Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **18**, 1–21; https://doi.org/10.14232/ejqtde.2026.1.18 www.math.u-szeged.hu/ejqtde/

Variational approaches for sixth-order boundary value problems

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Received 26 November 2025, appeared 1 June 2026

Communicated by Gabriele Bonanno

Abstract. In this work, we obtain the existence of multiple solutions for sixth-order differential equations with two parameters λ and μ, generally arising in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by the problems. Using a consequence of the local minimum theorem due to Bonanno we look into the existence of one solution under algebraic conditions on the nonlinear term and two solutions for the problem under algebraic conditions with the classical Ambrosetti–Rabinowitz condition on the nonlinear term. Furthermore, by employing two critical point theorems, one due to Averna and Bonanno, and another one due to Bonanno we guarantee the existence of two and three solutions for the problem in a special case.

Keywords: multiple solutions, sixth-order equations, variational methods, critical point theory.

2020 Mathematics Subject Classification: 34B15, 34B18, 35B38.

1 Introduction

In this paper, we study the following problem

$$\begin{cases} -u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), \ t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\mu \geq 0$, A, B and C are given real constants and $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The problem is motivated by the study for stationary solutions of the sixth-order parabolic differential equations

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial^6 t} + A\frac{\partial^4 u}{\partial^4 t} + B\frac{\partial^2 u}{\partial^2 t} + f(t,u(t)). \tag{1.2}$$

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This equation arises in the formation of spatially periodic patterns in bistable systems and serves as a model for describing the behavior of phase fronts in materials undergoing transitions between liquid and solid states. One of its most significant applications is in modeling phase-front dynamics in such materials.

Sixth-order boundary value problems (BVPs) also appear in astrophysics. In particular, the thin convective layer enclosed by stable layers, which is believed to surround A-type stars, can be modeled by sixth-order BVPs [11, 21, 22]. There exist many sixth-order differential equations similar to problem (1.1) that arise in engineering, material mechanics, and related fields. The study of sixth-order differential equations is therefore of considerable importance in engineering sciences. Consequently, numerous results have been established concerning the existence of multiple solutions for sixth-order boundary value problems; see, for instance, [6, 9, 12, 14–16, 20]. In [14], Gyulov et al. have established the existence and multiplicity of solutions of the following boundary value problem

$$\begin{cases} -u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda f(t, u(t)), \ 0 < t < L, \\ u(0) = u(L) = u''(0) = u''(L) = u^{(iv)}(0) = u^{(iv)}(L) = 0, \end{cases} \tag{1.3}$$

where A, B, C are given real constants and f is a continuous function on $\mathbb{R}^2$. In [15], Li obtained the existence and multiplicity of positive solutions for the sixth-order boundary value problem

$$\begin{cases} -u^{(vi)}(t) + A(t)u^{(iv)}(t) + B(t)u''(t) + C(t)u(t) + f(t, u(t)) = 0, \ t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1.4}$$

where $A, B, C \in C([0, 1])$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Bonanno et al. in [6], by using the critical point theory, have discussed the existence of at least one nontrivial solution for a nonlinear sixth-order ordinary differential equation for the following problem

$$-u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda f(t, u(t)), \quad t \in [a, b], \tag{1.5}$$

where $\lambda > 0$, A, B, C are given real constants and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function. Recently, Bonanno and Livrea in [9], by the variational methods under oscillating behavior on the nonlinear term, have obtained the existence of infinitely many solutions for the following nonlinear sixth-order differential equation

$$\begin{cases} -u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda f(t, u(t)), \ t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$, A, B and C are given real constants and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

For sixth-order problems, we also refer to [23]. Concerning fourth-order problems, relevant references include [13], as well as more recent contributions addressing fourth-order differential inclusions and hemivariational inequalities [8]. These works employ a variety of analytical techniques, including variational methods and fixed point theorems, and help situate the present study within the broader context of the existing literature.

Inspired by the above results, in this article we investigate the existence of one, two, and three weak solutions to the problem (1.1). To this end, we impose suitable conditions and introduce appropriate ranges for the two parameters λ and μ, which allow us to establish the existence of weak solutions to (1.1). We also provide illustrative examples demonstrating the applicability of the main theorems. Finally, we derive results concerning the existence and multiplicity of weak solutions in the particular case where the two parameters coincide.

Our approach is based on variational methods, and the principal tools are four local minimum theorems for differentiable functionals. More precisely, by applying a consequence of a local minimum theorem due to Bonanno, we establish the existence of at least one solution under suitable algebraic conditions on the nonlinear term, and the existence of two solutions under algebraic conditions combined with the classical Ambrosetti–Rabinowitz (AR) condition. Furthermore, by employing two critical point theorems one due to Averna and Bonanno and another due to Bonanno we prove the existence of two and three solutions to problem (1.1) in the case $\lambda = \mu$. In comparison with previous results, we introduce new assumptions that ensure the existence of nontrivial weak solutions to (1.1), thereby extending and generalizing several recent related works.

2 Preliminaries and basic notation

In this section, we introduce the tools that are necessary for our main results in the next section.

Set

$$X = \{u \in H^3(0,1) \cap H_0^1(0,1) \mid u''(0) = u''(1) = 0\}. \tag{2.1}$$

X is the Sobolev space with inner product

$$\langle u, v \rangle := \int_0^1 \Big(u'''(t)v'''(t) + u''(t)v''(t) + u'(t)v'(t) + u(t)v(t) \Big) dt$$

for all $u,\ v \in X$, and norm

$$\|u\| := \Big(|u'''|_2^2 + |u''|_2^2 + |u'|_2^2 + |u|_2^2 \Big)^{\frac{1}{2}} \tag{2.2}$$

for all $u \in X$.

Proposition 2.1 (see [9]). *If $k = \frac{1}{\pi^2}$, for every $u \in X$, we have*

$$\|u^{(i)}\|_2^2 \le k^{j-i} \|u^{(j)}\|_2^2, \quad i = 0,1,2,\ j = 1,2,3 \text{ with } i < j, \tag{2.3}$$

where $\|u\|_2 := (\int_0^1 |u(t)|^2 dt)^{\frac{1}{2}}$ is norm in $L^2(0,1)$.

We introduce the function $N : X \to \mathbb{R}$ as follows

$$N(u) := \|u'''\|_2^2 + A\|u''\|_2^2 + B\|u'\|_2^2 + C\|u\|_2^2, \quad \forall\, u \in X,$$

where $A, B, C \in \mathbb{R}$ and satisfy at least one of the following conditions:

(H_1) $A \ge 0$, $B \ge 0$, $C \ge 0$;

(H_2) $A \ge 0$, $B \ge 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$;

(H_3) $A \ge 0$, $B < 0$, $C \ge 0$ and $-Ak - Bk^2 < 1$;

(H_4) $A \ge 0$, $B < 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$;

(H_5) $A < 0$, $B \geq 0$, $C \geq 0$ and $-Ak < 1$;

(H_6) $A < 0$, $B \geq 0$, $C < 0$ and $\max\{-Ak, -Ak - Bk^2 - Ck^2\} < 1$;

(H_7) $A < 0$, $B < 0$, $C \geq 0$ and $-Ak - Bk^2 < 1$;

(H_8) $A < 0$, $B < 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$.

Moreover, fix $A, B, C \in \mathbb{R}$ and consider the following condition:

(H) $\max\{-Ak, -Ak - Bk^2, -Ak - Bk^2 - Ck^3\} < 1$.

Condition (H) is equivalent to every condition (H_1)–(H_8) (see [9, Proposition 2.2])

Proposition 2.2 ([9]). *Assume that (H) holds, and put $\|u\|_X = \sqrt{N(u)}$ for every $u \in X$. Then, $\|u\|_X$ is a norm equivalent to the usual one defined on (2.2) and $(X, \|\cdot\|_X)$ is a Hilbert space with inner product*

$$\langle u, v \rangle := \int_0^1 \Big(u'''(t)v'''(t) + Au''(t)v''(t) + Bu'(t)v'(t) + Cu(t)v(t) \Big) dt$$

for all u, $v \in X$.

Clearly $(X, \|\cdot\|_X) \hookrightarrow (C^0(0,1), \|\cdot\|_\infty)$ and the embedding is compact. For a qualitative estimate of the constant of this embedding it is useful to introduce the following number

$$\delta = \begin{cases} 1 & \text{if } (H_1) \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2 + Ck^3\} & \text{if } (H_2) \text{ or } (H_4) \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2\} & \text{if } (H_3) \text{ holds,} \\ 1 + Ak & \text{if } (H_5) \text{ holds,} \\ \min\{1 + Ak, 1 + Ak + Bk^2\} & \text{if } (H_6) \text{ holds,} \\ 1 + Ak + Bk^2 & \text{if } (H_7) \text{ holds,} \\ 1 + Ak + Bk^2 + Ck^3 & \text{if } (H_8) \text{ holds.} \end{cases} \tag{2.4}$$

Proposition 2.3 ([9]). *Assume that H holds, one has*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}} \|u\|_X$$

for all $u \in X$, where δ is given on (2.4).

We say that a function $u \in X$ is a weak solution of the problem (1.1) if

$$\int_0^1 \Big(u'''(t)v'''(t) + Au''(t)v''(t) + Bu'(t)v'(t) + Cu(t)v(t) \Big) dt \\ - \lambda \int_0^1 f(t, u(t))v(t)dt - \mu \int_0^1 g(t, u(t))v(t)dt = 0$$

holds for all $v \in X$. Corresponding to the functions f and g, we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, respectively, as follows:

$$F(t, x) = \int_0^x f(t, \xi) d\xi, \quad \text{for all } (t, x) \in [0,1] \times \mathbb{R}$$

and

$$G(t,x) = \int_0^x g(t,\xi)d\xi, \quad \text{for all } (t,x) \in [0,1] \times \mathbb{R}.$$

For $\gamma > 0$ and $\delta > 0$ set

$$G^\gamma = \int_{[0,1]} \max_{|x|<\gamma} G(t,x)dt, \quad \text{and} \quad G_\delta = \inf_{[0,1]\times[-\frac{\delta}{4},\frac{\delta}{4}]} G.$$

If g is sign-changing, then $G^\gamma \le 0$ and $G_\delta \ge 0$.

Definition 2.4. Assume that X is a real reflexive Banach space. We say that I satisfies the Palais–Smale condition (denoted by the (PS) condition for short) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Let $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$ and let $I = \Phi - \Psi$, where $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals. If every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ satisfying the following conditions

(j_1) $I(u_n)$ is bounded;

(j_2) $\lim\limits_{n\to\infty} |I'(u_n)|X^* = 0$;

$(j3)$ $r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$,

admits a convergent subsequence, then we say that the functional I satisfies the Palais–Smale condition cut off below at r_1 and above at r_2 (the ${}^{[r_1]}(PS)^{[r_2]}$-condition). Likewise, we define:
$(PS)^{[r_2]}$ when $r_1 = -\infty$ and $r_2 \in \mathbb{R}$;
${}^{[r_1]}(PS)$ when $r_1 \in \mathbb{R}$ and $r_2 = +\infty$.

By Definition 2.4, the classical (PS)-condition is recovered when the ${}^{[r_1]}(PS)^{[r_2]}$-condition holds with $r_1 = -\infty$ and $r_2 = +\infty$. Indeed, if Φ and Ψ are two continuously Gâteaux differentiable functionals defined on a real Banach space X and $r \in \mathbb{R}$ is fixed, the functional $I = \Phi - \Psi$ is said to satisfy the Palais–Smale condition cut off above at r (in short, $(PS)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

(1) $I(u_n)$ is bounded;

(2) $\lim_{n\to\infty} |I'(u_n)|X^* = 0$;

(3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$,

admits a convergent subsequence.

The proofs of our theorems are based on the following four theorems.

Theorem 2.5 ([5, Theorem 2.3])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that:*

(i_1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(i_2) *for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition.*

Then, for each $\lambda \in \Lambda := \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right)$*, there exists* $u_{0,\lambda} \in \Phi^{-1}(0,r)$ *such that* $I_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ *and* $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(0,r)$*.*

Theorem 2.6 ([5, Theorem 3.2])**.** *Let* X *be a real Banach space,* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$*. Fix* $r > 0$ *and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right),$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies (PS)-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right),$$

the functional I_λ *admits two distinct critical points.*

Theorem 2.7 ([2, Theorem A])**.** *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^* *and* $\Psi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(k_1) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$*, for all* $\lambda \in [0, \infty)$*;*

(k_2) *there is* $r \in \mathbb{R}$ *such that*

$$\inf_X \Phi < r$$

and

$$\varphi_1(r) < \varphi_2(r)$$

where

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}]-\infty,\bar{r}[} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r)}^{\omega}}\Psi}{r - \Phi(u)},$$

$$\varphi_2(\bar{r}) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r,\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\overline{\Phi^{-1}(-\infty,r)}^{\omega}$ *is the closure of* $\Phi^{-1}(-\infty,r)$ *in the weak topology. Then, for each* $\lambda \in \left(\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right)$ *the functional* $\Phi + \lambda\Psi$ *has at least three critical points in* X*.*

In Theorem 2.7, we consider that equivalent $\frac{1}{0}$ as ∞.

Theorem 2.8 ([3, Theorem 1.1])**.** *Let* X *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* Φ *is (strongly) continuous and satisfies*

$$\lim_{\|u\|\to\infty} \Phi(u) = \infty.$$

Assume also that there exist two constants r_1 *and* r_2 *such that*

(e_1) $\inf_X \Phi < r_1 < r_2$*;*

(e_2) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;

(e_3) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$, *where* φ_1 *is defined as in Theorem 2.7 and*

$$\varphi_2^*(r_1, r_2) = \inf_{u \in \Phi^{-1}(-\infty, r_1)} \sup_{v \in \Phi^{-1}[r_1, r_2]} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}.$$

Then, for each $\lambda \in \left(\frac{1}{\varphi_2^*(r_1,r_2)}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\right\}\right)$*, the functional* $\Phi + \lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(-\infty, r_1]$ *and* $\Phi^{-1}[r_1, r_2)$ *respectively.*

Various applications of the above theorems are presented in [7, 10].

3 Main results

In this section, we establish the main existence result of the paper. The following technical constant will be useful,

$$\eta = 4\delta\pi^4 \left(96\left(\frac{12}{5}\right)^5 + 4A\left(\frac{12}{5}\right)^4 + \frac{1248}{175}B + \frac{493}{756}C\right)^{-1} \tag{3.1}$$

where A, B, and C denote the real parameters appearing in problem (1.1) and satisfy the condition (H). Also, we consider

$$\omega(t) = \begin{cases} v(t), & t \in [0, \frac{5}{12}[, \\ 1, & t \in [\frac{5}{12}, \frac{7}{12}], \\ v(1-t), & t \in]\frac{7}{12}, 1], \end{cases} \tag{3.2}$$

where $v(t) = (\frac{12}{5})^4 t^4 - 2(\frac{12}{5})^3 t^3 + \frac{24}{5} t$ for every $t \in [0, \frac{5}{12}]$. We clearly observe that $\omega \in X$.

Theorem 3.1. *Assume that there exists a positive constant* γ *such that* $\frac{1}{2\eta} < \gamma^2$*, and that the following conditions hold:*

(A_1) $\frac{\int_0^1 \sup_{|x|\le\gamma} F(t,x(t))dt}{\gamma^2} < 2\eta \int_0^1 F(t, \omega(t))dt$;

(A_2) $\lim sup_{|\xi|\to\infty} \frac{F(t,\xi)}{|\xi|^2} \le 0$ *uniformly in* $[0, 1]$.

Then, for every

$$\lambda \in \Lambda := \left(\frac{2\delta\pi^4}{\eta \int_0^1 F(t, \omega(t))dt}, \frac{4\delta\gamma^2}{k^2 \int_0^1 \sup_{|x|\le\gamma} F(t, x(t))dt}\right)$$

and for every L^2*-Carathéodory function* $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$\lim sup_{|x|\to\infty} \frac{\sup_{x\in[0,1]} G(t, x)}{x^2} < \infty, \tag{3.3}$$

there exists $\delta_\lambda > 0$ *given by*

$$\min\left\{\frac{2\delta\pi^4 - \lambda\eta \int_0^1 F(t, \omega(t))dt}{\eta G_\delta}, \frac{4\delta\gamma^2 - \lambda k^2 \int_0^1 \sup_{|x|\le\gamma} F(t, x(t))dt}{k^2 G^\gamma}\right\} \tag{3.4}$$

such that for each $\mu \in [0, \delta_\lambda)$ *the problem* (1.1) *admits at least one weak solution* u_λ *in* X *such that* $\max_{t\in[0,1]} |u_\lambda(t)| < \gamma$.

Proof. We aim to apply Theorem 2.5 with respect to the space X equipped with the norm defined in (2.2), and we consider the functionals Φ and Ψ defined as follows:

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 \tag{3.5}$$

and

$$\Psi(u) = \int_0^1 \Big(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big)\,dt$$

for all $u \in X$. From the definition of Φ, we observe that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous. Its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, which is given by

$$\Phi'(u)(v) = \int_0^1 \Big(u'''(t)v'''(t) + Au''(t)v''(t) + Bu'(t)v'(t) + Cu(t)v(t)\Big)dt$$

for every $v \in X$. We have

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle = \|u - v\|_X^2.$$

Therefore, Φ' admits a continuous inverse on X. Moreover, the functional Ψ is in $C^1(X, \mathbb{R})$ and Ψ has compact derivative. Moreover, for $\lambda > 0$, we will show that the functional I_λ is coercive. Since $\mu < \delta_k$ and by (3.3), we can fix $\alpha > 0$ such that $\alpha\mu < \frac{2\delta}{k^2}$, and there exists $\rho_\alpha \in L^1(0,1)$ such that

$$G(t,x) \le \alpha x^2 + \rho_\alpha.$$

Now, we fix $\varepsilon < \frac{2\delta}{\lambda k^2} - \frac{\alpha\mu}{\lambda}$. From the assumption (A_2) there is a function $\rho_\varepsilon \in L^1(0,1)$ such that

$$F(t,x) \le \varepsilon x^2 + \rho_\varepsilon$$

for every $(t,x) \in [0,1] \times \mathbb{R}$. It follows that, for each $u \in X$,

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 - \int_0^1 \Big(\lambda F(t,u(t)) + \mu G(t,u(t))\Big)dt \\
&\ge \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon \int_0^1 u^2(t)dt - \lambda\|\rho_\varepsilon\|_1 - \alpha\mu \int_0^1 u^2(t)dt - \mu\|\rho_\alpha\|_1 \\
&\ge \left(\frac{1}{2} - \lambda\frac{k^2}{4\delta}\varepsilon - \mu\frac{k^2}{4\delta}\alpha\right)\|u\|_X^2 - \lambda\|\rho_\varepsilon\|_1 - \mu\|\rho_\alpha\|_1,
\end{aligned}$$

and thus

$$\lim_{\|u\|_X \to \infty} (\Phi(u) - \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi(u) - \lambda\Psi(u)$ is coercive. Therefore, by [4, Remark 2.1] the functional $I_\lambda = \Phi(u) - \lambda\Psi(u)$ verifies $(PS)^{[r]}$-condition for each $r > 0$, and so condition (i_2) of Theorem 2.5 is fulfilled. Put $r = \frac{4\delta}{k^2}\gamma^2$ and $\|\omega\|_X^2 = \frac{4\delta\pi^4}{\eta}$. Hence, by (3.5), we have

$$\Phi(\omega) = \frac{2\delta\pi^4}{\eta}. \tag{3.6}$$

Therefore, by the assumption $\frac{1}{2\eta} < \gamma^2$, we get $0 < \Phi(\omega) < r$. Moreover, by (2.3), we have

$$|u(t)|^2 \leq \|u\|_\infty^2 \leq \frac{k^2}{4\delta}\|u\|_X^2 \leq \frac{k^2}{4\delta}\Phi(u) \leq \frac{k^2}{4\delta}r = \gamma^2, \quad \forall\, t \in [0,1].$$

Thus,

$$\Phi^{-1}(-\infty, r] = \{u \in X; \Phi(u) \leq r\} \subseteq \{u \in X; |u(t)| \leq \gamma\}.$$

Therefore, one has

$$\sup_{u\in\Phi^{-1}(-\infty,r)} \int_0^1 F(t,u(t))dt \leq \sup_{|x|\leq\gamma} \int_0^1 F(t,x)dt,$$

and this in conjunction with the second inequality in (3.3) ensures

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r)} \int_0^1 \Big(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big)dt &\leq \sup_{u\in\Phi^{-1}(-\infty,r)} \left(\int_0^1 F(t,u(t)\right) dt + \frac{\mu}{\lambda}G^\gamma \\ &\leq \int_0^1 \sup_{|x|\leq\gamma} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma,\end{aligned}$$

for every $u \in X$ such that $\Phi(u) < r$. Thus,

$$\sup_{\Phi(u)\leq r} \Psi(u) \leq \int_0^1 \sup_{|x|\leq\gamma} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma.$$

On the other hand, we have

$$\Psi(\omega) = \int_0^1 \Big(F(t,\omega(t)) + \frac{\mu}{\lambda}G(t,\omega(t))\Big)\, dt \geq \int_0^1 F(t,\omega(t))dt + \frac{\mu}{\lambda}G_\delta.$$

Therefore

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \int_0^1 \Big(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big)dt}{r} \\ &\leq \frac{\int_0^1 \sup_{|x|\leq\gamma} F(t,x(t))dt + \frac{\mu}{\lambda}G^\gamma}{\frac{4\delta}{k^2}\gamma^2},\end{aligned} \tag{3.7}$$

and

$$\frac{\Psi(\omega)}{\Phi(\omega)} \geq \frac{\int_0^1 \Big(F(t,\omega(t)) + \frac{\mu}{\lambda}G(t,\omega(t))\Big)dt}{\frac{2\delta\pi^4}{\eta}} \geq \frac{\int_0^1 F(t,\omega(t))dt + \frac{\mu}{\lambda}G_\delta}{\frac{2\delta\pi^4}{\eta}}. \tag{3.8}$$

Since

$$\mu < \frac{4\delta\gamma^2 - \lambda k^2 \int_0^1 \sup_{|x|\leq\gamma} F(t,x(t))dt}{k^2 G^\gamma}$$

this means

$$\frac{\int_0^1 \sup_{|x|\leq\gamma} F(t,x(t))dt + \frac{\mu}{\lambda}G^\gamma}{\frac{4\delta}{k^2}\gamma^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{2\delta\pi^4 - \lambda\eta \int_0^1 F(t,\omega(t))dt}{\eta G_\delta},$$

this means

$$\frac{\int_0^1 F(t,\omega(t))dt + \frac{\mu}{\lambda} G_\delta}{\frac{2\delta\pi^4}{\eta}} > \frac{1}{\lambda}.$$

Then

$$\frac{\int_0^1 \sup_{|x|\le\gamma} F(t,x(t))dt + \frac{\mu}{\lambda} G^\gamma}{\frac{4\delta}{k^2}\gamma^2} < \frac{1}{\lambda} < \frac{\int_0^1 F(t,\omega(t))dt + \frac{\mu}{\lambda} G_\delta}{\frac{2\delta\pi^4}{\eta}}. \tag{3.9}$$

Hence, from (3.7) to (3.9), the condition (i_1) of Theorem 2.5 is fulfilled. Since

$$\lambda \in \left(\frac{\Phi(\omega)}{\Psi(\omega)}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)} \right).$$

Theorem 2.5 with $\overline{u} = \omega$ guarantees the existence of a local minimum point u_λ for the functional I_λ such that $0 < \Phi(u_\lambda) < r$ and so u_λ is a nontrivial weak solution of the problem (1.1) such that $\max_{t\in[0,1]} |u_\lambda(t)| < \gamma$. □

We now present an example to demonstrate the use of Theorem 3.1.

Example 3.2. As an application of Theorem 3.1 we consider the problem

$$\begin{cases} -u^{(vi)}(t) + 2u^{(iv)}(t) - u''(t) + 2u = \lambda f(t,u(t)) + \mu g(t,u(t)), \ t \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{iv}(0) = u^{iv}(1) = 0, \end{cases} \tag{3.10}$$

where $A = 2$, $B = 1$, $C = 2$, so $\delta = 1$ and $\eta \simeq .0005\pi^4$. Put $g(t,x) = x$ for all $(t,x) \in [0,1] \times \mathbb{R}$, thus $\limsup_{|\xi|\to\infty} \frac{\sup G(t,\xi)}{|\xi|^2} = 0$. Now, define

$$f(t,x) = \begin{cases} t^2, & x \le 1, \\ \frac{t^2}{x}, & x > 1, \end{cases}$$

for every $t \in [0,1]$. By the expression of f, we obtain

$$F(t,x) = \begin{cases} t^2 x, & x \le 1, \\ t^2(\ln x + 1), & x > 1. \end{cases}$$

Consequently, $\lim_{|\xi|\to\infty} \frac{F(t,\xi)}{|\xi|^2} = 0$, and and thus the condition (A_2) is satisfied. Let us take $\gamma = 10^6$. Then $\frac{1}{2\eta} \simeq \frac{1}{2\times.0005\times\pi^4} < 10^{12} = \gamma^2$, and moreover,

$$\frac{\int_0^1 \sup_{|x|\le\gamma} F(t,x)dt}{\gamma^2} = \frac{7}{3\times 10^{12}} < 2 \times .0005 \times \pi^4 \times \frac{1}{6} = 2\eta \int_0^1 F(t,\omega(t))dt$$

Therefore, the condition (A_1) also holds. Hence, all the assumptions of Theorem 3.1 are satisfied. It follows that, for each each $\lambda \in \left(\frac{2}{.0005\times\frac{1}{6}}, \frac{4\times\pi^4\times10^{12}}{7}\right)$ and for every

$$0 \le \mu < \min\left\{ \frac{2\times\pi^4 - \frac{1}{6}\times .0005\times\pi^4\lambda}{\frac{-1}{4}\times .0005\times\pi^4}, \frac{8\times 10^{12} - \frac{14}{\pi^4}\lambda}{10^6 \times \frac{1}{\pi^4}} \right\},$$

the problem (3.10)admits at least one weak solutions in X.

In what follows, we proceed to apply Theorem 2.6 in order to establish the existence of at least two weak solutions for problem (1.1).

Theorem 3.3. *Assume that there exists a positive constant γ such that $\frac{1}{2\eta} < \gamma^2$, and that the following conditions hold:*

(A_3) *there exist $\nu > 2$ and $T > 0$ such that*

$$0 < \nu F(t,\xi) < \xi f(t,\xi)$$

for all $|\xi| > T$ and $t \in [0,1]$,

(A_3') *there exist $\nu \geq \nu' > 2$ and $T' > 0$ such that*

$$0 < \nu' G(t,\xi) < \xi g(t,\xi)$$

for all $|\xi| > T'$ and $t \in [0,1]$.

Then, for each

$$\lambda \in \left(0, \frac{4\delta\gamma^2}{k^2 \int_0^1 \sup_{|x|\leq\gamma} F(t,x)dt}\right)$$

there exists $\delta_\lambda > 0$ given by (3.4) such that, for each $\mu \in [0, \delta_\lambda[$, the problem (1.1) admits at least two weak solutions u_1 and u_2 in X such that $\max_{t\in[0,1]} |u_1(t)| < \gamma$.

Proof. We aim to verify the hypotheses of Theorem 2.6 in the space X, equipped with the norm defined in (2.2). Considering the functionals Φ and Ψ as introduced in Theorem 2.6, we first verify the (PS) condition for the functional I_λ. To this end, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $I_\lambda(u_n)$ is bounded and $I_\lambda'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|I_\lambda(u_n)| \leq c_0$ and $|I_\lambda'(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Therefore, by the assumptions (A_3), (A_4) and definition of I_λ', we have

$$\begin{aligned} c_0 + c_1\|u_n\|_X &\geq \nu I_\lambda(u_n) - I_\lambda'(u_n)(u_n) && (3.11)\\ &\geq \left(\frac{\nu}{2}-1\right)\|u_n\|_X^2 + \lambda\int_0^1 \Big(f(t,u_n(t))u_n(t) - \nu F(t,u_n(t))\Big)dt && (3.12)\\ &+ \mu\int_0^1 \Big(g(t,u_n(t))u_n(t) - \nu G(t,u_n(t))\Big)dt && (3.13)\\ &\geq \left(\frac{\nu}{2}-1\right)\|u_n\|_X^2, && (3.14) \end{aligned}$$

for some $c_1 > 0$. Since $\nu \geq \nu' > 2$, this implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Next, we prove that there exists $u \in X$ such that $u_n \to u$ in X, as $n \to \infty$. By Proposition 2.3, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$ and a function u in X such that

$$u_n \rightharpoonup u \quad \text{in X,} \quad \text{and} \quad u_n \to u \quad \text{in } C^1([0,1]). \qquad (3.15)$$

By definition $I_\lambda'(u)$, we get

$$\begin{aligned} \langle I_\lambda'(u_n), u_n - u\rangle = &\int_0^1 \Big(u_n'''(t)(u_n'''(t) - u'''(t)) + Au_n''(t)(t)(u_n''(t) - u''(t))\\ &+ Bu_n'(t)(u_n'(t) - u(t)) + Cu_n(t)(u_n(t) - u(t))\Big)dt\\ &- \int_0^1 \Big(\lambda f(t,u_n(t))(u_n(t) - u(t)) + \mu g(t,u_n(t))(u_n(t) - u(t))\Big)dt. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle &= \int_0^1 \Big(u'''_n(t)(u'''_n(t) - u'''(t)) + Au''_n(t)(u''_n(t) - u''(t)) \\ &\quad + Bu'_n(t)(u'_n(t) - u(t)) + Cu_n(t)(u_n(t) - u(t))\Big)dt \\ &\quad - \int_0^1 \Big(\lambda f(t, u_n(t))(u_n(t) - u(t)) + \mu g(t, u_n(t))(u_n(t) - u(t))\Big)dt \\ &\quad - \Big(\int_0^1 \Big(u'''(t)(u'''_n(t) - u'''(t)) + Au''(t)(u''_n(t) - u''(t)) \\ &\quad + Bu'(t)(u'_n(t) - u'(t)) + Cu(t)(u_n(t) - u(t))\Big)dt \\ &\quad - \int_0^1 \Big(\lambda f(t, u(t))(u_n(t) - u(t)) + \mu g(t, u(t))(u_n(t) - u(t))\Big)dt \\ &= \int_0^1 \Big((u'''_n(t) - u'''(t))^2) + A(u''_n(t) - u''(t))^2 \\ &\quad + B(u'_n(t) - u'(t))^2 + C(u_n(t) - u(t))^2\Big)dt \\ &\quad - \int_0^1 \Big(\lambda f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t)) \\ &\quad + \mu(g(t, u_n(t)) - g(t, u(t))(u_n(t) - u(t))\Big)dt \\ &\geq \|u_n - u\|_X^2 - \int_0^1 \Big(\lambda(f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t)) \\ &\quad + \mu(g(t, u_n(t)) - g(t, u(t)))(u_n(t) - u(t))\Big)dt.\end{aligned}$$

From the continuity of f and g we get

$$\begin{aligned}\int_0^1 \Big((u'''_n(t) - u'''(t))^2) + A(u''_n(t) - u''(t))^2 + B(u'_n(t) - u'(t))^2 \\ + C(u_n(t) - u(t))^2\Big)dt \to 0, \quad n \to \infty, \end{aligned} \tag{3.16}$$

$$\lambda \int_0^1 (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt \to 0, \quad n \to \infty, \tag{3.17}$$

$$\mu \int_0^1 (g(t, u_n(t)) - g(t, u(t)))(u_n(t) - u(t))dt \to 0, \quad n \to \infty. \tag{3.18}$$

From (3.11) and (3.15) it is easy to see that

$$\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \to 0.$$

Therefore, by (3.16) to (3.18), we have

$$\|u_n - u\|_X^2 \to 0.$$

Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in X. Consequently, the functional I_λ satisfies the (PS) condition. Moreover, by combining assumptions (A_3) and $(A'3)$, there exist positive constants $a1, a_2, b_1, b_2 > 0$ such that

$$F(t, x) \geq a_1|x|^\nu - a_2 \tag{3.19}$$

for all $t \in [0,1]$ and $x \in \mathbb{R}$, and

$$G(t,x) \geq b_1|x|^{\nu'} - b_2 \tag{3.20}$$

for all $t \in [0,1]$ and $x \in \mathbb{R}$. Now, choosing any $u \in X \setminus \{0\}$, for each $\tau > 0$ one has

$$I_\lambda(\tau u) \leq \frac{1}{2}\|\tau u\|_X^2 - \lambda \int_0^1 F(t,\tau u(t))dt - \mu \int_0^1 G(t,\tau u(t))dt \tag{3.21}$$

$$\leq \frac{\tau^2}{2}\|u\|_X^2 - \lambda\tau^\nu \int_0^1 a_1|u(t)|^\nu dt + \lambda a_2 - \mu\tau^{\nu'} \int_0^1 b_1|u(t)|^{\nu'} dt + \mu b_2. \tag{3.22}$$

Since $\nu > 2$, this condition guarantees that I_λ is unbounded from below. Thus, all hypotheses of Theorem 2.6 are verified. Therefore, for each

$$\lambda \in \left(0, \frac{4\delta\gamma^2}{k^2 \int_0^1 \sup_{|x|\leq\gamma} F(t,x(t))dt}\right)$$

the functional I_λ admits two critical points that are weak solutions of the problem (1.1). □

Remark 3.4. In Theorem 2.5 we observe that, if $f(t,0) \neq 0$, then Theorem 3.6 ensures the existence of two nontrivial weak solutions for the problem (1.1). If the condition $f(t,0) \neq 0$ for all $t \in [0,1]$ does not hold, the second solution u_2 of the problem (1.1) may be trivial, but the problem has at least a nontrivial solution.

An example is given to demonstrate the application of Theorem 3.3.

Example 3.5. We consider the problem

$$\begin{cases} -u^{(vi)}(t) + u^{(iv)}(t) - u''(t) + 3u = \lambda f(t,u(t)) + \mu g(t,u(t)), \ t \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{iv}(0) = u^{iv}(1) = 0, \end{cases} \tag{3.23}$$

where $A = 1$, $B = -1$, $C = 3$. Hence, $\delta = \min\{1, 1 + Ak + Bk^2\} = 1$ and $\eta \simeq .0005\pi^4$. Let $g(t,x) = 4x^3 + 2x + t$ for all $(t,x) \in [0,1] \times \mathbb{R}$. Then thus $G(t,x) = x^4 + x^2 + tx$, and $\lim_{\xi\to+\infty} \frac{\xi g(t,\xi)}{G(t,\xi)} = 4$. Therefore, by choosing $\nu' = 4$ and $T' = 1$, the condition (A_3') is satisfied. Next, define

$$f(t,x) = \begin{cases} x^4 + 6, & x \leq 1 \\ 7x^6, & x > 1. \end{cases}$$

From this definition, we obtain

$$F(t,x) = \begin{cases} \frac{1}{5}x^5 + 6x, & x \leq 1 \\ x^7 + \frac{26}{5}, & x > 1. \end{cases}$$

Consequently, $\lim_{\xi\to+\infty} \frac{\xi f(t.\xi)}{F(t,\xi)} = 7 < \infty$ and $\lim_{\xi\to-\infty} \frac{\xi f(t.\xi)}{F(t,\xi)} = 5 < \infty$. Thus, by choosing $\nu = 5 > 2$ and $T = 1$, condition (A_3) is satisfied. Taking $\gamma = 10^3$, we observe that $\frac{1}{2\eta} = \frac{1}{.001\times\pi^4} < 10^6 = \gamma^2$. Hence, all the assumptions of Theorem 3.3 are fulfilled. Therefore, for each $\lambda \in \left(0, \frac{4\times10^6\pi^4}{10^{21}+\frac{26}{5}}\right)$ the problem (3.23) has at least two nontrivial weak solution.

We now turn to the case $\lambda = \mu$, and in this framework we study the existence of at least two and three weak solutions for the problem (1.1).

Theorem 3.6. *Assume that there exists a positive constant $\bar{\gamma}$ such that*

$$\bar{\gamma}^2 > \frac{1}{2\eta}, \tag{3.24}$$

and suppose that assumption (A_2) in Theorem 3.1 holds. Moreover, let

(A_4) $\int_0^1 F(t,\omega(t))dt \geq 0$;

(A_5)
$$\frac{\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}}{\frac{2\delta}{k^2}\bar{\gamma}^2} < \frac{G^{\bar{\gamma}} - \int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt}{\frac{\delta\pi^4}{\eta}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{2\delta\pi^4}{\eta}}{G^{\bar{\gamma}} - \int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt}, \frac{\frac{4\delta}{k^2}\bar{\gamma}^2}{\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}} \right)$$

and for the function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying in (3.3), the problem (1.1), in the case $\lambda = \mu$ admits at least three weak solutions in X.

Proof. Put $I_\lambda = \Phi(u) + \lambda\Psi(u)$, where

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 \tag{3.25}$$

and

$$\Psi(u) = -\int_0^1 \Big(F(t,u(t)) + G(t,u(t))\Big)dt$$

for all $u \in X$. Standard arguments show that Φ and Ψ are Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_0^1 \Big(u'''(t)v'''(t) + Au''(t)v''(t) + Bu'(t)v'(t) + Cu(t)v(t)\Big)dt$$

and

$$\Psi'(u)v = -\int_0^1 \Big(f(t,u(t))v(t)dt + g(t,u(t))v(t)\Big)dt$$

for all $u, v \in X$, respectively. We know that a critical point of the functional $\Phi(u) + \lambda\Psi(u)$ corresponds to a weak solution of the problem (1.1) in the case $\lambda = \mu$. Our goal is to apply Theorem 2.7 to the functionals Φ and Ψ. By the sequential weak lower semicontinuity of the norm, the functional Φ is sequentially weakly lower semicontinuous. Moreover, since Φ is continuously Gâteaux differentiable, its Gâteaux derivative admits a continuous inverse on X^*. The functional $\Psi : X \to \mathbb{R}$ is well-defined, continuously Gâteaux differentiable, and its Gâteaux derivative is compact. Therefore, it suffices to verify that Φ and Ψ satisfy conditions (k_1) and (k_2) in Theorem 2.7. Since $\mu < \delta_k$ and by assumption 3.3, we can fix $\alpha > 0$ such that the $\alpha\mu < \frac{2\delta}{k^2}$, and there exists a function $\rho_\alpha \in L^1(0,1)$ such that

$$G(t,x) \leq \alpha x^2 + \rho_\alpha.$$

Now, we fix $\varepsilon < \frac{2\delta}{\lambda k^2} - \alpha$. From the assumption (A_2) there is a function $\rho_\varepsilon \in L^1(0,1)$ such that

$$F(t,x) \le \varepsilon x^2 + \rho_\varepsilon$$

for every $(t,x) \in [0,1] \times \mathbb{R}$. Noting that $\lambda = \mu$, it follows that, for each $u \in X$,

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 - \int_0^1 \Big(\lambda F(t,u(t)) + \lambda G(t,u(t))\Big)dt \\ &\ge \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon \int_0^1 u^2(t)dt - \lambda\|\rho_\varepsilon\|_1 - \alpha\lambda \int_0^1 u^2(t)dt - \mu\|\rho_\alpha\|_1 \\ &\ge (\frac{1}{2} - \lambda\frac{k^2}{4\delta}\varepsilon - \lambda\frac{k^2}{4\delta}\alpha)\|u\|_X^2 - \lambda\|\rho_\varepsilon\|_1 - \lambda\|\rho_\alpha\|_1,\end{aligned}$$

and thus

$$\lim_{\|u\|_X \to \infty} (\Phi(u) + \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi(u) + \lambda\Psi(u)$ is coercive. Now it remains to show (k_2) of Theorem 2.7. Put $\bar{r} = \frac{4\delta}{k^2}\bar{\gamma}^2$. We clearly observe that $\omega \in X$ and $\|\omega\|_X^2 = \frac{4\delta\pi^4}{\eta}$. Hence, by (3.5), we have

$$\Phi(\omega) = \frac{2\delta\pi^4}{\eta}. \tag{3.26}$$

Thus by (3.24), $\Phi(\omega) > \bar{r}$. Moreover, by (A_4)

$$\Psi(\omega) = -\int_0^1 \Big(F(t,\omega(t)) + G(t,\omega(t))\Big)dt \le G^{\bar{\gamma}}.$$

Taking (2.3) into account, for every $u \in X$ such that $\Phi(u) < \bar{r}$, we have

$$\sup_{t\in[0,1]} |u(t)| \le \bar{\gamma}. \tag{3.27}$$

Thus

$$\begin{aligned}&\sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \int_0^1 \Big(F(t,u(t)) + G(t,u(t))\Big)dt \\ &\le \sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \int_0^1 F(t,u(t))dt + G^{\bar{\gamma}} \\ &\le \int_0^1 \sup_{|x|\le\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}\end{aligned} \tag{3.28}$$

for every $u \in X$ with $\Phi(u) < \bar{r}$. Thus

$$\sup_{\Phi(u)\le\bar{r}} \Psi(u) \le \int_0^1 \sup_{|x|\le\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}.$$

By simple calculations and from the definition of $\varphi_1(\bar{r})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega} = \Phi^{-1}(-\infty,\bar{r})$ one has

$$\begin{aligned}\varphi_1(\bar{r}) &= \inf_{u\in\Phi^{-1}]-\infty,\bar{r}[} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega}} \Psi}{\bar{r} - \Phi(u)} \le \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega}} \Psi}{\bar{r}} \\ &\le \frac{\int_0^1 \sup_{|x|\le\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}}{\frac{4\delta}{k^2}\bar{\gamma}^2}.\end{aligned}$$

On the other hand, by (3.28) one has

$$\begin{aligned}\varphi_2(\bar{r}) &= \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \sup_{v\in\Phi^{-1}[\bar{r},\infty)} \frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)} \geq \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \frac{\Psi(u)-\Psi(\omega)}{\Phi(\omega)-\Phi(u)} \\ &\geq \frac{\inf_{u\in\Phi^{-1}(-\infty,\bar{r})}\Psi(u)-\Psi(\omega)}{\Phi(\omega)-\Phi(u)} \\ &\geq \frac{-\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}}{\Phi(\omega)-\Phi(u)} \\ &\geq \frac{G^{\bar{\gamma}} - \int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt}{\frac{2\delta\pi^4}{\eta}}.\end{aligned}$$

Hence from (A_5) one has

$$\varphi_1(\bar{r}) < \varphi_2(\bar{r}).$$

Therefore, from Theorem 2.7, taking also into account that

$$\frac{1}{\varphi_2(\bar{r})} \leq \frac{\frac{2\delta\pi^4}{\eta}}{G^{\bar{\gamma}} - \int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt}$$

and

$$\frac{1}{\varphi_1(\bar{r})} \geq \frac{\frac{4\delta}{k^2}\bar{\gamma}^2}{\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + G^{\bar{\gamma}}}$$

we obtain the desired conclusion. □

Remark 3.7. In Theorem 3.6, if the condition

(A_6) $\frac{k^2\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + k^2 G^{\bar{\gamma}}}{2\bar{\gamma}^2} < \frac{\eta G^{\bar{\gamma}}}{\pi^4}$

is replaced by condition (A_5), and assumptions (3.24), (A_4), and (A_6) hold, with the function $g : \mathbb{R} \to \mathbb{R}$ satisfying (3.3), then for each

$$\lambda \in \left(\frac{\delta\pi^4}{\eta G^{\bar{\gamma}}}, \frac{\delta\bar{\gamma}^2}{k^2\int_0^1 \sup_{|x|\leq\bar{\gamma}} F(t,x)dt + k^2 G^{\bar{\gamma}}} \right)$$

the problem (1.1) in the case $\lambda = \mu$ admits at least three weak solutions.

To demonstrate the application of Theorem 3.6, we consider the following example.

Example 3.8. Consider the problem

$$\begin{cases} -u^{(vi)}(t) - u^{(iv)}(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), \ t \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.29}$$

where $A = -1$, $B = C = 0$. By (H_5), we have $\delta = 1 - \frac{1}{\pi^2}$ and $\eta \simeq .045$. Let $g(t,x) = 2x + t^2$ for all $(t,x) \in [0,1] \times \mathbb{R}$. Then $G(t,x) = x^2 + t^2 x$, and $\lim_{|\xi|\to\infty} \frac{G(t,\xi)}{\xi^2} = 1$. Now, let $f(t,x) = te^{-x}$, for $(t,x) \in [0,1] \times \mathbb{R}$. From this we have $F(t,x) = -te^{-x}$, for $(t,x) \in [0,1] \times \mathbb{R}$. Hence, $\lim_{|\xi|\to\infty} \frac{F(t,\xi)}{\xi^2} = 0$, so the condition (A_2) is satisfied. Taking $\bar{\gamma} = \sqrt{21}$, the assumptions (3.24) and (A_4) are satisfied, and we have $G^{\gamma} = 22$. Moreover,

$$\frac{-e^{-\sqrt{21}} + 22}{2\pi^4(1-\frac{1}{\pi^2}) \times 21} < \frac{e^{-\sqrt{21}} + 22}{\pi^4(1-\frac{1}{\pi^2}) \times 22},$$

which implies that assumption (A_5) holds. Therefore, by Theorem 2.7 and Remark 3.7, for each

$$\lambda \in \left(\frac{\pi^4(1-\frac{1}{\pi^2}) \times 22}{e^{-\sqrt{21}} + 22}, \frac{2\pi^4(1-\frac{1}{\pi^2}) \times 21}{-e^{-\sqrt{21}} + 22} \right)$$

the problem (3.29) admits at least three nontrivial weak solutions.

At this point, we aim to establish the existence and multiplicity of solutions for the problem (1.1) by applying Theorem 2.8 in the case $\lambda = \mu$.

Theorem 3.9. *Assume that there exist three positive constants $\bar{\gamma}_1$, η, and $\bar{\gamma}_2$ satisfying*

$$2\bar{\gamma}_1^2 < \frac{1}{\eta} < 2\bar{\gamma}_2^2, \tag{3.30}$$

such that assumption (A_5) in Theorem 2.7 holds and

(A_7) $$\frac{k^2}{4\delta} \max \left\{ \frac{\int_0^1 \sup_{|x|\le\bar{\gamma}_1} F(t,x)dt + G^{\bar{\gamma}_1}}{\bar{\gamma}_1^2}, \frac{\int_0^1 \sup_{|x|\le\bar{\gamma}_2} F(t,x)dt + G^{\bar{\gamma}_2}}{\bar{\gamma}_2^2} \right\} < \frac{\eta}{2\delta\pi^4}.$$

Then, for each

$$\lambda \in \Lambda = \left(\frac{2\delta\pi^4}{\eta}, \min\left\{ \frac{\frac{4\delta}{k^2}\bar{\gamma}_1^2}{\int_0^1 \sup_{|x|\le\bar{\gamma}_1} F(t,x)dt + G^{\bar{\gamma}_1}}, \frac{\frac{4\delta}{k^2}\bar{\gamma}_2^2}{\int_0^1 \sup_{|x|\le\bar{\gamma}_2} F(t,x)dt + G^{\bar{\gamma}_2}} \right\} \right)$$

and for function $g : \mathbb{R} \to \mathbb{R}$ such that satisfying in (3.3), the problem (1.1) in the case $\lambda = \mu$ admits at least two weak solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{t\in[0,1]} |u_{1,\lambda}(t)| < \bar{\gamma}_1$ and $\max_{t\in[0,1]} |u_{2,\lambda}(t)| < \bar{\gamma}_2$.

Proof. Put

$$\bar{f}(t,x) = \begin{cases} f(t,-\bar{\gamma}_2), & \text{if } (t,x) \in [0,1] \times (-\infty, -\bar{\gamma}_2] \\ f(t,x), & \text{if } (t,x) \in [0,1] \times [-\bar{\gamma}_2, \bar{\gamma}_2] \\ f(t,\bar{\gamma}_2), & \text{if } (t,x) \in [0,1] \times (\bar{\gamma}_2, \infty). \end{cases} \tag{3.31}$$

Clearly, $\bar{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Now, put $\bar{F}(t,x) = \int_0^1 \bar{f}(t,x)dx$ for all $(t,\xi) \in [0,1] \times \mathbb{R}$ and take X and Φ as in (2.1) and (3.25), respectively, and

$$\Psi(u) = -\int_0^1 \Big(\bar{F}(t,u(t)) + G(t,u(t)) \Big) dt$$

for all $u \in X$. Our goal is to apply Theorem 2.8 to Φ and Ψ. It is well known that $\lim_{\|u\|_X\to\infty} \Phi(u) = \infty$ and Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)v = -\int_0^1 \Big(\bar{f}(t,u(t))v(t) + g(t,u(t))v(t) \Big) dt$$

for any $v \in X$ as well as it is sequentially weakly lower semicontinuous. Furthermore Ψ' : $X \to X^*$. is a compact operator. Thus, it is enough to show that Φ and Ψ satisfy in the conditions (e_1), (e_2) and (e_3) in Theorem 2.8. Let

$$\bar{r}_1 = \frac{4\delta}{k^2}\bar{\gamma}_1^2, \qquad \bar{r}_2 = \frac{4\delta}{k^2}\bar{\gamma}_2^2$$

and $\omega \in X$ as in the proof of Theorem 2.8 due to the assumptions (3.30) and (3.6) we have $\bar{r}_1 < \Phi(\omega) < \bar{r}_2$ and $\inf_X \Phi < \bar{r}_1 < \bar{r}_2$. Moreover, arguing as in the proof of Theorem 3.6 and taking also into account Remark 3.7 we obtain

$$\varphi(\bar{r}_1) \leq \frac{\int_0^1 \sup_{|x|\leq \bar{\gamma}_1} F(t,x)dt + G^{\bar{\gamma}_1}}{\frac{4\delta}{k^2}\bar{\gamma}_1^2},$$

$$\varphi(\bar{r}_2) \leq \frac{\int_0^1 \sup_{|x|\leq \bar{\gamma}_2} F(t,x)dt + G^{\bar{\gamma}_2}}{\frac{4\delta}{k^2}\bar{\gamma}_2^2}$$

and

$$\varphi_2^*(\bar{r}_2, \bar{r}_2) \geq \frac{\eta}{2\delta\pi^4}.$$

Therefore, in view of (A_3) and (A_6), conditions (e_2) and (e_3) required by Theorem 2.8 are fulfilled. Therefore, from Theorem 2.8 we obtain that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda\Big(\bar{f}(t,u(t)) + g(u(t))\Big), \ t \in [0,1] \\ u(0) = u'(0) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{3.32}$$

admits at least two weak solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{t\in[0,1]} |u_{1,\lambda}(t)| < \bar{\gamma}_1^2$ and $\max_{t\in[0,1]} |u_{2,\lambda}(t)| < \bar{\gamma}_2^2$. Observing that these solutions are also solutions for the problem (1.1) in the case $\lambda = \mu$, the conclusion follows. □

Finally, we examine the case in which the variables of the function f are separable, so that the problem (1.1) takes the form:

$$\begin{cases} -u^{(vi)}(t) + Au^{(iv)}(t) - Bu''(t) + Cu(t) = \lambda\Big(\theta(t)f(u(t)) + g(t,u(t))\Big), \ t \in [0,1] \\ u(0) = u'(0) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.33}$$

where $\theta : [0,1] \to \mathbb{R}$ is a non-negative and non-zero function such that $\theta \in L^1[0,1]$ and $f : \mathbb{R} \to \mathbb{R}$ is a non-negative and continuous function.

Put

$$F(\xi) = \int_0^{\xi} f(x)dx, \quad \text{for all } \xi \in \mathbb{R}.$$

We apply the results of Theorems 3.6 and 3.9 to the case where $f(t,x) = \theta(t)f(x) \ \forall \ (t,x) \in [0,1] \times \mathbb{R}$.

Theorem 3.10. *Assume that there exist two positive constants $\bar{\gamma}$ and η such that*

$$\bar{\gamma}^2 > \frac{1}{2\eta},$$

and suppose that the assumption (A_2) and the following condition hold:

(A_8) $\frac{k^2\|\theta\|_{L^1[0,1]}F(\bar{\gamma})+k^2G^{\bar{\gamma}}}{2\bar{\gamma}^2} < \frac{\eta}{\pi^4}$.

Moreover, let the function $g : \mathbb{R} \to \mathbb{R}$ satisfy the condition (3.3). Then, for every

$$\lambda \in \left(\frac{2\delta\pi^4}{\eta}, \frac{4\delta\bar{\gamma}^2}{k^2\|\theta\|_{L^1[0,1]}F(\bar{\gamma}) + k^2G^{\bar{\gamma}}} \right)$$

the problem (1.1) in the case $\lambda = \mu$ admits at least three weak solutions in X.

Theorem 3.11. *Assume that there exist three positive constants $\bar{\gamma}_1$, η and $\bar{\gamma}_2$ such that*

$$\bar{\gamma}_1^2 < \frac{1}{2\eta} < \bar{\gamma}_2^2,$$

and suppose that the following condition holds:

$$(A_9) \qquad \frac{k^2}{4\delta} \max\left\{ \frac{\|\theta\|_{L^1[0,1]}F(\bar{\gamma}_1) + G^{\bar{\gamma}_1}}{\bar{\gamma}_1^2}, \frac{\|\theta\|_{L^1[0,1]}F(\bar{\gamma}_2) + G^{\bar{\gamma}_2}}{\bar{\gamma}_2^2} \right\} < \frac{\eta}{2\delta\pi^4}.$$

Moreover, let the function $g : \mathbb{R} \to \mathbb{R}$ satisfy the condition (3.3). Then, for each

$$\lambda \in \Lambda = \left(\frac{2\delta\pi^4}{\eta}, \min\left\{ \frac{4\delta\bar{\gamma}_1^2}{k^2\|\theta\|_{L^1[0,1]}F(\bar{\gamma}_1) + k^2G^{\bar{\gamma}_1}}, \frac{4\delta\bar{\gamma}_2^2}{k^2\|\theta\|_{L^1[0,1]}F(\bar{\gamma}_2) + k^2G^{\bar{\gamma}_2}} \right\} \right)$$

the problem (1.1) in the case $\lambda = \mu$ admits at least two weak solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{t\in[0,1]} |u_{1,\lambda}(t)| < \bar{\gamma}_1$ and $\max_{t\in[0,1]} |u_{2,\lambda}(t)| < \bar{\gamma}_2$.

Acknowledgment

The authors sincerely thank the referee for their careful reading of the manuscript and for providing constructive comments and insightful suggestions, which have greatly enhanced the quality and presentation of this work.

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