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Well-posedness of solutions for a double-diffusive convection system with anisotropic non-Newtonian operators and damping terms

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Abstract. This paper investigates the initial-boundary value problem for a double-diffusive convection system that incorporates anisotropic non-Newtonian operators and damping terms in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. The primary goal of this work is to establish the existence of weak solutions for this system. To achieve this, we first construct approximate solutions utilizing the Galerkin method. Subsequently, uniform estimates for these approximations are derived through an energy method. Finally, by combining compactness and monotonicity arguments, we prove the existence of weak solutions for the problem.

Keywords: double-diffusive convection, anisotropic non-Newtonian operators, damping terms, weak solution, existence.

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1 Introduction and main result

Double-diffusive convection describes a fluid phenomenon driven by the combined effects of two scalar fields having different diffusion rates. The core principle lies in the differing diffusion velocities of these fields, which leads to density stratification and, consequently, convective motion. Within the framework of fluid mechanics, particularly for incompressible double-diffusive convection systems subject to the Oberbeck–Boussinesq (OB) approximation and incorporating damping terms, their dynamics are mathematically described by a set of equations

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\boldsymbol{\tau} + \dfrac{1}{\rho_0}\nabla P + \kappa|\boldsymbol{u}|^{\sigma-2}\boldsymbol{u} = \big(\beta_\theta(\theta-\theta_c) + \beta_\psi(\psi-\psi_c)\big)\,\boldsymbol{g} + \boldsymbol{Q}_1, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \theta_t + (\boldsymbol{u}\cdot\nabla)\theta - \alpha_1\Delta\theta = Q_2, \\ \psi_t + (\boldsymbol{u}\cdot\nabla)\psi - \alpha_2\Delta\psi = Q_3, \end{cases} \tag{1.1}$$

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here, $\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$, $\theta(\boldsymbol{x},t) \in \mathbb{R}$, $\psi(\boldsymbol{x},t) \in \mathbb{R}$, and $P(\boldsymbol{x},t) \in \mathbb{R}$ denote the unknown velocity field, temperature, solute concentration, and pressure, respectively, at position $\boldsymbol{x} \in \Omega$ and time $t \in [0,T]$. Furthermore, θ_c and ψ_c represent the characteristic temperature and concentration, respectively, while ρ_0 is the mean density. β_θ and β_ψ are constants which is chosen for convenience in matching the OB equation of state $\rho = \rho(\theta, \psi, P)$. The parameter α_1 denotes the thermal conductivity coefficient, and α_2 is the diffusion coefficient. $\boldsymbol{Q}_1(\boldsymbol{x},t)$, $Q_2(\boldsymbol{x},t)$, $Q_3(\boldsymbol{x},t)$, and $\boldsymbol{g}(\boldsymbol{x},t)$ are prescribed source terms (typically, $\boldsymbol{g} = g\boldsymbol{e}$, where g is the free-fall acceleration and $\boldsymbol{e} = (0,0,1)$). The stress tensor $\boldsymbol{\tau}$ is related to the strain rate tensor $\mathcal{D}\boldsymbol{u}$ by the constitutive law $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$, where $\mathcal{D}\boldsymbol{u} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\right)$. For a comprehensive derivation and physical discussion of double-diffusive convection systems under the OB approximation, readers may refer to [12], for instance. In (1.1), the damping term $\kappa|\boldsymbol{u}|^{\sigma-2}\boldsymbol{u}$ ($\kappa > 0$, $\sigma > 1$), often referred to as an absorption term, lacks direct physical justification in Fluid Mechanics, though it can be conceptualized as part of the external body forces field (see [1–3]).

For Newtonian fluids, the relationship between $\boldsymbol{\tau}$ and $\mathcal{D}\boldsymbol{u}$ takes a linear form

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = \nu\mathcal{D}\boldsymbol{u}, \tag{1.2}$$

with $\nu > 0$ is the kinematic viscosity coefficient. When $\kappa = 0$ (i.e. without the damping term), there have been numerous relevant research outcomes. Rojas-Medar and Lorca, for instance, initially investigated the existence, uniqueness, and regularity of local strong solutions for this system using the Galerkin approximation method [21,22]. Subsequently, they established the existence of global strong solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ [23]. Further extending their work, in [14] they proved the existence and uniqueness of weak solutions for the equations in a bounded domain of $\mathbb{R}^N$, where $N \geq 2$. Chen and Guo [8] studied the well-posedness of the initial-boundary value problem for this system in a bounded domain. They then addressed the well-posedness of the Cauchy problem for the double-diffusive convection system in [9], encompassing the existence, uniqueness, and global stability of solutions. Building upon these findings, they later proved the global existence of weak solutions for the three-dimensional Cauchy problem in [7], employing regularization approximation and compactness theory. Wu [27] demonstrated several Serrin-type regularity criteria for the system, expressed in terms of the velocity field or its gradient. Furthermore, in [26], the authors extended these results to the middle eigenvalue of the strain tensor in anisotropic Lebesgue spaces. Ragusa–Wu [20] established global weak solutions for the non-dimensional system in a domain Ω under Navier boundary conditions, as well as local strong solutions with uniqueness. Moreover, they obtained the classical Serrin-type blow-up criterion for the local strong solution in the Lorentz space. Very recently, Wu [28] proved continuation criteria for strong solutions of the 3D double-diffusive convection system, specifically considering cases involving the deformation tensor in Vishik spaces. When $\kappa \neq 0$, the authors in [6] focused on the Navier–Stokes equations (specifically, with $\theta = 0$ and $\psi = 0$) and investigated the existence and uniqueness of solutions in the whole space $\mathbb{R}^3$. They established the existence of a global-in-time weak solution for any $\sigma \geq 1$, and global-in-time strong solutions for $\sigma \geq \frac{7}{2}$. Furthermore, uniqueness was demonstrated for the range $\frac{7}{2} \leq \sigma \leq 5$. Their methodology involved employing the Galerkin method to construct approximate solutions and deriving a priori estimates crucial for the compactness argument. Subsequently, Zhong [29,30] extended these findings to encompass any $\sigma \geq 1$ and $\kappa > 0$. For additional related results, one may also refer to [17–19].

For non-Newtonian fluids, the relationship between the stress tensor $\boldsymbol{\tau}$ and the strain rate tensor $\mathcal{D}\boldsymbol{u}$ is typically nonlinear (see e.g., [24]). One widely adopted constitutive approach is the power-law model (see e.g., [16]).

De Oliveira [10] first investigated an initial-boundary value problem for system (1.1) equipped with a non-Newtonian constitutive law, specifically in a simplified setting where $\theta = 0$ and $\psi = 0$. Assuming the stress tensor took the specific form $\boldsymbol{\tau}(\nabla \boldsymbol{u}) = |\nabla \boldsymbol{u}|^{q-2}\nabla \boldsymbol{u}$, the author proved the existence of global weak solutions for any constant exponent $q > \frac{2N}{N+2}$ and $\sigma > 1$. Their methodology involved regularization techniques, compactness arguments, coupled with local decomposition of the pressure and the Lipschitz-truncation method. The study of anisotropic non-Newtonian problems, particularly within the context of fluid dynamics, was first explored by Antontsev and Oliveira [4,5]. Specifically, [4] addressed the existence of weak and very weak solutions for the stationary problem, while [5] investigated the evolutionary problem, focusing on the existence of weak solutions and their asymptotic behavior. In these two papers, the authors considered a specific anisotropic stress tensor given by

$$\boldsymbol{\tau} = \nu \sum_{i=1}^{N} D_i(|D_i \boldsymbol{u}|^{q_i-2} D_i \boldsymbol{u}). \tag{1.3}$$

However, to date, a dedicated study of the full anisotropic non-Newtonian double-diffusive convection system originating from the merger of equations (1.1) and the anisotropic power-law constitutive relation (1.3) remains absent in the literature. This paper aims to address this research gap. More specifically, we consider the following initial-boundary value problem

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \displaystyle\sum_{i=1}^{3} \kappa_i |u_i|^{\sigma_i-2} u_i \boldsymbol{e}_i - \nu \sum_{i=1}^{3} D_i(|D_i \boldsymbol{u}|^{q_i-2} D_i \boldsymbol{u}) + \frac{1}{\rho_0}\nabla P & \\ \qquad\qquad = (\beta_\theta \theta + \beta_\psi \psi)\boldsymbol{g} + \boldsymbol{Q}_1, & \text{in } Q_T, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } Q_T, \\ \theta_t + (\boldsymbol{u}\cdot\nabla)\theta - \alpha_1 \Delta\theta = Q_2, & \text{in } Q_T, \\ \psi_t + (\boldsymbol{u}\cdot\nabla)\psi - \alpha_2 \Delta\psi = Q_3, & \text{in } Q_T, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(x), \quad \theta|_{t=0} = \theta_0(x), \quad \psi|_{t=0} = \psi_0(x), & \text{in } \Omega, \\ \boldsymbol{u} = 0, \quad \theta = 0, \quad \psi = 0, & \text{in } \Gamma_T, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a compact boundary $\partial\Omega$, and $0 < T < \infty$, $Q_T = \Omega \times [0,T]$, $\Gamma_T = \partial\Omega \times [0,T]$. $D_i \boldsymbol{u} = (\partial_i u_1, \partial_i u_2, \partial_i u_3)$, $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. The exponents σ_i and q_i are assumed to be constant with possible distinct values ranging in $(1,\infty)$, and κ_i are non-negative constants.

The aim of this paper is to establish the existence of weak solutions for system (1.4). Our methodology comprises three key stages: constructing approximate solutions using the Galerkin method, deriving uniform a priori estimates for these approximations through energy methods, and finally proving the existence of weak solutions by applying compactness and monotonicity arguments.

To facilitate our analysis, we introduce the vectors $\vec{\boldsymbol{\sigma}} := (\sigma_1, \sigma_2, \sigma_3)$ and $\vec{\boldsymbol{q}} := (q_1, q_2, q_3) \in \mathbb{R}^3$, whose components correspond to the exponents of the anisotropic damping and diffusion terms respectively in system (1.4). We assume $1 < \sigma_i, q_i < \infty$ for all $i \in \{1,2,3\}$. Furthermore, we define

$$\sigma^+ := \max_{i\in\{1,2,3\}} \sigma_i, \qquad \sigma^- := \min_{i\in\{1,2,3\}} \sigma_i, \qquad q^+ := \max_{i\in\{1,2,3\}} q_i, \qquad q^- := \min_{i\in\{1,2,3\}} q_i.$$

We then define the following anisotropic Lebesgue and Sobolev spaces, which are essential for our functional analytic framework

$$L^{\vec{q}}(\Omega) = \{ \boldsymbol{u} \mid \boldsymbol{u} \in L^{q_i}(\Omega), \quad \forall i = 1,2,3 \},$$
$$W^{1,\vec{q}}(\Omega) = \left\{ \boldsymbol{u} \mid \boldsymbol{u} \in W^{1,1}(\Omega),\ D_i \boldsymbol{u} \in L^{q_i}(\Omega), \quad \forall\, i = 1,2,3 \right\}.$$

These spaces are Banach spaces endowed with the norms

$$\|\boldsymbol{u}\|_{L^{\vec{q}}(\Omega)} = \sum_{i=1}^{3} \|\boldsymbol{u}\|_{L^{q_i}(\Omega)}, \qquad \|\boldsymbol{u}\|_{W^{1,\vec{q}}(\Omega)} = \|\boldsymbol{u}\|_{L^1(\Omega)} + \sum_{i=1}^{3} \|D_i \boldsymbol{u}\|_{L^{q_i}(\Omega)}.$$

Lemma 1.1 ([11]). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with Lipschitz boundary. If $\sum_{j=1}^{3} q_j^{-1} > 1$, then the following embedding relations hold*

$$W^{1,\vec{q}}(\Omega) \hookrightarrow L^s(\Omega), \qquad 1 \le s \le q_a^*, \tag{1.5}$$
$$W^{1,\vec{q}}(\Omega) \hookrightarrow\hookrightarrow L^s(\Omega), \qquad 1 \le s < q_a^*, \tag{1.6}$$

where $q_a^* = \max\{\overline{q}^*,\ q^+\}$ and $\overline{q}^* = \frac{3}{\sum_{j=1}^{3} q_j^{-1} - 1}$.

Remark 1.2. By Lemma 1.1, we assume q_i $(i = 1,2,3)$ satisfy $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} > 1$.

To rigorously define the notion of weak solutions considered in this paper, we first introduce the following specialized functional spaces

$$\mathcal{V} := \{ \boldsymbol{u} \in C_0^\infty(\Omega) : \operatorname{div} \boldsymbol{u} = 0 \},$$
$$H := \text{closure of } \ \mathcal{V} \text{ in the norm} \| \cdot \|_{L^2(\Omega)},$$
$$V_q := \text{closure of } \ \mathcal{V} \text{ in the norm} \| \cdot \|_{W^{1,q}(\Omega)}.$$

Furthermore, we define the anisotropic analogue of $V_{\vec{q}}$ as

$$V_{\vec{q}} := \text{closure of } \ \mathcal{V} \text{ in the norm } \| \cdot \|_{W^{1,\vec{q}}(\Omega)}.$$

The natural parabolic anisotropic space relevant to our problem is then defined as

$$L^{\vec{q}}\left(0,T;V_{\vec{q}}\right) := \left\{ \boldsymbol{u} \mid \boldsymbol{u} : [0,T] \to V_{\vec{q}}\ ,\ \boldsymbol{u}, |D_i \boldsymbol{u}|^{q_i} \in L^1\left(Q_T\right),\ \forall\, i = 1,2,3 \right\}.$$

This space is a Banach space with the norm defined by

$$\|\boldsymbol{u}\|_{L^{\vec{q}}\left(0,T;V_{\vec{q}}\right)} = \|\boldsymbol{u}\|_{L^1(Q_T)} + \sum_{i=1}^{3} \|D_i \boldsymbol{u}\|_{L^{q_i}(Q_T)}.$$

Moreover, for a bounded domain Ω and a finite T, the following continuous embeddings hold

$$L^{q^+}(0,T;V_{q^+}) \hookrightarrow L^{\vec{q}}(0,T;V_{\vec{q}}) \hookrightarrow L^{q^-}(0,T;V_{q^-}). \tag{1.7}$$

Consequently, as a closed subspace of $L^{q^-}(0,T;V_{q^-})$, the anisotropic parabolic space $L^{\vec{q}}(0,T;V_{\vec{q}})$ is both separable and reflexive. We denote by $L^{\vec{q}\,'}(0,T;V_{\vec{q}}')$ the dual spaces of $L^{\vec{q}}(0,T;V_{\vec{q}})$, where $V_{\vec{q}}'$ represents the dual space of $V_{\vec{q}}$.

Definition 1.3. Assume that $\boldsymbol{Q}_1 \in L^{\vec{q}'}(0,T;V'_{\vec{q}})$, $Q_2, Q_3 \in L^2(0,T;H^{-1})$, $\boldsymbol{g} \in L^\infty(Q_T)$, $q^- \geq 2$, $q_a^* \geq 2$. Then a triple $(\boldsymbol{u}, \theta, \psi)$ is called a weak solution to the system (1.4) if

(i). $\boldsymbol{u} \in L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}})$, $\theta, \psi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1)$, $\boldsymbol{e}_i \cdot \boldsymbol{u} \in L^{\sigma_i}(\Omega)$ for all $i \in \{1,2,3\}$;

(ii). $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\theta(0) = \theta_0$, $\psi(0) = \psi_0$;

(iii). For every $\boldsymbol{\varphi} \in V_{\vec{q}} \cap L^\delta(\Omega)$, $\boldsymbol{e}_i \cdot \boldsymbol{\varphi} \in L^{\sigma_i}(\Omega)$ for all $i \in \{1,2,3\}$, $\phi \in H^1$, and for a.e. $t \in [0,T]$,

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\varphi}\,dx + \sum_{i=1}^3 \kappa_i \int_\Omega |u_i|^{\sigma_i-2}u_i\boldsymbol{e}_i\cdot\boldsymbol{\varphi}\,dx + \nu\sum_{i=1}^3\int_\Omega |D_i\boldsymbol{u}|^{q_i-2}D_i\boldsymbol{u}\cdot D_i\boldsymbol{\varphi}\,dx \\ \qquad + \displaystyle\int_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{Q}_1\cdot\boldsymbol{\varphi}\,dx + \beta_\theta\int_\Omega \theta\boldsymbol{g}\cdot\boldsymbol{\varphi}\,dx + \beta_\psi\int_\Omega \psi\boldsymbol{g}\cdot\boldsymbol{\varphi}\,dx, \\ \dfrac{d}{dt}\displaystyle\int_\Omega \theta\cdot\phi\,dx + \int_\Omega [(\boldsymbol{u}\cdot\nabla)\theta]\cdot\phi\,dx + \alpha_1\int_\Omega \nabla\theta\cdot\nabla\phi\,dx = \int_\Omega Q_2\cdot\phi\,dx, \\ \dfrac{d}{dt}\displaystyle\int_\Omega \psi\cdot\phi\,dx + \int_\Omega [(\boldsymbol{u}\cdot\nabla)\psi]\cdot\phi\,dx + \alpha_2\int_\Omega \nabla\psi\cdot\nabla\phi\,dx = \int_\Omega Q_3\cdot\phi\,dx, \end{cases} \tag{1.8}$$

where δ is defined by

$$\frac{1}{q_a^*} + \frac{1}{q^-} + \frac{1}{\delta} = 1. \tag{1.9}$$

Remark 1.4. The pressure term P is not included in Definition 1.3. In fact, once the triple $(\boldsymbol{u}, \theta, \psi)$ has been determined, P can be uniquely recovered by de Rham's theorem.

Remark 1.5. As usual, the condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\theta(0) = \theta_0$, $\psi(0) = \psi_0$ is interpreted in the following sense:

$$\lim_{t\to 0^+}\int_\Omega \boldsymbol{u}(t)\cdot\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\varphi}\,dx, \qquad \forall\boldsymbol{\varphi}\in V_{\vec{q}}\cap L^\delta(\Omega), \tag{1.10}$$

$$\lim_{t\to 0^+}\int_\Omega \theta(t)\cdot\phi\,dx = \int_\Omega \theta_0\cdot\phi\,dx, \qquad \lim_{t\to 0^+}\int_\Omega \psi(t)\cdot\phi\,dx = \int_\Omega \psi_0\cdot\phi\,dx, \quad \forall\phi\in L^2(\Omega). \tag{1.11}$$

Remark 1.6. If $\sigma^+ \leq q^- + 2 - 2q^-/q_a^*$, then the continuous embedding $L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}}) \hookrightarrow L^{\sigma^+}(Q_T)$ holds. Consequently, in Definition 1.3, we only need to require $\boldsymbol{u} \in L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}})$. It is important to note that $\boldsymbol{\varphi} \in L^\delta(\Omega)$ is necessary to ensure the boundedness of the convective integral term when $\boldsymbol{u}$ is only assumed to belong to $L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}})$. However, if $\delta \leq q_a^*$ or if $\delta \leq \sigma^-$, then only $\boldsymbol{\varphi} \in V_{\vec{q}}$ and $\boldsymbol{e}_i \cdot \boldsymbol{\varphi} \in L^{\sigma_i}(\Omega)$ (for all $i = 1,2,3$) are required. This is due to the fact that, under these conditions, the embeddings $V_{\vec{q}} \hookrightarrow L^\delta(\Omega)$ and $L^{\sigma_i}(\Omega) \hookrightarrow L^\delta(\Omega)$ (for all $i = 1,2,3$) are valid.

Remark 1.7. If we denote $V := V_{\vec{q}} \cap L^\delta(\Omega) \cap \prod_{i=1}^3 L^{\sigma_i}(\Omega)$, then when $q_a^* \geq 2$, the following continuous embeddings holds

$$V \hookrightarrow V_{\vec{q}} \hookrightarrow H \cong H' \hookrightarrow V'_{\vec{q}} \hookrightarrow V'. \tag{1.12}$$

Our main existence result concerning weak solutions is the following.

Theorem 1.8. *Let Ω be a bounded domain in $\mathbb{R}^3$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that $\boldsymbol{u}_0 \in H$, $\theta_0, \psi_0 \in L^2(\Omega)$, $\boldsymbol{Q}_1 \in L^{\vec{q}\,'}(0,T;V'_{\vec{q}})$, $Q_2, Q_3 \in L^2(0,T;H^{-1})$, $\boldsymbol{g} \in L^\infty(Q_T)$, if $q^- > 2$ and $q_a^* \geq q^*$, where*

$$q^* = \begin{cases} \dfrac{2q^-(q^- - 1)}{(q^- + 1)(q^- - 2)}, & 2 < q^- < 3, \\ \dfrac{2q^-}{q^- - 1}, & q^- \geq 3. \end{cases} \tag{1.13}$$

Then, for any $\vec{\sigma}$ with $\sigma_i > 1$ where $i \in \{1,2,3\}$, system (1.4) has at least one weak solutions $(\boldsymbol{u}, \theta, \psi)$ in the sense of Definition 1.3, satisfying

$$\boldsymbol{u} \in C_w([0,T];H) \cap L^{\vec{q}}(0,T;V_{\vec{q}}), \qquad \theta,\ \psi \in C_w([0,T];L^2(\Omega)) \cap L^2(0,T;H^1),$$

where a function $v \in C_w([0,T];X)$ means that $v : [0,T] \to X$ is weakly continuous.

2 Preliminary lemmas

In this section, we recall some basic facts which will be used later.

Lemma 2.1 ([13])**.** *For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)$, define*

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} = \sum_{i,j=1}^{3} \int_\Omega u_i(x)\frac{\partial v_j(x)}{\partial x_i} w_j(x)\,dx, \tag{2.1}$$

if $\operatorname{div}\boldsymbol{u} = 0$, *then $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$ and $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$.*

Lemma 2.2 ([13])**.** *Let B_0, B, B_1 be three Banach spaces satisfying $B_0 \subset B \subset B_1$, where B_0 and B_1 are reflexive, and the embedding $B_0 \hookrightarrow B$ is compact. Let*

$$W = \left\{ \boldsymbol{u} \,\middle|\, \boldsymbol{u} \in L^{p_0}(0,T;B_0),\ \boldsymbol{u}' = \frac{d\boldsymbol{u}}{dt} \in L^{p_1}(0,T;B_1) \right\},$$

if $1 < p_i < \infty$, for $i = 0,1$, then the embedding $W \hookrightarrow L^{p_0}(0,T;B)$ is compact.

Lemma 2.3 ([25])**.** *Let X and Y be two Banach spaces, with X continuously embedded into Y. If the function $\boldsymbol{u} \in L^\infty(0,T;X)$ and $\boldsymbol{u} : [0,T] \to Y$ is weakly continuous, then $\boldsymbol{u} : [0,T] \to X$ is weakly continuous.*

3 The proof of the Theorem 1.8

The primary objective of this section is to prove Theorem 1.8, we will use the method of the Galerkin approximation combined with the theory of monotone operators.

3.1 Existence of approximative solutions

We begin by defining the space $\widetilde{V} :=$ closure of $\mathcal{V}$ in $W^{3,2}(\Omega)$. Let $\{\boldsymbol{\varphi}_r\}_{r\in\mathbb{N}}$ be a family of non-trivial eigenfunctions $\boldsymbol{\varphi}_j$ corresponding to the positive eigenvalues $\lambda_j > 0$ of the following spectral problem:

$$\sum_{|k|=3} (D^k\boldsymbol{\varphi}_j, D^k v) = \lambda_j(\boldsymbol{\varphi}_j, v), \qquad \forall\, v \in \widetilde{V}. \tag{3.1}$$

It follows from Theorem 4.11 in [15] that this family $\{\boldsymbol{\varphi}_r\}_{r\in\mathbb{N}}$ forms an orthogonal basis in $\widetilde{V}$ and can be chosen to be orthonormal in H. Similarly, let $\{\phi_r\}_{r\in\mathbb{N}}$ be the family of eigenfunctions of the Laplace operator $-\triangle$, serving as a basis for $H^1(\Omega)$. For any given $m \in \mathbb{N}$, we then define the finite-dimensional approximation spaces V^m and W^m as follows:

$$V^m = \mathrm{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_m\}, \qquad W^m = \mathrm{span}\{\phi_1, \phi_2, \dots, \phi_m\}.$$

These spaces will be used for constructing Galerkin approximations of the velocity and scalar fields, respectively.

We seek the Galerkin approximations of system (1.4) in the following forms:

$$\boldsymbol{u}^m(x,t) = \sum_{k=1}^m c_k^m(t)\boldsymbol{\varphi}_k(x), \qquad \theta^m(x,t) = \sum_{k=1}^m h_k^m(t)\phi_k(x), \qquad \psi^m(x,t) = \sum_{k=1}^m d_k^m(t)\phi_k(x),$$

where $\boldsymbol{\varphi}_k \in V^m$ and $\phi_k \in W^m$ for $k = 1, 2, \dots, m$. The time-dependent coefficients $c_k^m(t)$, $h_k^m(t)$ and $d_k^m(t)$ are determined by the following system of ordinary differential equations

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial \boldsymbol{u}^m}{\partial t}\cdot\boldsymbol{\varphi}_k\,dx + \sum_{i=1}^3 \kappa_i \int_\Omega |u_i^m|^{\sigma_i-2}u_i^m\cdot\boldsymbol{e}_i\cdot\boldsymbol{\varphi}_k dx + \nu\sum_{i=1}^3\int_\Omega |D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\cdot D_i\boldsymbol{\varphi}_k\,dx \\ \qquad\displaystyle + \int_\Omega [(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{u}^m]\cdot\boldsymbol{\varphi}_k\,dx = \int_\Omega \boldsymbol{Q}_1\cdot\boldsymbol{\varphi}_k\,dx + \beta_\theta\int_\Omega \theta^m\boldsymbol{g}\cdot\boldsymbol{\varphi}_k dx + \beta_\psi\int_\Omega \psi^m\boldsymbol{g}\cdot\boldsymbol{\varphi}_k dx, \\ \displaystyle\int_\Omega \frac{\partial\theta^m}{\partial t}\cdot\phi_k\,dx + \int_\Omega [(\boldsymbol{u}^m\cdot\nabla)\theta^m]\cdot\phi_k\,dx + \alpha_1\int_\Omega \nabla\theta^m\cdot\nabla\phi_k\,dx = \int_\Omega Q_2\cdot\phi_k\,dx, \\ \displaystyle\int_\Omega \frac{\partial\psi^m}{\partial t}\cdot\phi_k\,dx + \int_\Omega [(\boldsymbol{u}^m\cdot\nabla)\psi^m]\cdot\phi_k\,dx + \alpha_2\int_\Omega \nabla\psi^m\cdot\nabla\phi_k\,dx = \int_\Omega Q_3\cdot\phi_k\,dx, \end{cases} \tag{3.2}$$

for $k = 1, \dots, m$. The initial conditions for these approximations are chosen such that as $m \to \infty$

$$\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m \to \boldsymbol{u}_0 \text{ strongly in } H; \quad \theta^m(0) = \theta_0^m \to \theta_0,\ \psi^m(0) = \psi_0^m \to \psi_0 \text{ strongly in } L^2(\Omega). \tag{3.3}$$

Due to the Caratheodory theorem [15, Theorem 3.4], there exists a short time interval $[0, T_m]$ (with $0 < T_m < T$) for each m, such that the system (3.2) and (3.3) possesses unique classical solutions $c_k^m(t), h_k^m(t), d_k^m(t) \in C^1[0, T_m]$ for $k = 1, 2, \dots, m$. By employing the uniform a priori estimates derived in the subsequent section, we can demonstrate that these solutions can be extended to the entire interval $[0, T]$, ensuring $T_m = T$.

3.2 Uniform a priori estimates

Multiplying (3.2) by $c_k^m(t)$, $h_k^m(t)$, $d_k^m(t)$, respectively, summing over k form 1 to m, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^m\|^2_{L^2(\Omega)} + b(\boldsymbol{u}^m, \boldsymbol{u}^m, \boldsymbol{u}^m) + \nu\sum_{i=1}^3\int_\Omega |D_i\boldsymbol{u}^m|^{q_i}dx + \sum_{i=1}^3\kappa_i\int_\Omega |u_i^m|^{\sigma_i}dx \\ = \int_\Omega \boldsymbol{Q}_1\cdot\boldsymbol{u}^m dx + \beta_\theta\int_\Omega \theta^m\boldsymbol{g}\boldsymbol{u}^m dx + \beta_\psi\int_\Omega \psi^m\boldsymbol{g}\boldsymbol{u}^m dx, \end{aligned} \tag{3.4}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta^m\|^2_{L^2(\Omega)} + b(\boldsymbol{u}^m, \theta^m, \theta^m) + \alpha_1\|\nabla\theta^m\|^2_{L^2(\Omega)} = \int_\Omega Q_2\cdot\theta^m dx, \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\|\psi^m\|^2_{L^2(\Omega)} + b(\boldsymbol{u}^m, \psi^m, \psi^m) + \alpha_2\|\nabla\psi^m\|^2_{L^2(\Omega)} = \int_\Omega Q_3\cdot\psi^m dx. \tag{3.6}$$

Using Lemma 2.1, Hölder's inequality, Young's inequality, we obtain from (3.4)–(3.6)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^m\|^2_{L^2(\Omega)} + \nu\sum_{i=1}^{3}\int_\Omega |D_i\boldsymbol{u}^m|^{q_i}dx + \sum_{i=1}^{3}\kappa_i\int_\Omega |u_i^m|^{\sigma_i}dx \\
&\le \|\boldsymbol{Q}_1\|_{L^{\vec{q}'}(\Omega)}\|\boldsymbol{u}^m\|_{L^{\vec{q}}(\Omega)} + \beta_\theta\|\theta^m\|_{L^2(\Omega)}\|\boldsymbol{g}\|_\infty\|\boldsymbol{u}^m\|_{L^2(\Omega)} + \beta_\psi\|\psi^m\|_{L^2(\Omega)}\|\boldsymbol{g}\|_\infty\|\boldsymbol{u}^m\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\sum_{i=1}^{3}\|D_i\boldsymbol{u}^m\|^{q_i}_{L^{q_i}(\Omega)} + \sum_{i=1}^{3}2^{\frac{q_i}{q_i-1}}\|D_i\boldsymbol{Q}_1\|^{q_i'}_{L^{q_i'}(\Omega)} + \frac{1}{2}\beta_\theta\|\boldsymbol{g}\|_\infty\left(\|\theta^m\|^2_{L^2(\Omega)} + \|\boldsymbol{u}^m\|^2_{L^2(\Omega)}\right) \\
&\quad + \frac{1}{2}\beta_\psi\|\boldsymbol{g}\|_\infty\left(\|\psi^m\|^2_{L^2(\Omega)} + \|\boldsymbol{u}^m\|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{3.7}
$$

$$
\frac{1}{2}\frac{d}{dt}\|\theta^m\|^2_{L^2(\Omega)} + \alpha_1\|\nabla\theta^m\|^2_{L^2(\Omega)} \le \|Q_2\|_{H^{-1}}\|\theta^m\|_{W^{1,2}} \le \frac{1}{4}\|\nabla\theta^m\|^2_{L^2(\Omega)} + \|Q_2\|^2_{H^{-1}}, \tag{3.8}
$$

$$
\frac{1}{2}\frac{d}{dt}\|\psi^m\|^2_{L^2(\Omega)} + \alpha_2\|\nabla\psi^m\ |^2_{L^2(\Omega)} \le \|Q_3\|_{H^{-1}}\|\psi^m\|_{W^{1,2}} \le \frac{1}{4}\|\nabla\psi^m\|^2_{L^2(\Omega)} + \|Q_3\|^2_{H^{-1}}, \tag{3.9}
$$

adding (3.7)–(3.9), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\boldsymbol{u}^m\|^2_{L^2(\Omega)} + \|\theta^m\|^2_{L^2(\Omega)} + \|\psi^m\|^2_{L^2(\Omega)}\right) + (2\nu-1)\sum_{i=1}^{3}\int_\Omega |D_i\boldsymbol{u}^m|^{q_i}dx \\
&\qquad + 2\sum_{i=1}^{3}\kappa_i\int_\Omega |u_i^m|^{\sigma_i}dx + \left(2\alpha_1 - \frac{1}{2}\right)\|\nabla\theta^m\|^2_{L^2(\Omega)} + \left(2\alpha_2 - \frac{1}{2}\right)\|\nabla\psi^m\|^2_{L^2(\Omega)} \\
&\quad \le \sum_{i=1}^{3}2^{\frac{q_i}{q_i-1}}\|D_i\boldsymbol{Q}_1\|^{q_i'}_{L^{q_i'}(\Omega)} + 2\|Q_2\|^2_{H^{-1}} + 2\|Q_3\|^2_{H^{-1}} \\
&\qquad + \beta_\theta\|\boldsymbol{g}\|_\infty\left(\|\theta^m\|^2_{L^2(\Omega)} + \|\boldsymbol{u}^m\|^2_{L^2(\Omega)}\right) + \beta_\psi\|\boldsymbol{g}\|_\infty\left(\|\psi^m\|^2_{L^2(\Omega)} + \|\boldsymbol{u}^m\|^2_{L^2(\Omega)}\right) \\
&\quad \le \sum_{i=1}^{3}2^{\frac{q_i}{q_i-1}}\|D_i\boldsymbol{Q}_1\|^{q_i'}_{L^{q_i'}(\Omega)} + 2\|Q_2\|^2_{H^{-1}} + 2\|Q_3\|^2_{H^{-1}} \\
&\qquad + (\beta_\theta + \beta_\psi)\|\boldsymbol{g}\|_\infty\left(\|\boldsymbol{u}^m\|^2_{L^2(\Omega)} + \|\theta^m\|^2_{L^2(\Omega)} + \|\psi^m\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{3.10}
$$

Integrating (3.10) over $(0,t)$, $0 \le t \le T$, we obtain

$$
\begin{aligned}
&\|\boldsymbol{u}^m(t)\|^2_{L^2(\Omega)} + \|\theta^m(t)\|^2_{L^2(\Omega)} + \|\psi^m(t)\|^2_{L^2(\Omega)} + (2\nu-1)\sum_{i=1}^{3}\|D_i\boldsymbol{u}^m\|^{q_i}_{L^{q_i}(0,t;L^{q_i}(\Omega))} \\
&\quad + 2\sum_{i=1}^{3}\kappa_i\|u_i^m\|^{\sigma_i}_{L^{\sigma_i}(0,t;L^{\sigma_i}(\Omega))} + \left(2\alpha_1 - \frac{1}{2}\right)\int_0^t \|\nabla\theta^m\|^2_{L^2(\Omega)}ds \\
&\quad + \left(2\alpha_2 - \frac{1}{2}\right)\int_0^t \|\nabla\psi^m\|^2_{L^2(\Omega)}ds \\
&\le \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\psi_0\|^2_{L^2(\Omega)} + \sum_{i=1}^{3}2^{\frac{q_i}{q_i-1}}\|D_i\boldsymbol{Q}_1\|^{q_i'}_{L^{q_i'}(Q_T)} + 2\|Q_2\|^2_{L^2(0,T;H^{-1})} \\
&\quad + 2\|Q_3\|^2_{L^2(0,T;H^{-1})} + (\beta_\theta + \beta_\psi)\|\boldsymbol{g}\|_\infty\int_0^t\left(\|\boldsymbol{u}^m\|^2_{L^2(\Omega)} + \|\theta^m\|^2_{L^2(\Omega)} + \|\psi^m\|^2_{L^2(\Omega)}\right)ds,
\end{aligned} \tag{3.11}
$$

by setting

$$
\begin{aligned}
C_0 = {}& \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\psi_0\|^2_{L^2(\Omega)} + \sum_{i=1}^{3}2^{\frac{q_i}{q_i-1}}\|D_i\boldsymbol{Q}_1\|^{q_i'}_{L^{q_i'}(Q_T)} \\
& + 2\|Q_2\|^2_{L^2(0,T;H^{-1})} + 2\|Q_3\|^2_{L^2(0,T;H^{-1})},
\end{aligned} \tag{3.12}
$$

then the Gronwall's inequality yields that

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|\boldsymbol{u}^m(t)\|^2_{L^2(\Omega)}+\|\theta^m(t)\|^2_{L^2(\Omega)}+\|\psi^m(t)\|^2_{L^2(\Omega)}\right)+(2\nu-1)\sum_{i=1}^{3}\|D_i\boldsymbol{u}^m\|^{q_i}_{L^{q_i}(0,T;L^{q_i}(\Omega))}\\&\quad+2\sum_{i=1}^{3}\kappa_i\|u_i^m\|^{\sigma_i}_{L^{\sigma_i}(0,T;L^{\sigma_i}(\Omega))}+\left(2\alpha_1-\frac{1}{2}\right)\int_0^T\|\nabla\theta^m\|^2_{L^2(\Omega)}ds\\&\quad+\left(2\alpha_2-\frac{1}{2}\right)\int_0^T\|\nabla\psi^m\|^2_{L^2(\Omega)}ds\\&\le C_0(1+(\beta_\theta+\beta_\psi)\|\boldsymbol{g}\|_\infty Te^{(\beta_\theta+\beta_\psi)\|\boldsymbol{g}\|_\infty T}).\end{aligned}\tag{3.13}$$

Moreover, for all $\boldsymbol{\varphi}\in L^{q_i}(0,T;V_{q_i}(\Omega))$, we have

$$\begin{aligned}-\iint_{Q_T}D_i\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right)\boldsymbol{\varphi}dxdt&=\iint_{Q_T}\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right)\cdot D_i\boldsymbol{\varphi}dxdt\\&\le\int_0^T\|D_i\boldsymbol{u}^m\|^{q_i-1}_{L^{q_i}(\Omega)}\|D_i\boldsymbol{\varphi}\|_{L^{q_i}(\Omega)}dt\le\|D_i\boldsymbol{u}^m\|^{q_i-1}_{L^{q_i}(0,T;L^{q_i}(\Omega))}\|\boldsymbol{\varphi}\|_{L^{q_i}(0,T;V_{q_i})},\end{aligned}\tag{3.14}$$

noticing (3.13), we know that

$$\left\|D_i\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right)\right\|_{L^{q_i'}(0,T;V'_{q_i})}\le C.\tag{3.15}$$

So, for each $i=\{1,2,3\}$, there exists $V_i\in \mathrm{L}^{q_i'}(0,T;V'_{q_i})$ such that

$$D_i\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right)\rightharpoonup V_i\quad\text{weakly in }\mathrm{L}^{q_i'}(0,T;V'_{q_i}),\qquad\text{as }m\to\infty.$$

Next, we derive a priori estimates for $\frac{\partial\boldsymbol{u}^m}{\partial t}$, $\frac{\partial\theta^m}{\partial t}$ and $\frac{\partial\psi^m}{\partial t}$.

In the sense of distributions, we rewrite equation $(3.2)_1$ as follows

$$\begin{aligned}\frac{\partial\boldsymbol{u}^m}{\partial t}&=\nu\sum_{i=1}^{3}\mathcal{P}_m^*D_i\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right)+\sum_{i=1}^{3}\mathcal{P}_m^*\kappa_i\left(|u_i^m|^{\sigma_i-2}u_i^m\cdot\boldsymbol{e}_i\right)-\mathcal{P}_m^*\left(\boldsymbol{u}^m\cdot\nabla\right)\boldsymbol{u}^m\\&\quad+\mathcal{P}_m^*\boldsymbol{Q}_1+\mathcal{P}_m^*\beta_\theta\boldsymbol{g}\theta^m+\mathcal{P}_m^*\beta_\psi\boldsymbol{g}\psi^m:=\sum_{i=1}^{6}J_i,\end{aligned}\tag{3.16}$$

where $\mathcal{P}_m:V\to V^m$ is the orthogonal projection operator defined by (see [13]),

$$\mathcal{P}_m\boldsymbol{u}=\sum_{j=1}^{m}(\boldsymbol{u},\boldsymbol{\varphi}_j)\boldsymbol{\varphi}_j,\quad\text{for all }\boldsymbol{u}\in V,$$

and $\mathcal{P}_m^*:V'\to V'$ is the adjoint operator of $\mathcal{P}_m$ satisfying $\mathcal{P}_m^*\frac{\partial\boldsymbol{u}^m}{\partial t}=\frac{\partial\boldsymbol{u}^m}{\partial t}$ and

$$\|\mathcal{P}_m\|_{\mathcal{L}(V,V)}\le1,\qquad\|\mathcal{P}_m^*\|_{\mathcal{L}(V',V')}\le1.$$

Denote

$$A_i(\boldsymbol{u}^m)=D_i\left(|D_i\boldsymbol{u}^m|^{q_i-2}D_i\boldsymbol{u}^m\right),\qquad A(\boldsymbol{u}^m)=\sum_{i=1}^{3}A_i(\boldsymbol{u}^m).$$

We derive the estimation of each term J_i, $(i=1,2,3,4,5,6)$.

For J_1, since $W^{3,2}(\Omega) \hookrightarrow W^{1,\vec{q}}$ for all $q_i\,(i=1,2,3,\ 1<q_i<\infty)$, it holds that for all $\boldsymbol{\eta} \in L^{\vec{q}}(0,T;\widetilde{V})$

$$\nu \iint_{Q_T} \mathcal{P}_m^* A(\boldsymbol{u}^m)\boldsymbol{\eta} dxdt = \nu \iint_{Q_T} A(\boldsymbol{u}^m)\cdot(\mathcal{P}_m\boldsymbol{\eta})dxdt \le \nu \sum_{i=1}^{3}\int_0^T \|D_i\boldsymbol{u}^m\|_{L^{q_i}(\Omega)}^{q_i-1}\cdot\|\mathcal{P}_m\boldsymbol{\eta}\|_{\widetilde{V}}dt$$
$$\le \nu \sum_{i=1}^{3}\int_0^T \|D_i\boldsymbol{u}^m\|_{L^{q_i}(\Omega)}^{q_i-1}\cdot\|\boldsymbol{\eta}\|_{\widetilde{V}}dt \le \nu \sum_{i=1}^{3}\|D_i\boldsymbol{u}^m\|_{L^{q_i}(0,T;L^{q_i}(\Omega))}^{q_i-1}\cdot\|\boldsymbol{\eta}\|_{L^{\vec{q}}(0,T;\widetilde{V})}. \tag{3.17}$$

The term J_3 could be estimate as follows

$$\begin{aligned} -\iint_{Q_T} \mathcal{P}_m^*(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{u}^m\cdot\boldsymbol{\eta} dxdt &= -\iint_{Q_T}(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{u}^m\cdot(\mathcal{P}_m\boldsymbol{\eta})dxdt \\ &= \frac{1}{2}\iint_{Q_T}|\boldsymbol{u}^m|^2\cdot\nabla(\mathcal{P}_m\boldsymbol{\eta})dxdt \le \frac{1}{2}\int_0^T\|\boldsymbol{u}^m\|_{L^2(\Omega)}^2\cdot\|\nabla\boldsymbol{\eta}\|_{L^\infty(\Omega)}dt \\ &\le \frac{1}{2}\int_0^T\|\boldsymbol{u}^m\|_{L^2(\Omega)}^2\cdot\|\boldsymbol{\eta}\|_{\widetilde{V}}dt \le \frac{1}{2}\|\boldsymbol{u}^m\|_{L^\infty(0,T;H)}^2\cdot\|\boldsymbol{\eta}\|_{L^{\vec{q}}(0,T;\widetilde{V})}. \end{aligned} \tag{3.18}$$

The estimate of J_4 is trivial, and we can clearly get

$$\iint_{Q_T}\mathcal{P}_m^*\boldsymbol{Q}_1\boldsymbol{\eta}\,dxdt \le \iint_{Q_T}|\boldsymbol{Q}_1\cdot(\mathcal{P}_m\boldsymbol{\eta})|dt \le \|\boldsymbol{Q}_1\|_{L^{\vec{q}'}(0,T;V'_{\vec{q}})}\|\boldsymbol{\eta}\|_{L^{\vec{q}}(0,T;\widetilde{V})}. \tag{3.19}$$

By applying Hölder's inequality, we have for J_5

$$\iint_{Q_T}\mathcal{P}_m^*\beta_\theta\theta^m\boldsymbol{g}\boldsymbol{\eta}\,dxdt = \beta_\theta\iint_{Q_T}\theta^m\boldsymbol{g}\cdot(\mathcal{P}^m\boldsymbol{\eta})\,dxdt \le \beta_\theta\int_0^T\|\theta^m\|_{L^2(\Omega)}\cdot\|\boldsymbol{g}\|_{L^\infty(\Omega)}\cdot\|\boldsymbol{\eta}\|_{L^2(\Omega)}\,dt$$
$$\le \beta_\theta\int_0^T\|\theta^m\|_{W^{1,2}(\Omega)}\cdot\|\boldsymbol{g}\|_{L^\infty(\Omega)}\cdot\|\boldsymbol{\eta}\|_{\widetilde{V}}\,dt \le \beta_\theta\|\boldsymbol{g}\|_{L^\infty(Q_T)}\cdot\|\theta^m\|_{L^2(0,T;H^1)}\cdot\|\boldsymbol{\eta}\|_{L^{\vec{q}}(0,T;\widetilde{V})}. \tag{3.20}$$

Similarly, for J_6

$$\iint_{Q_T}\mathcal{P}_m^*\beta_\psi\psi^m\boldsymbol{g}\boldsymbol{\eta}\,dxdt = \beta_\psi\iint_{Q_T}\psi^m\boldsymbol{g}\cdot(\mathcal{P}^m\boldsymbol{\eta})\,dxdt \le \beta_\psi\int_0^T\|\psi^m\|_{L^2(\Omega)}\cdot\|\boldsymbol{g}\|_{L^\infty(\Omega)}\cdot\|\boldsymbol{\eta}\|_{L^2(\Omega)}\,dt$$
$$\le \beta_\psi\int_0^T\|\psi^m\|_{W^{1,2}(\Omega)}\cdot\|\boldsymbol{g}\|_{L^\infty(\Omega)}\cdot\|\boldsymbol{\eta}\|_{\widetilde{V}}\,dt \le \beta_\psi\|\boldsymbol{g}\|_{L^\infty(Q_T)}\cdot\|\psi^m\|_{L^2(0,T;H^1)}\cdot\|\boldsymbol{\eta}\|_{L^{\vec{q}}(0,T;\widetilde{V})}. \tag{3.21}$$

Finally, for J_2, we have that for all $\boldsymbol{\omega} \in L^{\vec{\sigma}}(0,T;V_{\vec{\sigma}})$

$$\begin{aligned} \iint_{Q_T}\sum_{i=1}^{3}\mathcal{P}_m^*\kappa_i|u_i^m|^{\sigma_i-2}u_i^m\cdot\boldsymbol{e}_i\cdot\boldsymbol{\omega}\,dxdt &= \iint_{Q_T}\sum_{i=1}^{3}\kappa_i|u_i^m|^{\sigma_i-2}u_i^m\cdot\boldsymbol{e}_i\cdot(\mathcal{P}_m\boldsymbol{\omega})\,dxdt \\ \le \sum_{i=1}^{3}\kappa_i\int_0^T\|u_i^m\|_{L^{\sigma_i}(\Omega)}^{\sigma_i-1}\cdot\|\mathcal{P}_m\boldsymbol{\omega}\|_{W^{1,\sigma_i}(\Omega)}\,dt &\le \kappa\|\boldsymbol{u}^m\|_{L^{\vec{\sigma}}(0,T;L^{\vec{\sigma}}(\Omega))}^{\vec{\sigma}-1}\cdot\|\boldsymbol{\omega}\|_{L^{\vec{\sigma}}(0,T;V_{\vec{\sigma}})}. \end{aligned} \tag{3.22}$$

Combining (3.17)–(3.22) and using (3.13), (3.15), we obtain

$$\frac{\partial\boldsymbol{u}^m}{\partial t} \in L^\infty(0,T;\widetilde{V}')\cap L^{\vec{q}'}(0,T;\widetilde{V}')\cap\left(\bigcup_{i=1}^{3}L^{\sigma_i'}(Q_T)\right).$$

Since for any finite time T, it holds

$$L^\infty(0,T;\widetilde{V}') \hookrightarrow L^{\vec{q}'}(0,T;\widetilde{V}') \hookrightarrow L^{(q^+)'}(0,T;\widetilde{V}'),$$

and

$$L^{\sigma_i'}(Q_T) \hookrightarrow L^{\sigma_i'}(0,T;\widetilde{V}') \hookrightarrow L^{(\sigma^+)'}(0,T;\widetilde{V}'),$$

therefore

$$\frac{\partial \boldsymbol{u}^m}{\partial t} \in L^{\iota'}(0,T;\widetilde{V}'), \qquad \iota' := \min\{(q^+)',\ (\sigma^+)'\}. \tag{3.23}$$

Analogously, we define the orthogonal projection operator $\mathcal{R}_m : H^1 \to W^m$ as

$$\mathcal{R}_m\theta = \sum_{j=1}^{m}(\theta,\phi_j)\phi_j, \quad \text{for all } \theta \in H^1,$$

its adjoint operator $\mathcal{R}_m^* : H^{-1} \to H^{-1}$, satisfies $\mathcal{R}_m^* \frac{\partial\theta^m}{\partial t} = \frac{\partial\theta^m}{\partial t}$. Furthermore, it holds

$$\|\mathcal{R}_m\|_{\mathcal{L}(H,H)} \le 1, \qquad \|\mathcal{R}_m^*\|_{\mathcal{L}(H^{-1},H^{-1})} \le 1.$$

In the sense of distributions, we rewrite equation $(3.2)_2$ as

$$\frac{\partial\theta^m}{\partial t} = -\mathcal{R}_m^*(u^m\cdot\nabla)\theta^m + \alpha_1\mathcal{R}_m^*\Delta\theta^m + \mathcal{R}_m^*Q_2 =: \sum_{j=1}^{3} I_j. \tag{3.24}$$

We now estimate each term I_j for $j = 1,2,3$, let $\mu \in L^2(0,T;H^2)$ be a test function.

Estimation of I_1: From $N = 3$ we derive $H^1 \hookrightarrow L^6(\Omega)$. By the definition of the adjoint operator $\mathcal{R}_m^*$ and Hölder's inequality, we have

$$\begin{aligned}
-\iint_{Q_T} \mathcal{R}_m^*\left(\boldsymbol{u}^m\cdot\nabla\right)\theta^m\cdot\mu\,dxdt &= -\iint_{Q_T}\left(\boldsymbol{u}^m\cdot\nabla\right)\theta^m\cdot\left(\mathcal{R}_m\mu\right)dxdt\\
&\le \int_0^T \|\boldsymbol{u}^m\|_{L^2(\Omega)}\left\|\nabla\left(\mathcal{R}_m\mu\right)\right\|_{L^3(\Omega)}\|\theta^m\|_{L^6(\Omega)}\,dt \le \int_0^T \|\boldsymbol{u}^m\|_H\cdot\|\mu\|_{H^2}\cdot\|\theta^m\|_{H^1}\,dt\\
&\le \|\boldsymbol{u}^m\|_{L^\infty(0,T;H)}\cdot\|\mu\|_{L^2(0,T;H^2)}\cdot\|\theta^m\|_{L^2(0,T;H^1)}\,.
\end{aligned} \tag{3.25}$$

Estimation of I_2: By the definition of the adjoint operator and Hölder's inequality, we obtain

$$\begin{aligned}
\iint_{Q_T}\alpha_1\mathcal{R}_m^*\Delta\theta^m\cdot\mu\,dxdt &= \alpha_1\iint_{Q_T}\Delta\theta^m\cdot\left(\mathcal{R}_m\mu\right)dxdt \le \alpha_1\int_0^T\|\nabla\theta^m\|_{L^2(\Omega)}\cdot\|\nabla\mu\|_{L^2(\Omega)}\,dt\\
&\le \alpha_1\int_0^T\|\theta^m\|_{H^1}\cdot\|\mu\|_{H^2}\,dt \le \alpha_1\|\theta^m\|_{L^2(0,T;H^1)}\cdot\|\mu\|_{L^2(0,T;H^2)}.
\end{aligned} \tag{3.26}$$

Estimation of I_3: By the definition of the adjoint operator and Hölder's inequality, we have

$$\iint_{Q_T}\mathcal{R}_m^*Q_2\cdot\mu\,dxdt \le \iint_{Q_T}|Q_2\cdot\mathcal{R}_m\mu|\,dxdt \le \|Q_2\|_{L^2(0,T;H^{-1})}\cdot\|\mu\|_{L^2(0,T;H^2)}. \tag{3.27}$$

Combining estimates (3.25)–(3.27) and utilizing (3.13), we conclude that

$$\frac{\partial\theta^m}{\partial t} \in L^2(0,T;H^{-2}). \tag{3.28}$$

Similarly, we reformulate equation $(3.2)_3$ by defining the terms Y_j $(j = 1,2,3)$ as:

$$\frac{\partial\psi^m}{\partial t} = -\mathcal{R}_m^*(u^m\cdot\nabla)\psi^m + \alpha_2\mathcal{R}_m^*\Delta\psi^m + \mathcal{R}_m^*Q_3 =: \sum_{j=1}^{3} Y_j. \tag{3.29}$$

We now estimate Y_1. For all $\mu \in L^2(0,T;H^2)$, we have

$$
\begin{aligned}
-\iint_{Q_T} \mathcal{R}_m^* \left(\boldsymbol{u}^m \cdot \nabla\right) \psi^m \cdot \mu \, dxdt &= -\iint_{Q_T} \left(\boldsymbol{u}^m \cdot \nabla\right) \psi^m \cdot \left(\mathcal{R}_m \mu\right) dxdt \\
&\leq \int_0^T \|\boldsymbol{u}^m\|_{L^2(\Omega)} \|\nabla\left(\mathcal{R}_m\mu\right)\|_{L^3(\Omega)} \|\psi^m\|_{L^6(\Omega)} \, dt \leq \int_0^T \|\boldsymbol{u}^m\|_H \cdot \|\mu\|_{H^2} \cdot \|\psi^m\|_{H^1} \, dt \\
&\leq \|\boldsymbol{u}^m\|_{L^\infty(0,T;H)} \cdot \|\mu\|_{L^2(0,T;H^2)} \cdot \|\psi^m\|_{L^2(0,T;H^1)} .
\end{aligned}
\tag{3.30}
$$

For Y_2, by the definition of the adjoint operator and Hölder's inequality

$$
\begin{aligned}
\iint_{Q_T} \alpha_2 \mathcal{R}_m^* \Delta\psi^m \cdot \mu \, dxdt &= \alpha_2 \iint_{Q_T} \Delta\psi^m \cdot \left(\mathcal{R}_m\mu\right) dxdt \\
&\leq \alpha_2 \int_0^T \|\nabla\psi^m\|_{L^2(\Omega)} \cdot \|\nabla\mu\|_{L^2(\Omega)} \, dt \leq \alpha_2 \int_0^T \|\psi^m\|_{H^1} \cdot \|\mu\|_{H^2} \, dt \\
&\leq \alpha_2 \|\psi^m\|_{L^2(0,T;H^1)} \cdot \|\mu\|_{L^2(0,T;H^2)}.
\end{aligned}
\tag{3.31}
$$

Finally, for Y_3, using the adjoint property and Hölder's inequality

$$
\iint_{Q_T} \mathcal{R}_m^* Q_3 \cdot \mu \, dxdt \leq \iint_{Q_T} |Q_3 \cdot \mathcal{R}_m \mu| \, dxdt \leq \|Q_3\|_{L^2(0,T;H^{-1})} \cdot \|\mu\|_{L^2(0,T;H^2)}. \tag{3.32}
$$

Combining estimates (3.30)–(3.32) and utilizing (3.13), we arrive at

$$
\frac{\partial \psi^m}{\partial t} \in L^2(0,T;H^{-2}). \tag{3.33}
$$

3.3 Proof of the existence of solutions

From the uniform estimates (3.13), (3.15), (3.23), (3.28), and (3.33), there exist a subsequence (still denoted by m) such that the following convergences hold as $m \to \infty$

$$
\begin{aligned}
\boldsymbol{u}^m &\rightharpoonup \boldsymbol{u} && \text{weakly-* in } L^\infty(0,T;H); \\
\boldsymbol{u}^m &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^{\vec{q}}(0,T;V_{\vec{q}}); \\
u_i^m &\rightharpoonup u_i && \text{weakly in } L^{\sigma_i}(Q_T),\ i=1,2,3; \\
\frac{\partial \boldsymbol{u}^m}{\partial t} &\rightharpoonup \frac{\partial \boldsymbol{u}}{\partial t} && \text{weakly in } L^{\iota'}(0,T;\widetilde{V}'); \\
A(\boldsymbol{u}^m) &\rightharpoonup V && \text{weakly in } L^{\vec{q}\,'}(0,T;V_{\vec{q}}'); \\
\theta^m &\rightharpoonup \theta && \text{weakly-* in } L^\infty(0,T;L^2(\Omega)); \\
\theta^m &\rightharpoonup \theta && \text{weakly in } L^2(0,T;H^1); \\
\frac{\partial \theta^m}{\partial t} &\rightharpoonup \frac{\partial \theta}{\partial t} && \text{weakly in } L^2(0,T;H^{-2}); \\
\psi^m &\rightharpoonup \psi && \text{weakly-* in } L^\infty(0,T;L^2(\Omega)); \\
\psi^m &\rightharpoonup \psi && \text{weakly in } L^2(0,T;H^1); \\
\frac{\partial \psi^m}{\partial t} &\rightharpoonup \frac{\partial \psi}{\partial t} && \text{weakly in } L^2(0,T;H^{-2}).
\end{aligned}
$$

Moreover, since $L^{\vec{q}}(0,T;V_{\vec{q}}) \hookrightarrow L^{q^-}(0,T;V_{\vec{q}})$, due to (3.13), we also have $\boldsymbol{u}^m \in L^{q^-}(0,T;V_{\vec{q}})$.

Now, we observe that by the definition of the space $\widetilde{V}$, the continuous embedding $H \hookrightarrow \widetilde{V}'$ holds. Combining this with the anisotropic compactness embedding (1.6), which states $V_{\vec{q}} \hookrightarrow\hookrightarrow H$, by the Aubin–Lions Lemma, we deduce that

$$\boldsymbol{u}^m \to \boldsymbol{u} \quad \text{strongly in } L^{q^-}(0,T;H). \tag{3.34}$$

Using the interpolation inequality, we obtain from (3.13) and (3.34) that

$$\boldsymbol{u}^m \to \boldsymbol{u} \quad \text{strongly in } L^r(0,T;H), \quad \text{for any } r \geq 1, \tag{3.35}$$

consequently, for any $i \in \{1,2,3\}$, we also have

$$u_i^m \to u_i \quad \text{strongly in } L^r(Q_T), \quad \text{for any } r \geq 1.$$

Furthermore, noting that $H^1 \hookrightarrow\hookrightarrow L^2(\Omega)$, we can again use the Aubin–Lions Lemma to deduce that

$$\begin{aligned} \theta^m &\to \theta \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \\ \psi^m &\to \psi \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

By fixing k and passing the system $(3.2)_1$ to the limit as $m \to \infty$, we obtain for all $\boldsymbol{\varphi}_k \in V^m$ and for a.e. $t \in [0,T]$

$$\begin{aligned} &\int_\Omega \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{\varphi}_k dx + \nu \sum_{i=1}^3 \int_\Omega V_i \cdot D_i \boldsymbol{\varphi}_k dx + \int_\Omega [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \boldsymbol{\varphi}_k dx + \sum_{i=1}^3 \kappa_i \int_\Omega |u_i|^{\sigma_i - 2} u_i \cdot \boldsymbol{e}_i \cdot \boldsymbol{\varphi}_k dx \\ &\quad = \int_\Omega \boldsymbol{Q}_1 \cdot \boldsymbol{\varphi}_k dx + \beta_\theta \int_\Omega \theta \boldsymbol{g} \cdot \boldsymbol{\varphi}_k dx + \beta_\psi \int_\Omega \psi \boldsymbol{g} \cdot \boldsymbol{\varphi}_k dx. \end{aligned} \tag{3.36}$$

Since $\widetilde{V} = \bigcup_{m=1}^\infty V^m$, this equation holds for all $\boldsymbol{\varphi} \in \widetilde{V}$. By a continuity argument, the equation (3.36) holds for all $\boldsymbol{\varphi} \in V_{\vec{q}}$ as long as the integrals there remain bounded for $\boldsymbol{u} \in L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}})$ and $u_i \in L^{\sigma_i}(Q_T)$. The only difficulty here is to show the boundedness of the convective integral term. For this, by Hölder's inequality, we get

$$\int_\Omega [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \boldsymbol{\varphi}\, dx = -\int_\Omega \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi}\, dx \leq \|\boldsymbol{u}\|^2_{L^{2(q^-)'}(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{L^{q^-}(\Omega)}, \tag{3.37}$$

since $q_a^* > \frac{2q^-}{q^- - 1}$, the embedding $V_{\vec{q}} \hookrightarrow V_{q^-}$ holds, then using Lemma 1.1, we obtain

$$\int_\Omega [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \boldsymbol{\varphi}\, dx \leq \|\boldsymbol{u}\|^2_{V_{\vec{q}}} \|\boldsymbol{\varphi}\|_{V_{\vec{q}}}.$$

Similarly, by passing to the limit as $m \to \infty$ in $(3.2)_2$, $(3.2)_3$, we obtain, for any $\phi \in H^1$

$$\begin{aligned} \int_\Omega \frac{\partial \theta}{\partial t} \cdot \phi\, dx + \int_\Omega [(\boldsymbol{u} \cdot \nabla)\theta] \cdot \phi\, dx + \alpha_1 \int_\Omega \nabla\theta \cdot \nabla\phi\, dx &= \int_\Omega Q_2 \cdot \phi\, dx, \\ \int_\Omega \frac{\partial \psi}{\partial t} \cdot \phi\, dx + \int_\Omega [(\boldsymbol{u} \cdot \nabla)\psi] \cdot \phi\, dx + \alpha_2 \int_\Omega \nabla\psi \cdot \nabla\phi\, dx &= \int_\Omega Q_3 \cdot \phi\, dx. \end{aligned} \tag{3.38}$$

Moreover, by observing (1.12) and utilizing (1.8), we can write

$$\frac{d}{dt}\langle \boldsymbol{u}(t), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{w}(t), \boldsymbol{\varphi} \rangle,$$

$$\boldsymbol{w} := \boldsymbol{Q}_1 + \sum_{i=1}^3 D_i \left(|D_i \boldsymbol{u}|^{q_i - 2} D_i \boldsymbol{u} \right) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \sum_{i=1}^3 \kappa_i |u_i|^{\sigma_i - 2} u_i \boldsymbol{e}_i + (\beta_\theta \theta + \beta_\psi \psi)\boldsymbol{g}, \tag{3.39}$$

for any $\varphi \in V$, in the distribution sense in $\mathcal{D}(0,T)$. Consequently, by using Definition 1.3(1) and (1.9), together with the assumptions that $q_a^* \geq 2$ and $q^- \geq 2$, it can be shown that $w \in L^1(0,T;\mathbf{V}')$, and therefore (see [25, Lemma III-1.1])

$$\frac{\partial u}{\partial t} \in L^1(0,T;V').$$

Similarly, we could obtain

$$\frac{\partial \theta}{\partial t} \in L^1(0,T;H^{-1}), \qquad \frac{\partial \psi}{\partial t} \in L^1(0,T;H^{-1}).$$

Since $u \in L^\infty(0,T;H)$, $\theta,\psi \in L^\infty(0,T;L^2(\Omega))$, and $H \hookrightarrow V'$, $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, from Lemma 2.3, we conclude that $u \in C_w([0,T];H)$, and $\theta,\psi \in C_w([0,T];L^2(\Omega))$.

Next, to complete the proof of the Theorem 1.8, it suffices to prove that

$$A_i(u) = V_i, \qquad A(u) = \sum_{i=1}^{3} V_i = V. \tag{3.40}$$

Since for all $\xi,\eta \in L^{\vec{q}}(0,T;V_{\vec{q}})$ with $\xi \neq \eta$, we have

$$\sum_{i=1}^{3}\langle |D_i\xi|^{q_i-2}D_i\xi - |D_i\eta|^{q_i-2}D_i\eta, D_i\xi - D_i\eta\rangle_{L^{q_i'}(Q_T)\times L^{q_i}(Q_T)} > 0,$$

which implies that A_i and A are strictly monotone operators. This monotonicity property allows us to apply the monotonicity method to establish the validity of (3.40).

Furthermore, to support the subsequent analysis, we need to demonstrate that

$$\int_\Omega [(u\cdot\nabla)u]\cdot v\,dx = -\int_\Omega u\otimes u : \nabla v\,dx \in L^1(0,T), \tag{3.41}$$

for all $u \in L^\infty(0,T;H) \cap L^{\vec{q}}(0,T;V_{\vec{q}})$ and $v \in L^{\vec{q}}(0,T;V_{\vec{q}})$. To prove (3.41), we first note that, by the definition of q^- and Lemma 1.1, it follows that

$$L^{\vec{q}}(0,T;V_{\vec{q}}) \hookrightarrow L^{q^-}(0,T;V_{\vec{q}}) \hookrightarrow L^{q^-}(0,T;L^{q_a^*}).$$

Subsequently, applying the interpolation inequality yields

$$L^{\vec{q}}(0,T;V_{\vec{q}}) \cap L^\infty(0,T;H) \subset L^{q^-}(0,T;L^{q_a^*}) \cap L^\infty(0,T;L^2(\Omega)) \subset L^\rho(Q_T),$$

where $\rho = 2 + q^- - 2q^-/q_a^*$. From (1.13), we can deduce that $2/\rho + 1/q^- \leqslant 1$, which enables us to apply Hölder's inequality to establish (3.41).

Next, following the methodology in [13], for a.e., $t \in (0,T)$, we obtain

$$\begin{aligned} \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \nu\int_0^t (V,u)ds \geq& \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2 - \int_0^t\left(\sum_{i=1}^{3}\kappa_i|u_i|^{\sigma_i-2}u_i, u_i\right)ds \\ &+ \int_0^t (Q_1,u)ds + \beta_\theta\iint_{Q_t}\theta g u dxds + \beta_\psi\iint_{Q_t}\psi g u dxds. \end{aligned} \tag{3.42}$$

For every $\eta \in L^{\vec{q}}(0,T;V_{\vec{q}})$, we define

$$\chi_m = \nu\int_0^t (A(u^m) - A(\eta), u^m - \eta)\,ds + \frac{1}{2}\|u^m(t)\|_{L^2(\Omega)}^2.$$

By the monotonicity of A, we deduce that

$$\lim_{m\to\infty}\inf \chi_m \geq \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)}. \tag{3.43}$$

From (3.2), we express χ_m as

$$\chi_m = \frac{1}{2}\|\boldsymbol{u}^m(0)\|^2_{L^2(\Omega)} - \int_0^t \left(\sum_{i=1}^{3}\kappa_i|u_i^m|^{\sigma_i-2}u_i^m, u_i^m\right) ds - \nu\int_0^t (A(\boldsymbol{\eta}), \boldsymbol{u}^m - \boldsymbol{\eta})\, ds$$
$$-\nu\int_0^t (A(\boldsymbol{u}^m), \boldsymbol{\eta})\, ds + \int_0^t (\boldsymbol{Q}_1, \boldsymbol{u}^m) ds + \beta_\theta \iint_{Q_t} \theta^m \boldsymbol{g}\boldsymbol{u}^m dxds + \beta_\psi \iint_{Q_t} \psi^m \boldsymbol{g}\boldsymbol{u}^m dxds.$$

Taking the limit as $m \to \infty$, we obtain

$$\chi = \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)} - \int_0^t \left(\sum_{i=1}^{3}\kappa_i|u_i|^{\sigma_i-2}u_i, u_i\right) ds - \nu\int_0^t (A(\boldsymbol{\eta}), \boldsymbol{u} - \boldsymbol{\eta})\, ds$$
$$-\nu\int_0^t (A(\boldsymbol{u}), \boldsymbol{\eta})\, ds + \int_0^t (\boldsymbol{Q}_1, \boldsymbol{u}) ds + \beta_\theta \iint_{Q_t} \theta \boldsymbol{g}\boldsymbol{u} dxds + \beta_\psi \iint_{Q_t} \psi \boldsymbol{g}\boldsymbol{u} dxds.$$

Combining (3.42) and (3.43), we then have

$$\int_0^t (V - A(\boldsymbol{\eta}), \boldsymbol{u} - \boldsymbol{\eta}) ds \geq 0, \quad \text{a.e., } t \in (0, T). \tag{3.44}$$

Let $\boldsymbol{\eta} = \boldsymbol{u} - \lambda\boldsymbol{\xi}$, for any $\lambda \geq 0$ and $\boldsymbol{\xi} \in L^{\vec{q}}(0, T; V_{\vec{q}})$. Substituting this into the inequality, we get

$$\int_0^t (V - A(\boldsymbol{u} - \lambda\boldsymbol{\xi}), \boldsymbol{\xi}) ds \geq 0. \tag{3.45}$$

Taking the limit as $\lambda \to 0$ in formula (3.45), we conclude that

$$\int_0^t (V - A(\boldsymbol{u}), \boldsymbol{\xi}) ds \geq 0, \quad \text{for any } \boldsymbol{\xi} \in L^{\vec{q}}(0, T; V_{\vec{q}}).$$

Thus, $V = A(\boldsymbol{u})$, a.e. in Q_T, and Theorem 1.8 is proved.

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