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On positive solutions for classes of quasilinear singular problems

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Abstract. We study classes of quasilinear singular problems, involving the p-Laplacian operator and a potential, with homogeneous Dirichlet boundary condition. First, we prove the existence and regularity of positive solutions to a purely singular problem. Then, we investigate the existence of positive solutions with respect to a parameter depending on the behavior of the nonlinearities at infinity and at the origin. Finally, we extend some of these results to coupled system of equations. We use sub-super solution techniques to establish our results.

Keywords: singular problem, potential, sublinear, positive solution, sub- and supersolution method.

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1 Introduction

We first consider singular problems of the form

$$\begin{cases} -\Delta_p w + a(x)|w|^{p-2}w = \dfrac{1}{w^\alpha} & \text{in } \Omega\,, \\ w = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$ is the p-Laplacian of z with $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 2$ with $C^{2,\beta}$ boundary $\partial\Omega$ with $\beta \in (0,1)$ or a bounded open interval if $N = 1$. We assume $\alpha \in [0,1)$, and the potential function $a \in L^\infty(\Omega)$ satisfies:

(H1) $\operatorname{ess\,inf} a > -\lambda_p$, where λ_p is the principal eigenvalue of $-\Delta_p$ with Dirichlet boundary condition on $\partial\Omega$.

Our first result establishes the existence and regularity of a weak solution of the purely singular problem (1.1).

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Theorem 1.1. *Let $p \leqslant N$ and $\alpha \in (0, \frac{p}{N})$. If* **(H1)** *holds, then the singular problem* (1.1) *has at least one positive solution $w \in W_0^{1,p}(\Omega) \cap C_0^1(\overline{\Omega})$. Moreover,*

$$\frac{\partial w}{\partial \eta} < 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where η is the unit outer normal to the boundary $\partial\Omega$.

Theorem 1.1 generalizes the result in [16, Lemma 2.3] for the $p = 2$ case, to the case $p > 1$. See [8, Lemma 3.1] for discussions of such a result for the case when $a = 0$.

Next, we consider singular problems of the form

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda \dfrac{f(u)}{u^\alpha} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.3}$$

Here $\lambda > 0$ is a parameter, and the nonlinearity $f : [0, \infty) \to \mathbb{R}$ is continuous. We now introduce the following hypotheses:

(H2) There exist $\sigma > \alpha$ and $A > 0$ such that $f(s) \geqslant As^\sigma$ for $s \gg 1$,

(H3) $\lim_{s\to+\infty} \frac{f(s)}{s^{\alpha+p-1}} = 0\,.$

The hypothesis **(H2)** requires $\frac{f(s)}{s^\alpha} \to +\infty$ as $s \to +\infty$, while hypothesis **(H3)** ensures that $\frac{f(s)}{s^\alpha}$ has a $(p-1)$ *sublinear growth at infinity*.

We prove the following results, depending on the behavior of f near the origin. We include both $f(0) > 0$ (with $f : [0, \infty) \to (0, \infty)$) and $f(0) < 0$ cases giving rise to two distinct types of singularities $\frac{f(s)}{s^\alpha} \to +\infty$ and $\frac{f(s)}{s^\alpha} \to -\infty$ as $s \to 0^+$, respectively. The latter is termed in the literature as *infinite semipositone problems* and are technically challenging, especially to study positive solutions.

Theorem 1.2. *Suppose that $p \leqslant N$ and $\alpha \in (0, \frac{p}{N})$. If $f : [0, \infty) \to (0, \infty)$, and assumptions* **(H1)** *and* **(H3)** *hold, then for each $\lambda > 0$, problem* (1.3) *has at least one positive solution $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Theorem 1.3. *Let $p \leqslant N$ and $\alpha \in (0, \frac{p}{N})$. Assume that $f(0) < 0$ and* **(H1)–(H3)** *hold. Then there exists $\lambda^* > 0$ such that, for all $\lambda > \lambda^*$, problem* (1.3) *admits at least one positive solution $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Next, we study positive solutions for infinite semipositone problems of the form

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda g(u) - \dfrac{1}{u^\alpha} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.4}$$

where $g : [0, \infty) \to \mathbb{R}$ is a continuous function that satisfies:

(H4) There exist k_1, $k_2 > 0$ and $s_0 > 0$ such that $g(s) \geqslant k_1 s^{p-1} - k_2$ for every $0 \leqslant s \leqslant s_0$

(H5) $\lim_{s\to\infty} \frac{g(s)}{s^{p-1}} = k_3$ for some $k_3 > 0$.

Note that (1.4) has infinite semipositone structure due to the presence of $-\frac{1}{u^\alpha}$ in the reaction term, and **(H5)** requires the reaction term to have $(p-1)$ linear growth at infinity.

In this case, we establish the following existence result which extends the result in [14] with $a = 0$ to $a \in L^\infty(\Omega)$.

Theorem 1.4. *Assume that* **(H1)** *and* **(H4)–(H5)** *hold. Then there exist positive constants $s_0^*(k_1, \Omega)$, $\rho(\Omega)$, $\hat{\lambda}$ and $\underline{\lambda}$ such that if $s_0 \geqslant s_0^*$ and $\frac{k_1}{k_3} > \rho$, (1.4) has a positive solution for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$.*

Finally, we state a nonexistence result which will complement some of the previous existence results.

Theorem 1.5. *Assume $\alpha \in [0, 1)$, $a \in L^\infty(\Omega)$ and there exists $k^* > 0$ such that*

$$f(s) \leqslant k^* s^{p-1+\alpha} \quad \textit{for all } s \geqslant 0\,. \tag{1.5}$$

Then for $\lambda < \frac{\lambda_1}{k^*}$, (1.3) does not have a positive solution.

In [15], the author studied the existence of positive solutions to the problem (1.3) in the case $\alpha \in (0, 1)$, $a \neq 0$ and $p = 2$. See [3, 12, 20, 21], where authors deal with $a \equiv 0$ and $p = 2$, [2] for a nonconstant weight and [9, 13, 19] for the p-Laplacian case. We also refer the reader to [6, 11] for reviews of singular problems in bounded and unbounded domains.

In Section 2, we discuss the existence and regularity of solutions to several auxiliary problems. In particular, since we use the method of sub- and supersolution to establish our existence results, we adapt the proof available for singular problems for the p-Laplacian with $a \equiv 0$ to our case and establish Theorem 2.1. In Section 3, we prove Theorem 1.1 by utilizing [1, Thm. 1.1]. In Sections 4–7, we prove Theorems 1.2–1.5, respectively. In Section 8, we extend some of our single equation results to cooperative systems.

2 Preliminaries and auxiliary problems

We begin with a discussion of a sub-and supersolution method for problems of the form

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = h(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where $h : (0, +\infty) \to \mathbb{R}$ is a continuous function.

By a weak solution of (2.1) we mean a function $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $u > 0$ in Ω and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \mathrm{d}x + \int_\Omega a(x)|u|^{p-2}u\varphi \mathrm{d}x = \int_\Omega h(u)\varphi \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(\Omega)\,.$$

By a weak subsolution of (2.1) we mean a function $\underline{u} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\underline{u} > 0$ in Ω and

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla \varphi \mathrm{d}x + \int_\Omega a(x)|\underline{u}|^{p-2}\underline{u}\varphi \mathrm{d}x \leqslant \int_\Omega h(\underline{u})\varphi \mathrm{d}x \quad \text{for all } 0 \leqslant \varphi \in C_c^\infty(\Omega)\,.$$

A weak supersolution is a function $\overline{u} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\overline{u} > 0$ in Ω and satisfies the reverse inequality in the definition of subsolution above.

Then, the following existence result holds.

Theorem 2.1. *Assume $a \in L^\infty(\Omega)$ and for all $\theta_1, \theta_2 \in (0, \infty)$ with $\theta_1 < \theta_2$, there exists $C := C(\theta_1, \theta_2) > 0$, such that*

$$|h(s)| \leqslant C \quad \textit{on } [\theta_1, \theta_2]. \tag{2.2}$$

Suppose that there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (2.1) such that $\underline{u} \leqslant \overline{u}$, then there exists a solution u of (2.1) such that $\underline{u} \leqslant u \leqslant \overline{u}$.

The proof follows as outlined in [19, Lemma 1.8] for the p-Laplacian case. Below we mention key modifications and show that their arguments hold for Theorem 2.1 as well.

Proof. Take a sequence of subdomains of Ω with $C^{2,\beta}$-boundaries, say $\{\Omega_j\}_{j=1}^{\infty}$, such that

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_j \subset\subset \Omega_{j+1} \subset\subset \cdots \quad \text{and} \quad \bigcup_{j=1}^{\infty} \Omega_j = \Omega\,. \tag{2.3}$$

For each $j = 1, 2, \ldots$, consider the problem

$$\begin{cases} -\Delta_p u = h_1(x,u) & \text{in } \Omega_j\,, \\ \quad u = \underline{u} & \text{on } \partial\Omega_j\,, \end{cases} \tag{P_j}$$

where $h_1(x,s) := h(s) - a(x)|s|^{p-2}s$. Let $a_j := \min_{\overline{\Omega}_j} \underline{u}$ and $b_j := \max_{\overline{\Omega}_j} \overline{u}$, and define $\tilde{h} : \Omega_j \times [0,\infty) \to \mathbb{R}$ by

$$\tilde{h}(x,s) := \begin{cases} h_1\left(x, a_j\right), & 0 \le s < a_j\,, \\ h_1(x,s), & a_j \leqslant s \leqslant b_j\,, \\ h_1\left(x, b_j\right), & b_j < s\,. \end{cases}$$

Note that h_1 and hence $\tilde{h}$ is a Carathéodory function, and for all $s \in [a_j, b_j]$ and $x \in \Omega_j$, we have

$$\begin{aligned} |\tilde{h}(x,s)| &\leqslant \max\{|h_1(x,s)|, |h_1(x,a_j)|, |h_1(x,b_j)|\} \\ &\leqslant \|h_1\|_{L^\infty(\Omega_j \times [a_j,b_j])}\,, \end{aligned}$$

since $a \in L^\infty(\Omega)$ and h satisfies (2.2).

The restrictions of the functions $\underline{u}$ and $\overline{u}$ on Ω_j are sub- and supersolutions of the following nonhomogeneous problem respectively:

$$\begin{cases} -\Delta_p u = \tilde{h}(x,u) & \text{in } \Omega_j, \\ \quad u = \underline{u} & \text{on } \partial\Omega_j\,. \end{cases} \tag{$\tilde{P}_j$}$$

Further, by [17, Thm. 6.8.13], we have that $T\underline{u} \in W^{1-1/p,p}(\partial\Omega_j)$, where T is the trace operator. Therefore, $\underline{u} \in W^{1-\frac{1}{p},p}(\partial\Omega_j) \cap L^\infty(\Omega_j)$. Hence, it follows from [4, Prop. 4.1] that the nonhomogeneous problem ($\tilde{P}_j$) has a solution $u_j \in W^{1,p}\left(\Omega_j\right) \cap C(\overline{\Omega}_j)$ such that $\underline{u} \leqslant u_j \leqslant \overline{u}$ on $\overline{\Omega}_j$, which is also a solution of (P_j). Then the rest of the arguments in the proof of [16, Lemma 1.8] can be carried out verbatim to complete the proof. □

Next, we discuss several auxiliary problems whose solutions are used in the paper. First, for $a \in L^\infty(\Omega)$, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta_p \varphi + a(x)|\varphi|^{p-2}\varphi = \lambda|\varphi|^{p-2}\varphi & \text{in } \Omega\,, \\ \qquad\qquad \varphi = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.4}$$

The following result holds:

Lemma 2.2. *The eigenvalue problem* (2.4) *has a principal eigenvalue* λ_1 *and a corresponding positive eigenfunction* $\varphi_1 \in W_0^{1,p}(\Omega) \cap C_0^1(\overline{\Omega})$ *satisfying* $\|\varphi_1\|_\infty = 1$ *and* $\frac{\partial \varphi_1}{\partial \eta} < 0$ *on* $\partial\Omega$*. Moreover, there exist* $\delta > 0$, $d > 0$, $m^* > 0$, *and* $m > 0$ *such that*

$$|\nabla \varphi_1| \geqslant m \quad \text{in } \Omega_\delta \quad \text{and} \quad \varphi_1 \geqslant d \quad \text{in } \Omega \setminus \overline{\Omega}_\delta \,, \tag{2.5}$$

and

$$\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla \varphi_1|^p - \left(a(x)\frac{1-C_{p\alpha}}{C_{p\alpha}} + \lambda_1\right)\varphi_1^p \geq m^* \quad \text{in } \Omega_\delta \,, \tag{2.6}$$

where $\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) < \delta\}$ *and* $C_{p\alpha} := \left[\frac{p}{p-1+\alpha}\right]^{p-1} (> 1)$. *If in addition, a satisfies* **(H1)** *then* $\lambda_1 > 0$.

Proof. The existence of a principal eigenvalue

$$\lambda_1 := \inf\left\{\int_\Omega (|\nabla \varphi|^p + a(x)|\varphi|^p)\, \mathrm{d}x : \varphi \in W_0^{1,p}(\Omega) \text{ and } \|\varphi\|_p = 1\right\}$$

and a corresponding positive eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \cap C_0^1(\overline{\Omega})$ satisfying $\frac{\partial \varphi_1}{\partial \eta} < 0$ on $\partial\Omega$ and $\|\varphi_1\|_\infty = 1$ are given by [7, Thm. 6.4.5].

Consequently, we get $0 < \left|\frac{\partial \varphi_1}{\partial \eta}\right| \leq |\nabla \varphi_1|$ on $\partial\Omega$ and $\varphi_1 > 0$ in Ω. Using the continuity of the functions $|\nabla \varphi_1|$ and φ_1, we have that there exist $\delta_1 > 0$, $d > 0$ and $m > 0$ such that (2.5) hold.

Let $b(x) = a(x)\frac{1-C_{p\alpha}}{C_{p\alpha}} + \lambda_1$ and assume without loss of generality that $\|b\|_\infty \neq 0$. Then there exists $\delta_2 \in (0, \delta_1)$ such that $\varphi_1(x) < m\left(\frac{(1-\alpha)(p-1)}{(p-1+\alpha)\|b\|_\infty}\right)^{1/p}$ for all $x \in \overline{\Omega}_{\delta_2}$. Hence, for all $x \in \overline{\Omega}_{\delta_2}$

$$\varphi_1^p(x) b(x) \leq \varphi_1^p(x)\|b\|_\infty < \frac{(1-\alpha)(p-1)}{p-1+\alpha} m^p < \frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla \varphi_1|^p.$$

Taking $\delta := \frac{\delta_2}{2}$, (2.5) and (2.6) hold.

Finally, suppose **(H1)** holds, and let $\lambda_p > 0$ be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_p \varphi = \lambda |\varphi|^{p-2}\varphi & \text{in } \Omega\,, \\ \varphi = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.7}$$

Then, it is known that

$$\lambda_p := \inf\left\{\int_\Omega |\nabla \varphi|^p \, \mathrm{d}x : \varphi \in W_0^{1,p}(\Omega) \text{ and } \|\varphi\|_p = 1\right\}, \tag{2.8}$$

see, e.g. [7, Prop. 6.2.2] for details. Hence, using $a(x) > -\lambda_p$ a.e. in Ω, we have

$$\begin{aligned} \lambda_1 &= \int_\Omega (|\nabla \varphi_1|^p + a(x)|\varphi_1|^p)\, \mathrm{d}x \\ &> \int_\Omega |\nabla \varphi_1|^p \, \mathrm{d}x - \lambda_p \int_\Omega |\varphi_1|^p \, \mathrm{d}x \geqslant 0\,. \end{aligned}$$

Thus, we conclude that $\lambda_1 > 0$, as desired. This proves the lemma. □

Second, we consider a problem of the form

$$\begin{cases} -\Delta_p e + a(x)|e|^{p-2}e = 1 & \text{in } \Omega\,, \\ e = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.9}$$

and obtain the following result.

Lemma 2.3. *Suppose a satisfies* **(H1)**. *Then* (2.9) *has a unique solution* $e \in W_0^{1,p}(\Omega) \cap C_0^{1,\beta}(\overline{\Omega})$ *satisfying*

$$e > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial e}{\partial \eta} < 0 \quad \text{on } \partial\Omega\,. \tag{2.10}$$

Proof. By Lemma 2.2, $\lambda_1 > 0$. Then by [7, Thm. 6.4.6(e)], there exists a unique positive solution $e \in W_0^{1,p}(\Omega)$ of (2.9). Moreover, $e \in L^\infty(\Omega)$ by [7, Thm 6.2.6], and by [7, Thm. 6.2.7] there exists $0 < \beta < 1$ such that $e \in C_0^{1,\beta}(\overline{\Omega})$. Finally, using [7, Thm. 6.2.8], we see that $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$. □

Third, we recall the anti-maximum principle for the p-Laplacian operator.

Remark 2.4. Consider the problem

$$\begin{cases} -\Delta_p z - \mu|z|^{p-2}z = -1 & \text{in } \Omega\,, \\ z = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.11}$$

Let λ_p be the principal eigenvalue of $-\Delta_p$ as defined in (2.8). By [10, Thm. 5.1] and [5, Thm. 7.3], there exists $\epsilon > 0$ such that if $\mu \in (\lambda_p, \lambda_p + \epsilon)$, the solution z_μ of (2.11) satisfies:

$$z_\mu > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial z_\mu}{\partial \eta} < 0 \quad \text{on } \partial\Omega\,.$$

Consequently, there exist $m_\mu > 0, A_\mu > 0, \delta_\mu > 0$ such that

$$\left|\nabla z_\mu\right| \geqslant m_\mu \quad \text{in } \Omega_{\delta_\mu} \quad \text{and} \quad z_\mu \geqslant A_\mu \quad \text{in } \Omega \setminus \Omega_{\delta_\mu}\,, \tag{2.12}$$

where $\Omega_{\delta_\mu} := \{x \in \Omega : d(x, \partial\Omega) < \delta_\mu\}$.

Next, to establish Theorem 1.1, we will use the following result in [1].

Proposition 2.5 ([1, Thm. 1.1]). *Consider*

$$\begin{cases} -\Delta_p u = \overline{g}(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.13}$$

where $1 < p < \infty$, Ω *is a bounded* C^2 *domain and the reaction term* $\overline{g} : \Omega \times (0,\infty) \to \mathbb{R}$ *is a Carathéodory function satisfying the following:*

(a) *There exist* $\gamma, t_0 > 0$, *nontrivial nonnegative measurable functions* $\underline{a}$ *and* $\overline{a}$ *satisfying*

$$\underline{a}(x) \leqslant \overline{g}(x,t) \leqslant \overline{a}(x)t^{-\gamma} \quad \text{for } (x,t) \in \Omega \times (0, t_0)\,, \tag{2.14}$$

and

$$M_T := \sup_{(x,t)\in\Omega\times[t_0,T]} |\overline{g}(x,t)| < \infty \quad \forall T > t_0 \tag{2.15}$$

(b) *There exists a nonnegative function $\zeta \in C_0^1(\overline{\Omega})$ such that $\bar{a}\zeta^{-\gamma} \in L^q(\Omega)$ for some $q > N/p$ if $p \leqslant N$ and $q = 1$ if $p > N$*

(c) $\frac{\bar{g}(x,t)}{t^{p-1}} \leqslant \lambda$ *for $(x,t) \in \Omega \times (T, \infty)$ for some $\lambda < \lambda_p$ and $T > t_0$.*

Then (2.13) has a positive solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, if $q > N$ then $u \in C_0^1(\overline{\Omega})$.

Finally, we derive a computation that will be used in the construction of our subsolutions in the proofs of Theorem 1.3 and Theorems 8.2–8.3.

Lemma 2.6. *Let $\psi := \lambda^r \varphi_1^{p/(p-1+\alpha)}$ with $\lambda, r > 0$ and $\alpha \in (0,1)$, where $\varphi_1 > 0$ is the eigenfunction corresponding to the principal eigenvalue λ_1 of (2.4). Then*

$$\begin{aligned} &-\Delta_p \psi + a(x)|\psi|^{p-2}\psi \\ &\quad = \frac{\lambda^{r(p-1+\alpha)}}{\psi^\alpha} C_{p\alpha} \left[\left(a(x)\frac{C_{p\alpha}-1}{C_{p\alpha}} + \lambda_1 \right) \varphi_1^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p \right]. \end{aligned} \tag{2.16}$$

(Here $C_{p\alpha}$ is as in Lemma 2.2.)

Proof. We note that since

$$\nabla \psi = \lambda^r \left[\frac{p}{p-1+\alpha} \right] \varphi_1^{\frac{1-\alpha}{p-1+\alpha}} \nabla \varphi_1,$$

we get

$$|\nabla \psi|^{p-2} \nabla \psi = \lambda^{r(p-1)} C_{p\alpha} \varphi_1^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \left(|\nabla \varphi_1|^{p-2} \nabla \varphi_1 \right).$$

Then,

$$\begin{aligned} \Delta_p \psi &= \lambda^{r(p-1)} C_{p\alpha} \left[\varphi_1^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \Delta_p \varphi_1 + \left[\frac{(1-\alpha)(p-1)}{p-1+\alpha} \right] \varphi_1^{\frac{-\alpha p}{p-1+\alpha}} |\nabla \varphi_1|^p \right] \\ &= \frac{\lambda^{r(p-1)}}{\varphi_1^{\frac{\alpha p}{p-1+\alpha}}} C_{p\alpha} \left[\varphi_1^{\frac{(1-\alpha)(p-1)+\alpha p}{p-1+\alpha}} \Delta_p \varphi_1 + \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p \right] \\ &= \frac{\lambda^{r(p-1+\alpha)}}{\left(\lambda^r \varphi_1^{p/(p-1+\alpha)} \right)^\alpha} C_{p\alpha} \left[a(x)\varphi_1^p - \lambda_1 \varphi_1^p + \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p \right], \end{aligned} \tag{2.17}$$

where the last equality holds since (λ_1, φ_1) is the eigenpair of the eigenvalue problem (2.4). On the other hand, we have

$$a(x)|\psi|^{p-2}\psi = a(x)\lambda^{r(p-1)} \varphi_1^{\frac{p(p-1)}{p-1+\alpha}}. \tag{2.18}$$

Therefore, it follows from (2.17) and (2.18) that

$$\begin{aligned} &-\Delta_p \psi + a(x)|\psi|^{p-2}\psi \\ &\quad = \frac{\lambda^{r(p-1+\alpha)}}{\psi^\alpha} C_{p\alpha} \left[\lambda_1 \varphi_1^p - a(x)\varphi_1^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p + \frac{a(x)\varphi_1^p}{C_{p\alpha}} \right] \\ &\quad = \frac{\lambda^{r(p-1+\alpha)}}{\psi^\alpha} C_{p\alpha} \left[\left(a(x)\frac{1}{C_{p\alpha}} - a(x) + \lambda_1 \right) \varphi_1^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p \right], \end{aligned}$$

establishing (2.16). $\square$

3 Proof of Theorem 1.1

We use Proposition 2.5 to establish the result. Thus, it suffices to verify its assumptions. Let $\overline{g}(x,t) := \frac{1-a(x)t^{p-1+\alpha}}{t^\alpha}$ for $(x,t) \in \Omega \times (0,+\infty)$, which is clearly a Carathéodory function. Without loss of generality, we assume that $\|a\|_\infty \neq 0$, and let $0 < t_0 < \frac{1}{\|a\|_\infty^{1/(p-1+\alpha)}}$. Define

$$\underline{a}(x) := \begin{cases} \frac{1}{t_0^\alpha}, & \text{if } a(x) < 0, \\ \frac{1-\|a\|_\infty t_0^{p-1+\alpha}}{t_0^\alpha}, & \text{if } a(x) \geqslant 0, \end{cases}$$

and

$$\overline{a}(x) := 1 + \|a\|_\infty t_0^{p-1+\alpha}.$$

Clearly, if $a(x) < 0$ then $1 < -a(x)t^{p-1+\alpha} + 1$, and consequently for $t \in (0, t_0)$

$$\underline{a}(x) = \frac{1}{t_0^\alpha} < \frac{1}{t^\alpha} < \overline{g}(x,t). \tag{3.1}$$

On the other hand, if $a(x) \geqslant 0$, using the fact that $t_0 < \frac{1}{\|a\|_\infty^{1/(p-1+\alpha)}}$, for $t \in (0, t_0)$, we have

$$\underline{a}(x) = \frac{1-\|a\|_\infty t_0^{p-1+\alpha}}{t_0^\alpha} \leqslant \frac{1-a(x)t^{p-1+\alpha}}{t_0^\alpha} \leqslant \frac{1-a(x)t^{p-1+\alpha}}{t^\alpha} = \overline{g}(x,t). \tag{3.2}$$

Next, for $t \in (0, t_0)$

$$\frac{\overline{a}(x)}{t^\alpha} = \frac{1+\|a\|_\infty t_0^{p-1+\alpha}}{t^\alpha} \geqslant \frac{1-a(x)t^{p-1+\alpha}}{t^\alpha} = \overline{g}(x,t). \tag{3.3}$$

Therefore, it follows from (3.1), (3.2) and (3.3) that $\underline{a}(x) \leqslant \overline{g}(x,t) \leqslant \overline{a}(x)t^{-\alpha}$ for $t \in (0, t_0)$ and for a.e. $x \in \Omega$. Moreover, since $t_0 > 0$

$$\sup_{(x,t)\in\Omega\times[t_0,T]} |\overline{g}(x,t)| < \infty \quad \text{for all } T \geqslant t_0$$

verifying hypothesis (a) of Proposition 2.5.

Now, to verify part (b) of Proposition 2.5, let $\zeta = e \in C_0^{1,\beta}(\overline{\Omega})$, where e is the solution of (2.9). Then, since $\partial\Omega$ is sufficiently smooth, the arguments in the proof of [18, Lemma] show that $\int_\Omega e^\theta < +\infty$ if and only if $\theta > -1$. Since $\alpha \in (0,1)$, we have $-\alpha > -1$ and hence $\overline{a}e^{-\alpha} \in L^1(\Omega)$, in particular when $p > N$. If $p \leqslant N$, there exists $q \in (N, \frac{1}{\alpha})$ such that $e^{-\alpha} \in L^q(\Omega)$, and hence $\overline{a}e^{-\alpha} \in L^q(\Omega)$.

Finally, note that there exists $T > t_0$ such that $\frac{1}{T^{\alpha+p-1}} < \operatorname{ess\,inf} a + \lambda_p$. Since **(H1)** holds, for a.e. $x \in \Omega$ and $t > T$

$$\frac{\overline{g}(x,t)}{t^{p-1}} = \frac{1-a(x)t^{p-1+\alpha}}{t^{p-1+\alpha}} \leqslant \frac{1}{T^{\alpha+p-1}} - a(x) < \lambda_p\,,$$

verifying hypothesis (c) of Proposition 2.5. Then by Proposition 2.5 the problem (1.1) has a positive solution $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and that $w \in C_0^1(\overline{\Omega})$ whenever $p \leqslant N$.

Moreover, it is clear from the proof of Proposition 2.5, that is of [1, Thm. 1.1], that whenever $p \leqslant N$, w satisfies

$$w > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial w}{\partial \eta} < 0 \quad \text{on } \partial\Omega\,. \tag{3.4}$$

This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

We construct an ordered pair of subsolution and supersolution of (1.3) for each fixed $\lambda > 0$. Without loss of generality, we take $\lambda = 1$.

Let $\underline{u} := m\varphi_1$, where $\varphi_1 > 0$ is the eigenfunction given in Lemma 2.2 and $m > 0$. We note that since $f > 0$, we have $\lim_{s\to 0^+} \frac{f(s)}{s^{\alpha+p-1}} = +\infty$. Then there exists m sufficiently small such that

$$\lambda_1 (m\varphi_1)^{p-1} \leqslant \frac{f(m\varphi_1)}{(m\varphi_1)^\alpha}.$$

Therefore,

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} = \lambda_1 (m\varphi_1)^{p-1} \leqslant \frac{f(m\varphi_1)}{(m\varphi_1)^\alpha} = \frac{f(\underline{u})}{\underline{u}^\alpha} \quad \text{in } \Omega.$$

Also, $\underline{u} = 0$ on $\partial\Omega$, hence, $\underline{u}$ is a subsolution of (1.3).

Next, let $w > 0$ be as given by Theorem 1.1 and $\overline{u} := Mw$ for $M > 0$. Recall that f is not necessarily monotone. However, $f^*(s) := \max_{t\in[0,s]} f(t)$ is nondecreasing, $f(s) \leqslant f^*(s)$ for $s \geqslant 0$, and by **(H3)**, we also have

$$\lim_{s\to+\infty} \frac{f^*(s)}{s^{\alpha+p-1}} = 0. \tag{4.1}$$

Then by (4.1), there exists $M \gg 1$ such that

$$\frac{f^*(M\|w\|_\infty)}{(M\|w\|_\infty)^{p-1+\alpha}} \leqslant \frac{1}{\|w\|_\infty^{p-1+\alpha}} \quad \text{or equivalently} \quad \frac{f^*(M\|w\|_\infty)}{M^\alpha} \leqslant M^{p-1}. \tag{4.2}$$

Using (4.2) and the fact that f^* is nondecreasing, we get

$$-\Delta_p \overline{u} + a(x)|\overline{u}|^{p-2}\overline{u} = \frac{M^{p-1}}{w^\alpha} \geqslant \frac{f^*(M\|w\|_\infty)}{(Mw)^\alpha} \geqslant \frac{f^*(Mw)}{(Mw)^\alpha} \geqslant \frac{f(\overline{u})}{\overline{u}^\alpha} \quad \text{in } \Omega. \tag{4.3}$$

Clearly $\overline{u} = 0$ on $\partial\Omega$, and hence $\overline{u}$ is a supersolution of (1.3).

Finally, using the facts that $\frac{\partial w}{\partial \eta} < 0$ and $\frac{\partial \varphi_1}{\partial \eta} < 0$ on $\partial\Omega$, we choose $M > 0$ large so that $\overline{u} \geqslant \underline{u}$ in Ω. By Theorem 2.1, the problem (1.3) admits a positive solution $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\underline{u} \leqslant u \leqslant \overline{u}$. This completes the proof.

5 Proof of Theorem 1.3

We construct an ordered pair of sub- and supersolution and apply Theorem 2.1 to establish our result. First, we show that

$$\underline{u} := \lambda^r \varphi_1^{p/(p-1+\alpha)} \quad \text{with } r \in \left(\frac{1}{p-1+\alpha}, \frac{1}{p-1+\alpha-\sigma}\right) \tag{5.1}$$

is a subsolution for $\lambda > 0$ large. By Lemma 2.6, $\underline{u}$ satisfies

$$\begin{aligned} &-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \\ &\quad = \frac{\lambda^{r(p-1+\alpha)}}{\underline{u}^\alpha} C_{p\alpha} \left[\left(a(x)\frac{C_{p\alpha}-1}{C_{p\alpha}} + \lambda_1 \right) \varphi_1^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \varphi_1|^p \right], \end{aligned} \tag{5.2}$$

where $C_{p\alpha}$ is given by Lemma 2.2. Then it follows from (2.6), that

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \leqslant -\frac{m^* \lambda^{r(p-1+\alpha)}}{\underline{u}^\alpha} C_{p\alpha} \quad \text{in } \Omega_\delta,$$

where $\delta > 0$ and m^* are as in Lemma 2.2. Since $r > \frac{1}{p-1+\alpha}$, for λ sufficiently large, we have

$$-m^*\lambda^{r(p-1+\alpha)}C_{p\alpha} \leqslant \lambda \underline{f}\,, \tag{5.3}$$

where $\underline{f} := \min_{[0,\infty)} f < 0$. Therefore,

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \leqslant \lambda \frac{\underline{f}}{\underline{u}^\alpha} \leqslant \lambda \frac{f(\underline{u})}{\underline{u}^\alpha} \quad \text{on } \Omega_\delta\,. \tag{5.4}$$

Next, since $C_{p\alpha} > 1$, we have

$$K := \lambda_1 + \frac{C_{p\alpha}-1}{C_{p\alpha}}\|a\|_\infty \geqslant \frac{C_{p\alpha}-1}{C_{p\alpha}} a(x) + \lambda_1\,.$$

Then (5.2) implies

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \leqslant \frac{K\varphi_1^p \lambda^{r(p-1+\alpha)}}{\underline{u}^\alpha} C_{p\alpha} \quad \text{in } \Omega \setminus \overline{\Omega}_\delta. \tag{5.5}$$

Since $\varphi_1 \geqslant d$ in $\Omega \setminus \overline{\Omega}_\delta$, by (2.5), and $r < 1/(p-1+\alpha-\sigma)$, there exists λ large such that

$$K\lambda^{r(p-1+\alpha)}\varphi_1^p C_{p\alpha} \leqslant A\lambda^{1+r\sigma}\varphi_1^{\frac{p\sigma}{p-1+\alpha}}\,, \tag{5.6}$$

where $A > 0$ is as in hypothesis **(H2)**. Then, by (5.5), (5.6) and hypothesis **(H2)**, it follows that

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \leqslant \frac{A\lambda(\varphi_1^{\frac{p}{p-1+\alpha}}\lambda^r)^\sigma}{\underline{u}^\alpha} \leqslant \frac{\lambda f(\underline{u})}{\underline{u}^\alpha} \quad \text{on } \Omega \setminus \overline{\Omega}_\delta\,. \tag{5.7}$$

Let $\lambda^* > 0$ be such that (5.3) and (5.6) hold for any $\lambda > \lambda^*$. Then, it follows from (5.4) and (5.7) that $\underline{u}$ defined in (5.1) is a subsolution of (1.3) for $\lambda > \lambda^*$.

Now, we will show that $\overline{u} = Mw$ is a supersolution of (1.3), where $w > 0$ is as given by Theorem 1.1, and $M > 0$ to be determined. By repeating the argument in the proof of Theorem 1.2 and noting that $M = M(\lambda)$ in this case, we get

$$-\Delta_p \overline{u} + a(x)|\overline{u}|^{p-2}\overline{u} = \frac{M^{p-1}}{w^\alpha} \geqslant \frac{\lambda f^*(M\|w\|_\infty)}{(Mw)^\alpha} \geqslant \frac{\lambda f^*(Mw)}{(Mw)^\alpha} \geqslant \frac{\lambda f(\overline{u})}{\overline{u}^\alpha} \quad \text{in } \Omega\,, \tag{5.8}$$

where $f^*(s) := \max_{t\in[0,s]} f(t)$. Clearly $\overline{u} = 0$ on $\partial\Omega$, hence $\overline{u}$ is a supersolution of (1.3).

Since $\frac{\partial w}{\partial \eta} < 0$ on $\partial\Omega$ and $w > 0$, we can choose M larger, if necessary, so that $\overline{u} \geqslant \underline{u}$ in Ω. By Theorem 2.1, problem (1.3) admits a solution $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\underline{u} \leqslant u \leqslant \overline{u}$ for $\lambda > \lambda^*$. This completes the proof. □

6 Proof of Theorem 1.4

Here also we construct an ordered pair of sub- and supersolution to apply Theorem 2.1 to establish the result. In this case, we first determine a λ interval where the existence will be guaranteed. For this, let $\mu \in (\lambda_p, \lambda_p + \epsilon)$ be fixed, and let z_μ denote the solution of (2.11). Define $\rho(\Omega) := 2\|e\|_\infty^{p-1}\big[\mu\big(\frac{p}{p-1+\alpha}\big)^{p-1} + \|a\|_\infty\big]$, where e is the solution of (2.9). Let k_1 and k_3 in **(H4)** and **(H5)**, respectively, be such that $\frac{k_1}{k_3} > \rho$. Then

$$\underline{\lambda} := \frac{\mu\left(\frac{p}{p-1+\alpha}\right)^{p-1} + \|a\|_\infty}{k_1} < \hat{\lambda} := \frac{1}{2k_3\,\|e\|_\infty^{p-1}}\,. \tag{6.1}$$

Then the interval $(\underline{\lambda}, \hat{\lambda})$ is non-empty by construction.

Now, let $\underline{u} = k_0 z_\mu^{\frac{p}{p-1+\alpha}}$ and $s_0^* := k_0 \left\|z_\mu\right\|_\infty^{\frac{p}{p-1+\alpha}}$, where the constant $k_0 > 0$ be such that

$$\frac{1}{k_0^{p-1+\alpha}} \left(1 + \frac{k_2 k_0^\alpha}{2k_3 \left\|e\right\|_\infty^{p-1}} z_\mu^{\frac{\alpha p}{p-1+\alpha}}\right) \leqslant \min\left\{\frac{p^{p-1}(1-\alpha)(p-1)m_\mu^p}{(p-1+\alpha)^p}, \left(\frac{p}{p-1+\alpha}\right)^{p-1} A_\mu\right\}. \quad (6.2)$$

Here the positive constants m_μ and A_μ are as discussed in Remark 2.12. Noting that

$$\nabla \underline{u} = k_0 \left(\frac{p}{p-1+\alpha}\right) z_\mu^{\frac{1-\alpha}{p-1+\alpha}} \nabla z_\mu ,$$

we have

$$\begin{aligned}
&- \Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \\
&\quad = - k_0^{p-1} \left(\frac{p}{p-1+\alpha}\right)^{p-1} \left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha} z_\mu^{\frac{-\alpha p}{p-1+\alpha}} \left|\nabla z_\mu\right|^p z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \left(1 - \mu z_\mu^{p-1}\right)\right\} \\
&\qquad + a(x) k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \\
&\quad = k_0^{p-1}\left[\left(\frac{p}{p-1+\alpha}\right)^{p-1} \mu + a(x)\right] z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - k_0^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \\
&\qquad - \frac{k_0^{p-1} p^{p-1}(1-\alpha)(p-1)\left|\nabla z_\mu\right|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p}. \quad (6.3)
\end{aligned}$$

Now, we will show that $\underline{u}$ is a weak subsolution. For $\lambda \geq \underline{\lambda}$, we have

$$k_0^{p-1}\left[\left(\frac{p}{p-1+\alpha}\right)^{p-1} \mu + a(x)\right] z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \leqslant \lambda k_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \quad (6.4)$$

and

$$\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k_2 \leqslant \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \frac{k_2}{2k_3 \left\|e\right\|_\infty^{p-1}} = \frac{k_0^{p-1}}{k_0^{p-1+\alpha} z_\mu^{\frac{\alpha p}{p-1+\alpha}}} \left(1 + \frac{k_2 k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2k_3 \left\|e\right\|_\infty^{p-1}}\right). \quad (6.5)$$

Using $|\nabla z_\mu| \geq m_\mu$, (6.2), and (6.5), we get

$$\begin{aligned}
\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k_2 &\leqslant \frac{k_0^{p-1} p^{p-1}(1-\alpha)(p-1)m_\mu^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p} \\
&\leqslant \frac{k_0^{p-1} p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p} \quad \text{in } \Omega_{\delta_\mu}. \quad (6.6)
\end{aligned}$$

Then (6.3), (6.4), (6.6), **(H4)**, and the fact that $k_0^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} > 0$ yield

$$-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} \leqslant \lambda k_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - \lambda k_2 - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1-\alpha}}} \leqslant \lambda g(\underline{u}) - \frac{1}{(\underline{u})^\alpha} \quad \text{in } \Omega_{\delta_\mu}. \quad (6.7)$$

Next, since $z_\mu \geq A_\mu$ on $\Omega \setminus \Omega_{\delta_\mu}$, (6.2) and (6.5) yield

$$\begin{aligned}\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k &\leqslant \left(\frac{p}{p-1+\alpha}\right)^{p-1} A_\mu \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}} \\ &\leqslant k_0^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \quad \text{in } \Omega \setminus \overline{\Omega}_{\delta_\mu}. \qquad (6.8)\end{aligned}$$

Then, it follows from (6.3), (6.4), (6.8) and hypothesis **(H4)** that

$$\begin{aligned}-\Delta_p \underline{u} + a(x)|\underline{u}|^{p-2}\underline{u} &\leqslant \lambda k_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - \lambda k_2 - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1-\alpha}}} \\ &\leqslant \lambda g(\underline{u}) - \frac{1}{(\underline{u})^\alpha} \quad \text{in } \Omega \setminus \overline{\Omega}_{\delta_\mu}. \qquad (6.9)\end{aligned}$$

Therefore, combining (6.7) and (6.9), we conclude that $\underline{u}$ is a weak subsolution of (1.4).

Next, let $\overline{u} = Me$, where e is the solution of the problem (2.9) and $M > 0$ to be chosen. Let $g^*(s) := \max_{t\in[0,s]} g(t)$. Then g^* is nondecreasing and by hypothesis **(H5)**, we get

$$\lim_{s\to+\infty} \frac{g^*(s)}{s^{p-1}} = k_3\,, \qquad (6.10)$$

which in turn implies that there exists $M \gg 1$ so that

$$2k_3 \geq \frac{g^*\left(M\left\|e\right\|_\infty\right)}{\left(M\left\|e\right\|_\infty\right)^{p-1}}\,. \qquad (6.11)$$

Then, using (6.11) and the fact that $\lambda \leq \hat{\lambda}$, $\overline{u}$ satisfies

$$-\Delta_p \overline{u} + a(x)|\overline{u}|^{p-2}\overline{u} = M^{p-1} \geq \frac{g^*\left(M\left\|e\right\|_\infty\right)}{\left\|e\right\|_\infty^{p-1} 2k_3} = \hat{\lambda} g^*\left(M\left\|e\right\|_\infty\right) \geq \lambda g\left(\overline{u}\right) \quad \text{in } \Omega\,.$$

Thus, $\overline{u}$ is a weak supersolution of problem (1.4). Finally, using (2.10), we can choose $M \gg 1$ such that $\underline{u} \leq \overline{u}$. Then, for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$, by Theorem 2.1, there exists a positive weak solution such that $k_0 z_\mu^{\frac{p}{p-1+\alpha}} \leq u \leq Me$ in Ω. This completes the proof of Theorem 1.4. □

7 Proof of Theorem 1.5

Let f satisfy (1.5) and $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ be a positive solution of (1.3). Then, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x + \int_\Omega a(x)|u|^{p-2}u\varphi \mathrm{d}x = \lambda \int_\Omega \frac{f(u)}{u^\alpha}\varphi \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(\Omega)\,.$$

Taking $\varphi = u$ as the test function, we have

$$\int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Omega a(x)|u|^p \mathrm{d}x = \lambda \int_\Omega \frac{f(u)}{u^{\alpha-1}}\mathrm{d}x\,.$$

By (1.5) and Lemma 2.2, we get

$$\lambda_1 \int_\Omega u^p \mathrm{d}x \leqslant \int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Omega a(x) u^p \mathrm{d}x = \lambda \int_\Omega \frac{f(u)}{u^{\alpha-1}}\mathrm{d}x \leqslant \lambda k^* \int_\Omega u^p\,.$$

Therefore,

$$(\lambda_1 - \lambda k^*) \int_\Omega u^p \mathrm{d}x \leqslant 0\,,$$

and since u is positive, we must have $\lambda \geqslant \frac{\lambda_1}{k^*}$. Hence, (1.3) does not have a positive solution for $0 < \lambda < \frac{\lambda_1}{k^*}$. □

8 Coupled systems

In this section, we extend the existence results established for the scalar case in Theorem 1.3 and Theorem 1.4 to the corresponding coupled system of equations. For this purpose, we first discuss a sub- and supersolution theorem for systems of the form:

$$\begin{cases} -\Delta_p u_1 + a(x)|u_1|^{p-2}u_1 = h_1(u_1,u_2) & \text{in } \Omega, \\ -\Delta_p u_2 + a(x)|u_2|^{p-2}u_2 = h_2(u_1,u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

where $\Omega \subset \mathbb{R}^N$ is as before, $h_i : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ are continuous functions for each $i = 1,2$ and $a \in L^\infty(\Omega)$.

By a weak solution of (8.1) we mean a pair (u_1,u_2) where $u_1,u_2 \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, $u_1,u_2 > 0$ in Ω, and satisfy:

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi \mathrm{d}x + \int_\Omega a(x)|u_1|^{p-2}u_1\varphi \mathrm{d}x = \int_\Omega h_1(u_1,u_2)\varphi \mathrm{d}x\,,$$
$$\int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla\varphi \mathrm{d}x + \int_\Omega a(x)|u_2|^{p-2}u_2\varphi \mathrm{d}x = \int_\Omega h_2(u_1,u_2)\varphi \mathrm{d}x\,,$$

for all $\varphi \in C_c^\infty(\Omega)$.

By a weak subsolution of (8.1), we mean a pair $(\underline{u}_1,\underline{u}_2)$ where $\underline{u}_1,\underline{u}_2 \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, $\underline{u}_1,\underline{u}_2 > 0$ in Ω, and satisfy:

$$\int_\Omega |\nabla \underline{u}_1|^{p-2}\nabla \underline{u}_1 \nabla\varphi \mathrm{d}x + \int_\Omega a(x)|\underline{u}_1|^{p-2}\underline{u}_1\varphi \mathrm{d}x \leqslant \int_\Omega h_1(\underline{u}_1,\underline{u}_2)\varphi \mathrm{d}x$$
$$\int_\Omega |\nabla \underline{u}_2|^{p-2}\nabla \underline{u}_2 \nabla\varphi \mathrm{d}x + \int_\Omega a(x)|\underline{u}_2|^{p-2}\underline{u}_2\varphi \mathrm{d}x \leqslant \int_\Omega h_2(\underline{u}_1,\underline{u}_2)\varphi \mathrm{d}x\,,$$

for all $0 \leqslant \varphi \in C_c^\infty(\Omega)$. A pair of supersolution is defined analogously by reversing the inequalities above.

Next, for each $i = 1,2$, define Carathéodory functions $\tilde{h}_i : \Omega \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$\tilde{h}_i(x,s_1,s_2) := h_i(s_1,s_2) - a(x)|s_i|^{p-2}s_i \tag{8.2}$$

We assume that the $\tilde{h}_i$ satisfy:

(h1) $\tilde{h}_1$ is nondecreasing in s_2 for all fixed $x \in \Omega, s_1 \in (0,\infty)$, and $\tilde{h}_2$ is nondecreasing in s_1 for all fixed $x \in \Omega, s_2 \in (0,\infty)$ (*quasimonotone nondecreasing*).

Then, we establish:

Theorem 8.1. *Assume* **(h1)** *hold, and for any $\Omega' \subset\subset \Omega$ and for all $a_i, b_i \in (0,\infty)$ with $a_i < b_i$, there exists $K(\overline{\Omega'}, a_i, b_i) > 0$, such that*

$$|\tilde{h}_i(x, s_1, s_2)| \leqslant K \quad \text{on } \Omega' \times [a_1, b_1] \times [a_2, b_2], \tag{8.3}$$

$i = 1,2$. Let $(\underline{u}_1, \underline{u}_2)$, $(\bar{u}_1, \bar{u}_2)$ be weak sub- and supersolutions, respectively, of problem (8.1) such that $(\underline{u}_1, \underline{u}_2) \leq (\bar{u}_1, \bar{u}_2)$, and assume there exists $M^* > 0$ such that:

(h2) *$\tilde{h}_1(x, s_1, s_2) + M^*\psi(s_1)$ is nondecreasing in s_1 for all $(\underline{u}_1, \underline{u}_2) \leqslant (s_1, s_2) \leqslant (\bar{u}_1, \bar{u}_2)$ for all $x \in \Omega$.*

(h3) *$\tilde{h}_2(x, s_1, s_2) + M^*\psi(s_2)$ is nondecreasing in s_2 for all $(\underline{u}_1, \underline{u}_2) \leqslant (s_1, s_2) \leqslant (\bar{u}_1, \bar{u}_2)$ for all $x \in \Omega$*

hold, where $\psi(s) := |s|^{p-2}s$. Then there exists a weak solution (u_1, u_2) of (8.1) satisfying $(\underline{u}_1, \underline{u}_2) \leqslant (u_1, u_2) \leqslant (\bar{u}_1, \bar{u}_2)$. (Here, by $(a,b) \leqslant (c,d)$ we mean $a \leqslant c$ and $b \leqslant d$, that is, ordering is understood in componentwise sense.)

In the rest of this section, for $i = 1,2$, $f_i : [0,\infty) \to \mathbb{R}$ are assumed to be continuous and nondecreasing functions.

First, we consider a system of the form:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda \dfrac{f_1(v)}{u^{\alpha_1}} & \text{in } \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \lambda \dfrac{f_2(u)}{v^{\alpha_2}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.4}$$

where $\alpha_i \in (0,1)$ and f_i satisfy the following assumptions for $i = 1,2$:

(S1) $f_i(0) < 0$;

(S2) There exist $\sigma_i > 0$ and $A_i > 0$ such that $\alpha_i < \sigma_i < \alpha_i + (p-1)$ and $f_i(s) > A_i s^{\sigma_i}$ for $s \gg 1$;

(S3) $\lim_{s\to\infty} f_2(s) = \infty$ and $\lim_{s\to\infty} \frac{f_1\left(L(f_2(s))^{\frac{1}{p-1}}\right)}{s^{p-1+\alpha_1}} = 0$ for all $L > 0$.

We establish the following existence result.

Theorem 8.2. *Let $p \leqslant N$ and $\alpha_1, \alpha_2 \in (0, \frac{p}{N})$. Assume that* **(H1)** *and* **(S1)–(S3)** *hold. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ (8.4) has a positive solution (u,v) with $u, v \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Next, we study a coupled system of the form:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda \left[f_1(v) - \dfrac{1}{u^{\alpha_1}} \right] & \text{in } \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \lambda \left[f_2(u) - \dfrac{1}{v^{\alpha_2}} \right] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.5}$$

where for $\alpha_i \in (0,1)$ and f_i satisfy the following for $i = 1,2$:

(S4) $f_i(s) > 0$ for $s \geq 0$; $i = 1,2$;

(S5) $\lim_{s\to\infty} f_2(s) = \infty$ and $\lim_{s\to\infty} \frac{f_1(L(f_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0$ for all $L > 0$.

Then we prove the following existence result:

Theorem 8.3. *Let $\alpha_1, \alpha_2 \in (0,1)$. Assume that* **(H1)** *and* **(S4)–(S5)** *hold. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ (8.5) has a positive solution (u, v) with $u, v \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Finally, we study a singular system of the form:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f_1(v) - \dfrac{1}{u^\alpha} & \text{in } \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \lambda f_2(u) - \dfrac{1}{v^\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.6}$$

where $\alpha \in (0,1)$ and f_i satisfy the following assumptions for each $i = 1, 2$:

(S6) There exist σ_i and $s_i > 0$ such that $f_i(s) \geqslant \sigma_i s^{p-1} - k_i$ for every $0 \leqslant s \leqslant s_i$;

(S7) $\lim_{s\to\infty} \frac{f_1(f_2(s)^{\frac{1}{p-1}})}{s^{p-1}} = \hat{\sigma}$ for some $\hat{\sigma} > 0$;

(S8) There exists $\tau > 0$ such that for each $P > 0$, $f_1(Ps) \leqslant P^\tau f_1(s)$ for $s \gg 1$.

Then we establish:

Theorem 8.4. *Assume that* **(H1)**, **(S6)–(S8)** *hold and $\alpha \in (0,1)$. Then there exist positive constants $s_0(\hat{\sigma}, \Omega)$, $\rho_s(\Omega)$, $\hat{\lambda}$ and $\underline{\lambda}$ such that if $\min\{s_1, s_2\} \geqslant s_0$ and $\frac{\min\{\sigma_1,\sigma_2\}}{\hat{\sigma}^{\frac{p-1}{p-1+\tau}}} > \rho_s$, (8.6) has a positive solution for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$.*

Proof of Theorem 8.1

In [19], it was noted that Theorem 8.1 holds for the case $a \equiv 0$, see [19, Lemma 1.8]. Their approach can be applied to our case $a \not\equiv 0$, which we briefly outline below.

Take a sequence of subdomains of Ω with $C^{2,\beta}$-boundaries, say $\{\Omega_j\}_{j=1}^\infty$, such that

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_j \subset\subset \Omega_{j+1} \subset\subset \cdots \quad \text{and} \quad \bigcup_{j=1}^\infty \Omega_j = \Omega. \tag{8.7}$$

Now, for each fixed $j = 1, 2, \ldots$, consider the nonhomogeneous problem:

$$\begin{cases} -\Delta_p u_1 = \tilde{h}_1(x, u_1, u_2) & \text{in } \Omega_j, \\ -\Delta_p u_2 = \tilde{h}_2(x, u_1, u_2) & \text{in } \Omega_j, \\ u_1 = \underline{u_1} \text{ and } u_2 = \underline{u_2} & \text{on } \partial\Omega_j. \end{cases} \tag{8.8}$$

We note that due to hypotheses **(h1)–(h3)**, we can construct a monotonically increasing sequence using [4, Prop. 4.1]

$$(\underline{u_1}, \underline{u_2}) = (u_{1,1}, u_{2,1}) \leqslant \cdots \leqslant (u_{1,n}, u_{2,n}) \leqslant \cdots (\overline{u_1}, \overline{u_2})$$

satisfying:

$$\begin{cases} -\Delta_p u_{1,n} + M^*\psi(u_{1,n}) = \tilde{h}_1(x, u_{1,n-1}, u_{2,n-1}) + M^*\psi(u_{1,n-1}) & \text{in } \Omega_j, \\ -\Delta_p u_{2,n} + M^*\psi(u_{2,n}) = \tilde{h}_2(x, u_{1,n-1}, u_{2,n-1}) + M^*\psi(u_{2,n-1}) & \text{in } \Omega_j, \\ u_{1,n} = \underline{u_1} \text{ and } u_{2,n} = \underline{u_2} & \text{on } \partial\Omega_j. \end{cases} \tag{8.9}$$

Note that the system (8.9) is decoupled, hence the existence of the iterated solutions $(u_{1,n}, u_{2,n})$ easily follows. Then, since the monotone sequence is bounded, using the regularity theory [22, Thm. 1] we can conclude that (8.8) has a weak solution $\left(u_{1(j)}, u_{2(j)}\right)$ such that $u_{i(j)} \in W^{1,p}(\Omega_j) \cap C(\overline{\Omega}_j)$ for $i = 1, 2$ and $\left(\underline{u}_1, \underline{u}_2\right) \leqslant \left(u_{1(j)}, u_{2(j)}\right) \leqslant \left(\bar{u}_1, \bar{u}_2\right)$. The rest of the proof follows from the arguments in the proof of [16, Lemma 1.8]. □

Proofs of Theorem 8.2–Theorem 8.4

Here we construct weak sub- and supersolution pairs $(\underline{u}_1, \underline{u}_2)$ and $(\overline{u}_1, \overline{u}_2)$ such that $(\underline{u}_1, \underline{u}_2) \leq (\overline{u}_1, \overline{u}_2)$. Constructions of $(\underline{u}_1, \underline{u}_2)$ for each theorem follow straightforwardly by using the arguments used in the single equation cases componentwise. Hence we emphasize on the construction of $(\overline{u}_1, \overline{u}_2)$, which rely on the hypotheses **(S3)**, **(S5)** and **(S7)**.

Recall functions φ_1, e and z_μ discussed in Section 2.

Proof of Theorem 8.2: Here, as in the scalar case (see the proof of Theorem 1.3), there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, the pair

$$(\underline{u}_1, \underline{u}_2) = (\lambda^r \varphi_1^{p/(p-1+\alpha_1)}, \lambda^r \varphi_1^{p/(p-1+\alpha_2)}),$$

where

$$r \in \left(\frac{1}{p-1+\alpha_1}, \frac{1}{p-1+\alpha_1-\sigma_1}\right) \cap \left(\frac{1}{p-1+\alpha_2}, \frac{1}{p-1+\alpha_2-\sigma_2}\right)$$

is a weak subsolution of (8.4).

Next, we construct a weak supersolution pair $(\overline{u}_1, \overline{u}_2) \geq (\underline{u}_1, \underline{u}_2)$. Let

$$(\overline{u}_1, \overline{u}_2) = \left(M w_1, \left(f_2\left(M\left\|w_1\right\|_\infty\right)\right)^{\frac{1}{p-1}} w_2\right),$$

where w_1 and w_2 are the solutions for the problems (1.1) for $\alpha_1, \alpha_2 \in (0,1)$, respectively, and $M > 0$. Using **(S3)**, for each $\lambda > \lambda^*$, there exists $M = M(\lambda) \gg 1$ such that

$$\frac{1}{\lambda \left\|w_1\right\|_\infty^{p-1+\alpha_1}} \geqslant \frac{f_1\left(\left\|w_2\right\|_\infty \left(f_2\left(M\left\|w_1\right\|_\infty\right)\right)^{\frac{1}{p-1}}\right)}{\left\|w_1\right\|_\infty^{p-1+\alpha_1} M^{p-1+\alpha_1}}$$

and

$$\frac{1}{\lambda} \geqslant \frac{1}{f_2(M\|w_1\|_\infty)^{\frac{\alpha_2}{p-1}}}.$$

Then

$$-\Delta_p \overline{u}_1 + a(x)|\overline{u}_1|^{p-2}\overline{u}_1 = \frac{M^{p-1}}{w_1^{\alpha_1}} \geqslant \lambda \frac{f_1\left(\left\|w_2\right\|_\infty \left(f_2\left(M\left\|w_1\right\|_\infty\right)\right)^{\frac{1}{p-1}}\right)}{(M w_1)^{\alpha_1}} \geq \lambda \frac{f_1(\overline{u}_2)}{\overline{u}_1^{\alpha_1}} \quad \text{in } \Omega$$

and

$$-\Delta_p \overline{u}_2 + a(x)|\overline{u}_2|^{p-2}\overline{u}_2 = \frac{f_2(M\|w_1\|_\infty)}{w_2^{\alpha_2}} \geqslant \lambda \frac{f_2(M w_1)}{(f_2(M\|w_1\|_\infty)^{\frac{1}{p-1}})^{\alpha_2} w_2^{\alpha_2}} = \lambda \frac{f_2(\overline{u}_1)}{\overline{u}_2^{\alpha_2}} \quad \text{in } \Omega.$$

Further, choosing M larger if necessary, using (1.2) and **(S3)**, we have $(\underline{u}_1, \underline{u}_2) \leq (\overline{u}_1, \overline{u}_2)$ componentwise. By Theorem 8.1, there exists a weak solution (u_1, u_2) of (8.4) satisfying $(\underline{u}_1, \underline{u}_2) \leqslant (u_1, u_2) \leqslant (\bar{u}_1, \bar{u}_2)$, completing the proof. □

Proof of Theorem 8.3: Again, it is easy to show that the pair

$$(\underline{u}_1, \underline{u}_2) := \left(\lambda^r \varphi_1^{p/(p-1+\alpha_1)}, \lambda^r \varphi_1^{p/(p-1+\alpha_2)}\right),$$

is a weak subsolution of (8.5) for λ sufficiently large, where $r \in \left(\frac{1}{p-1+\alpha_1}, \frac{1}{p-1}\right) \cap \left(\frac{1}{p-1+\alpha_2}, \frac{1}{p-1}\right)$. Next, let

$$(\overline{u}_1, \overline{u}_2) = \left(Me, (\lambda f_2(M\|e\|_\infty))^{\frac{1}{p-1}} e\right),$$

where $M > 0$. Then for each $\lambda > 0$, using **(S5)**, there exists $M = M(\lambda) > 0$ such that

$$\frac{1}{\lambda \|e\|_\infty^{p-1}} \geq \frac{f_1\left(\left(\lambda \|e\|_\infty^{p-1} f_2(M\|e\|_\infty)\right)^{\frac{1}{p-1}}\right)}{(M\|e\|_\infty)^{p-1}}.$$

Then

$$\begin{aligned}
-\Delta_p \overline{u}_1 + a(x)|\overline{u}_1|^{p-2}\overline{u}_1 &= M^{p-1} \\
&\geq \lambda f_1\left((\lambda f_2(M\|e\|_\infty))^{\frac{1}{p-1}} \|e\|_\infty\right) \\
&\geq \lambda f_1\left((\lambda f_2(M\|e\|_\infty))^{\frac{1}{p-1}} e\right) \\
&\geq \lambda \left[f_1(\overline{u}_2) - \frac{1}{\overline{u}_1^{\alpha_1}}\right] \quad \text{in } \Omega.
\end{aligned}$$

Using **(S5)** again,

$$\begin{aligned}
-\Delta_p \overline{u}_2 + a(x)|\overline{u}_2|^{p-2}\overline{u}_2 &= \lambda(f_2(M\|e\|_\infty)) \\
&\geq \lambda f_2(Me) \\
&\geq \lambda \left[f_2(\overline{u}_1) - \frac{1}{\overline{u}_2^{\alpha_2}}\right] \quad \text{in } \Omega.
\end{aligned}$$

Therefore, $(\overline{u}_1, \overline{u}_2)$ is a weak supersolution of (8.5). Since e satisfies (2.10), choosing M larger if necessary and using **(S5)**, we get $(\underline{u}_1, \underline{u}_2) \leq (\overline{u}_1, \overline{u}_2)$ componentwise. Then Theorem 8.1 guarantees the existence of a weak solution (u_1, u_2) of (8.5) satisfying $(\underline{u}_1, \underline{u}_2) \leqslant (u_1, u_2) \leqslant (\bar{u}_1, \bar{u}_2)$, as desired. □

Proof of Theorem 8.4: Define $\rho_s(\Omega) := 2^{\frac{p-1}{p-1+\tau}} \|e\|_\infty^{p-1} \left[\mu\left(\frac{p}{p-1+\alpha}\right)^{p-1} + \|a\|_\infty\right]$ and let $\hat{\sigma}, \sigma_1, \sigma_2$ and τ be as in **(S6)–(S8)** such that $\frac{\min\{\sigma_1, \sigma_2\}}{\hat{\sigma}^{\frac{p-1}{p-1+\tau}}} > \rho_s$. Let

$$\underline{\lambda} := \frac{(p/(p-1+\tau))^{p-1}\mu + \|a\|_\infty}{\min\{\sigma_1, \sigma_2\}} < \hat{\lambda} := \frac{1}{(2\hat{\sigma})^{\frac{p-1}{p-1+\tau}} \|e\|_\infty^{p-1}}$$

and $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. Let $(\overline{u}_1, \overline{u}_2) = \left(Me, [\lambda f_2(M\|e\|_\infty)]^{\frac{1}{p-1}} e\right)$, where, using **(S7)**, $M = M(\lambda) \gg 1$ is chosen such that

$$2\hat{\sigma} \geq \frac{f_1\left((f_2(M\|e\|_\infty))^{\frac{1}{p-1}}\right)}{(M\|e\|_\infty)^{p-1}}.$$

Then, using $\lambda \leq \hat{\lambda}$, **(S7)**, and **(S8)**, we obtain

$$\begin{aligned}-\Delta_p \overline{u}_1 + a(x)|\overline{u}_1|^{p-2}\overline{u}_1 &= M^{p-1} \\ &\geq \frac{f_1\left(\left[f_2\left(M\|e\|_\infty\right)\right]^{\frac{1}{p-1}}\right)}{\|e\|_\infty^{p-1} 2\hat{\sigma}} \\ &\geq \lambda\lambda^{\frac{\tau}{p-1}}\|e\|_\infty^{\tau} f_1\left(\left[f_2\left(M\|e\|_\infty\right)\right]^{\frac{1}{p-1}}\right) \\ &\geq \lambda f_1\left(\left[\lambda f_2\left(M\|e\|_\infty\right)\right]^{\frac{1}{p-1}}\|e\|_\infty\right) \geq \lambda f_1\left(\overline{u}_2\right) - \frac{1}{\overline{u}_1^{\alpha}} \quad \text{in } \Omega.\end{aligned}$$

It is easy to see that

$$-\Delta_p \overline{u}_2 + a(x)|\overline{u}_2|^{p-2}\overline{u}_2 = \lambda f_2\left(M\|e\|_\infty\right) \geq \lambda f_2\left(Me\right) \geq \lambda f_2\left(\overline{u}_1\right) - \frac{1}{\overline{u}_2^{\alpha}} \quad \text{in } \Omega,$$

and thus $(\overline{u}_1, \overline{u}_2)$ is a weak supersolution of (8.6).

To construct a weak subsolution pair, we let $k = \max\{k_1, k_2\}$ and $k_0 > 0$ be such that

$$\frac{1}{k_0^{p-1+\alpha}}\left(1 + \frac{k k_0^{\alpha} z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{(2\hat{\sigma})^{\frac{p-1}{p-1+\tau}}\|e\|_\infty^{p-1}}\right) \leq \min\left\{\frac{p^{p-1}(1-\alpha)(p-1)m_\mu^p}{(p-1+\alpha)^p}, \left(\frac{p}{p-1+\alpha}\right)^{p-1} A_\mu\right\},$$

where $m_\mu, A_\mu > 0$ are given by (2.12). Let $s^* = k_0\|z_\mu^{\frac{p}{p-1+\alpha}}\|_\infty$. Now let s_1 and s_2 in **(S6)** such that $\min\{s_1, s_2\} \geq s_0$. Then, as in the proof of Theorem 1.4, $(\underline{u}_1, \underline{u}_2) = \left(k_0 z_\mu^{\frac{p}{p-1+\alpha}}, k_0 z_\mu^{\frac{p}{p-1+\alpha}}\right)$ is a weak subsolution for (8.6). As before, by taking M larger, if necessary, $(\underline{u}_1, \underline{u}_2) \leq (\overline{u}_1, \overline{u}_2)$ componentwise. Then Theorem 8.1 yields the existence of a weak positive solution for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$, completing the proof. □

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