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Sign-changing solution for a logarithmic Kirchhoff type problem

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Abstract. This paper is devoted to study the following logarithmic Kirchhoff type problem

$$\left(a+b\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)dx\right)(-\Delta u+V(x)u)=|u|^{p-2}u\log|u|,$$

where $a,b>0$, $2<p<4$. Under some assumptions on the potential function $V(x)$, we prove the existence of a sign-changing solution to this problem for $b>0$ small enough by using the truncated technique and constraint variational method.

Keywords: Kirchhoff type problem, logarithmic nonlinearity, sign-changing solution, truncated technique, variational method.

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1 Introduction

In this paper, we discuss about the existence of a sign-changing solution for the following logarithmic Kirchhoff type problem

$$\left(a+b\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)dx\right)(-\Delta u+V(x)u)=|u|^{p-2}u\log|u|,\quad x\in\mathbb{R}^3, \tag{1.1}$$

where $a,b>0$, $2<p<4$. Moreover, the potential function $V(x)$ satisfies

(V_1) $V(x)$ is continuous and $\lim_{|x|\to\infty}V(x)=+\infty$;

(V_2) There exists a constant V_0 such that $\inf_{x\in\mathbb{R}^3}V(x)\geq V_0>0$.

Problem (1.1) has a strong physical background. In fact, as a special case, the following Kirchhoff type problem

$$\left(a+b\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)dx\right)(-\Delta u+V(x)u)=f(x,u),\quad x\in\mathbb{R}^3, \tag{1.2}$$

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is closely related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial y} \right|^2 dy \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which was proposed by Kirchhoff in [8] to describe the transversal oscillations of a stretched string, where ρ is the mass density, P_0 is the initial tension, h represents the cross-sectional area, E is the Young's modulus of the material and L is the length of the string. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations, so the nonlocal term appears. We want to point out that such nonlocal problems also appear in other fields, such as population density, model suspension bridges and biological systems. For more physical background on Kirchhoff type problems, as well as studies on elliptic equations, wavelet analysis and its generalization to fractional calculus, we refer the readers to [1, 2, 5, 6, 9, 13, 15, 16] and the references therein. After Lions in [10] established an abstract functional analysis framework, Kirchhoff type problems have received much attention. In particular, the existence and multiplicity of sign-changing solutions for Kirchhoff type problems similar to problem (1.2) are established, see for example [7, 11, 17–21, 23, 24] and the references therein.

Another interesting aspect of problem (1.1) is the presence of a logarithmic nonlinearity. In the past years, there have been increasing interests in studying logarithmic nonlinearity due to its relevance in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Instein consideration (see [25] and the references therein). Suppose the potential function $V(x)$ satisfies (V_1) and (V_2), Gao, Jing and Liu et al. [4] proved that problem (1.1) has only trivial solution for large $b > 0$ and two positive solutions for small $b > 0$ and $2 < p < 4$. Subsequently, for the case $4 < p < 6$, the existence of positive ground state solutions, ground state sign-changing solutions and sequence of solutions for problem (1.1) are obtained by Yang and Liao in [22].

To the best of our knowledge, there are very few results for the existence of sign-changing solutions of problem (1.1) for $2 < p < 4$. Recently, we only know that Fan, Squassina and Zhang [3] have considered the following Kirchhoff type problem

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^3, \tag{1.3}$$

where $2 < p < 4$. Under the condition that the potential function $V(x)$ is radial and satisfies (V_2), they obtained the existence of multiple radial sign-changing solutions for problem (1.3) with $3 < p < 4$ and $b > 0$ small enough. Inspired by the above literature, we are interested in the existence of sign-changing solutions of problem (1.1) with $2 < p < 4$.

The energy functional corresponding to problem (1.1) is defined by

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \ln |u| dx + \frac{1}{p^2} \int_{\mathbb{R}^3} |u|^p dx, \quad u \in X,$$

with $J(0) = 0$, where Sobolev space X is defined as follows

$$X := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \right\},$$

endowed with the inner product

$$\langle u, v \rangle := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx, \quad \forall u, v \in X$$

and endowed with the norm

$$\|u\|^2 := \langle u, u\rangle = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx.$$

Denote $|u|_k = (\int_{\mathbb{R}^3} |u|^k dx)^{1/k}$ the norm of $u \in L^k(\mathbb{R}^3)$ for $k > 1$, the $C, C_1, C_2, \dots$ represent several different positive constants. For fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t^{p-1} \ln |t|| < \varepsilon |t| + C_\varepsilon |t|^{q-1}, \quad t > 0, \tag{1.4}$$

for $2 < p < q < 4$. Define

$$S_k = \inf_{u \in X\setminus\{0\}} \frac{\|u\|^k}{|u|_k^k}, \quad k \in [2, 6].$$

From Lemma 2.1 in [4], we have

$$|u|_k^k \le S_k^{-1} \|u\|^k, \quad \forall u \in X. \tag{1.5}$$

By (V_1), (V_2) and $2 < p < 4$, we deduce from (1.4) that $J \in C^1(X, \mathbb{R})$ and the Fréchet derivative of J is given by

$$\langle J'(u), v\rangle = \left(a + b\|u\|^2\right) \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\, dx - \int_{\mathbb{R}^3} |u|^{p-2} uv \ln |u| dx, \quad \text{for all } u, v \in X.$$

Obviously, the weak solutions to problem (1.1) are the critical points of J in X. Moreover, we call $u \in X$ is a sign-changing solution to problem (1.1) if u is a solution of problem (1.1) with $u^\pm \neq 0$, where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

Now we state the main results.

Theorem 1.1. *Assume that (V_1)–(V_2) hold and $2 < p < 4$, then problem (1.1) has at least a sign-changing solution for $b > 0$ small enough.*

Remark 1.2. For $p \in (2, 4)$, it seems to be very difficult to show the boundedness of $(PS)_c$ sequence of J, which is a key point in the proof of [7] and [22].

2 The proof of the main results

In order to overcome the difficulty mentioned in Remark 1.2, motivated by [14], we use a cut-off function $\phi \in C^\infty([0, +\infty), \mathbb{R})$ satisfying

$$\begin{cases} \phi(t) = 1, & t \le 1, \\ \phi(t) = 0, & t > 2, \\ -2 \le \phi'(t) \le 0, & t \ge 0, \end{cases}$$

and consider the following truncated functional

$$J_T(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\Phi_T(u)\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p |\ln |u| dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |u|^p dx, \quad u \in X,$$

with $J_T(0) = 0$, where for any $T > 0$,

$$\Phi_T(u) = \phi\left(\frac{\|u\|^2}{T^2}\right).$$

It follows that $J_T(u)$ is well defined and $J_T \in C^1(X, \mathbb{R})$. Moreover, for any $u, v \in X$, we have

$$\begin{aligned}\langle J_T'(u), v\rangle = &\left(a + b\Phi_T(u)\|u\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4\right)\int_{\mathbb{R}^3}(\nabla u \nabla v + V(x)uv)dx \\ &- \int_{\mathbb{R}^3}|u|^{p-2}uv\ln|u|dx.\end{aligned} \tag{2.1}$$

It is easy to see that if u is a critical point of J_T satisfying $\|u\| < T$, then u is also a critical point of J. To prove Theorem 1.1, we only need to find a minimizer u_* of J_T such that $u_* \in \mathcal{M}_T, J_T(u_*) = m_T = \inf\{J_T(u) : u \in \mathcal{M}_T\}$, and then derive $\|u_*\| < T$, where

$$\mathcal{M}_T = \left\{u \in X : u^{\pm} \neq 0,\ \langle J_T'(u), u^{\pm}\rangle = 0\right\}.$$

Now we present some lemmas which are needed to prove our main results.

Lemma 2.1. *For any $u \in X$ with $u^{\pm} \neq 0$, there exists a positive numbers pair (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}_T$.*

Proof. For any $u \in X$ with $u^{\pm} \neq 0$, let

$$g(s,t) = \langle J_T'(su^+ + tu^-),\ su^+\rangle, h(s,t) = \langle J_T'(su^+ + tu^-),\ tu^-\rangle.$$

From (2.1), one has

$$\begin{aligned}g(s,t) = &\Bigg(a + b\Phi_T(su^+ + tu^-)\|su^+ + tu^-\|^2 \\ &+ \frac{b}{2T^2}\phi'\Big(\frac{\|su^+ + tu^-\|^2}{T^2}\Big)\|su^+ + tu^-\|^4\Bigg)\|su^+\|^2 - \int_{\mathbb{R}^3}|su^+|^p\ln|su^+|dx;\end{aligned} \tag{2.2}$$

$$\begin{aligned}h(s,t) = &\Bigg(a + b\Phi_T(su^+ + tu^-)\|su^+ + tu^-\|^2 \\ &+ \frac{b}{2T^2}\phi'\Big(\frac{\|su^+ + tu^-\|^2}{T^2}\Big)\|su^+ + tu^-\|^4\Bigg)\|tu^-\|^2 - \int_{\mathbb{R}^3}|tu^-|^p\ln|tu^-|dx.\end{aligned} \tag{2.3}$$

Let $t = s$ in (2.2) and (2.3),

$$\begin{aligned}g(s,s) = &\left(a + b\Phi_T(su)\|su\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|su\|^2}{T^2}\right)\|su\|^4\right)\|su^+\|^2 - \int_{\mathbb{R}^3}|su^+|^p\ln|su^+|dx \\ = &\, as^2\|u^+\|^2 + bs^4\Phi_T(su)\left(\|u^+\|^4 + \|u^+\|^2\|u^-\|^2\right) \\ &+ \frac{b}{2T^2}s^6\phi'\left(\frac{\|su\|^2}{T^2}\right)\left(\|u^+\|^6 + 2\|u^+\|^4\|u^-\|^2 + \|u^+\|^2\|u^-\|^4\right) \\ &- s^p\int_{\mathbb{R}^3}|u^+|^p\ln|u^+|\,dx - s^p\ln s\int_{\mathbb{R}^3}|u^+|^p\,dx,\end{aligned}$$

$$\begin{aligned}h(t,t) = &\left(a + b\Phi_T(tu)\|tu\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|tu\|^2}{T^2}\right)\|tu\|^4\right)\|tu^-\|^2 - \int_{\mathbb{R}^3}|tu^-|^p\ln|tu^-|dx \\ = &\, at^2\|u^-\|^2 + bt^4\Phi_T(tu)\left(\|u^-\|^4 + \|u^-\|^2\|u^+\|^2\right) \\ &+ \frac{b}{2T^2}t^6\phi'\left(\frac{\|tu\|^2}{T^2}\right)\left(\|u^-\|^6 + 2\|u^-\|^4\|u^+\|^2 + \|u^-\|^2\|u^+\|^4\right) \\ &- t^p\int_{\mathbb{R}^3}|u^-|^p\ln|u^-|\,dx - t^p\ln t\int_{\mathbb{R}^3}|u^-|^p\,dx.\end{aligned}$$

Note that for large $s, t > 0$ and $2 < p < 4$, one has

$$\Phi_T(su) = \phi'\left(\frac{\|su\|^2}{T^2}\right) = 0, \quad \Phi_T(tu) = \phi'\left(\frac{\|tu\|^2}{T^2}\right) = 0,$$

we can conclude that $\lim_{s\to 0^+} g(s,s) = 0$ and $\lim_{s\to+\infty} g(s,s) = -\infty$. Similarly, we have $\lim_{t\to 0^+} h(t,t) = 0$ and $\lim_{t\to+\infty} h(t,t) = -\infty$. Therefore, there exists $0 < r < R$ such that

$$\begin{aligned} g(r,r) > 0, \quad & h(r,r) > 0; \\ g(R,R) < 0, \quad & h(R,R) < 0. \end{aligned}$$

It follows that

$$\begin{aligned} g(r,t) > 0, \quad g(R,t) < 0, \quad \forall t \in [r,R], \\ h(s,r) > 0, \quad h(s,R) < 0, \quad \forall s \in [r,R]. \end{aligned}$$

Based on Miranda's Theorem [12], there exist $r < s_u, t_u < R$ such that $g(s_u,t_u) = h(s_u,t_u) = 0$, which implies that $s_u u^+ + t_u u^- \in \mathcal{M}_T$. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *The functional J_T is bounded from below and coercive on $\mathcal{M}_T$.*

Proof. For $u \in \mathcal{M}_T$, by the definition of ϕ, we obtain

$$\begin{aligned} J_T(u) &= J_T(u) - \frac{1}{p}\langle J_T'(u), u\rangle \\ &= \left(\frac{a}{2} - \frac{a}{p}\right)\|u\|^2 - \left(\frac{b}{p} - \frac{b}{4}\right)\Phi_T(u)\|u\|^4 - \frac{b}{2pT^2}\phi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^6 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u|^p dx \\ &\geq \left(\frac{a}{2} - \frac{a}{p}\right)\|u\|^2 - \left(\frac{b}{p} - \frac{b}{4}\right)\Phi_T(u)\|u\|^4 \\ &\geq \left(\frac{a}{2} - \frac{a}{p}\right)\|u\|^2 - \left(\frac{b}{p} - \frac{b}{4}\right)4T^4. \end{aligned} \tag{2.4}$$

Therefore, we have J_T is bounded from below and coercive on $\mathcal{M}_T$. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let $b \in \left(0, \frac{a}{16T^2}\right)$. Then there exists a constant $\mu > 0$ such that $\|u\| \geq \mu > 0$ for all $u \in \mathcal{M}_T$.*

Proof. Assume the conclusion is invalid, there exists a sequence $\{u_n\} \subset \mathcal{M}_T$ such that

$$\|u_n^+\| \to 0 \quad \text{or} \quad \|u_n^-\| \to 0, \quad \text{as } n \to \infty. \tag{2.5}$$

According to $u_n \in \mathcal{M}_T$, we deduce from (1.4) and (1.5) that

$$\begin{aligned} a\|u_n^\pm\|^2 + b\Phi_T(u_n)\|u_n\|^2\|u_n^\pm\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\|u_n^\pm\|^2 \\ = \int_{\mathbb{R}^3}|u_n^\pm|^p \ln|u_n^\pm| dx \\ \leq \varepsilon S_2^{-1}\|u_n^\pm\|^2 + C_\varepsilon S_q^{-1}\|u_n^\pm\|^q. \end{aligned} \tag{2.6}$$

Notice that $\phi(t) = 0$ for $t > 2$ and $-2 \le \phi'(t) \le 0$ for $|t| \le 2$, let $\varepsilon = \frac{a}{2} S_2$, for $b \in \left(0, \frac{a}{16T^2}\right)$, it follows from (2.5) and (2.6) that

$$0 < \frac{a}{4} \le a + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4 \le S_q^{-1} C_\varepsilon \|u_n^{\pm}\|^{q-2} \to 0, \tag{2.7}$$

which is impossible. Therefore, the conclusion is valid. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Let $b \in \left(0, \frac{a}{16T^2}\right)$. Then J_T satisfies the $(PS)_{m_T}$ condition.*

Proof. Let $\{u_n\} \subset X$ be a $(PS)_{m_T}$ sequence for J_T, that is

$$J_T(u_n) \to m_T \quad \text{and} \quad J_T'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Similar to (2.4), it is easy to get that $\{u_n\}$ is bounded in X. Up to a subsequence, we may assume that for some $u \in X$,

$$\begin{cases} u_n \rightharpoonup u, & \text{in } X, \\ u_n \to u, & \text{in } L^k(\mathbb{R}^3), \quad 2 \le k < 6, \\ u_n \to u, & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.9}$$

It follows from (2.1) that

$$\begin{aligned} &\left(a + b\Phi_T(u_n)\|u_n\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)\int_{\mathbb{R}^3}(\nabla u_n \nabla(u_n - u) + V(x)u_n(u_n - u))\,dx \\ &\quad = \int_{\mathbb{R}^3}|u_n|^{p-2}u_n(u_n - u)\ln|u_n|dx + \langle J_T'(u_n), u_n - u\rangle. \end{aligned} \tag{2.10}$$

By the Hölder inequality, (1.4) and (2.9), we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3}|u_n|^{p-2}u_n(u_n - u)\ln|u_n|dx\right| &\le \int_{\mathbb{R}^3}\left(\varepsilon|u_n| + C_\varepsilon|u_n|^{q-1}\right)|u_n - u|dx \\ &\le \varepsilon\,|u_n|_2\,|u_n - u|_2 + C_\varepsilon\,|u_n|_q^q\,|u_n - u|_q = o(1). \end{aligned} \tag{2.11}$$

Moreover, for $b \in \left(0, \frac{a}{16T^2}\right)$, it follows from (2.7) that

$$\begin{aligned} a + b\Phi_T(u_n)\|u_n\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4 &\ge a + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4 \\ &\ge \frac{a}{4} > 0. \end{aligned} \tag{2.12}$$

From (2.8), (2.10)–(2.12), we obtain

$$\int_{\mathbb{R}^3}(\nabla u_n \nabla(u_n - u) + V(x)u_n(u_n - u))\,dx \to 0,$$

which yields $\|u_n\| \to \|u\|$. Together with $u_n \rightharpoonup u$ in X implies that $u_n \to u$ in X. The proof of Lemma 2.4 is complete. □

Lemma 2.5. *There exist $b_0 > 0$ and $u_* \in \mathcal{M}_T$ such that $J_T(u_*) = m_T$ and u_* is a sign-changing critical point of J_T.*

Proof. In view of Lemmas 2.2 and 2.3, for $b \in (0, \frac{a}{16T^2})$, we can use Ekeland's variational principle to obtain a minimizing sequence $\{u_n\} \subset \mathcal{M}_T$ satisfying

$$m_T \leq J_T(u_n) < m_T + \frac{1}{n},$$

and

$$J_T(v) \geq J_T(u_n) - \frac{1}{n}\|u_n - v\|, \quad \text{for any } v \in \mathcal{M}_T. \tag{2.13}$$

It then follows that $\{u_n\}$ is bounded in X. For any fixed $\varphi \in C^\infty(\mathbb{R}^3)$ and each $n \in \mathbb{N}$, we study the functions $g_n^\pm \in C^1(\mathbb{R}^3, \mathbb{R})$ defined by

$$\begin{aligned}
&g_n^\pm(r,k,l)\\
&\quad = a\|(u_n + r\varphi + ku_n^+ + lu_n^-)^\pm\|^2\\
&\quad\quad - \int_{\mathbb{R}^3} |(u_n + r\varphi + ku_n^+ + lu_n^-)^\pm|^p \ln|(u_n + r\varphi + ku_n^+ + lu_n^-)^\pm| dx\\
&\quad\quad + b\Phi_T(u_n + r\varphi + ku_n^+ + lu_n^-)\|u_n + r\varphi + ku_n^+ + lu_n^-\|^2\|(u_n + r\varphi + ku_n^+ + lu_n^-)^\pm\|^2\\
&\quad\quad + \frac{b}{2T^2}\phi'\left(\frac{\|u_n + r\varphi + ku_n^+ + lu_n^-\|^2}{T^2}\right)\|(u_n + r\varphi + ku_n^+ + lu_n^-)^\pm\|^2\|u_n + r\varphi + ku_n^+ + lu_n^-\|^4.
\end{aligned}$$

Since $\langle J_T'(u_n), u_n^\pm\rangle = 0$, we have $g_n^\pm(0,0,0) = 0$, which implies that

$$a\|u_n^\pm\|^2 = \int_{\mathbb{R}^3} |u_n^\pm|^p \ln|u_n^\pm|\, dx - b\Phi_T(u_n)\|u_n\|^2\|u_n^\pm\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\|u_n^\pm\|^2. \tag{2.14}$$

After direct calculation, it can be inferred from (2.14) that

$$\begin{aligned}
\frac{\partial g_n^+(0,0,0)}{\partial k} &= 2b\Phi_T(u_n)A_{n,1}^2 + \frac{4b}{T^2}\phi'\left(\frac{A_n}{T^2}\right)A_nA_{n,1}^2 + \frac{b}{T^4}\phi''\left(\frac{A_n}{T^2}\right)A_n^2A_{n,1}^2\\
&\quad - (p-2)\int_{\mathbb{R}^3} |u_n^+|^p \ln|u_n^+|\, dx - \int_{\mathbb{R}^3} |u_n^+|^p\, dx,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\frac{\partial g_n^-(0,0,0)}{\partial l} &= 2b\Phi_T(u_n)A_{n,2}^2 + \frac{4b}{T^2}\phi'\left(\frac{A_n}{T^2}\right)A_nA_{n,2}^2 + \frac{b}{T^4}\phi''\left(\frac{A_n}{T^2}\right)A_n^2A_{n,2}^2\\
&\quad - (p-2)\int_{\mathbb{R}^3} |u_n^-|^p \ln|u_n^-|\, dx - \int_{\mathbb{R}^3} |u_n^-|^p\, dx,
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\frac{\partial g_n^+(0,0,0)}{\partial l} = \frac{\partial g_n^-(0,0,0)}{\partial k} &= 2b\Phi_T(u_n)A_{n,1}A_{n,2} + \frac{4b}{T^2}\phi'\left(\frac{A_n}{T^2}\right)A_nA_{n,1}A_{n,2}\\
&\quad + \frac{b}{T^4}\phi''\left(\frac{A_n}{T^2}\right)A_n^2A_{n,1}A_{n,2},
\end{aligned} \tag{2.17}$$

where

$$A_n = \|u_n\|^2, \quad A_{n,1} = \|u_n^+\|^2, \quad A_{n,2} = \|u_n^-\|^2.$$

From (2.7), (2.14) and Lemma 2.3, we have for $b < \frac{a}{16T^2}$,

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_n^\pm|^p \ln|u_n^\pm|\, dx &\geq a\|u_n^\pm\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\|u_n^\pm\|^2\\
&\geq (a - 4bT^2)\|u_n^\pm\|^2\\
&\geq \frac{a}{4}\mu^2.
\end{aligned} \tag{2.18}$$

Combining (2.15)–(2.18) and the boundedness of $\{u_n\}$, we can choose $b_0 > 0$ sufficiently small $\big(b_0 < \frac{a}{16T^2}\big)$ such that for any $b \in (0, b_0)$,

$$\begin{aligned}\begin{vmatrix} \frac{\partial g_n^+(0,0,0)}{\partial k} & \frac{\partial g_n^+(0,0,0)}{\partial l} \\ \frac{\partial g_n^-(0,0,0)}{\partial k} & \frac{\partial g_n^-(0,0,0)}{\partial l} \end{vmatrix} &\geq \frac{(p-2)^2}{2}\int_{\mathbb{R}^3} |u_n^+|^p \ln|u_n^+|\,dx \int_{\mathbb{R}^3} |u_n^-|^p \ln|u_n^-|\,dx \\ &\geq \frac{(p-2)^2}{32}a^2\mu^4.\end{aligned}$$

Thus, we can use implicit function theorem to obtain $r_n > 0$ and $k_n, l_n \in C^1((-r_n, r_n), \mathbb{R})$ such that $k_n(0) = l_n(0) = 0$ and $g_n^{\pm}(r, k_n(r), l_n(r)) = 0$, $\forall r \in (-r_n, r_n)$. It implies that

$$u_n + r\varphi + k_n(r)u_n^+ + l_n(r)u_n^- \in \mathcal{M}_T.$$

Set $\tilde{w}_n = r\varphi + k_n(r)u_n^+ + l_n(r)u_n^-$. From (2.13), we have

$$J_T(u_n + \tilde{w}_n) - J_T(u_n) \geq -\frac{1}{n}\|\tilde{w}_n\|. \tag{2.19}$$

By using Taylor expansion and $\langle J_T'(u_n), u_n^{\pm}\rangle = 0$, one has

$$J_T(u_n + \tilde{w}_n) - J_T(u_n) = \langle J_T'(u_n), \tilde{w}_n\rangle + o(\|\tilde{w}_n\|) = r\langle J_T'(u_n), \varphi\rangle + o(\|\tilde{w}_n\|).$$

It follows from (2.19) that

$$\langle J_T'(u_n), \varphi\rangle + \frac{o(\|\tilde{w}_n\|)}{r} \geq -\frac{1}{n}\|\varphi\| - \frac{1}{n}\left\|\frac{k_n(r)}{r}u_n^+ + \frac{l_n(r)}{r}u_n^-\right\|. \tag{2.20}$$

According to (2.15), (2.18) and the boundedness of $\{u_n\}$, it is easy to get there exists $C_1 > 0$ such that for $b \in (0, b_0)$,

$$|k_n'(0)| = \left|\frac{\partial g_n^+(0,0,0)}{\partial r} \Big/ \frac{\partial g_n^+(0,0,0)}{\partial k}\right| \leq C_1.$$

Similarly, we can obtain $|l_n'(0)| \leq C_2$. For fixed n, letting $r \to 0$ in (2.20), we can deduce that

$$\|J_T'(u_n)\| \leq \frac{C_3}{n}.$$

That is $J_T'(u_n) \to 0$. From Lemma 2.4, there exists $u_* \in X$ such that $u_n \to u_*$ in X. Since $u_n \in \mathcal{M}_T$, one has

$$\begin{aligned}\int_{\mathbb{R}^3} |u_*^{\pm}|^p \ln|u_*^{\pm}|\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n^{\pm}|^p \ln|u_n^{\pm}|\,dx \\ &= \lim_{n\to\infty}\left(a\|u_n^{\pm}\|^2 + b\Phi_T(u_n)\|u_n\|^2\|u_n^{\pm}\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\|u_n^{\pm}\|^2\right) \\ &= a\|u_*^{\pm}\|^2 + b\Phi_T(u_*)\|u_*\|^2\|u_*^{\pm}\|^2 + \frac{b}{2T^2}\phi'\left(\frac{\|u_*\|^2}{T^2}\right)\|u_*\|^4\|u_*^{\pm}\|^2.\end{aligned}$$

In other words, $\langle J_T'(u_*), u_*^{\pm}\rangle = 0$. By Lemma 2.3, we know for $b \in (0, b_0)$

$$\|u_*^{\pm}\| = \lim_{n\to\infty}\|u_n^{\pm}\| \geq \mu > 0,$$

which implies $u_*^{\pm} \neq 0$). Hence, we have $u_* \in \mathcal{M}_T$. Based on the fact $u_n \to u_*$ in X, from (2.13) and $J_T'(u_n) = 0$, one has $J_T(u_*) = m_T$ and $J_T'(u_*) = 0$, namely, u_* is a sign-changing critical point of J_T. The proof of Lemma 2.5 is complete. □

Lemma 2.6. *Let u_* be the sign-changing critical point of J_T obtained in Lemma 2.5. Then for $T > 0$ large enough, there exists $b_1 > 0$ such that $\|u_*\| < T$ for any $b \in (0, b_1)$.*

Proof. Choose $\varphi \in X$ with $\varphi \neq 0$. By Lemma 2.1, there exists a positive numbers pair (s_φ, t_φ) such that $\bar{\varphi} = s_\varphi \varphi^+ + t_\varphi \varphi^- \in \mathcal{M}_T$. Define

$$F(s,t) = \frac{a}{2}\|s\varphi^+ + t\varphi^-\|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |s\varphi^+ + t\varphi^-|^p \ln|s\varphi^+ + t\varphi^-|\,dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |s\varphi^+ + t\varphi^-|^p dx,$$

similarly to the analysis in the proof of Lemma 2.1, it is easy to know that

$$\theta = \max_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+} F(s,t) < +\infty.$$

Therefore,

$$\begin{aligned} m_T \leq J_T(\bar{\varphi}) &= \frac{a}{2}\|\bar{\varphi}\|^2 + \frac{b}{4}\Phi_T(\bar{\varphi})\|\bar{\varphi}\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |\bar{\varphi}|^p \ln|\bar{\varphi}|\,dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |\bar{\varphi}|^p\,dx \\ &\leq \frac{a}{2}\|\bar{\varphi}\|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |\bar{\varphi}|^p \ln|\bar{\varphi}|\,dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |\bar{\varphi}|^p\,dx + bT^4 \\ &\leq \theta + bT^4. \end{aligned} \tag{2.21}$$

By (2.4), we have

$$J_T(u_*) \geq \left(\frac{a}{2} - \frac{a}{p}\right)\|u_*\|^2 - \left(\frac{b}{p} - \frac{b}{4}\right)T^4. \tag{2.22}$$

Now, suppose to the contrary that $\|u_*\| \geq T$. It follows from (2.21) and (2.22) that

$$\left(\frac{a}{2} - \frac{a}{p}\right)T^2 \leq \left(\frac{a}{2} - \frac{a}{p}\right)\|u_*\|^2 \leq J_T(u_*) + \left(\frac{b}{p} - \frac{b}{4}\right)T^4 \leq \theta + \frac{5}{4}bT^4,$$

which is not true for large T and $b < \frac{4a}{5T^4}$. The proof of Lemma 2.6 is complete. □

Proof of Theorem 1.1. Let T, b_1 be as in Lemma 2.6. By Lemmas 2.5 and 2.6, we obtain that J_T has a sign-changing critical point u_* with $J_T(u_*) = m_T$ and $\|u_*\| < T$. It follows that $J(u_*) = J_T(u_*)$ and u_* is a sign-changing critical point of J with $J(u_*) = m_T$. The proof of Theorem 1.1 is complete. □

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