



# Variational approach to nonhomogeneous elliptic systems with Hardy potentials

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**Abstract.** This paper investigates the existence of weak solutions to a nonlinear elliptic system driven by distinct nonhomogeneous operators, where the nonlinearities exhibit superlinear growth and involve Hardy-type singular terms. To overcome the lack of compactness caused by the presence of the Hardy term, we establish a general version of the Brezis–Lieb convergence theorem and apply the Mountain Pass Theorem under the Cerami condition to prove the existence of a nontrivial, nonnegative weak solution. Our results extend previous works by addressing a broader class of nonhomogeneous operators.

**Keywords:** nonhomogeneous operator, singular elliptic system, superlinear growth conditions, Hardy potential, variational methods.

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## 1 Introduction

In this paper, we study the existence of solutions for the following nonlinear system involving a generalized nonhomogeneous operator together with Hardy-type potentials:

$$\begin{cases} -\operatorname{div}(a_1(\nabla u)) + A(x)|u|^{p-2}u = H_u(x, u, v) + \lambda \frac{|u|^{p-2}u}{|x|^p}, & \text{in } \Omega, \\ -\operatorname{div}(a_2(\nabla v)) + B(x)|v|^{q-2}v = H_v(x, u, v) + \mu \frac{|v|^{q-2}v}{|x|^q}, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  is a regular bounded domain such that  $0 \in \Omega$ , and assume  $p < N$  with  $1 < q \leq p < q^* = \frac{Nq}{N-q}$ . The weight functions  $A, B \in L_{\text{loc}}^\infty(\Omega)$  satisfying

$$0 \leq A(x) \leq \alpha_1 d(x)^{-\beta_1}, \quad (1.2)$$

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$$0 \leq B(x) \leq \alpha_2 d(x)^{-\beta_2}, \quad (1.3)$$

for all  $x \in \Omega_\epsilon = \{x \in \Omega : d(x) < \epsilon\}$ , where  $\epsilon > 0$  is sufficiently small and  $d$  is the distance map defined for all  $x \in \bar{\Omega}$  by  $d(x) := \inf_{y \in \partial\Omega} d(x, y)$ , and  $\alpha_1, \alpha_2 > 0$ ,  $0 \leq \beta_1 < p$ ,  $0 \leq \beta_2 < q$ .

The nonlinearities  $H_u$  and  $H_v$  denote the partial derivatives of  $H$  with respect to its second and third variables respectively, where  $H$  is assumed to satisfy the hypotheses stated below. Furthermore,  $a_1, a_2 : \mathbb{R}^N \rightarrow \mathbb{R}^N$  are continuous operators subject to additional regularity conditions, which cover many classical differential operators.

Numerous studies have investigated Problem (1.1) in the setting of a homogeneous operator. An early contribution is [30], which established the existence of two positive solutions for a semilinear elliptic systems involving the Laplacian operator and with concave-convex nonlinearities and sign-changing weight function with the help of the Nehari manifold, this work is a natural extension of earlier studies on the same class of systems; we refer the reader to [1–3, 7, 11, 14, 23]. In recent years, researchers have attempted to generalize this class of systems by investigating problems with singular terms, either of the form  $u^{-\gamma}$ , see [4, 5, 10] or potential Hardy, see [17, 28]. We note that this class has also been studied for fractional differential operators, especially in [31] by using the mountain pass theorem, the authors proved the existence of nontrivial and nonnegative solutions for a quasilinear Schrödinger type system driven by the fractional  $p$ -Laplacian that the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition, in [12], the authors addressed problems involving Hardy-type terms and nonlinearities not necessarily satisfying the Ambrosetti–Rabinowitz condition. They further studied systems with critical nonlinear terms, as recently investigated in the literature, and established radial versions of the main theorems. In addition, they showed that the results could be extended to more general elliptic nonlocal integro-differential operators beyond the fractional Laplacian. In [13], the authors extended the results of [12] to the  $p$ -fractional Kirchhoff case with the presence of the Hardy terms as well as critical nonlinearities. For more references, we refer to [9, 27, 29].

The structural conditions imposed on the mappings  $a_i$  are formulated in Hypotheses ( $H_{a_i}$ ) below. These assumptions are sufficiently general to cover a wide range of nonlinear differential operators, including the  $p$ -Laplacian and the  $(p, q)$ -Laplacian. In contrast to the  $p$ -Laplacian etc. However, the operators associated with problem (1.1) are not necessarily homogeneous. This lack of homogeneity introduces substantial analytical difficulties. The hypotheses on  $a_i(\cdot)$  are motivated by the global regularity theory developed by Lieberman [21] and by the nonlinear maximum principle of Pucci–Serrin [26]. Related, albeit slightly more restrictive, conditions have also been employed in the literature, for instance by Öztürk–Papageorgiou [24] in the study of Robin problems, and by Papageorgiou–Rădulescu [25] in the context of Neumann problems.

The novelty of our work lies in the study of a system involving two differential operators that are not necessarily homogeneous, together with the nonlinearities satisfy a growth condition but do not necessarily satisfy the Ambrosetti–Rabinowitz condition, in addition, our system contains a singular Hardy potential, where the origin belongs to the domain  $\Omega$ . To the best of our knowledge, this problem has not been previously addressed in the literature, particularly since such a generalization gives rise to substantial technical difficulties that can only be overcome through nonstandard methods.

We now present the hypotheses concerning the operators  $a_i$   $i = 1, 2$ . We set

$$\theta_1(t) = t^{p-2}, \quad \theta_2(t) = t^{q-2}, \quad \forall t > 0.$$

Then, suppose that the maps  $a_i(\cdot)$ ,  $i = 1, 2$ , satisfy:

$$(\mathbf{H}_{a_i}) \quad \begin{cases} (1) & a_i(x) = a_i^0(|x|)x, \quad \forall x \in \mathbb{R}^N \text{ with } a_i^0 \in C^1(]0, +\infty[; \mathbb{R}_+), \\ (2) & c_i \theta_i(|x|)y^2 \leq (\nabla a_i(x)y, y)_{\mathbb{R}^N}, \quad \forall x, y \in \mathbb{R}^N \setminus \{0\}, \\ (3) & |a_i(x)| \leq c'_i(1 + |x|\theta_i(|x|)), \quad |\nabla a_i(x)| \leq c'_i \left( \frac{1}{|x|} + \theta_i(|x|) \right), \quad \forall x \in \mathbb{R}^N \setminus \{0\} \end{cases}$$

for some constants  $c_1, c_2, c'_1, c'_2 > 0$ .

We set  $G_i(x) = \int_0^{|x|} a_i^0(s)s \, ds$ , for  $x \in \mathbb{R}^N$ . The mappings  $G_i$  are differentiable and we have

$$\nabla G_i(x) = a_i^0(|x|)|x| \cdot \frac{x}{|x|} = a_i(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Moreover, we suppose that the maps  $G_i$  satisfy

$$(\mathbf{H}_{G_i}) \quad \exists \gamma_1, \gamma_2 \in \mathbb{R} : \begin{cases} \gamma_1 \leq pG_1(x) - a_1(x).x, \\ \gamma_2 \leq qG_2(x) - a_2(x).x, \end{cases} \quad \forall x \in \mathbb{R}^N.$$

The above assumptions yield some useful properties of the mappings  $a_i$  and  $G_i$  (see [25]). We summarize these properties in the following lemma.

**Lemma 1.1.** *If hypotheses  $(H_{a_i})$  hold, then*

- (i) *The functions  $a_i(\cdot)$  are strictly monotone and  $G_i(\cdot)$  are strictly convex.*
- (ii)  $\frac{c_1}{p-1}|x|^p \leq a_1(x).x \leq c'_1(|x| + |x|^p), \quad \frac{c_2}{q-1}|x|^q \leq a_2(x).x \leq c'_2(|x| + |x|^q).$
- (iii)  $\frac{c_1}{p(p-1)}|x|^p \leq G_1(x) \leq c_3(1 + |x|^p), \quad \frac{c_2}{q(q-1)}|x|^q \leq G_2(x) \leq c'_3(1 + |x|^q).$

There are several differential operators that satisfy hypotheses  $(H_{a_i})$ , and therefore fall within our framework. We illustrate this with examples for  $a_1$  and the same for  $a_2$  (see [25]).

- (a)  $a_1(y) = |y|^{p-2}y$  with  $1 < p < \infty$ . This map corresponds to the  $p$ -Laplace operator  $\Delta_p$ .
- (b)  $a_1(y) = |y|^{p-2}y + |y|^{s-2}y$  with  $1 < s < p < \infty$ . This map corresponds to the  $(p, s)$ -Laplace differential operator  $\Delta_p + \Delta_s$ .
- (c)  $a_1(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$  with  $2 \leq p < \infty$ . This map corresponds to the generalized  $p$ -mean curvature differential operator.

Concerning the nonlinearity  $H$ , we consider the following hypothesis:

(H1) For all  $(u, v) \in \mathbb{R}^2$ ,  $H(\cdot, u, v) : \Omega \rightarrow \mathbb{R}$  is measurable such that  $H(x, \cdot, \cdot) \in C^1(\mathbb{R}^2)$ ,  $H(x, 0, 0) = 0$  for a.e.  $x \in \Omega$  and

$$\exists C_1, C_2 > 0 : |H_z(x, z)| \leq C_1|z|^{r-1} + C_2 \quad \text{a.e. } x \in \Omega \quad \text{and for any } z \in \mathbb{R}^2.$$

where  $r \in (p, q^*)$ ,  $z = (z_1, z_2) \in \mathbb{R}^2$  and  $|z| = \sqrt{z_1^2 + z_2^2}$ .

(H2)  $H(x, 0, 0) = 0$ , for all  $x \in \Omega$ ,  $H_{z_2}(x, z_1, z_2) = 0$  if  $x \in \Omega$  and  $z_1 \leq 0$ , and  $H_{z_1}(x, z_1, z_2) = 0$  if  $x \in \Omega$  and  $z_2 \leq 0$ .

(H3) There exists  $\beta, C > 0$ , and  $\tau \in (\frac{N(r-q)}{q}, r]$  such that:

$$F(x, z) \geq \beta|z|^\tau - C, \quad \text{a.e. } x \in \Omega, \quad (1.4)$$

where

$$F(x, z) = H_{z_1}(x, z_1, z_2)z_1 + H_{z_2}(x, z_1, z_2)z_2 - pH(x, z_1, z_2).$$

(H4)  $\lim_{|z| \rightarrow \infty} \frac{H(x, z)}{|z|^p} = \infty$ , uniformly a.e.  $x \in \Omega$ .

(H5)  $\limsup_{|z| \rightarrow 0} \frac{H(x, z)}{|z|^p} < \frac{\nu_p}{2^{p-1}} \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\}$ , uniformly a.e.  $x \in \Omega$ , where  $\nu_p$  is given in (2.3) and  $\alpha^\pm := \max\{\pm\alpha, 0\}$ ,  $\forall \alpha \in \mathbb{R}$ .

**Remark 1.2.** The assumption (H3) serves as a replacement for the classical Ambrosetti–Rabinowitz condition.

This paper is organized as follows: In the next section, we present the functional framework and establish some useful results necessary for our analysis. In Section 3, we state and prove our main results using the Mountain Pass Theorem under the Cerami condition.

## 2 Preliminaries

The analysis of Problem (1.1) is carried out within the classical Sobolev framework. Since these spaces are well-known, we recall only the essential facts. Let  $1 < m < +\infty$ . Denote by  $L^m(\Omega)$  the Lebesgue space equipped with the norm

$$\|u\|_{L^m} = \left( \int_{\Omega} |u|^m dx \right)^{1/m}.$$

The Sobolev space  $W^{1,m}(\Omega)$  is endowed with the norm

$$\|u\|_{W^{1,m}} = \|u\|_{L^m} + \|\nabla u\|_{L^m}.$$

and  $W_0^{1,m}(\Omega)$  be the closure of  $C_c^\infty(\Omega)$  in  $W^{1,m}(\Omega)$ , it is a Banach subspace of  $W^{1,m}(\Omega)$  with the equivalent norm

$$\|u\|_{W_0^{1,m}} = \|\nabla u\|_{L^m}.$$

We recall the classical Hardy's inequality (see [22])

$$\exists C > 0 : \left\| \frac{u}{d} \right\|_{L^m} \leq C \|\nabla u\|_{L^m}, \quad \forall u \in W_0^{1,m}(\Omega). \quad (2.1)$$

We present another version of Hardy's inequality, which can be found in [15]. For any  $u \in W_0^{1,m}(\Omega)$  we have

$$\mathcal{H}_m \left\| \frac{u}{x} \right\|_{L^m}^m \leq \|\nabla u\|_{L^m}^m, \quad \text{with } \mathcal{H}_m = \left( \frac{m}{N-m} \right)^{-m} \quad \text{and} \quad \left\| \frac{u}{x} \right\|_{L^m}^m = \int_{\Omega} \frac{|u|^m}{|x|^m} dx. \quad (2.2)$$

Define

$$E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$$

endowed with the norm:

$$\|(u, v)\|_E = \|u\|_{W_0^{1,p}} + \|v\|_{W_0^{1,q}}.$$

It is well known that the space  $(E, \|\cdot\|_E)$  is a separable and reflexive Banach space. Moreover we have:

**Lemma 2.1.** *The embedding  $E \hookrightarrow L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$  is continuous for any  $s \in [1, q^*]$  and compact for any  $s \in [1, q^*)$ , with  $q^* = \frac{Nq}{N-q}$ , being  $1 < q \leq p < q^*$ . Moreover the constant*

$$v_s = \inf \left\{ \|\nabla u\|_{L^p}^s + \|\nabla u\|_{L^q}^s : \int_{\Omega} |(u, v)|^s dx = 1 \right\} \quad (2.3)$$

is well defined and strictly positive.

We give some useful properties concerning the operators  $a_1, a_2$ .

**Lemma 2.2.** *Let  $p > 1$ . There exists a constant  $C > 0$  such that*

$$\forall x, y \in \mathbb{R}^N : (a_1(x) - a_1(y)) \cdot (x - y) \geq \begin{cases} C|x - y|^p & \text{if } p \geq 2, \\ C|x - y|^2(|x| + |y|)^{p-2} & \text{if } p < 2. \end{cases} \quad (2.4)$$

An analogous inequality is valid for  $a_2$ .

*Proof.* We have:

$$\begin{aligned} (a_1(x) - a_1(y)) \cdot (x - y) &= \int_0^1 \frac{d}{dt} a_1(x + t(x - y)) dt \cdot (x - y) \\ &= \int_0^1 (\nabla a_1(x + t(x - y))(x - y), (x - y))_{\mathbb{R}^N} dt \\ &\geq c_1 |x - y|^2 \int_0^1 |x + t(x - y)|^{p-2} dt, \quad (\text{by } (H_{a_1})_2). \end{aligned}$$

- If  $1 < p < 2$ , we have  $|x + t(x - y)| \leq 2(|x| + |y|)$ . It follows that

$$(a_1(x) - a_1(y)) \cdot (x - y) \geq 2^{p-2} c_1 |x - y|^2 \int_0^1 (|x| + |y|)^{p-2} dt = C|x - y|^2(|x| + |y|)^{p-2}.$$

- For  $p \geq 2$ , It is enough to show that

$$A := \int_0^1 \frac{|y + t(x - y)|^{p-2}}{|x - y|^{p-2}} dt \geq C, \quad \text{for some constant } C > 0.$$

- If  $y = 0$ , then:

$$A = \int_0^1 t^{p-2} dt = \frac{1}{p-1}. \quad (2.5)$$

- If  $y \neq 0$ , we distinguish two cases:

- 1) If  $|x - y| \leq |y|$ , then:

$$|y + t(x - y)|^{p-2} \geq (|y| - t|x - y|)^{p-2} \geq (1 - t)^{p-2} |y|^{p-2}.$$

Also, we have:

$$\frac{1}{|x - y|^{p-2}} \geq \frac{1}{|y|^{p-2}}.$$

Hence we obtain:

$$A \geq \int_0^1 (1 - t)^{p-2} dt = \frac{1}{p-1}. \quad (2.6)$$

2) If  $|y| < |x - y|$ , then:

$$\begin{aligned}
A &\geq \int_0^1 \frac{||y| - t|x - y||^{p-2}}{|x - y|^{p-2}} dt \\
&= \int_0^{\frac{|y|}{|x-y|}} \left( \frac{|y| - t|x - y|}{|x - y|^{p-2}} \right)^{p-2} dt + \int_{\frac{|y|}{|x-y|}}^1 \left( \frac{t|x - y| - |y|}{|x - y|^{p-2}} \right)^{p-2} dt \\
&= \frac{1}{(p-1)|x - y|^{p-1}} \left( |y|^{p-1} + (|x - y| - |y|)^{p-1} \right) \geq \frac{2^{2-p}}{p-1}. \tag{2.7}
\end{aligned}$$

From (2.5), (2.6) and (2.7), we deduce that:

$$A \geq C = \frac{2^{2-p}}{p-1} > 0. \quad \square$$

**Lemma 2.3.** *Let  $(u_n)_n$  be a bounded sequence in  $L^p(\Omega)$  with  $1 < p < 2$ . Then there exists  $c > 0$  such that:*

$$\int_{\Omega} |u_n - u|^2 (|u_n| + |u|)^{p-2} dx \geq c \|u_n - u\|_{L^p}^2, \quad \forall n.$$

*Proof.* We have:

$$\|u_n - u\|_{L^p}^p = \int_{\Omega} |u_n - u|^p dx = \int_{\Omega} \frac{|u_n - u|^p}{(|u_n| + |u|)^{\gamma}} (|u_n| + |u|)^{\gamma} dx$$

where  $\gamma = \frac{p}{2}(2 - p)$ . Using Hölder's inequality with the exponent  $\frac{p}{2}$  and  $\frac{2}{2-p}$ , we obtain

$$\begin{aligned}
\|u_n - u\|_{L^p}^p &\leq \left( \int_{\Omega} \frac{|u_n - u|^2}{(|u_n| + |u|)^{2-p}} dx \right)^{\frac{p}{2}} \left( \int_{\Omega} (|u_n| + |u|)^p dx \right)^{1-\frac{p}{2}} \\
&\leq C \left( \int_{\Omega} |u_n - u|^2 (|u_n| + |u|)^{p-2} dx \right)^{\frac{p}{2}}.
\end{aligned}$$

Hence, we obtain the result. □

The following important result follows from a direct application of the preceding two lemmas.

**Corollary 2.4.** *Let  $(U_n, V_n)_n$  be a bounded sequence in  $(L^p(\Omega) \times L^q(\Omega))^N$ . Then, there exists a constant  $C > 0$ , such that*

$$\int_{\Omega} (a_1(U_n) - a_1(U)) \cdot (U_n - U) dx \geq \begin{cases} C \|U_n - U\|_{L^p(\Omega)}^p & \text{if } p \geq 2, \\ C \|U_n - U\|_{L^p(\Omega)}^2 & \text{if } p < 2, \end{cases} \tag{2.8}$$

$$\int_{\Omega} (a_2(V_n) - a_2(V)) \cdot (V_n - V) dx \geq \begin{cases} C \|V_n - V\|_{L^q(\Omega)}^q & \text{if } q \geq 2, \\ C \|V_n - V\|_{L^q(\Omega)}^2 & \text{if } q < 2. \end{cases} \tag{2.9}$$

In order to prove our result, we need to apply the Mountain Pass theorem under the Cerami condition (see [19]).

**Definition 2.5.** Let  $X$  be a real Banach space and  $X'$  its dual. We say that a  $C^1$ -functional  $J : X \rightarrow \mathbb{R}$  satisfies the Cerami condition if from any sequence  $(u_n)_n \subset X$  such that:

$$(J(u_n))_n \text{ is bounded in } \mathbb{R} \quad \text{and} \quad (1 + \|u_n\|_X) \|J'(u_n)\|_{X'} \rightarrow 0, \quad (2.10)$$

as  $n \rightarrow +\infty$ , we can extract a strangely convergent subsequence in  $E$ . A sequence satisfying (2.10) is called Cerami sequence for  $J$ .

**Theorem 2.6** (Mountain pass theorem). *Let  $X$  be a real Banach space and  $X'$  its dual. Suppose that  $J \in C^1(X, \mathbb{R})$  satisfies*

$$\max\{J(0), J(e)\} \leq \alpha < \beta \leq \inf_{\|u\|_X=\rho} J(u),$$

for some  $\alpha, \beta \in \mathbb{R}$ ,  $\rho > 0$  and  $e \in X$  with  $\|e\|_X > \rho$ . Let  $c \geq \beta$  be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where  $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$  is the set of continuous paths joining 0 and  $e$ . Then there exists a sequence  $\{u_n\}_n \subset X$  such that

$$J(u_n) \rightarrow c \geq \beta \quad \text{and} \quad \|J'(u_n)\|_{X'}(1 + \|u_n\|_X) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

### 3 Main result

In this section, we state and prove our main result. We start by introducing the notion of a weak solution to our Problem (1.1).

**Definition 3.1.** We say that  $(u, v) \in E$  is a weak solution of problem (1.1) if:

$$\begin{aligned} & \int_{\Omega} a_1(\nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} a_2(\nabla v) \cdot \nabla \psi \, dx + \int_{\Omega} A(x) |u|^{p-2} u \varphi \, dx + \int_{\Omega} B(x) |v|^{q-2} v \psi \, dx \\ & - \int_{\Omega} H_u(x, u, v) \varphi \, dx - \int_{\Omega} H_v(x, u, v) \psi \, dx - \lambda \int_{\Omega} \frac{|u|^{p-2} u}{|x|^p} \varphi \, dx - \mu \int_{\Omega} \frac{|v|^{q-2} v}{|x|^q} \psi \, dx = 0, \end{aligned}$$

for all  $(\varphi, \psi) \in E$ .

**Remark 3.2.**

- Thanks to the Hardy inequalities (2.1) and (2.2), the singular terms above are well defined. That is,

$$A(x) |u|^{p-2} u, \frac{|u|^{p-2} u}{|x|^p} \in \left(W_0^{1,p}(\Omega)\right)', \quad B(x) |v|^{q-2} v, \frac{|v|^{q-2} v}{|x|^q} \in \left(W_0^{1,q}(\Omega)\right)'.$$

Indeed, for the first term, we have for all  $\varphi \in W_0^{1,p}(\Omega)$ ,

$$\begin{aligned} \left| \int_{\Omega} A(x) |u|^{p-2} u \varphi \, dx \right| & \leq \int_{\Omega} A(x)^{1/p'} |u|^{p-1} A(x)^{1/p} |\varphi| \, dx \\ & \leq \left( \int_{\Omega} A(x) |u|^p \, dx \right)^{1/p'} \left( \int_{\Omega} A(x) |\varphi|^p \, dx \right)^{1/p} \\ & \leq \left( \int_{\Omega} A(x) d^p \left| \frac{u}{d} \right|^p \, dx \right)^{1/p'} \left( \int_{\Omega} A(x) d^p \left| \frac{\varphi}{d} \right|^p \, dx \right)^{1/p} \\ & \leq C \|u\|_{W_0^{1,p}}^{p-1} \|\varphi\|_{W_0^{1,p}}. \end{aligned}$$

Here we used condition (1.2) and the Hardy inequality (2.1). A similar argument applies to the other terms.

- The remaining terms in the above definition are well defined due to the growth assumptions on  $a_i$  and  $H$ .

The main result is the following.

**Theorem 3.3.** *Let  $p < N$  and  $1 < q < p < q^*$ . Assume that (H1)–(H5),  $(H_{a_i})$  and  $(H_{G_i})$  hold. Then for any  $\lambda, \mu$  satisfying*

$$\lambda < \frac{c_1 \mathcal{H}_p}{p-1}, \quad \mu < \frac{c_2 \mathcal{H}_q}{q-1}, \quad (3.1)$$

system (1.1) has at least one nontrivial and nonnegative solution  $(u, v) \in E$ .

The associated functional of the system (1.1) is:

$$\begin{aligned} J(u, v) = & \int_{\Omega} G_1(\nabla u) dx + \int_{\Omega} G_2(\nabla v) dx + \frac{1}{p} \int_{\Omega} A(x)|u|^p dx \\ & + \frac{1}{q} \int_{\Omega} B(x)|v|^q dx - \int_{\Omega} H(x, u, v) dx - \frac{\lambda}{p} \left\| \frac{u}{x} \right\|_{L^p}^p - \frac{\mu}{q} \left\| \frac{v}{x} \right\|_{L^q}^q. \end{aligned}$$

Under (H1) and using Lemma 1.1(iii) and the classical Hardy inequality, we have that the functional  $J$  is well defined in  $E$ , of class  $\mathcal{C}^1(E)$  and for any  $(u, v) \in E$  we have:

$$\begin{aligned} \langle J'(u, v), (\varphi, \psi) \rangle = & \int_{\Omega} a_1(\nabla u) \nabla \varphi dx + \int_{\Omega} a_2(\nabla v) \nabla \psi dx \\ & + \int_{\Omega} A(x)|u|^{p-2} u \varphi dx + \int_{\Omega} B(x)|v|^{q-2} v \psi dx \\ & - \int_{\Omega} H_u(x, u, v) \varphi dx - \int_{\Omega} H_v(x, u, v) \psi dx - \lambda \int_{\Omega} \frac{|u|^{p-2} u}{|x|^p} \varphi dx - \mu \int_{\Omega} \frac{|v|^{q-2} v}{|x|^q} \psi dx, \end{aligned}$$

for any  $(u, v), (\varphi, \psi) \in E$ . Then, any critical point of  $J$  is a weak solution of the system (1.1).

To establish Theorem 3.3, we will apply the Mountain-Pass theorem 2.6 on the functional  $J$ . To verify the Cerami condition, we first establish the boundedness of any Cerami sequence for  $J$ , as stated in the following lemma.

**Lemma 3.4.** *Under the assumptions of Theorem 3.3, any Cerami sequence for  $J$  is bounded in  $E$ .*

*Proof.* Let  $(u_n, v_n)_n \subset E$  be a Cerami sequence of  $J$ ; that is

$$|J(u_n, v_n)| \leq C, \quad (3.2)$$

$$\varepsilon_n := (1 + \|(u_n, v_n)\|_E) J'(u_n, v_n) \rightarrow 0 \quad \text{in } E' \quad (3.3)$$

as  $n \rightarrow +\infty$ , with  $E'$  is the dual space of  $E$ . From (3.3), for any  $(\varphi, \psi) \in E$  we have

$$|\langle J'(u_n, v_n), (\varphi, \psi) \rangle| \leq \|J'(u_n, v_n)\|_{E'} \|(\varphi, \psi)\|_E \leq \frac{\varepsilon_n \|(\varphi, \psi)\|_E}{1 + \|(u_n, v_n)\|_E}.$$

By substituting, we get

$$\begin{aligned} & \left| \int_{\Omega} a_1(\nabla u_n) \nabla \varphi dx + \int_{\Omega} a_2(\nabla v_n) \nabla \psi dx + \int_{\Omega} A(x)|u_n|^{p-2} u_n \varphi dx \right. \\ & \quad + \int_{\Omega} B(x)|v_n|^{q-2} v_n \psi dx - \int_{\Omega} H_u(x, u_n, v_n) \varphi dx - \int_{\Omega} H_v(x, u_n, v_n) \psi dx \\ & \quad \left. - \lambda \int_{\Omega} \frac{|u_n|^{p-2} u_n}{|x|^p} \varphi dx - \mu \int_{\Omega} \frac{|v_n|^{q-2} v_n}{|x|^q} \psi dx \right| \leq \frac{\varepsilon_n \|(\varphi, \psi)\|_E}{1 + \|(u_n, v_n)\|_E}. \quad (3.4) \end{aligned}$$

Taking  $(\varphi, \psi) = (u_n, v_n) \in E$  in (3.4), for all  $n$ . we obtain that

$$\begin{aligned} & - \int_{\Omega} a_1(\nabla u_n) \nabla u_n dx - \int_{\Omega} a_2(\nabla v_n) \nabla v_n dx - \int_{\Omega} A(x) |u_n|^p dx - \int_{\Omega} B(x) |v_n|^q dx \\ & + \int_{\Omega} H_u(x, u_n, v_n) u_n dx + \int_{\Omega} H_v(x, u_n, v_n) v_n dx + \lambda \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx + \mu \int_{\Omega} \frac{|v_n|^q}{|x|^q} dx \leq \varepsilon_n. \end{aligned} \quad (3.5)$$

On other hand, since  $p \geq q > 1$ , by (3.2) we get

$$\begin{aligned} & p \int_{\Omega} G_1(\nabla u_n) dx + p \int_{\Omega} G_2(\nabla v_n) dx + \int_{\Omega} A(x) |u_n|^p dx + \frac{p}{q} \int_{\Omega} B(x) |v_n|^q dx \\ & - p \int_{\Omega} H(x, u_n, v_n) dx - \lambda \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx - \mu \frac{p}{q} \int_{\Omega} \frac{|v_n|^q}{|x|^q} dx \leq C. \end{aligned} \quad (3.6)$$

By adding (3.5) and (3.6), we find

$$\begin{aligned} & \int_{\Omega} (pG_1(\nabla u_n) - a_1(\nabla u_n) \nabla u_n) + (qG_2(\nabla v_n) - a_2(\nabla v_n) \nabla v_n) dx + (p - q) \int_{\Omega} G_2(\nabla v_n) dx \\ & + \int_{\Omega} (H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n - pH(x, u_n, v_n)) dx + \mu \left(1 - \frac{p}{q}\right) \int_{\Omega} \left|\frac{v_n}{x}\right|^q dx \leq C. \end{aligned}$$

Thanks to hypotheses  $(H_{G_i})$  and Lemma 1.1(iii), we obtain

$$\begin{aligned} & \int_{\Omega} [H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n - pH(x, u_n, v_n)] dx \\ & + (p - q) \frac{c_2}{q(q - 1)} \int_{\Omega} |\nabla v_n|^q dx - \mu \frac{p - q}{q} \int_{\Omega} \left|\frac{v_n}{x}\right|^q dx \leq C, \end{aligned}$$

for a different constant  $C > 0$ . Therefore, using (H3) and Hardy inequality, we obtain

$$\beta \int_{\Omega} |(u_n, v_n)|^{\tau} dx + \frac{p - q}{q} \left( \frac{c_2}{q - 1} - \frac{\mu^+}{\mathcal{H}_q} \right) \int_{\Omega} |\nabla v_n|^q dx \leq C.$$

Consequently, by (3.1), we have  $\frac{c_2}{q-1} - \frac{\mu}{\mathcal{H}_q} > 0$  and then the sequence  $(u_n, v_n)_n$  is bounded in  $L^{\tau}(\Omega) \times L^{\tau}(\Omega)$ .

From the assumptions (H1), (H3), we have  $\tau \leq r < q^*$ . Hence, there exists  $t \in [0, 1)$ , such that

$$\frac{1}{r} = \frac{1 - t}{\tau} + \frac{t}{q^*}. \quad (3.7)$$

The interpolation inequality and the Sobolev embedding yield

$$\begin{aligned} \|u_n\|_{L^r} & \leq \|u_n\|_{L^{\tau}}^{1-t} \|u_n\|_{L^{q^*}}^t \leq C \|u_n\|_{W_0^{1,p}}^t, \\ \|v_n\|_{L^r} & \leq \|v_n\|_{L^{\tau}}^{1-t} \|v_n\|_{L^{q^*}}^t < C \|v_n\|_{W_0^{1,q}}^t. \end{aligned}$$

It follows that

$$\|(u_n, v_n)\|_{L^r \times L^r}^r \leq C \|(u_n, v_n)\|_E^{tr}, \quad \forall n \geq 1. \quad (3.8)$$

Taking  $(\varphi, \psi) = (u_n, v_n)$  in (3.4) we get

$$\begin{aligned} & \int_{\Omega} a_1(\nabla u_n) \nabla u_n dx + \int_{\Omega} A(x) |u_n|^p dx + \int_{\Omega} a_2(\nabla v_n) \nabla v_n dx + \int_{\Omega} B(x) |v_n|^q dx \\ & - \int_{\Omega} (H_u(x, u_n, v_n), H_v(x, u_n, v_n)) \cdot (u_n, v_n) dx - \lambda \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx - \mu \int_{\Omega} \frac{|v_n|^q}{|x|^q} dx \leq \varepsilon_n. \end{aligned}$$

Using Hardy inequality, (H1) and Lemma 1.1-(ii), we deduce that

$$\min \left\{ \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p}, \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right\} (\|\nabla u_n\|_{L^p}^p + \|\nabla v_n\|_{L^q}^q) \leq \varepsilon_n + C_1 \|(u_n, v_n)\|_{L^r \times L^r}^r + C_2 |\Omega|. \quad (3.9)$$

Combining (3.8) and (3.9) and using (3.1), we get

$$\|\nabla u_n\|_{L^p}^p + \|\nabla v_n\|_{L^q}^q \leq C(1 + \|u_n\|_r^r + \|v_n\|_r^r) \leq C(1 + \|(u_n, v_n)\|_E^{tr}), \quad (3.10)$$

for some constant  $C > 0$ . From the hypothesis (H3), we have

$$\tau > (r - q) \frac{N}{q} \quad (3.11)$$

which is equivalent to  $tr < q$ . Indeed, from (3.7), we have

$$t = \frac{1 - \frac{\tau}{r}}{1 - \frac{\tau}{q^*}} \quad \text{and} \quad tr = \frac{r - \tau}{1 - \frac{\tau}{q^*}},$$

thus

$$\tau > (r - q) \frac{N}{q} \iff tr < q.$$

Hence, by applying Young's inequality to the last term in (3.10), it follows that  $(u_n, v_n)_n$  is bounded in  $E$ .  $\square$

We are now ready to prove our important compactness result, which concerns the Cerami condition. This part involves two major difficulties: the nonhomogeneity of the operators  $a_i$  and the presence of the Hardy potential, which prevents the use of standard techniques. To overcome these challenges, we use a convexity inequality (given Corollary 2.4) and a generalized Brezis–Lieb convergence theorem for the operators  $a_i$  presented in Step 02 of the proof of the following proposition.

**Proposition 3.5.** *Under the assumptions of Theorem 3.3, the functional  $J$  satisfies the Cerami condition.*

*Proof.* Let  $(u_n, v_n)_n$  be a Cerami sequence of  $J$ , Lemma 3.4 ensure the boundedness of  $(u_n, v_n)_n$  in  $E$ . Hence, by passing to a suitable subsequence if necessary, we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } E, \quad (3.12)$$

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } L^s(\Omega) \times L^s(\Omega), \text{ for all } 1 < s < q^*, \quad (3.13)$$

$$(u_n, v_n) \rightarrow (u, v) \quad \text{a.e. in } \Omega. \quad (3.14)$$

For clarity, we organize the proof in three steps:

• **Step 01.** We claim that up to a subsequence

$$(\nabla u_n, \nabla v_n) \longrightarrow (\nabla u, \nabla v) \quad \text{a.e. in } \Omega. \quad (3.15)$$

Let  $\varepsilon > 0$ ,  $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$  be a cut-off function satisfying  $0 \leq \psi_\varepsilon \leq 1$ ,  $\psi_\varepsilon = 0$  in  $B_\varepsilon := B_\varepsilon(0, \varepsilon)$  and  $\psi_\varepsilon = 1$  in  $\mathbb{R}^N \setminus B_{2\varepsilon}$ . Thus, we have  $\psi_\varepsilon(u_n, v_n) \in E$ . Taking  $\varphi = \psi_\varepsilon(u_n - u)$ ,  $\psi = \psi_\varepsilon(v_n - v)$

in (3.4) and passing to the limit as  $n \rightarrow +\infty$ , we get

$$\begin{aligned}
& \int_{\Omega} a_1(\nabla u_n) \cdot [(\nabla u_n - \nabla u)\psi_{\varepsilon} + \nabla\psi_{\varepsilon}(u_n - u)] dx + \int_{\Omega} A(x)|u_n|^{p-2}u_n(u_n - u)\psi_{\varepsilon} dx \\
& + \int_{\Omega} a_2(\nabla v_n) \cdot [(\nabla v_n - \nabla v)\psi_{\varepsilon} + \nabla\psi_{\varepsilon}(v_n - v)] dx + \int_{\Omega} B(x)|v_n|^{q-2}v_n(v_n - v)\psi_{\varepsilon} dx \\
& - \int_{\Omega} [H_u(x, u_n, v_n)\psi_{\varepsilon}(u_n - u) + H_v(x, u_n, v_n)\psi_{\varepsilon}(v_n - v)] dx \\
& - \lambda \int_{\Omega} \frac{|u_n|^{p-2}u_n}{|x|^p}\psi_{\varepsilon}(u_n - u) dx - \mu \int_{\Omega} \frac{|v_n|^{q-2}v_n}{|x|^q}\psi_{\varepsilon}(v_n - v) dx \longrightarrow 0. \tag{3.16}
\end{aligned}$$

Choose  $\varepsilon > 0$  sufficiently enough such that  $B_{2\varepsilon} \subset \Omega$ . Then each of the above terms tends to 0 or to a non negative limit. Indeed, by  $(H_{a_i})_3$ , we have

$$\begin{aligned}
\left| \int_{\Omega} a_1(\nabla u_n) \cdot \nabla\psi_{\varepsilon}(u_n - u) dx \right| & \leq C_{\varepsilon} \int_{\Omega} (1 + |\nabla u_n|)^{p-1} |u_n - u| dx \\
& \leq C_{\varepsilon} \left( \int_{\Omega} (1 + |\nabla u_n|^p) dx \right)^{p/p'} \left( \int_{\Omega} |u_n - u|^p dx \right)^{1/p} \\
& \leq C_{\varepsilon} \|u_n - u\|_{L^p}^{1/p} \longrightarrow 0 \quad \text{by (3.13)}.
\end{aligned}$$

With a generic constant  $C_{\varepsilon} > 0$ . Similarly for the corresponding term involving  $a_2$ . By a convexity inequality, we obtain that

$$\begin{aligned}
\int_{\Omega} A(x)|u_n|^{p-2}u_n(u_n - u)\psi_{\varepsilon} dx & = \int_{\Omega} A(x) [ (|u_n|^{p-2}u_n - |u|^{p-2}u) + |u|^{p-2}u ] (u_n - u)\psi_{\varepsilon} dx \\
& \geq \int_{\Omega} A(x)|u|^{p-2}u(u_n - u)\psi_{\varepsilon} dx \longrightarrow 0 \text{ as } n \rightarrow +\infty. \tag{3.17}
\end{aligned}$$

Here we used the fact that  $A(x)|u|^{p-2}u \in (W_0^{1,p}(\Omega))'$  (see Remark 3.2) and  $u_n \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$ . Similarly for the term involving  $B$ . Using (H1) and Hölder inequality, (3.13), we can show that

$$\int_{\Omega} [H_u(x, u_n, v_n)(u_n - u)\psi_{\varepsilon} + H_v(x, u_n, v_n)(v_n - v)\psi_{\varepsilon}] dx \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Since  $\psi_{\varepsilon} = 0$  in  $B_{\varepsilon}$ , we have

$$\begin{aligned}
& \left| \lambda \int_{\Omega} \frac{|u_n|^{p-2}u_n}{|x|^p}(u_n - u)\psi_{\varepsilon} dx + \mu \int_{\Omega} \frac{|v_n|^{q-2}v_n}{|x|^q}(v_n - v)\psi_{\varepsilon} dx \right| \\
& \leq \frac{|\lambda|}{\varepsilon^p} \int_{\Omega} |u_n|^{p-1}|u_n - u| dx + \frac{|\mu|}{\varepsilon^q} \int_{\Omega} |v_n|^{q-1}|v_n - v| dx \\
& \leq C_{\varepsilon} (\|v_n\|_{L^p}^{p/p'} \|v_n - v\|_{L^p} + \|v_n\|_{L^q}^{q/q'} \|v_n - v\|_{L^q}) \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.
\end{aligned}$$

Therefore, using (3.16) and the fact that  $\psi_{\varepsilon} = 1$  in  $\Omega \setminus B_{2\varepsilon}$ , we obtain

$$\limsup_{n \rightarrow +\infty} \left( \int_{\Omega \setminus B_{2\varepsilon}} a_1(\nabla u_n) \cdot (\nabla u_n - \nabla u) dx + \int_{\Omega \setminus B_{2\varepsilon}} a_2(\nabla v_n) \cdot (\nabla v_n - \nabla v) dx \right) \leq 0. \tag{3.18}$$

On the other hand, from  $(H_{a_i})_{(iii)}$  we have  $a_1(\nabla u) \in (W_0^{1,p}(\Omega))'$ ,  $a_2(\nabla v) \in (W_0^{1,q}(\Omega))'$ . Therefore

$$\lim_{n \rightarrow +\infty} \left( \int_{\Omega \setminus B_{2\varepsilon}} a_1(\nabla u) \cdot (\nabla u_n - \nabla u) dx + \int_{\Omega \setminus B_{2\varepsilon}} a_2(\nabla v) \cdot (\nabla v_n - \nabla v) dx \right) = 0. \tag{3.19}$$

Subtracting (3.19) from (3.18), we get

$$\limsup_{n \rightarrow +\infty} \left( \int_{\Omega \setminus B_{2\varepsilon}} (a_1(\nabla u_n) - a_1(\nabla u)) \cdot (\nabla u_n - \nabla u) \, dx + \int_{\Omega \setminus B_{2\varepsilon}} a_2(\nabla v_n - a_1(\nabla u)) \cdot (\nabla v_n - \nabla v) \, dx \right) \leq 0. \quad (3.20)$$

Hence, by Corollary 2.4, we obtain

$$\begin{cases} \limsup_{n \rightarrow +\infty} C \|\nabla u_n - \nabla u\|_{L^p(\Omega \setminus B_{2\varepsilon})}^p = 0 & \text{if } p \geq 2, \\ \limsup_{n \rightarrow +\infty} C \|\nabla u_n - \nabla u\|_{L^p(\Omega \setminus B_{2\varepsilon})}^2 = 0 & \text{if } p < 2 \end{cases}$$

and

$$\begin{cases} \limsup_{n \rightarrow +\infty} C \|\nabla v_n - \nabla v\|_{L^q(\Omega \setminus B_{2\varepsilon})}^q = 0 & \text{if } q \geq 2, \\ \limsup_{n \rightarrow +\infty} C \|\nabla v_n - \nabla v\|_{L^q(\Omega \setminus B_{2\varepsilon})}^2 = 0 & \text{if } q < 2. \end{cases}$$

Therefore, applying the Lebesgue inverse theorem, we obtain up to a subsequence

$$(\nabla u_n, \nabla v_n) \longrightarrow (\nabla u, \nabla v), \quad \text{a.e. in } \Omega \setminus B_{2\varepsilon}.$$

Since  $\varepsilon > 0$  is arbitrary, we infer by diagonalization argument that

$$(\nabla u_n, \nabla v_n) \longrightarrow (\nabla u, \nabla v), \quad \text{a.e. in } \Omega.$$

• **Step 02.** We claim that up to a subsequence

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |a_1(\nabla u_n) \cdot \nabla u_n - a_1(\nabla u) \cdot \nabla u - a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u)| \, dx = 0, \quad (3.21)$$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |a_2(\nabla v_n) \cdot \nabla v_n - a_2(\nabla v) \cdot \nabla v - a_2(\nabla v_n - \nabla v) \cdot (\nabla v_n - \nabla v)| \, dx = 0. \quad (3.22)$$

It suffices to establish (3.21), since (3.22) follows by a similar argument. First, we prove the following inequality:  $\forall \varepsilon > 0, \exists C_\varepsilon > 0$  :

$$|a_1(x+y) \cdot (x+y) - a_1(x) \cdot x - a_1(y) \cdot y| \leq \varepsilon a_1(x) \cdot x + \varepsilon + C_\varepsilon a_1(y) \cdot y, \quad \forall x, y \in \mathbb{R}^N. \quad (3.23)$$

Let  $\varepsilon > 0$ . Consider the function  $g(t) := a_1(x+ty) \cdot (x+ty)$ ,  $t \in \mathbb{R}$ . We have  $g$  is of class  $\mathcal{C}^1$  and by the mean value theorem

$$\begin{aligned} |a_1(x+y) \cdot (x+y) - a_1(x) \cdot x - a_1(y) \cdot y| &\leq |a_1(x+y) \cdot (x+y) - a_1(x) \cdot x| + a_1(y) \cdot y \\ &\leq |\nabla a_1(x+\theta y) \cdot (x+\theta y) + a_1(x+\theta y) \cdot y| + a_1(y) \cdot y, \quad \theta \in ]0, 1[ \\ &\leq c'_1(1+|x+\theta y|^{p-1})|y| + c'_1(1+|x+\theta y|^{p-1})|y| + a_1(y) \cdot y, \quad \text{by } (H_{a_i})_3, \\ &\leq C|x|^{p-1}|y| + C|y| + C|y|^p + a_1(y) \cdot y, \quad \text{for some constant } C > 0 \\ &\leq \varepsilon \frac{p-1}{c_1} |x|^p + \varepsilon + C'_\varepsilon |y|^p + a_1(y) \cdot y, \quad \text{(by Young's inequality)} \\ &\leq \varepsilon a_1(x) \cdot x + \varepsilon + C_\varepsilon a_1(y) \cdot y, \quad \text{by Lemma 1.1(ii)}. \end{aligned}$$

Now, take

$$U_n := |a_1(\nabla u_n) \nabla u_n - a_1(\nabla u_n - \nabla u) (\nabla u_n - \nabla u) - a_1(\nabla u) \nabla u|$$

and

$$V_n := (U_n - \varepsilon a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u) - \varepsilon)^+.$$

Since  $\nabla u_n \rightarrow \nabla u$  as  $n \rightarrow +\infty$ , and by (3.23), we have

$$V_n \rightarrow 0, \text{ and } \forall n : V_n \leq C_\varepsilon a_1(\nabla u) \nabla u \in L^1(\Omega).$$

Hence by Lebesgue convergence theorem, we obtain  $V_n \rightarrow 0$  in  $L^1(\Omega)$ . On the other hand, we have  $U_n \leq \varepsilon a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u) + \varepsilon + V_n$ . Then

$$\begin{aligned} \int_{\Omega} U_n dx &\leq \int_{\Omega} (\varepsilon a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u) + \varepsilon) dx + \int_{\Omega} V_n dx \\ &\leq \varepsilon \int_{\Omega} C(1 + |\nabla u_n - \nabla u|^p) dx + \|V_n\|_{L^1(\Omega)} \leq \varepsilon M + \|V_n\|_{L^1(\Omega)}. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we deduce that  $\int_{\Omega} U_n dx \rightarrow 0$  as  $n \rightarrow +\infty$ , which gives (3.21).

• **Step 03.** We are now in a position to prove the convergence of  $(u_n, v_n)_n$  in  $E$ . Taking  $\varphi = (u_n - u)$ ,  $\psi = (v_n - v)$  in (3.4), we obtain

$$\begin{aligned} &\int_{\Omega} a_1(\nabla u_n) \cdot \nabla(u_n - u) dx + \int_{\Omega} a_2(\nabla v_n) \cdot \nabla(v_n - v) dx + \int_{\Omega} A(x) |u_n|^{p-2} u_n (u_n - u) dx \\ &+ \int_{\Omega} B(x) |v_n|^{q-2} v_n (v_n - v) dx - \int_{\Omega} H_u(x, u_n, v_n) (u_n - u) dx - \int_{\Omega} H_v(x, u_n, v_n) (v_n - v) dx \\ &- \lambda \int_{\Omega} \frac{|u_n|^{p-2} u_n}{|x|^p} (u_n - u) dx - \mu \int_{\Omega} \frac{|v_n|^{p-2} v_n}{|x|^p} (v_n - v) dx = o(1). \end{aligned} \quad (3.24)$$

By the same arguments as in Step 1, the terms containing  $A, B, H$ , converge either to 0 or to nonnegative limits. For the remaining terms, we have

$$a_1(\nabla u_n) \rightarrow a_1(\nabla u), \quad \text{a.e. in } \Omega$$

and

$$(a_1(\nabla u_n))_n \text{ is bounded in } L^{p'}(\Omega)^N \text{ by } (H_{a_i})_3.$$

Hence by Lemma 4.8 of [18], we obtain  $a_1(\nabla u_n) \rightarrow a_1(\nabla u)$  in  $L^{p'}(\Omega)$  and consequently

$$\begin{aligned} \int_{\Omega} a_1(\nabla u_n) \cdot \nabla(u_n - u) dx &= \int_{\Omega} a_1(\nabla u_n) \cdot \nabla u_n - a_1(\nabla u) \cdot \nabla u dx + o(1) \\ &\stackrel{(3.21)}{=} \int_{\Omega} a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u) dx + o(1). \end{aligned} \quad (3.25)$$

Similarly, we get

$$\begin{aligned} \int_{\Omega} a_2(\nabla v_n) \cdot \nabla(v_n - v) dx &= \int_{\Omega} a_2(\nabla v_n) \cdot \nabla v_n - a_2(\nabla v) \cdot \nabla v dx + o(1) \\ &\stackrel{(3.22)}{=} \int_{\Omega} a_2(\nabla v_n - \nabla v) \cdot (\nabla v_n - \nabla v) dx + o(1). \end{aligned} \quad (3.26)$$

For the Hardy terms, using (3.14) we get :

$$\frac{|u_n|^{p-2} u_n}{|x|^{p-1}} \rightarrow \frac{|u|^{p-2} u}{|x|^{p-1}}, \quad \text{a.e. in } \Omega$$

and by Hardy's inequality, we obtain

$$\left( \frac{|u_n|^{p-2} u_n}{|x|^{p-1}} \right)_n \text{ is bounded in } L^{p'}(\Omega) \text{ and } \frac{u}{|x|} \in L^p(\Omega).$$

Hence by [18, Lemme 4.8, and Lemme 4.6], we obtain

$$\begin{aligned} \int_{\Omega} \frac{|u_n|^{p-2} u_n (u_n - u)}{|x|^p} dx &= \int_{\Omega} \frac{|u_n|^p}{|x|^p} - \frac{|u|^p}{|x|^p} dx + o(1) \\ &= \int_{\Omega} \frac{|u_n - u|^p}{|x|^p} + o(1). \end{aligned} \quad (3.27)$$

Similarly, we have

$$\begin{aligned} \int_{\Omega} \frac{|v_n|^{p-2} v_n (v_n - v)}{|x|^p} dx &= \int_{\Omega} \frac{|v_n|^p}{|x|^p} - \frac{|v|^p}{|x|^p} dx + o(1) \\ &= \int_{\Omega} \frac{|v_n - v|^p}{|x|^p} dx + o(1). \end{aligned} \quad (3.28)$$

We substitute (3.25)–(3.28) in (3.24), we obtain

$$\begin{aligned} \int_{\Omega} a_1(\nabla u_n - \nabla u) \cdot (\nabla u_n - \nabla u) dx - \lambda \int_{\Omega} \frac{|u_n - u|^p}{|x|^p} dx \\ + \int_{\Omega} a_2(\nabla v_n - \nabla v) \cdot (\nabla v_n - \nabla v) dx - \mu \int_{\Omega} \frac{|v_n - v|^p}{|x|^p} dx + o(1) \leq 0. \end{aligned}$$

Using Lemma 1.1(ii) and Hardy's inequality, we obtain

$$\left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right) \int_{\Omega} |\nabla u_n - \nabla u|^p dx + \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \int_{\Omega} |\nabla v_n - \nabla v|^q dx + o(1) = 0.$$

This concludes the proof.  $\square$

Now, we show the necessary geometric conditions of the functional  $J$ .

**Lemma 3.6.** *Assume that  $(H_{a_i})$  and (H5) hold. Then, there exist  $\rho, \alpha > 0$  such that  $J(u, v) \geq \alpha$  for any  $(u, v) \in E$  with  $\|(u, v)\|_E = \rho$ .*

*Proof.* By virtue of hypothesis (H5), for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$H(x, z) \leq (\ell + \varepsilon)|z|^p, \quad \text{for a.a. } x \in \Omega \text{ and all } |z| \leq \delta, \quad (3.29)$$

where  $\ell = \limsup_{|z| \rightarrow 0} \frac{H(x, z)}{|z|^p}$ . Let  $(u, v) \in E$  with  $0 < \|(u, v)\|_E = \rho < \min\{\delta, 1\}$ . Using (3.29), Lemma 1.1(iii) and Lemma 2.1, we get:

$$\begin{aligned} J(u, v) &\geq \frac{c_1}{p(p-1)} \int_{\Omega} |\nabla u|^p dx + \frac{c_2}{q(q-1)} \int_{\Omega} |\nabla v|^q dx + \frac{1}{p} \int_{\Omega} A(x) |u|^p dx \\ &\quad + \frac{1}{q} \int_{\Omega} B(x) |v|^q dx - (\ell + \varepsilon) \int_{\Omega} |(u, v)|^p dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx - \frac{\mu}{q} \int_{\Omega} \frac{|v_n|^q}{|x|^q} dx \\ &\geq \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\} (\|\nabla u\|_{L^p}^p + \|\nabla v\|_{L^q}^q) - \frac{\ell + \varepsilon}{\nu_p} \|(u, v)\|_E^p. \end{aligned}$$

Since  $\|\nabla u\|_{L^q} \leq \|(u, v)\|_E \leq 1$  and  $q \leq p$ , we get

$$\begin{aligned} J(u, v) &\geq \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\} (\|\nabla u\|_{L^p}^p + \|\nabla v\|_{L^q}^q) - \frac{\ell + \varepsilon}{\nu_p} \|(u, v)\|_E^p \\ &\geq \left( \frac{1}{2^{p-1}} \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\} - \frac{\ell + \varepsilon}{\nu_p} \right) \|(u, v)\|_E^p. \end{aligned}$$

By hypothesis (H5), we can choose  $\varepsilon > 0$  small enough such that

$$\frac{1}{2^{p-1}} \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\} - \frac{\ell + \varepsilon}{\nu_p} > 0.$$

Therefore,

$$J(u, v) \geq \left( \frac{1}{2^{p-1}} \min \left\{ \frac{1}{p} \left( \frac{c_1}{p-1} - \frac{\lambda^+}{\mathcal{H}_p} \right), \frac{1}{q} \left( \frac{c_2}{q-1} - \frac{\mu^+}{\mathcal{H}_q} \right) \right\} - \frac{\ell + \varepsilon}{\nu_p} \right) \rho^p =: \alpha > 0$$

for any  $(u, v) \in E$ , with  $\|(u, v)\|_E = \rho$ .  $\square$

We show the second geometric condition.

**Lemma 3.7.** *Suppose that the assumptions  $(H_{a_i})$  and (H4) are satisfied. Then there exists  $(e_1, e_2) \in E$  such that  $\|(e_1, e_2)\|_E > \rho$  and  $J(e_1, e_2) < 0$ .*

*Proof.* Using (H4), for any  $\zeta > 0$ , we can find  $C_\zeta > 0$  such that

$$H(x, z) \geq \zeta |z|^p - C_\zeta, \quad \text{for a.a. } x \in \Omega \text{ and all } z \in \mathbb{R}^2. \quad (3.30)$$

Let  $(u^*, v^*) \in E$  be a fixed pair of positive functions. Then, by virtue of Lemma 1.1(iii) and using (3.30), for  $t > 0$ , we get:

$$\begin{aligned} J(tu^*, tv^*) &\leq c_5 (|\Omega| + t^p \|\nabla u^*\|_{L^p}^p) + c'_5 (|\Omega| + t^q \|\nabla v^*\|_{L^q}^q) \\ &\quad + \frac{t^p}{p} \int_{\Omega} A(x) |u^*|^p dx + \frac{t^q}{q} \int_{\Omega} B(x) |v^*|^q dx - \zeta \|(u_0, v_0)\|_{L^p \times L^p}^p + C_\zeta |\Omega| \\ &\quad - \frac{\lambda}{p} t^p \int_{\Omega} \frac{|u^*|^p}{|x|^p} dx - \frac{\mu}{q} t^q \int_{\Omega} \frac{|v^*|^q}{|x|^q} dx \\ &\leq t^p \left( c_5 \|\nabla u^*\|_{L^p}^p + \frac{1}{p} \int_{\Omega} A(x) |u^*|^p dx - \zeta \|(u^*, v^*)\|_{L^p \times L^p}^p - \frac{\lambda}{p} \int_{\Omega} \frac{|u^*|^p}{|x|^p} dx \right) \\ &\quad + \frac{c'_5 t^q}{q} \|\nabla v^*\|_{L^q}^q + \frac{t^q}{q} \int_{\Omega} B(x) |v^*|^q dx - \frac{\mu}{q} t^q \int_{\Omega} \frac{|v^*|^q}{|x|^q} dx + \left( \frac{c_5}{p} + \frac{c'_5}{q} + C_\zeta \right) |\Omega|. \end{aligned}$$

Choosing  $\zeta > 0$  large enough one has  $J(tu^*, tv^*) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Hence there exists some  $t_0 > 0$  such that  $\|(t_0 u^*, t_0 v^*)\|_E > \rho$  and  $J(t_0 u^*, t_0 v^*) < 0$ .  $\square$

*Proof of Theorem 3.3.* We know that  $J \in C^1(E)$  and  $J(0, 0) = 0$ . Combining this fact with Proposition 3.5, Lemma 3.6 and Lemma 3.7, an application of Theorem 2.6 gives the existence of a critical point  $(u, v) \in E$  of  $J$ , with  $(u, v) \neq (0, 0)$ . Hence, we obtain the existence of nontrivial weak solution of Problem (1.1); that is

$$\begin{aligned} &\int_{\Omega} a_1(\nabla u) \nabla \varphi dx + \int_{\Omega} a_2(\nabla v) \nabla \psi dx + \int_{\Omega} A(x) |u|^{p-2} u \varphi dx + \int_{\Omega} B(x) |v|^{q-2} v \psi dx \\ &\quad - \int_{\Omega} H_u(x, u, v) \varphi dx - \int_{\Omega} H_v(x, u, v) \psi dx - \lambda \int_{\Omega} \frac{|u|^{p-2} u}{|x|^p} \varphi dx - \mu \int_{\Omega} \frac{|v|^{q-2} v}{|x|^q} \psi dx = 0, \end{aligned}$$

for all  $(\varphi, \psi) \in E$ , taking  $(\varphi, \psi) = (-u^-, -v^-)$  and by using (ii) in Lemma 1.1 and (H2), we get

$$\begin{aligned} 0 &= - \int_{\Omega} H_u(x, u, v) u^- dx - \int_{\Omega} H_v(x, u, v) v^- dx - \int_{\Omega} a_1(-\nabla u^-) \nabla(-\nabla u^-) dx \\ &\quad + \int_{\Omega} a_2(-\nabla v^-) \nabla(-\nabla v^-) dx + \int_{\Omega} A(x) |u^-|^p dx + \int_{\Omega} B(x) |v^-|^q dx \\ &\geq \frac{c_1}{p-1} \|\nabla u^-\|_{L^p}^p + \frac{c_2}{q-1} \|\nabla v^-\|_{L^q}^q \geq 0. \end{aligned}$$

Therefore,  $(u^-, v^-) = (0, 0)$  a.e. in  $\Omega \times \Omega$ . In conclusion, the weak solution of (1.1) is nonnegative.  $\square$

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