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Existence and multiplicity for degenerate fractional Kirchhoff problems with nonsmooth potential

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Abstract. This paper investigates a class of nonlocal variable-order fractional $p(\cdot)$-Kirchhoff type equations with nonsmooth nonlinearities

$$\begin{cases} \mathscr{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) \in \partial F(x,u) \quad \text{in } \Omega, \\ u = 0, \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and $\partial F(x,u)$ is the partial generalized gradient of $F(x,\cdot)$ at the point u. Using nonsmooth critical point theory, Krasnoselskii genus theory and variational methods, we establish the existence of multiple nontrivial solutions under suitable conditions on the functions $p, s, \mathscr{M}$ and F. Our findings extend the scope of degenerate $p(\cdot)$ fractional cases to encompass nonsmooth scenarios.

Keywords: fractional $p(\cdot)$-equation, differential inclusion, variational methods, multiple solutions.

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1 Introduction

We consider the following variable $s(\cdot)$-order fractional $p(\cdot)$-Kirchhoff type problem

$$\begin{cases} \mathscr{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) \in \partial F(x,u) \quad \text{in } \Omega, \\ u = 0, \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\mathscr{M}$ is a model of Kirchhoff coefficient, Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\partial F(x,u)$ is the partial generalized gradient of $F(x,\cdot)$ at the point u. The operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is called variable $s(\cdot)$-order fractional $p(\cdot)$-Laplacian, given $s(\cdot) : \bar{\Omega} \times \Omega \to (0,1)$ and $p(\cdot) : \bar{\Omega} \times \Omega \to (1,+\infty)$ with $s(x,y)p(x,y) < N$ for all $(x,y) \in \bar{\Omega} \times \bar{\Omega}$, which can be defined as

$$(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) = P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+p(x,y)s(x,y)}} \mathrm{d}y, \quad x \in \mathbb{R}^N$$

[✉] Corresponding author. Email: junjyuan@sina.com along any $u \in C_0^\infty(\mathbb{R}^N)$, where P.V. stands for the Cauchy principle value.

In recent years, nonlocal problems, such as those represented by equation (1.1), have attracted significant attention due to their broad applications in scientific disciplines. These operators arise naturally in the modeling of physical processes, population dynamics, optimization, and mathematical finance, among others. Nonlocal and fractional operators, such as the complex integro-differential operator

$$\mathscr{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy\right)(-\Delta)_{p(\cdot)}^{s(\cdot)}u(x),$$

serve as infinitesimal generators for Lévy stable diffusion processes. Their study not only extends the mathematical framework of quantum mechanics but also provides insights into systems involving elastic particle interactions, turbulent fluid dynamics, and material transport in fractured media.

The literature on nonlocal fractional operators and their applications is vast, as evidenced by comprehensive monographs like [23] and detailed studies such as [19]. Much of this work focuses on variational methods for analyzing problem (1.1) with differentiable $F(x,u)$. For instance, in [1], the authors investigated a nonlocal variable order fractional problem and the existence and multiplicity of solutions via variational techniques. Similarly, the authors in[8] examined the existence of nontrivial weak solutions for a variant of (1.1) with a power type term $\lambda|u(x)|^{r(x)-2}u(x)$, employing direct variational methods and Ekeland's principle. Further contributions include multiplicity results for Schrödinger equations driven by variable order fractional Laplacians [32] and the study of infinitely many solutions for Kirchhoff-type problems using critical point theory [31]. Additional related results can be found in [4,5,7,12,18,21,28] et al.

However, existing results typically assume that the associated potential $F(x,u)$ is differentiable (a condition that often fails in practical applications). Many real world problems, such as free boundary and obstacle problems, involve nonsmooth or even discontinuous potentials, rendering classical C^1-variational methods inapplicable. This raises a fundamental question: How can problem (1.1) be analyzed when $F(x,u)$ lacks differentiability?

To address this challenge, the theory of discontinuous differential inclusions has emerged as a powerful tool, offering new approaches to nonsmooth and nonlocal elliptic problems. Nonsmooth analysis, particularly the critical point theory for locally Lipschitz functionals based on Clarke's generalized gradient [15], provides a natural extension of classical variational methods. For a thorough treatment of these techniques, we refer to the monographs [20,24] and the works [2,3,14,17,25–27,29,30,33–35].

In this paper, we study problem (1.1) in the absence of differentiability assumptions on $F(x,u)$, leveraging tools from nonsmooth analysis to derive new existence and multiplicity results.

To give our main results, we first make the following hypotheses on the functions $\mathscr{M}$, s and p. For any $x \in \mathbb{R}^N$, set $\bar{p}(x) := p(x,x)$ and $\bar{s}(x) := s(x,x)$.

(M1) $\mathscr{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function and satisfies

$$m_1\zeta^{\theta-1} \le \mathscr{M}(\zeta) \le m_2\zeta^{\theta-1} \quad \text{for any } \zeta \ge 0,$$

where $0 < m_1 \le m_2$ are real numbers and $\theta > 1$;

(S) $s(x,y)$ is a symmetric function, i.e., $s(x,y) = s(y,x)$, and

$$0 < s^- := \inf_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} s(x,y) \le s^+ := \sup_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} s(x,y) < 1;$$

(P) $p(x,y)$ is also a symmetric function, i.e., $s(x,y) = s(y,x)$, and

$$1 < p^- := \inf_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} p(x,y) \le p^+ := \sup_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} s(x,y) < \infty.$$

(F0) $F(\cdot,\zeta)$ is measurable, $F(x,\cdot)$ is locally Lipschitz;

(F1) There exist $C_1 > 0$ and $1 < r(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N-\bar{s}(x)\bar{p}(x)}$ for all $x \in \Omega$, all $w^* \in \partial F(x,\zeta)$ such that

$$|\omega^*| \le C_1(1 + |\zeta|^{r(x)-1}) \quad \text{for all } (x,\zeta) \in \Omega \times \mathbb{R};$$

(F2) there exist $C_2 > 0$, $\eta_0 \in (1, qp^-)$ such that

$$F(x,\zeta) \ge C_2|\eta|^{\eta_0} \quad \text{for all } x \in \Omega, \text{ and } |\zeta| \ge r_0;$$

(F3) For all $(x,\zeta) \in \Omega \times \mathbb{R}$,

$$F(x,-\zeta) = F(x,\zeta);$$

(F4) There exists $r_0 \ge 0$ such that

$$F(x,\zeta) = 0, \quad \text{for all } x \in \Omega \text{ and } |\zeta| \le r_0.$$

Remark 1.1. It is straightforward to observe that numerous functions can fulfill the specified hypotheses. Here we give a Weierstrass-type singular function, which is nowhere differentiable. Define

$$F_3(x,\zeta) = \sum_{n=0}^{\infty} a^n \left(|\zeta|^{r(x)} - r_0^{r(x)}\right)_+ \cdot \cos(b^n\pi\zeta),$$

where $a \in (0,1)$, b is an odd integer satisfying $ab > 1 + \frac{3\pi}{2}$, $r(x) \in (1, p_s^*(x))$ is measurable, $(f)_+ = \max(f, 0)$.

The main result of this work is the following theorem.

Theorem 1.2. *Suppose that (M), (S), (P) and (F1)–(F4) hold. If $r^+ < qp^-$, then problem (1.1) has infinitely many solutions.*

The structure of this paper is organized as follows: Section 2 presents the fundamental preparatory work that serves as the foundation for subsequent proofs. In Section 3, we employ nonsmooth critical point theory, Krasnoselskii genus theory and variational methods to prove Theorem 1.2.

2 Preliminaries

Let us give some notations. We denote a real Banach space as $(X, \|\cdot\|)$, with its topological dual being $(X^*, \|\cdot\|_*)$. The constants C and C_i (where $i = 1, 2, \ldots$) represent estimated values that may vary from line to line. The symbol '$\to$' denotes strong convergence within the space X, whereas '$\rightharpoonup$' signifies weak convergence. The norm in the space $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. Set $\bar{r} = \max \frac{r(x)-1}{r(x)}$, $\underline{r} = \min \frac{r(x)-1}{r(x)}$, $\bar{q} = \max \frac{q(x)-1}{q(x)}$ and $\underline{q} = \min \frac{q(x)-1}{q(x)}$. Define $p_s^*(x) = \frac{N\bar{p}(x)}{N-\bar{s}(x)\bar{p}(x)}$, $U_{p,s}(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y$, and

$$C_+(\Omega) := \{g : g \in C(\bar{\Omega}) \text{ and } g(x) > 1, \forall x \in \bar{\Omega}\}.$$

For any $g(\cdot) \in C_+(\Omega)$, the variable exponent Lebesgue space $L^{g(\cdot)}(\Omega)$ is defined as

$$L^{g(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{g(x)} \mathrm{d}x < \infty \right\}.$$

This space is endowed with the Luxemburg norm, which is given by

$$\|u\|_{L^{g(x)}(\Omega)} = \|u\|_{g(\cdot)} := \inf \left\{ \varrho > 0 : \int_\Omega \left| \frac{u(x)}{\varrho} \right|^{g(x)} \mathrm{d}x \leq 1 \right\}$$

and $(L^{g(\cdot)}(\Omega), \|u\|_{L^{g(\cdot)}(\Omega)})$ is termed a Banach space, specifically a variable exponent Lebesgue space.

Denote $Q = \mathbb{R}^{2N} \setminus (\complement^{\Omega}_{\mathbb{R}^N} \times \complement^{\Omega}_{\mathbb{R}^N})$. We introduce the fractional Sobolev space with variable exponents through the Gagliardo approach as follows:

$$W := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u|_\Omega \in L^{\bar{p}(x)}(\Omega), \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\varrho^{p(x,y)} |x - y|^{N + s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y < \infty \text{ for some } \varrho > 0 \right\}.$$

The space W is equipped with the norm

$$\|u\|_W := \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_W,$$

where $[u]_W$ denotes the seminorm defined by

$$[u]_W = \inf \left\{ \varrho > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\varrho^{p(x,y)} |x - y|^{N + s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y < 1 \right\}.$$

Consequently, $(W, \| \cdot \|_W)$ forms a separable reflexive Banach space. We further define the subspace W_0 of W as

$$W_0 := \left\{ u \in W \mid u = 0 \ \text{ a.e. in } C^{\Omega}_{\mathbb{R}^N} \right\},$$

endowed with the norm

$$\|u\|_{W_0} = [u]_W.$$

Proposition 2.1 ([10]). *Assume $s(\cdot)$ and $p(\cdot)$ fulfill conditions (S) and (P) with $s(x,y)p(x,y) < N$ for every $(x,y) \in \bar{\Omega} \times \Omega$. Then, for any $g \in C_+(\bar{\Omega})$ such that $1 < g^- \leq g(x) < p_s^*(x) := \frac{N\bar{p}(x)}{N - \bar{s}(x)\bar{p}(x)}$ for every $x \in \bar{\Omega}$, there exists a positive constant $C = C(N, s, p, g, \Omega)$ ensuring that*

$$\|u\|_{L^{g(x)}(\Omega)} \leq C\|u\|_{W_0},$$

for all $u \in W_0$. Furthermore, the embedding $W_0 \hookrightarrow L^{g(x)}(\Omega)$ is compact.

Proposition 2.2 ([12]). *Assume that $h \in L^\infty_+(\Omega)$, $g \in C_+(\bar{\Omega})$. If $|u|^{k(x)} \in L^{g(x)}(\Omega)$, then*

$$\min\{\|u\|^{k^-}_{k(x)g(x)}, \|u\|^{k^+}_{k(x)g(x)}\} \leq \||u|^{k(x)}\|_{g(x)} \leq \max\{\|u\|^{k^-}_{k(x)g(x)}, \|u\|^{k^+}_{k(x)g(x)}\}.$$

Let us set the fractional modular function $\rho_{s,p} : W_0 \to \mathbb{R}$ as

$$\rho_{s,p}(u) := \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + s(x,y)p(x,y)}} \mathrm{d}x\mathrm{d}y. \tag{2.1}$$

Proposition 2.3 ([10]). *Let $u, u_n \in W_0$ and $\rho_{s,p}$ be defined as in (2.1). Then we have the following results:*

(i) $\|u\|_{W_0} < 1\ (= 1; > 1)$ *if and only if* $\rho_{s,p} < 1\ (= 1; > 1)$;

(ii) *If* $\|u\|_{W_0} > 1$, *then* $\|u\|_{W_0}^{p^-} \le \rho_{s,p} \le \|u\|_{W_0}^{p^+}$;

(iii) *If* $\|u\|_{W_0} < 1$, *then* $\|u\|_{W_0}^{p^+} \le \rho_{s,p} \le \|u\|_{W_0}^{p^-}$;

(iv) $\lim_{n\to\infty} \|u_n - u\|_{W_0} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{s,p}(u_n - u) = 0$.

Proposition 2.4 ([10]). *$(W_0, \|\cdot\|_{W_0})$ is a separable, reflexive and uniformly convex Banach space.*

Proposition 2.5 ([1]). *For all $u, v \in W_0$, we consider the operator $A : W_0 \to W_0^*$ such that*

$$\langle A(u), v\rangle = \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+s(x,y)p(x,y)}} dxdy.$$

Then

(i) *A is a bounded and strictly monotone operator;*

(ii) *A is a mapping of type (S^+), that is, if $u_n \rightharpoonup u$ in W_0 and $\limsup\langle A(u_n) - A(u), u_n - u\rangle$, then $u_n \to u$ in W_0.*

Definition 2.6 ([20]). I satisfies the nonsmooth $(PS)_c$ condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \text{ and } m(u_n) \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence, where $m(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

The proof of the following theorem can be found in the literature [13].

Lemma 2.7. *If $u \in W_0$ and $w \in \partial J_1(u)$, then*

$$w(x) \in [\underline{f}(u(x)), \bar{f}(u(x))] \quad \text{a.e. in } \Omega.$$

We introduce fundamental concepts related to the Krasnoselskii genus, which are essential for proving our key theorems. Let $\mathcal{O}$ represent the collection of all closed subsets $\mathcal{K} \subset X \setminus \{0\}$ that are symmetric about the origin; that is, $u \in \mathcal{K}$ implies $-u \in \mathcal{K}$.

Definition 2.8. For $\mathcal{K} \in \mathcal{O}$, the Krasnoselskii genus $\gamma(\mathcal{K})$ is defined as the smallest positive integer k for which there exists an odd function $\phi \in C(\mathcal{K}, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for every $x \in \mathcal{K}$. If no such k exists, we define $\gamma(\mathcal{K}) = \infty$. Additionally, by convention, $\gamma(\emptyset) = 0$.

Subsequently, we will elucidate several basic properties of the genus that are pivotal in our analysis. For a deeper exploration of this topic, readers are referred to the extensive discussions in references [6,11,16,22].

Proposition 2.9. *Let $X = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 2.10. *$\gamma(\mathcal{S}^{N-1}) = N$, where $\mathcal{S}^{N-1}$ is a unit sphere of $\mathbb{R}^N$.*

Proposition 2.11. *If $\mathcal{K} \in \mathcal{O}$, $0 \notin \mathcal{K}$ and $\gamma(\mathcal{K}) \ge 2$, then K has infinitely many points.*

3 Proof of the main result

The objective of our proof is to establish that the set of critical points of the functional I is compact, symmetric, and does not contain zero, while also having genus greater than two. Our main result then follows from Proposition 2.11. In proving Theorem 1.2, we require several technical lemmas, which we now present.

In the light of the variational structure of (1.1), we look for critical points of the associated Euler Lagrange function $I : W_0 \to \mathbb{R}$ defined by

$$I(u) = \hat{\mathscr{M}}(U_{p,s}(u)) - \int_\Omega F(x,u)\mathrm{d}x \tag{3.1}$$

for all $u \in W_0$, where

$$U_{p,s}(u) = \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}\,\mathrm{d}x\,\mathrm{d}y, \quad \hat{\mathscr{M}}(t) = \int_0^t \mathscr{M}(s)\,\mathrm{d}s,$$

and

$$\langle u^*, v\rangle = \mathscr{M}(U_{p,s}(u)) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))((v(x)-v(y))}{|x-y|^{N+p(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y - \int_\Omega w^*(x)v\,\mathrm{d}x,$$

for any $v \in W_0$, where $u^* \in \partial I(u_n)$ and $w^*(x) \in \partial F(x,u)$. Thus, critical points of I are weak solutions of (1.1).

Lemma 3.1. *If the assumptions (M), (S), (P), (F1) hold, and $r^+ < qp^-$, then the functional I is coercive.*

Proof. Under the hypotheses of (M) and (F1), we can derive the following inequality:

$$\begin{aligned} I(u) &= \hat{\mathscr{M}}(U_{p,s}(u)) - \int_\Omega F(x,u)\mathrm{d}x \\ &\geq \frac{m_1}{q}(U_{p,s}(u))^q - \frac{C_1}{r^-}\int_\Omega |u|^{r(x)}\mathrm{d}x - C_1|\Omega| \end{aligned}$$

for every $u \in W_0$. Utilizing Proposition 2.3, we deduce that

$$\begin{aligned} I(u) &\geq \frac{m_1}{q(p^+)^q}\min\left\{\|u\|_{W_0}^{qp^+}, \|u\|_{W_0}^{qp^-}\right\} - \frac{C_1}{r^-}\max\left\{\|u\|_{r(x)}^{r^+}, \|u\|_{r(x)}^{r^-}\right\} - C_1|\Omega| \\ &\geq \frac{m_1}{q(p^+)^q}\min\left\{\|u\|_{W_0}^{qp^+}, \|u\|_{W_0}^{qp^-}\right\} - \frac{CC_1}{r^-}\max\left\{\|u\|_{W_0}^{r^+}, \|u\|_{W_0}^{r^-}\right\} - C_1|\Omega|. \end{aligned}$$

Since $r^+ < qp^-$, we infer that I is coercive. □

Lemma 3.2. *If hypotheses (M), (S), (P), (F1) and $r^+ < qp^-$, then I satisfies the nonsmooth $(PS)_c$ condition for all $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}$ be a sequence in W_0 such that

$$I(u_n) \to c \;\text{ and }\; m(u_n) \to 0.$$

From now on, we consider $\{u_n^*\} \subset \partial I(u_n) \subset W_0^*$ such that

$$m(u_n) = \|u_n^*\|_* = o_n(1).$$

From Lemma 3.1, it is known that the sequence $\{u_n\}$ is bounded. By considering a subsequence if necessary, we can assume that

$$\begin{aligned} u_n &\to u(x) \quad \text{in } L^{r(x)}(\Omega), \\ u_n(x) &\rightharpoonup u(x) \quad \text{in } W_0. \end{aligned}$$

Consequently,

$$\begin{aligned} &\langle u_n^*, u_n - u \rangle \\ &= \mathscr{M}(U_{p,s}(u_n)) \int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))((u_n(x) - u(x)) - (u_n(y) - u(y)))}{|x-y|^{N+p(x,y)s(x,y)}} \\ &\quad - \int_\Omega w_n^*(u_n - u)\, dx \to 0, \end{aligned}$$

where $w_n^*(x) \in \partial F(x, u_n)$.

Next, we estimate

$$\begin{aligned} \left| \int_\Omega w_n(x)(u_n - u)\, dx \right| &\le \left| \int_\Omega C(1 + |u_n|^{r(x)-1})|u_n - u|\, dx \right| \\ &\le C\|u_n - u\|_{L^2(\Omega)} + C\|u_n\|_{r(x)}\|u_n - u\|_{r(x)} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore,

$$\mathscr{M}(U_{p,s}(u_n)) \int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))((u_n(x) - u(x)) - (u_n(y) - u(y)))}{|x-y|^{N+p(x,y)s(x,y)}} \to 0.$$

Thus, we can assume that

$$U_{p,s}(u_n) \to d_1 \ge 0 \quad \text{as } n \to +\infty.$$

If $d_1 = 0$, then $\{u_n\}$ converges strongly to $u = 0$ in W_0, and the proof is complete.

If $d_1 > 0$, noting that the function $\mathscr{M}$ is continuous, we have

$$\mathscr{M}(U_{p,s}(u_n)) \to \mathscr{M}(d_1) \ge 0 \quad \text{as } n \to +\infty.$$

Then, from (M), for n sufficiently large, we have

$$0 < C_3 < \mathscr{M}(U_{p,s}(u_n)) < C_4,$$

which implies

$$\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))((u_n(x) - u(x)) - (u_n(y) - u(y)))}{|x-y|^{N+p(x,y)s(x,y)}} \to 0.$$

Then, by Proposition 2.5, we conclude that

$$u_n \to u \quad \text{in } W_0.$$

□

Let K_c be the set of critical points of I. More precisely,

$$K_c = \{u \in W_0 : I(u) = c \ \text{ and } \ 0 \in \partial I(u)\}.$$

By virtue of (F3), it follows that I is an even function, and K_c is symmetric. The following result plays a crucial role in proving our main theorems, and its proof can be found in [13].

Lemma 3.3. *If I satisfy the nonsmooth $(PS)_c$ condition, then K_c is compact.*

To demonstrate that K_c does not encompass the origin, we delineate a particular class of levels c. For every $k \in \mathbb{N}$, we introduce the set

$$\Gamma_k = \{E \subset W_0 : E \text{ is closed}, \, E = -E, \text{ and } \gamma(E) \geq k\}$$

and define the corresponding values

$$c_k = \inf_{E \in \Gamma_k} \sup_{u \in E} I(u).$$

Given the sequence

$$-\infty \leq c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_k \leq \cdots,$$

and since I is coercive, continuous, and bounded below, it follows that $c_1 > -\infty$. Using an approach similar to that in [9, Proposition 3.1], we can show that each c_k is a critical value of the functional I. □

Lemma 3.4. *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that $\gamma(I^{-\epsilon}) \geq k$, where $I^{-\epsilon} = \{u \in W_0 : I(u) \leq -\epsilon\}$.*

Proof. Fix $k \in \mathbb{N}$. Let E_k be a k-dimensional subspace of W_0. Since all norms are equivalent in finite-dimensional spaces, we have

$$C\|u\|_{W_0} \leq \|u\|_{\eta_0} \text{ for all } u \in E_k.$$

By (M), (F2), and Propositions 2.1–2.3, we derive

$$I(u) \leq \frac{m_2}{q}(U_{p,s}(u))^q - C_2 \int_\Omega |u|^{\eta_0} \, dx.$$

For $\|u\|_{W_0} \leq 1$, we obtain

$$\begin{aligned} I(u) &\leq \frac{m_2}{q}\|u\|_{W_0}^{qp^-} - C_3\|u\|_{W_0}^{\eta_0} \\ &= \|u\|_{W_0}^{\eta_0}\left(\frac{m_2}{q}\|u\|_{W_0}^{qp^- - \eta_0} - C_3\right). \end{aligned}$$

Consider $R > 0$ satisfying

$$R < \min\left\{1, \left(\frac{qC_3}{m_2}\right)^{\frac{1}{qp^- - \eta_0}}\right\}.$$

There exists $\epsilon = \epsilon(R) > 0$ such that

$$I(u) < -\epsilon < 0$$

for all $u \in S_R = \{u \in E_k : \|u\|_{W_0} = R\}$. Since E_k is isomorphic to $\mathbb{R}^k$ and S_R is homeomorphic to S^{k-1}, Corollary 2.10 implies that $\gamma(S_R) = \gamma(S^{k-1}) = k$. Moreover, as $S_R \subset I^{-\epsilon}$ is symmetric and closed, we obtain

$$k = \gamma(S_R) \leq \gamma(I^{-\epsilon}).$$

□

Lemma 3.5. *For a given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. For each $k \in \mathbb{N}$, by Lemma 3.4, there exists $\epsilon > 0$ such that $\gamma(I^{-\epsilon})$. Moreover, $0 \notin I^{-\epsilon}$ and $I^{-\epsilon} \in \Gamma_k$. Additionally, observe that $\sup_{u \in I^{-\epsilon}} \leq -\epsilon$, hence

$$-\infty < c_k = \inf_{E \in \Gamma_k} \sup_{u \in C} I(u) \leq \sup_{u \in I^{-\epsilon}} I(u) \leq -\epsilon < 0.$$

□

From the above lemma, it follows that $0 \notin K_{c_k}$. The following lemma is equally essential for proving our main result.

Lemma 3.6 ([13]). *Suppose that X is a reflexive Banach space and I is an even, locally Lipschitz function satisfying the $(PS)_c$ condition. If U is any neighborhood of K_c, then for any $\epsilon_0 > 0$, there exist $\epsilon \in (0, \epsilon_0)$ and an odd homeomorphism $\eta : X \to X$ such that*

(i) *$\eta(x) = x$ for $x \notin I^{c+\epsilon} \setminus I^{c-\epsilon}$;*

(ii) *$\eta(I^{c+\epsilon} \setminus U) \subset I^{c-\epsilon}$;*

(iii) *If $K_c = \emptyset$, then $\eta(I^{c+\epsilon}) \subset I^{c-\epsilon}$.*

Lemma 3.7. *If $c_k = c_{k+1} = \cdots = c_{k+l}$ for some $l \in \mathbb{N}$, then $\gamma(K_{c_k}) \geq l + 1$.*

Proof. Suppose, by contradiction, that $\gamma(K_{c_k}) \leq l$. Since K_{c_k} is compact and symmetric, there exists a symmetric open neighborhood $U \subset W_0$ such that $K_{c_k} \subset U$ and $\gamma(\overline{U}) = \gamma(K_{c_k}) \leq l$. Let $\Omega_0 = \overline{U}$, then Ω_0 is closed, symmetric and $\gamma(\Omega_0) \leq l$. By Lemma 3.6, there exists an odd homeomorphism $\eta : W_0 \to W_0$ and $\sigma > 0$ with $0 < \sigma < -c_k$ such that

$$\eta(I^{c_k+\sigma} \setminus U) \subset I^{c_k-\sigma}.$$

Since $c_k = c_{k+l}$, there exists $B \in \Gamma_{k+l}$ such that $\sup_{u \in B} I(u) < c_k + \sigma$, i.e., $B \subset I^{c_k+\sigma}$, and

$$\eta(B \setminus U) \subset \eta(I^{c_k+\sigma} \setminus U) \subset I^{c_k-\sigma}. \tag{3.2}$$

Now consider the set $B \setminus U$. Since B is closed and U is open, $B \setminus U = B \cap (W_0 \setminus U)$ is closed. Moreover, because both B and U are symmetric, $B \setminus U$ is symmetric. By the properties of Krasnoselskii genus, we have

$$\gamma(B \setminus U) \geq \gamma(B) - \gamma(\overline{U}) \geq (k + l) - l = k.$$

Since η is an odd homeomorphism, it preserves genus, therefore

$$\gamma(\eta(B \setminus U)) = \gamma(B \setminus U) \geq k.$$

However, from (3.2) we have $\eta(B \setminus U) \subset I^{c_k-\sigma}$, which implies

$$\sup_{u \in \eta(B \setminus U)} I(u) \leq c_k - \sigma.$$

But $\eta(B \setminus U)$ is closed, symmetric and $\gamma(\eta(B \setminus U)) \geq k$, by the definition of c_k we should have

$$c_k \leq \sup_{u \in \eta(B \setminus U)} I(u) \leq c_k - \sigma,$$

which is a contradiction. Therefore, $\gamma(K_{c_k}) \geq l + 1$. This completes the proof. □

We now proceed to prove Theorem 1.2.

Proof of Theorem 1.2. If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$ and noting that each c_k is a critical value of I, then we derive infinitely many critical points of I. Hence, problem (1.1) has infinitely many solutions.

Furthermore, if there exist two constants $c_k = c_{k+l}$, then $c_k = c_{k+1} = \cdots = c_{k+l}$. By Lemma 3.7, we have

$$\gamma(K_{c_k}) \geq l + 1 \geq 2.$$

From Proposition 2.11, K_{c_k} has infinitely many points.

Next, we prove that the set $\{x \in \Omega : |u_k(x)| \geq r_0\}$ has positive measure, where $r_0 > 0$ is a constant such that $F(x, \zeta) = 0$ for all $|\zeta| \leq r_0$ according to hypothesis (F4). The consideration of this inequality is crucial because it ensures the nontriviality of the solutions u_k. Suppose, by contradiction, that the set $\{x \in \Omega : |u_k(x)| \geq r_0\}$ has null measure. Then $|u_k(x)| < r_0$ for almost every $x \in \Omega$, and by (F4), we have $F(x, u_k(x)) = 0$ for almost every $x \in \Omega$. This implies that

$$\int_\Omega F(x, u_k(x))dx = 0.$$

On the other hand, from the definition of the functional I and the fact that u_k is a critical point with $I(u_k) = c_k < 0$, we have

$$I(u_k) = \hat{\mathscr{M}}(U_{p,s}(u_k)) - \int_\Omega F(x, u_k(x))dx = \hat{\mathscr{M}}(U_{p,s}(u_k)) < 0.$$

However, by hypothesis (M1), $\mathscr{M}(\zeta) \geq m_1 \zeta^{\theta-1} \geq 0$ for all $\zeta \geq 0$, which implies that $\hat{\mathscr{M}}(t) = \int_0^t \mathscr{M}(s)ds \geq 0$ for all $t \geq 0$. This contradicts the inequality $\hat{\mathscr{M}}(U_{p,s}(u_k)) < 0$. Therefore, the set $\{x \in \Omega : |u_k(x)| \geq r_0\}$ must have positive measure, ensuring that the solutions u_k are nontrivial. ☐

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References

[1] M. Allaoui, M. K. Hamdani, L. Mbarki, A degenerate Kirchhoff-type problem involving variable $s(\cdot)$-order fractional $p(\cdot)$-Laplacian with weights, *Period. Math. Hung.* **88**(2024), 396–411. `https://doi.org/10.1007/s10998-023-00563-0`; MR4751332; Zbl 1563.35646

[2] C. O. Alves, Z. Yuan, L. Huang, Existence and multiplicity of solutions for discontinuous elliptic problems in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A* **151**(2021), 548–572. `https://doi.org/10.1017/prm.2020.30`; MR4241291; Zbl 1465.35241

[3] C. O. Alves, J. Santos, R. Nemer, Multiple solutions for a problem with discontinuous nonlinearity, *Ann. Mat. Pura Appl. (4)* **197**(2018), 883–903. `https://doi.org/10.1007/s10231-017-0708-6`; MR3802698; Zbl 1396.35017

[4] V. Ambrosio, A Kirchhoff type equation in $\mathbb{R}^N$ involving the fractional (p,q)-Laplacian, *J. Geom. Anal.* **32**(2022), Paper No. 135, 46 pp. `https://doi.org/10.1007/s12220-022-00876-5`; MR4376646; Zbl 1485.35003

[5] V. Ambrosio, T. Isernia, V. D. Rădulescu, Concentration of positive solutions for a class of fractional p-Kirchhoff type equations, *Proc. Roy. Soc. Edinburgh Sect. A* **151**(2021), 601–651. `https://doi.org/10.1017/prm.2020.32`; MR4241293; Zbl 07342500

[6] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**(1973), 349–381. `https://doi.org/10.1016/0022-1236(73)90051-7`; MR370183; Zbl 0273.49063

[7] E. Azroul, A. Benkirane, M. Shimi, Existence and multiplicity of solutions for fractional $p(x,.)$-Kirchhoff-type problems in $\mathbb{R}^N$, *Appl. Anal.* **100**(2021), 2029–2048. `https://doi.org/10.1080/00036811.2019.1673373`; MR4269981; Zbl 1470.35386

[8] E. Azroul, A. Benkirane, M. Shimi, M. Srati, On a class of fractional $p(x)$-Kirchhoff type problems, *Appl. Anal.* **100**(2021), 383–402. `https://doi.org/10.1080/00036811.2019.1603372`; MR4203418; Zbl 1458.35445

[9] T. Bartsch, T. Weth, M. Willem, Partial symmetry of least energy nodal solution to some variational problems, *J. Anal. Math.* **96**(2005), 1–18. `https://doi.org/10.1007/BF02787822`; MR2177179

[10] R. Biswas, S. Tiwari, Variable order nonlocal Choquard problem with variable exponents, *Complex Var. Elliptic Equ.* **66**(2021), 853–875. `https://doi.org/10.1080/17476933.2020.1751136`; MR4243679; Zbl 1466.35153

[11] A. Castro, Métodos variacionales y análisis functional no lineal, edited by the Universidad Pedagógica y Tecnológ ca de Colombia and the Sociedad Colombiana de Matemáticas, 1980.

[12] B. Cekic, R. A. Mashiyev, Existence and localization results for $p(x)$-Laplacian via topological methods, *Fixed Point Theory Appl.* **2010**(2010), 1–7. `https://doi.org/10.1155/2010/120646`; MR2678552; Zbl 1198.35109

[13] K. C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80**(1981), 102–129. `https://doi.org/10.1016/0022-247X(81)90095-0`; MR614246

[14] K. C. Chang, S. Shao, D. Zhang, W. Zhang, Nonsmooth critical point theory and applications to the spectral graph theory, *Sci. China Math.* **64**(2021), 1–32. `https://doi.org/10.1007/s11425-019-1625-8`; MR4200982; Zbl 1462.58008

[15] F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983. `https://doi.org/10.1137/1.9781611971309`; MR0709590; Zbl 0582.49001

[16] D. G. Costa, *An invitation to variational methods in differential equations*, Birkhäuser, Boston, 2007. `https://doi.org/10.1007/978-0-8176-4536-6`; MR2308255; Zbl 1123.35001

[17] D. Davis, M. Díaz, D. Drusvyatskiy, Escaping strict saddle points of the Moreau envelope in nonsmooth optimization, *SIAM J. Optim.* **32**(2022), 1958–1983. `https://doi.org/10.1137/21M140319X`; MR4466664; Zbl 1509.65045

[18] L. M. Del Pezzo, A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, *J. Differential Equations* **263**(2017), 765–778. `https://doi.org/10.1016/j.jde.2017.02.051`; MR3631323; Zbl 1362.35061

[19] S. Dipierro, M. Medina, E. Valdinoci, *Fractional elliptic problems with critical growth in the whole of* $\mathbb{R}^N$, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), Vol. 15, Edizioni della Normale, Pisa, 2017. `https://doi.org/10.1007/978-88-7642-601-8`; MR3617721; Zbl 1375.49001

[20] L. Gasiński, N. S. Papageorgiou, *Nonsmooth critical point theory and nonlinear boundary value problems*, Chapman and Hall/CRC Press, Boca Raton, FL, 2005. `https://doi.org/10.1201/9781420028355`; MR2092433; Zbl 1058.58005

[21] D. Kang, C. Liu, X. Zhang, Existence of solutions for Kirchhoff-type fractional Dirichlet problem with p-Laplacian, *Mathematics* **8**(2020), 106. `https://doi.org/10.3390/math8010106`

[22] M. A. Krasnoselskii, Topological methods in the theory of nonlinear integral equations, MacMillan, New York, 1964. MR159197

[23] G. Molica Bisci, V. D. Radulescu, V. Servadei, Variational methods for nonlocal fractional problems, *Encyclopedia of Mathematics and its Applications*, Vol. 162 Cambridge University Press, 2016. `https://doi.org/10.1017/CBO9781316282397`; MR3445279; Zbl 1356.49003

[24] D. Motreanu, V. Rădulescu, Variational and non-variational methods in nonlinear analysis and boundary value problems, Nonconvex Optimization and its Applications, Vol. 67, Kluwer Academic Publishers, Dordrecht, 2003. `https://doi.org/10.1007/978-1-4613-0241-8`; MR1985870; Zbl 1040.49001

[25] J. A. Santos, C. O. Alves, E. Massa, A nonsmooth variational approach to semipositone quasilinear problems in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **527**(2023), 127432. `https://doi.org/10.1016/j.jmaa.2023.127432`; MR4595939; Zbl 1519.35179

[26] Q. Shi, M. Hong, Penalty dual decomposition method for nonsmooth nonconvex optimization. I. Algorithms and convergence analysis, *IEEE Trans. Signal Process.* **68**(2020), 4108–4122. `https://doi.org/10.1109/TSP.2020.3001906`; MR4128136; Zbl 1543.90239

[27] G. V. Smirnov, Introduction to the theory of differential inclusions, American Mathematical Society, Providence, RI, 2022. `https://doi.org/10.1090/gsm/041`; MR1867542; Zbl 0992.34001

[28] M. Tao, B. Zhang, Solutions for nonhomogeneous fractional (p, q)-Laplacian systems with critical nonlinearities, *Adv. Nonlinear Anal.* **11**(2022), 1332–1351. `https://doi.org/10.1515/anona-2022-0248`; MR4402489; Zbl 1489.35118

[29] M. A. Teixeira, Perturbation theory for non-smooth systems, in: G. Gaeta (Ed.), *Perturbation theory: mathematics, methods and applications*, Springer New York, 2022, pp. 503–517. `https://doi.org/10.1007/978-1-0716-2621-4_400`; MR4570714; Zbl 1548.34002

[30] V. Vijayaraj, C. Ravichandran, P. Sawangtong, K. S. Nisar, Existence results of Atangana-Baleanu fractional integro-differential inclusions of Sobolev type, *Alexandria Eng. J.* **66**(2023), 249–255. `https://doi.org/10.1016/j.aej.2022.11.037`

[31] L. Wang, B. Zhang, Infinitely many solutions for Kirchhoff-type variable-order fractional Laplacian problems involving variable exponents, *Appl. Anal.* **100**(2021), 2418–2435. `https://doi.org/10.1080/00036811.2019.1688790`; MR4291385; Zbl 1475.35403

[32] M. Xiang, B. Zhang, D. Yang, Multiplicity results for variable-order fractional Laplacian equations with variable growth, *Nonlinear Anal.* **178**(2019), 190–204. `https://doi.org/10.1016/j.na.2018.07.016`; MR3886611 ; Zbl 1402.35307

[33] Z. Yuan, Existence and concentration of solutions for a class of biharmonic Kirchhoff equations with discontinuous nonlinearity, *Math. Methods Appl. Sci.* **45**(2022), 0170–4214. `https://doi.org/10.1002/mma.7956`; MR4532641; Zbl 1529.35179

[34] Z. Yuan, X. Wang, Q. Zhang, An unbounded critical point theory for a class of non-differentiable functionals and its application, *J. Appl. Anal. Comput.* **12**(2022), 1104–1117. `https://doi.org/10.11948/20220143`; MR4432658; Zbl 1563.35732

[35] Z. Yuan, J. Yu, Existence of solutions for Dirichlet elliptic problems with discontinuous nonlinearity, *Nonlinear Anal.* **197**(2020), 111848. `https://doi.org/10.1016/j.na.2020.111848`; MR4079055; Zbl 1440.35169