x)|.$$

It follows from (V) that for $2 \leq s < 6$, H is compactly embedded into $L^s(\mathbb{R}^3)$, and it is well known that $H \hookrightarrow L^s(\mathbb{R}^3)$ continuously for $s \in [2,6]$. Therefore, for any $s \in [2,6]$ there exists a positive constant $S_s > 0$ such that $\|u\|_s \leq S_s \|u\|$ for $u \in H$.

- S is the best constant of the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, which is given by $S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|^2_{\mathcal{D}^{1,2}}}{\|u\|_6^2}$.
- $o(1) \to 0$, as $n \to \infty$.
- C, C_i are denoted positive constants possibly different in different places.

For deducing our results, we introduce the following result. Its proof can be done as in [17].

Lemma 2.1. *For any fixed $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which solves equation $\Delta\phi = (\omega + \phi)u^2$. Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \to \Phi[u] := \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and*

(i) $-\omega \leq \phi_u \leq 0$ on the set $\{x | u(x) \neq 0\}$;

(ii) $\|\phi_u\|_{\mathcal{D}^{1,2}} \leq C\|u\|^2_{H^1}$ and $\int_{\mathbb{R}^3} |\phi_u| u^2 dx \leq C\|u\|^4_{H^1}$.

Next we shall cite Hardy–Littlewood–Sobolev inequality [22], which plays a fundamental tool in our arguments:

Lemma 2.2. *Let $N > \alpha > 0$, $p, q > 1$ and $\infty > s > r \geq 1$ be such that*

$$\frac{\alpha}{N} + 1 = \frac{1}{q} + \frac{1}{p}, \qquad \frac{\alpha}{N} = \frac{1}{r} - \frac{1}{s}.$$

(i) For any $g \in L^q(\mathbb{R}^N)$ and $f \in L^p(\mathbb{R}^N)$, one has

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|y-x|^{N-\alpha}} dx dy \right| \leq C(\alpha, p, N) \|g\|_{L^q(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N)}.$$

(ii) For any $f \in L^r(\mathbb{R}^N)$, one has

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C(\alpha, r, N) \|f\|_{L^r(\mathbb{R}^N)}.$$

From a standard argument (see [8,27]), one can define the functional $J : H \to \mathbb{R}$ of system (1.1) as follows

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2) dx - \frac{1}{2p} \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx - \int_{\mathbb{R}^3} F(x,u) dx, \ \forall u \in H.$$

Obviously, $J \in C^1(H, \mathbb{R})$ and for any $u, v \in H$

$$\begin{aligned} \langle J'(u), v \rangle = & \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv] dx \\ & - \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^{p-2} uv dx - \int_{\mathbb{R}^3} f(x,u) v dx. \end{aligned}$$

It is well known to us that (u, ϕ) is a solution of system (1.1) if and only if u is a critical point of J and $\phi = \phi_u$.

3 Proofs of Theorem 1.1 and Theorem 1.2

Prior to proceeding to the proof of Theorem 1.1–1.2, we establish some useful preliminary results.

Lemma 3.1. *Suppose that (V), (F_1)–(F_4) hold, $0 < \alpha < 3$, $\frac{3+\alpha}{3} < p < 3 + \alpha$. Then the functional $J(u)$ satisfies $(PS)_c$ condition.*

Proof. Let $\{u_n\} \subset H$ be a $(PS)_c$ sequence of J, i.e.,

$$J(u_n) \to c, \quad J'(u_n) \to 0, \quad n \to \infty. \tag{3.1}$$

First, we prove that $\{u_n\}$ is bounded. Notice that $\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx \geq 0$. Combining this with (3.1), (F_4) and Lemma 2.1, we have

$$\begin{aligned} c + o(1)\|u_n\| &= J(u_n) - \frac{1}{\mu} \langle J'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 + \frac{2}{\mu} \int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2 dx + \frac{2p - \mu}{2p\mu} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 - \frac{2\omega^2 + c_2}{\mu} \int_{\mathbb{R}^3} u_n^2 dx. \end{aligned} \tag{3.2}$$

Arguing by contradiction, suppose that $\|u_n\| \to +\infty$. Let $v_n = u_n / \|u_n\|$, then $\|v_n\| = 1$. According to the compact embedding $H \hookrightarrow L^s(\mathbb{R}^3)$, passing to a subsequence if necessary, we may assume that there is a $v_0 \in H$ such that, for any $2 \leq s < 6$

$$v_n \rightharpoonup v_0 \quad \text{in } H, \qquad v_n(x) \to v_0(x) \quad \text{a.e. } x \in \mathbb{R}^3, \qquad v_n \to v_0 \quad \text{in } L^s(\mathbb{R}^3).$$

It follows from (3.2) that

$$(4\omega^2+2c_2)\|v_0\|_2^2 \geq (\mu-2)>0,$$

which implies that $v_0\neq 0$. Let $\Omega=\{y\in\mathbb{R}^3: v_0(y)\neq 0\}$, then $|u_n|=|v_n|\|u_n\|\to+\infty$ a.e. $x\in\Omega$ as $n\to\infty$. From (F_1)–(F_3), it is easy to see that there exists $C_1>0$ such that

$$F(x,u)+C_1|u|^2\geq 0, \qquad \forall (x,u)\in\mathbb{R}^3\times\mathbb{R}. \tag{3.3}$$

Thus, it follows from Lemma 2.1, (3.1) and (3.3) that

$$\begin{aligned} c+o(1)&=J(u_n)\\ &=\frac{1}{2}\|u_n\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx-\frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha*|u_n|^p)|u_n|^pdx-\int_{\mathbb{R}^3}F(x,u_n)dx\\ &\leq\frac{1}{2}\|u_n\|^2+\frac{\omega^2}{2}\int_{\mathbb{R}^3}u_n^2dx+C_1\int_{\mathbb{R}^3}|u_n|^2dx-\int_{\mathbb{R}^3}\big(F(x,u_n)+C_1|u_n|^2\big)dx\\ &\leq\frac{(\omega^2+2C_1)S_2^2+1}{2}\|u_n\|^2-\int_{\mathbb{R}^3}\big(F(x,u_n)+C_1|u_n|^2\big)dx. \end{aligned} \tag{3.4}$$

By (3.3)–(3.4), (F_3) and Fatou's lemma, one has

$$\begin{aligned} 0&\leq\frac{(\omega^2+2C_1)S_2^2+1}{2}-\liminf_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(x,u_n)+C_1|u_n|^2}{|u_n|^2}|v_n|^2dx\\ &\leq\frac{(\omega^2+2C_1)S_2^2+1}{2}-\int_{\mathbb{R}^3}\liminf_{n\to\infty}\frac{F(x,u_n)+C_1|u_n|^2}{|u_n|^2}|v_n|^2dx=-\infty, \end{aligned} \tag{3.5}$$

which is a contradiction. Hence, $\{u_n\}$ is bounded in H.

Next we prove the convergence of $\{u_n\}$. Since $\{u_n\}\subset H$ is bounded, there exists $u\in H$ such that, going if necessary to a subsequence,

$$u_n\rightharpoonup u \quad \text{in } H, \qquad u_n\to u \quad \text{in } L^p(\mathbb{R}^3)\ (2\leq p<6), \qquad u_n(x)\to u(x) \quad \text{a.e. } x\in\mathbb{R}^3.$$

Since

$$\begin{aligned} \|u_n-u\|^2&=\langle J'(u_n)-J'(u),u_n-u\rangle+2\omega\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)dx\\ &\quad+\int_{\mathbb{R}^3}\Big((I_\alpha*|u_n|^p)|u_n|^{p-2}u_n-(I_\alpha*|u|^p)|u|^{p-2}u\Big)(u_n-u)dx\\ &\quad+\int_{\mathbb{R}^3}(\phi_{u_n}^2u_n-\phi_u^2u)(u_n-u)dx+\int_{\mathbb{R}^3}\Big(f(x,u_n)-f(x,u)\Big)(u_n-u)dx, \end{aligned}$$

in view of the proof of Lemma 2.4 in [30], to prove $u_n\to u$, $n\to\infty$, it suffices to prove

$$\int_{\mathbb{R}^3}\Big((I_\alpha*|u_n|^p)|u_n|^{p-2}u_n-(I_\alpha*|u|^p)|u|^{p-2}u\Big)(u_n-u)dx\to 0, \qquad n\to\infty.$$

Indeed, using $0<\alpha<3$, $\frac{3+\alpha}{3}<p<3+\alpha$, then $2<\frac{6p}{3+\alpha}<6$. Therefore, for all $u\in H$, $u\in L^{\frac{6p}{3+\alpha}}(\mathbb{R}^3)$, $|u|^p\in L^{\frac{6}{3+\alpha}}(\mathbb{R}^3)$. This combing with Lemma 2.2, we get

$$\left(\int_{\mathbb{R}^3}(I_\alpha*|u|^p)^{\frac{6}{3-\alpha}}dx\right)^{\frac{3-\alpha}{6}}\leq C(\alpha,p,3)\left(\int_{\mathbb{R}^3}|u|^{\frac{6p}{3+\alpha}}dx\right)^{\frac{3+\alpha}{6}}=C(\alpha,p,3)\|u\|^p_{\frac{6p}{3+\alpha}},$$

i.e.

$$I_\alpha*|u|^p\in L^{\frac{6}{3-\alpha}}(\mathbb{R}^3). \tag{3.6}$$

So

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n(u_n-u)dx\right| &\le \|I_\alpha * |u_n|^p\|_{\frac{6}{3-\alpha}}\||u_n|^{p-2}u_n(u_n-u)\|_{\frac{6}{3+\alpha}}\\ &\le C(\alpha,p,3)\|u_n\|^{2p-1}_{\frac{6p}{3+\alpha}}\|u_n-u\|_{\frac{6p}{3+\alpha}}\\ &\to 0, \qquad n\to\infty.\end{aligned}$$

Similarly, it can be inferred that

$$\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}u(u_n-u)dx \to 0, \qquad n\to\infty.$$

Thus, we can obtain $u_n \to u$ as $n\to\infty$ in H and the proof is complete. □

Lemma 3.2. *Assume that* (V) *and* (F_1)–(F_4) *hold,* $0<\alpha<3$, $\frac{3+\alpha}{3}<p<3+\alpha$, *then*

(i) there exist $\rho,\gamma>0$ *such that* $J(u)\big|_{\|u\|=\rho}\ge\gamma>0$;

(ii) there exists $e\in H$ *such that* $\|e\|>\rho$ *and* $J(e)<0$.

Proof. (i) Since $0<\alpha<3$, $\frac{3+\alpha}{3}<p<3+\alpha$, then $2<\frac{6s}{3+\alpha}<6$. It follows from Lemma 2.2 and the Sobolev inequality that

$$\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \le C(\alpha,p,3)\|u\|^{2p}_{\frac{6p}{3+\alpha}} \le C_2\|u\|^{2p}. \tag{3.7}$$

By (F_1) and (F_2), for any $\varepsilon\in(0,\frac{1}{2S_2^2})$, there exists $c(\varepsilon)>0$ such that

$$|F(x,u)| \le \frac{\varepsilon}{2}|u|^2+\frac{c(\varepsilon)}{s}|u|^s \le \frac{1}{4S_2^2}|u|^2+\frac{c(\varepsilon)}{s}|u|^s. \tag{3.8}$$

Then for all $u\in H$, by (3.7)–(3.8), Lemma 2.1 and the Sobolev inequality, we have

$$\begin{aligned}J(u) &= \frac{1}{2}\|u\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2dx-\frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^pdx-\int_{\mathbb{R}^3}F(x,u)dx\\ &\ge \frac{1}{4}\|u\|^2-\frac{C_2}{2p}\|u\|^{2p}-\frac{c(\varepsilon)S_s^s}{s}\|u\|^s.\end{aligned} \tag{3.9}$$

From (3.9) and $\frac{3+\alpha}{3}<p<3+\alpha, 2<s<6$, one can easily see that there exist $\rho,\gamma>0$ such that $J(u)\big|_{\|u\|=\rho}\ge\gamma>0$.

(ii) Choosing a function $v\in H\setminus\{0\}$, then by Lemma 2.1, (3.3), Fatou's lemma and the fact $\int_{\mathbb{R}^3}(I_\alpha * |v|^p)|v|^pdx\ge 0$, one has

$$\begin{aligned}J(tv) &\le \frac{t^2}{2}\|v\|^2+\frac{t^2\omega^2}{2}\|v\|_2^2-\frac{t^{2p}}{2p}\int_{\mathbb{R}^3}(I_\alpha * |v|^p)|v|^pdx-\int_{\mathbb{R}^3}F(x,tv)dx\\ &\le \frac{t^2}{2}\|v\|^2+\frac{t^2\omega^2}{2}\|v\|_2^2+C_1t^2\|v\|_2^2-\frac{t^{2p}}{2p}\int_{\mathbb{R}^3}(I_\alpha * |v|^p)|v|^pdx\\ &\to -\infty, \qquad t\to+\infty.\end{aligned}$$

Thus, taking $e=t_0v$ with $t_0>0$ large, we have $\|e\|>\rho$ and $J(e)<0$. □

Lemma 3.3. *Assume that* (V) *and* (F_1)–(F_4) *hold,* $0<\alpha<3$, $\frac{3+\alpha}{3}<p<3+\alpha$, *then* *(i) There exist $\bar{\gamma}, \bar{\rho} > 0$ such that $J|_{\partial B_{\bar{\rho}} \cap Z_k} \geq \bar{\gamma} > 0$;*

(ii) for any finite dimensional subspace $\widetilde{H} \subset H$, there exists $R = R(\widetilde{H}) > 0$ such that $J|_{\widetilde{H} \setminus B_R} < 0$.

Proof. From (3.9), we can deduce that (i) holds. Proof by contradiction is now used to obtain (ii). We assume that there exists a sequence $\{u_n\} \subset \widetilde{H}$ such that $\|u_n\| \to +\infty$ and $J(u_n) \geq 0$ for any $n \in \mathbb{N}$. Let $v_n = u_n / \|u_n\|$, then $\|v_n\| = 1$. So, we may assume that, up to a subsequence, $v_n \rightharpoonup v$ in H. Because $\widetilde{H}$ is finite dimensional subspace, then $v_n \to v \in \widetilde{H}$ and $\|v\| = 1$. Set $\Omega = \{y \in \mathbb{R}^3 : v(y) \neq 0\}$, then $|u_n| = |v_n| \|u_n\| \to +\infty$ a.e. $x \in \Omega$ $(n \to \infty)$. Thus, it follows from $J(u_n) \geq 0$, $\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx \geq 0$, (3.3), Fatou's lemma, (F_3) and Lemma 2.1 that

$$\begin{aligned}
0 \leq & \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|^2} \\
\leq & \frac{(\omega^2 + 2C_1)S_2^2 + 1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x,u_n) + C_1|u_n|^2}{|u_n|^2} |v_n|^2 dx \\
\leq & \frac{(\omega^2 + 2C_1)S_2^2 + 1}{2} - \int_{\mathbb{R}^3} \liminf_{n\to\infty} \frac{F(x,u_n) + C_1|u_n|^2}{|u_n|^2} |v_n|^2 dx \\
= & -\infty.
\end{aligned} \tag{3.10}$$

which is a contradiction. This shows that (ii) holds. □

Proof of Theorem 1.1. By Lemma 3.1 and Lemma 3.2, all conditions of the mountain pass theorem (see [28, Theorem 2.2]) are satisfied. Thus, the system (1.1) has a nontrivial solution (u, ϕ_u) satisfying

$$\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2)dx - \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx - \int_{\mathbb{R}^3} F(x,u)dx > 0.$$

Proof of Theorem 1.2. By Lemma 3.1 and Lemma 3.3, all conditions of Z_2-version of the mountain pass theorem (see [28, Theorem 9.12]) are satisfied. Thus, the the system (1.1) possesses infinitely many nontrivial solutions $\{(u_n, \phi_{u_n})\}$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 - \omega\phi_{u_n} u_n^2)dx - \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx - \int_{\mathbb{R}^3} F(x,u_n)dx \to +\infty$$

as $n \to +\infty$.

4 Proof of Theorem 1.3

In this section, we will prove that the system (1.2) possesses a weak solution for $1 < \alpha < 2$, $\alpha + 1 \leq p < 3$ and $\lambda > 0$ small. Obviously, the functional of system (1.2) is not well defined unless $q = 6$ and then we cannot apply variational methods directly. To overcome this difficulty, we define the following truncation function inspired by reference [20].

For any given $M > 0$, let

$$\psi(t) = \begin{cases} |t|^{q-2}t, & |t| \leq M, \\ M^{q-2p}|t|^{2p-2}t, & |t| > M. \end{cases}$$

Then $\psi(t)$ admits the following properties:

Lemma 4.1. *For any given $M > 0$, then*

(i) $\psi(t) \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $|\psi(t)| \leq M^{q-2p}|t|^{2p-1}$, $\forall t \in \mathbb{R}$;

(ii) $\lim_{|t|\to 0} \frac{\psi(t)}{t} = 0$;

(iii) $\lim_{|t|\to +\infty} \frac{\Psi(t)}{t^2} = +\infty$, *where* $\Psi(t) = \int_0^t \psi(s)ds$;

(iv) $\psi(t)t \geq 2p\Psi(t) \geq 0$, $\forall t \in \mathbb{R}$.

Now, we consider the revised Klein–Gordon–Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = \lambda\psi(u) + (I_\alpha * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{4.1}$$

As discussed above, one can define the functional of system (4.1):

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right)dx - \frac{1}{2p}(I_\alpha * |u|^p)|u|^{p-2}u - \lambda\int_{\mathbb{R}^3} \Psi(u)dx, \quad \forall u \in H.$$

By Lemma 2.1, Lemma 2.2 and Lemma 4.1, J_λ is well defined in H, $J_\lambda \in C^1(H, \mathbb{R})$, and for all $u, v \in H$,

$$\begin{aligned} \langle J_\lambda'(u), v\rangle = &\int_{\mathbb{R}^3} \left[\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv\right]dx \\ &- \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^{p-2}uv dx - \lambda\int_{\mathbb{R}^3} \psi(u)v dx. \end{aligned}$$

Clearly, if $u \in H$ is a critical point of J_λ with $\|u\|_\infty \leq M$, then (u, ϕ_u) is a weak solution of system (1.2).

Using Lemma 4.1, $\lambda\psi(u)$ verifies the conditions (F_1)–(F_4). Hence, by Theorem 1.1, $\alpha \in (1,2) \subset (0,3)$ and $p \in [\alpha+1, 3) \subset \left(\frac{3+\alpha}{3}, 3+\alpha\right)$, we can obtain that the system (4.1) has a nontrivial solution $(u_\lambda, \phi_{u_\lambda})$ such that

$$J_\lambda'(u_\lambda) = 0, \qquad J_\lambda(u_\lambda) = c_\lambda,$$

where $c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t))$, $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}$.

Lemma 4.2. *Assume that (V) holds, $1 < \alpha < 2$, $\alpha + 1 \leq p < 3$, $q \geq 6$, then there exists a constant $C_3 > 0$ independent of λ, M such that $C_3 \geq \|u_\lambda\|$.*

Proof. Choosing a function $\xi \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$, then $\Psi(t\xi) \geq 0$ for any $t \geq 0$. By Lemma 2.1, for any $t \geq 0$, one has

$$\begin{aligned} \lim_{t\to+\infty} \frac{J_\lambda(t\xi)}{t^2} &\leq \frac{1}{2}\|\xi\|^2 - \lim_{t\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_{t\xi}\xi^2 dx - \lim_{t\to+\infty} \frac{t^{2p-2}}{2p}\int_{\mathbb{R}^3} (I_\alpha * |\xi|^p)|\xi|^p dx \\ &\leq \frac{1}{2}\|\xi\|^2 + \frac{\omega^2}{2}\|\xi\|_2^2 - \lim_{t\to+\infty} \frac{t^{2p-2}}{2p}\int_{\mathbb{R}^3} (I_\alpha * |\xi|^p)|\xi|^p dx = -\infty, \end{aligned}$$

which means that there exists $t_0 > 0$ independent of λ, M large enough such that $J_\lambda(t_0\xi) < 0$. Set $\gamma(t) = tt_0\xi, \quad t \in [0,1]$, then $\gamma(t) \in \Gamma$ and $\Psi(t_0 t\xi) \geq 0$. Therefore

$$\begin{aligned} c_\lambda &\leq \max_{t\in[0,1]} J_\lambda(\gamma(t)) \\ &\leq \max_{t\in[0,1]} \left(\frac{t^2}{2}\|t_0\xi\|^2 + \frac{t^2\omega^2}{2}\|t_0\xi\|_2^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^3}(I_\alpha * |t_0\xi|^p)|t_0\xi|^p dx \right) \\ &\leq \max_{t\geq 0} \left(\frac{t^2}{2}\|\xi\|^2 + \frac{t^2\omega^2}{2}\|\xi\|_2^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^3}(I_\alpha * |\xi|^p)|\xi|^p dx \right) =: A, \end{aligned}$$

where $A > 0$ is a constant independent of λ, M. Because $J_\lambda(u_\lambda) = c_\lambda, \quad \langle J_\lambda'(u_\lambda), u_\lambda\rangle = 0$, we have by Lemma 4.1 that

$$2pA \geq 2pc_\lambda = 2pJ_\lambda(u_\lambda) - \langle J_\lambda'(u_\lambda), u_\lambda\rangle \geq (p-1)\|u_\lambda\|^2.$$

Thus, there exists a constant $C_3 = \sqrt{\frac{2pA}{p-1}} > 0$ independent of λ, M such that $C_3 \geq \|u_\lambda\|$. □

Now we use Moser iteration to estimate $\|u_\lambda\|_\infty$.

Lemma 4.3. *Assume that (V) holds, $1 < \alpha < 2$, $\alpha + 1 \leq p < 3$, $q \geq 6$. Then there exist constants $C, C_4 > 0$ independent of λ, M such that*

$$\|u\|_\infty \leq C\left[\lambda M^{q-2p}\|u\|_6^{2p-2} + C_4\|u\|^{p-2}_{\frac{6(p-2)}{2p+\alpha-5}}\right]^{\frac{1}{6-2p}}\|u\|_6,$$

where (u, ϕ_u) is weak solution of system (4.1).

Proof. For any $m \in \mathbb{N}, \beta > 1$, set $A_m = \{x \in \mathbb{R}^3 : |u|^{\beta-1} \leq m\}$, $B_m = \{x \in \mathbb{R}^3 : |u|^{\beta-1} > m\}$. Define two sequences as follows

$$u_m = \begin{cases} u|u|^{2(\beta-1)}, & x \in A_m, \\ m^2 u, & x \in B_m; \end{cases} \qquad \eta_m = \begin{cases} u|u|^{(\beta-1)}, & x \in A_m, \\ mu, & x \in B_m. \end{cases}$$

Using the same computing to Lemma 3.2 in [32], one has

$$\int_{\mathbb{R}^3}|\nabla\eta_m|^2 dx \leq \beta^2\left(\lambda\int_{\mathbb{R}^3}\psi(u)u_m dx + \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}uu_m dx\right). \tag{4.2}$$

Using Lemma 4.1 and $\eta_m^2 = uu_m$, one has

$$|\lambda\psi(u)u_m| \leq \lambda M^{q-2p}|u|^{2p-2}\eta_m^2. \tag{4.3}$$

From (4.2) and (4.3), one has

$$\int_{\mathbb{R}^3}|\nabla\eta_m|^2 dx \leq \beta^2\lambda M^{q-2p}\int_{\mathbb{R}^3}|u|^{2p-2}\eta_m^2 dx + \beta^2\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}uu_m dx. \tag{4.4}$$

The Hölder inequality together with (3.6) implies that

$$\begin{aligned} \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}uu_m dx &= \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}\eta_m^2 dx \\ &\leq \|I_\alpha * |u|^p\|_{\frac{6}{3-\alpha}}\||u|^{p-2}\eta_m^2\|_{\frac{6}{3+\alpha}} \\ &\leq C_4\|\eta_m\|_{2\gamma}^2\|u\|^{p-2}_{\frac{6(p-2)}{2p+\alpha-5}}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}|u|^{2p-2}\eta_m^2dx &\leq \left(\int_{\mathbb{R}^3}(|u|^{(2p-2)})^{\frac{3}{p-1}}dx\right)^{\frac{p-1}{3}}\left(\int_{\mathbb{R}^3}(\eta_m^2)^{\frac{3}{4-p}}dx\right)^{\frac{4-p}{3}} \\ &= \|u\|_6^{2p-2}\|\eta_m\|_{2\gamma}^2,\end{aligned}\tag{4.6}$$

where $\gamma=\frac{3}{4-p}$, $C_4=\|I_\alpha*|u|^p\|_{\frac{6}{3-\alpha}}$ independent of λ, M. Because of $1<\alpha<2, \alpha+1\leq p<3$, we have $\frac{3}{2}<\gamma<3$, $2p+\alpha-5>0$ and $2\leq\frac{6(p-2)}{2p+\alpha-5}<6$. By (4.4)–(4.6) and the definition of η_m, one has

$$\begin{aligned}\left(\int_{A_m}|u|^{6\beta}dx\right)^{\frac{1}{3}} &= \left(\int_{A_m}|\eta_m|^6dx\right)^{\frac{1}{3}} \\ &\leq S^{-1}\int_{\mathbb{R}^3}|\nabla\eta_m|^2dx \\ &\leq S^{-1}\beta^2\|\eta_m\|_{2\gamma}^2\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right].\end{aligned}\tag{4.7}$$

Letting $m\to+\infty$ in (4.7), one has

$$\left(\int_{\mathbb{R}^3}|u|^{6\beta}dx\right)^{\frac{1}{3}}\leq S^{-1}\beta^2\|u\|_{2\beta\gamma}^{2\beta}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right],$$

i.e.

$$\|u\|_{6\beta}\leq C_5^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right]^{\frac{1}{2\beta}}\|u\|_{2\beta\gamma},\tag{4.8}$$

where $C_5=\max\{S^{-1},1\}$. In the subsequent step, the technique of Moser iteration is utilized to verify that the conclusion is indeed valid. Define $\sigma=\frac{3}{\gamma}$, then $1<\sigma<2$. Taking $\beta=\sigma$ in (4.8), one has

$$\|u\|_{6\sigma}\leq C_5^{\frac{1}{2\sigma}}\sigma^{\frac{1}{\sigma}}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right]^{\frac{1}{2\sigma}}\|u\|_6.\tag{4.9}$$

Taking $\beta=\sigma^2$ in (4.8), one has

$$\|u\|_{6\sigma^2}\leq C_5^{\frac{1}{2\sigma^2}}\sigma^{\frac{2}{\sigma^2}}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right]^{\frac{1}{2\sigma^2}}\|u\|_{6\sigma}.$$

Combing this with(4.9), one has

$$\|u\|_{6\sigma^2}\leq C_5^{\frac{1}{2\sigma^2}+\frac{1}{2\sigma}}\sigma^{\frac{2}{\sigma^2}+\frac{1}{\sigma}}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right]^{\frac{1}{2\sigma^2}+\frac{1}{2\sigma}}\|u\|_6.$$

Taking $\beta=\sigma^n$ $(n=1,2,\dots)$ in (4.8) and using Moser iteration, we get

$$\|u\|_{6\sigma^n}\leq C_5^{\sum\limits_{i=1}^{n}\frac{1}{2\sigma^i}}\sigma^{\sum\limits_{i=1}^{n}\frac{i}{\sigma^i}}\left[\lambda M^{q-2p}\|u\|_6^{2p-2}+C_4\|u\|_{\frac{6(p-2)}{2p+\alpha-5}}^{p-2}\right]^{\sum\limits_{i=1}^{n}\frac{1}{2\sigma^i}}\|u\|_6.\tag{4.10}$$

Because of $\sigma>1$, the convergence of the following two series is easily established and

$$\sum_{i=1}^{\infty}\frac{1}{2\sigma^i}=\frac{1}{2(\sigma-1)},\qquad\sum_{i=1}^{\infty}\frac{i}{\sigma^i}=\frac{\sigma}{(\sigma-1)^2}.\tag{4.11}$$

Letting $n \to +\infty$, we have from (4.10) and (4.11) that

$$\|u\|_\infty \le C\left[\lambda M^{q-2p}\|u\|_6^{2p-2} + C_4\|u\|^{p-2}_{\frac{6(p-2)}{2p+\alpha-5}}\right]^{\frac{1}{6-2p}}\|u\|_6,$$

where $C = C_5^{\frac{1}{2(\sigma-1)}}\sigma^{\frac{\sigma}{(\sigma-1)^2}}$ independent of λ, M. □

Proof of Theorem 1.3. We only need to prove $\|u_\lambda\|_\infty \le M$. By Lemma 4.2, $C_3 \ge \|u_\lambda\|^2$, and then there exist two constants $C_6, C_7 > 0$ independent of λ, M such that $\|u_\lambda\|^{p-2}_{\frac{6(p-2)}{2p+\alpha-5}} \le C_6$, $\|u_\lambda\|_6 \le C_7$. From Lemma 4.3, one has

$$\begin{aligned}\|u_\lambda\|_\infty &\le C\left[\lambda M^{q-2p}\|u\|_6^{2p-2} + C_4\|u\|^{p-2}_{\frac{6(p-2)}{2p+\alpha-5}}\right]^{\frac{1}{6-2p}}\|u\|_6\\ &\le C_8\left[\lambda M^{q-2p}C_7^{2p-2} + C_4C_6\right]^{\frac{1}{6-2p}},\end{aligned} \tag{4.12}$$

where $C_8 = CC_7$ independent of λ, M. Because of $C_4, C_6, C_8 > 0$ independent of λ, M, by taking $M > 0$ sufficiently large, we ensure that $C_8 \le \sqrt{M}$, $C_4C_6 \le \frac{M^{3-p}}{2}$. Observe that for the value M selected above, M^{q-2p} is a fixed constant and $C_7 > 0$ independent of λ, M. Consequently, it is possible to choose $\lambda_* > 0$ sufficiently small, ensuring the inequality $\lambda_* M^{q-2p}C_7^{2p-2} \le \frac{M^{3-p}}{2}$ holds. Therefore

$$\left[\lambda M^{q-2p}C_7^{2p-2} + C_4C_6\right]^{\frac{1}{6-2p}} \le \sqrt{M},$$

for $\lambda < \lambda_*$. For $\lambda < \lambda_*$, the combination of this result with (4.12) gives $\|u_\lambda\|_\infty \le C_8\sqrt{M} \le M$, i.e., $\{(u_\lambda, \phi_{u_\lambda})\}$ is a nontrivial solution of system (1.2). The proof is completed.

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