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Stability of oscillatory solutions of differential equations with a general piecewise constant argument.

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Abstract

We examine scalar differential equations with a general piecewise constant argument, in short DEPCAG, that is, the argument is a general step function. Criteria of existence of the oscillatory and nonoscillatory solutions of such equations are proposed. Necessary and sufficient conditions for stability of the zero solution are obtained. Appropriate examples are given to show our results.

Keywords: Piecewise constant argument; oscillation; stability of solutions, hybrid equations.

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1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integer, natural and real numbers, respectively.

We investigate the global asymptotic behavior as well as oscillation of the solution of differential equations with a general piecewise constant argument (DEPCAG):

$$y'(t) = a(t)y(t) + b(t)y(\gamma(t)), \quad y(\tau) = y_0, \tag{1.1}$$

where $a(t), b(t)$ are real-valued continuous functions of t defined on $[\tau, \infty)$. The deviation argument $\ell(t) = t - \gamma(t)$ is negative for $t_i < t < \gamma_i$ and positive for $\gamma_i < t < t_{i+1}$, $i \in \mathbb{Z}$. Therefore, equations (1.1) is of considerable interests: on each interval $[t_i,\ t_{i+1})$ it is of alternately advanced and retarded type. Eq.(1.1) are of advanced type on $I_i^+ = [t_i,\ \gamma_i]$ and retarded type on $I_i^- = (\gamma_i, t_{i+1})$.

Differential equations with piecewise constant argument (DEPCA) with argument deviation of fixed sign were the first to be investigated, see [2],[9],[11], [24],[28],[33],[36]. These equations are related to impulse and loaded equations and share the properties of certain models of vertically transmitted diseases, see [8]. The study of DEPCA of alternately of retarded and advanced type was initiated by A. R. Aftabizadeh and J. Wiener [1] in 1986, K. L. Cooke and J. Wiener [10] in 1987. They observed that the change of sign in the argument deviation led not only to interesting periodic properties but also to complications in the asymptotic and oscillatory behavior of solutions. It was then natural to try to study the oscillatory and the stability properties of DEPCA with a general deviation argument.

Criteria for the existence of oscillatory solutions of DEPCA have been derived by many authors [1]-[4],[6],[7],[10],[16]-[23],[25],[29]-[36]. It is therefore of interest to know what additional conditions are needed to yield stability of oscillatory solutions. While such questions have been dealt with in the area of differential equations. As an example, in [1], A. R. Aftabizadeh et al. established the following result: Let $a, b \in \mathbb{R}$ and $b \neq 0$ such that

$$a > 0 \quad \text{and} \quad \frac{-a(e^a+1)}{(e^{a/2}-1)^2} < b < \frac{-a}{e^{a/2}-1}e^{a/2},$$

$$a < 0 \quad \text{and} \quad b < \frac{-a}{e^{a/2}-1}e^{a/2} \quad \text{or} \quad b > \frac{-a(e^a+1)}{(e^{a/2}-1)^2}.$$

Then every oscillatory solution x of the following differential equation with piecewise constant argument

$$x'(t) = ax(t) + b[t + \tfrac{1}{2}], \quad x(0) = x_0, \tag{1.2}$$

tends to zero as $t \to \infty$.

To the best of our knowledge, there are some studies which are related to DEPCAG [5],[12]–[15],[26],[27], but does not have any results up to now to establish some simple criteria for the existence of oscillatory and nonoscillatory solutions of DEPCAG. The aim of this paper is to extend these classic results [1], [10] and [34] to DEPCAG (1.1).

For the reader's convenience we give some known definitions that are required later.

We understand a solution $y(t)$ of Eq.(1.1) as a continuous function on $[\tau, \infty)$ such that the derivative $y'(t)$ exists at each point $t \in [\tau, \infty)$, with the possible exception of the points $t_i,\ i \in \mathbb{Z}$ where one-sided derivative exists and Eq.(1.1) is satisfied by $y(t)$ on each interval (t_i, t_{i+1}) as well.

A function $y(t)$ defined on $[\tau, \infty)$ is said to be oscillatory if there exist two real valued sequences $(\nu_n)_{n\geq 0}$, $(\nu'_n)_{n\geq 0} \subset [\tau, \infty)$ such that $\nu_n \to \infty$, $\nu'_n \to \infty$ as $n \to \infty$ and $y(\nu_n) \leq 0 \leq y(\nu'_n)$ for $n \geq N$, where N is sufficiently large. Otherwise, the solution is called nonoscillatory.

A solution $\{x_n\}_{n\geq i(\tau)}$ of the difference equation is called oscillatory if the sequence $\{x_n\}_{n\geq i(\tau)}$ is neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

Our paper is organized in the following way: In the next section, criteria of existence of the oscillatory and nonoscillatory solutions of scalar differential equations with a general piecewise constant argument are established. In Section 3, the stability of the solutions of linear differential equations is treated. Furthermore, appropriate examples are provided in the last section.

2 Existence of the Oscillatory and Nonoscillatory solutions

In this section we establish sufficient conditions for the oscillatory and nonoscillatory solutions of scalar differential equations of alternately advanced and retarded type.

The following assumption will be needed throughout the paper:

(N) For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{Z}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$, $\lambda(\tau, \gamma_{i(\tau)}) \neq 0$, $\lambda\,(t_i, \gamma_i) \neq 0$ for all $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$, where

$$\lambda\,(t, s) := e^{\int_s^t a(\kappa)d\kappa} + \int_s^t e^{\int_u^t a(\kappa)d\kappa} b(u)du. \tag{2.1}$$

In the following theorem the conditions of existence and uniqueness of solutions on $[\tau, \infty)$ are established. The proof of the assertion is similar to that of Theorem 2.1 in [12].

Theorem 2.1 *Suppose that* ***(N)*** *holds. Then, Eq.(1.1) has a unique solution on* $[\tau, \infty)$ *with the initial condition* $y(\tau) = y_0$. *Moreover for* $t \in [t_n, t_{n+1})$, $n > i(\tau)$, y *has the form*

$$y(t) = \frac{\lambda(t, \gamma_n)}{\lambda(t_n, \gamma_n)} x_n \tag{2.2}$$

where $x_n = y(t_n)$ *and the sequence* $\{x_n\}_{n \geq i(\tau)}$ *is the unique solution of the difference equation*

$$x_{n+1} = \frac{\lambda(t_{n+1}, \gamma_n)}{\lambda(t_n, \gamma_n)} x_n, \tag{2.3}$$

for $n > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$.

Proof. Let $y_n(t)$ be a solution of equation (1.1) on the interval $t_n \leq t < t_{n+1}$. On this interval, we have

$$y_n'(t) = a(t)y(t) + b(t)y_n(\gamma_n).$$

The general solution of this equation on the given interval is

$$\begin{aligned} y_n(t) &= \left[e^{\int_{\gamma_n}^t a(\kappa)d\kappa} + \int_{\gamma_n}^t e^{\int_s^t a(\kappa)d\kappa} b(s)ds \right] y_n(\gamma_n) \\ &= \lambda(t, \gamma_n)\, y_n(\gamma_n). \end{aligned} \tag{2.4}$$

For $t = t_n$ and for $t \to t_{n+1}$ in (2.4), we have

$$y_n(\gamma_n) = \frac{y_n(t_n)}{\lambda(t_n, \gamma_n)} \quad \text{and} \quad y_n(t_{n+1}) = \lambda(t_{n+1}, \gamma_n)\, y_n(\gamma_n) \text{ for all } n > i(\tau). \tag{2.5}$$

Hence, replacing (2.5) in the previous relationship gives us:

$$y_n(t) = \left(\frac{\lambda(t, \gamma_n)}{\lambda(t_n, \gamma_n)} \right) y_n(t_n). \tag{2.6}$$

From (2.6), we obtain the difference equation (2.3). Considering the initial condition $x_{i(\tau)} = y(\tau) = y_0$, the solution of (2.3) can be obtained uniquely. So, the unique solution of (1.1) with the initial condition $y(\tau) = y_0$ is obtained as (2.2). ■

Note that in general, by recurrence relation, it is not difficult to see that the unique solution of Eq.(1.1) on $t \in [\tau, \infty)$ is given by

$$y(t) = y(\tau) \left(\frac{\lambda\left(t, \gamma_{i(t)}\right)}{\lambda\left(t_{i(t)}, \gamma_{i(t)}\right)} \right) \left(\prod_{j=i(\tau)+1}^{i(t)-1} \frac{\lambda(t_{j+1}, \gamma_j)}{\lambda(t_j, \gamma_j)} \right) \left(\frac{\lambda\left(t_{i(\tau)+1}, \gamma_{i(\tau)}\right)}{\lambda\left(\tau, \gamma_{i(\tau)}\right)} \right). \tag{2.7}$$

The next results are particular cases of Theorem 2.1.

Corollary 2.1 *Let* $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at} - 1)$, $\vartheta_i^+ = \gamma(t_i) - t_i$, $\vartheta_i^- = t_{i+1} - \gamma(t_i)$ *for all* $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$ *and assume that* $\hat{\lambda}\left(\tau - \gamma(t_{i(\tau)})\right) \neq 0$ *and* $\hat{\lambda}\left(-\vartheta_i^+\right) \neq 0$ *for all* $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. *For* $a(t) = a \neq 0$, $b(t) = b$ *constants, Eq.*(1.1) *has a unique solution* y *which is given by*

$$y(t) = \frac{\hat{\lambda}\left(t - \gamma_n\right)}{\hat{\lambda}(-\vartheta_n^+)} x_n, \quad t_n \leq t < t_{n+1} \tag{2.8}$$

where $x_n = y(t_n)$ *and the sequence* $\{x_n\}_{n\geq i(\tau)}$ *satisfies the difference equations*

$$x_{n+1} = \frac{\hat{\lambda}(\vartheta_n^-)}{\hat{\lambda}(-\vartheta_n^+)} x_n, \tag{2.9}$$

for $n > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$.

Corollary 2.2 *Let* $\beta(t) := \int_{\gamma(t)}^{t} b(s)ds$, $\beta_i^- := \int_{\gamma(t_i)}^{t_{i+1}} b(s)ds$, $\beta(\tau) \neq -1$ *and* $\beta(t_i) \neq -1$ *for all* $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. *Then* $u'(t) = b(t)u(\gamma(t))$ *with the initial condition* $u(\tau) = y_0$ *has a unique solution* u *which is given by*

$$u(t) = \frac{1 + \beta(t)}{1 + \beta(t_n)} u_n, \quad t_n \leq t < t_{n+1} \tag{2.10}$$

where $u_n = u(t_n)$ *and the sequence* $\{u_n\}_{n\geq i(\tau)}$ *satisfies the difference equations*

$$u_{n+1} = \frac{1 + \beta_n^-}{1 + \beta(t_n)} u_n, \tag{2.11}$$

for $n > i(\tau)$ *with the initial condition* $u_{i(\tau)} = y_0$.

The following theorem give some sufficient conditions for the existence of oscillatory and nonoscillatory solutions of Eq.(1.1).

Theorem 2.2 *Suppose that **(N)** holds and let* $y : [\tau, \infty) \to \mathbb{R}$ *be a solution of Eq.(1.1). Then*

a) If the solution $\{x_n\}_{n\geq i(\tau)}$ *of the difference equation (2.3) is oscillatory, then the solution* $y(t)$ *of Eq.(1.1) is also oscillatory.*

b) If the sequence $\{x_n\}_{n\geq i(\tau)}$ *is nonoscillatory, then* $y(t)$ *is nonoscillatory if and only if*

$$\int_t^{\gamma_i} b(s) e^{\int_s^{\gamma_i} a(\kappa)d\kappa} ds < 1 \tag{2.12}$$

holds true for $t_i \leq t < t_{i+1}$, $i \geq N$, *where* N *is sufficiently large.*

Proof. a) From (2.6), $y(t)$ can be written on the interval $t_n \le t < t_{n+1}$, $n \in \{i(\tau)+j\}_{j\in\mathbb{N}}$ as

$$y(t) = \left(\frac{\lambda\left(t,\gamma_n\right)}{\lambda\left(t_n,\gamma_n\right)}\right) x_n.$$

This implies $y(t) = y(t_n) = x_n$ for $t = t_n$. From the theory of the difference equations it is well known that x_n is oscillatory if and only if $x_n \cdot x_{n+1} \le 0$ for $n \ge N'$, where N' is a sufficiently large integer. Thus $y(t)$ is an oscillatory solution.

b) Now, let x_i be a nonoscillatory solution of the difference equation (2.3). According to this, we can assume that $x_i > 0$ for $i \ge N$, where N is large enough. If $y(t)$ is a nonoscillatory solution, then we can take $y(t) > 0$ for $t \ge T$ where T is sufficiently large. Hence, from (2.2), we have

$$y(t) = \frac{\lambda\left(t,\gamma_i\right)}{\lambda\left(t_i,\gamma_i\right)} x_i, \tag{2.13}$$

for $i \ge n$ where $n = max\{N,T\}$. Since $y(t) > 0$, we have

$$\frac{\lambda\left(t,\gamma_i\right)}{\lambda\left(t_i,\gamma_i\right)} > 0,$$

which implies (2.12). Now, let us assume that (2.12) is true. We should show that $y(t)$ is nonoscillatory. For a contradiction assume that $y(t)$ is an oscillatory solution. Therefore, there must exist two sequences (ν_n), (ν_n') such that $\nu_n \to \infty$, $\nu_n' \to \infty$ as $n \to \infty$ and $y(\nu_n) \le 0 \le y(\nu_n')$. Let $t_n < \nu_n < t_{n+1}$. It is clear that $\nu_n \to \infty$ as $n \to \infty$. So, from (2.2) we get

$$y(\nu_n) = \frac{\lambda\left(\nu_n,\gamma_{i(\nu_n)}\right)}{\lambda\left(t_{i(\nu_n)},\gamma_{i(\nu_n)}\right)} x_{i(\nu_n)}.$$

Since $y(\nu_n) \le 0$ and $x_{i(\nu_n)} = y(t_{i(\nu_n)}) > 0$, we have $\frac{\lambda\left(\nu_n,\gamma_{i(\nu_n)}\right)}{\lambda\left(t_{i(\nu_n)},\gamma_{i(\nu_n)}\right)} < 0$, which is a contradiction to (2.12). The proof is the same, if $x_i < 0$, for $i \ge N$. Hence the proof is completed. ■

By employing the similar technique as presented above, one can obtain the following results for the oscillation of Eq.(1.1).

Theorem 2.3 *Let $b(t)$ be locally integrable on $[\tau,\infty)$. Every solution of Eq.(1.1) is oscillatory if the sequence $\{\frac{\lambda(t_{n+1},\gamma_n)}{\lambda(t_n,\gamma_n)}\}_{n\ge i(\tau)}$ is not eventually positive.*

Proof. From (2.3), $\{x_n\}_{n\geq i(\tau)}$ can be written as

$$x_{n+1} = \left(\frac{\lambda(t_{n+1}, \gamma_n)}{\lambda(t_n, \gamma_n)} \right) x_n.$$

It is easy to see that the sequence $\{x_n\}_{n\geq i(\tau)}$ oscillates if $\{\frac{\lambda(t_{n+1},\gamma_n)}{\lambda(t_n,\gamma_n)}\}_{n\geq i(\tau)}$ is not eventually positive. Therefore, by Theorem 2.2 a), $y(t)$ oscillates if $\{x_n\}_{n\geq i(\tau)}$ oscillates. This completes the proof. ■

Theorem 2.4 *If either of the conditions*

$$\limsup_{n\to\infty} \int_{t_n}^{\gamma_n} b(s) e^{\int_s^{\gamma_n} a(\kappa)d\kappa} ds > 1, \tag{2.14}$$

$$\liminf_{n\to\infty} \int_{\gamma_n}^{t_{n+1}} b(s) e^{\int_s^{\gamma_n} a(\kappa)d\kappa} ds < -1 \tag{2.15}$$

holds true, then every solution of Eq.(1.1) is oscillatory.

Proof. Suppose that y is a solution of Eq.(1.1) such that $y(t) > 0$ (or $y(t) < 0$) for $t > t_j$, where $j \in \mathbb{N}$ is sufficiently large. If $t \in I_i$, $i > j$, then by (2.4) we have

$$y(t_i) = \left(e^{\int_{\gamma_i}^{t_i} a(\kappa)d\kappa} + \int_{\gamma_i}^{t_i} e^{\int_s^{t_i} a(\kappa)d\kappa} b(s)ds \right) y(\gamma_i) = \lambda(t_i, \gamma_i) y(\gamma_i).$$

Since $y(\gamma_i)$ and $y(t_i) > 0$, thus

$$0 < \lambda(t_i, \gamma_i) \quad \text{if and only if} \quad \int_{t_i}^{\gamma_i} e^{\int_s^{\gamma_i} a(\kappa)d\kappa} b(s)ds < 1,$$

or

$$\limsup_{i\to\infty} \int_{t_i}^{\gamma_i} e^{\int_s^{\gamma_i} a(\kappa)d\kappa} b(s)ds \leq 1,$$

which contradicts condition (2.14).
Similarly, with $t = t_{i+1}$ in (2.4), we get, after some simplifications and using the fact that $y(\gamma_i) > 0$ and that $y(t_{i+1}) > 0$,

$$\int_{\gamma_i}^{t_{i+1}} e^{\int_s^{\gamma_i} a(\kappa)d\kappa} b(s)ds > -1,$$

or

$$\liminf_{i\to\infty} \int_{\gamma_i}^{t_{i+1}} e^{\int_s^{\gamma_i} a(\kappa)d\kappa} b(s)ds \geq -1,$$

which contradicts (2.15). Thus, Eq.(1.1) has oscillatory solutions only. ■

Note that condition (2.14) or (2.15) is the classic hypothesis to verify the existence of oscillatory solutions for DEPCA. See [1], [10], [33] and [34].

In a similar way to Theorem 2.4 we obtain

Theorem 2.5 *If the conditions*

$$\limsup_{n\to\infty} \int_{t_n}^{\gamma_n} b(s) e^{\int_s^{\gamma_n} a(\kappa)d\kappa} ds < 1, \tag{2.16}$$

$$\liminf_{n\to\infty} \int_{\gamma_n}^{t_{n+1}} b(s) e^{\int_s^{\gamma_n} a(\kappa)d\kappa} ds > -1 \tag{2.17}$$

hold true, then the sequence $\{x_n\}_{n\geq i(\tau)}$ *of the difference equation (2.3) is nonoscillatory.*

Now, we establish some oscillation and nonoscillation results on DEPCAG with constant coefficients which will be deduced from the previous results. Let us consider the equation (1.1) with constant coefficients:

$$y'(t) = ay(t) + by(\gamma(t)), \quad y(\tau) = y_0, \tag{2.18}$$

where a, b are real constants.

Similar to Theorem 2.4 , we give the following result for Eq.(2.18).

Corollary 2.3 *If* $a \neq 0$ *each one of the conditions*

$$b > \limsup_{i\to\infty} \frac{a}{e^{a(\gamma_i - t_i)} - 1}, \quad b < -\liminf_{i\to\infty} \frac{ae^{a(t_{i+1}-\gamma_i)}}{e^{a(t_{i+1}-\gamma_i)} - 1} \tag{2.19}$$

implies that every solution of Eq.(2.18) is oscillatory.

Corollary 2.3 extends Theorem 2.3 of Aftabizadeh and Wiener [1] with $\gamma(t) = [t + \frac{1}{2}]$.

The following Corollary shows that (2.19) is "best posible" (sharp) condition.

Corollary 2.4 *If*

$$-\liminf_{i\to\infty} \frac{ae^{a(t_{i+1}-\gamma_i)}}{e^{a(t_{i+1}-\gamma_i)} - 1} < b < \limsup_{i\to\infty} \frac{a}{e^{a(\gamma_i - t_i)} - 1}, \tag{2.20}$$

then Eq.(2.18) has no oscillatory solution.

Proof. Condition (2.20) implies $\frac{\hat{\lambda}(\vartheta_n^-)}{\hat{\lambda}(-\vartheta_n^+)} > 0$ for all $n \geq i(\tau)$. So from (2.7) we deduce that the solution $y(t)$ of (2.18) is always of one sign on $[\tau, \infty)$. ■

Corollary 2.4 extends Theorem 2.4 of Aftabizadeh and Wiener [1] with $\gamma(t) = [t + \frac{1}{2}]$ and Theorem 3.2 of [34] with $\gamma(t) = m[\frac{t+k}{m}]$, $0 < k < m$.

3 Global asymptotic stability

Theorem 3.1 *Let $b(t)$ be locally integrable on $[\tau, \infty)$. The zero solution of Eq.(1.1) is global asymptotic stability as $t \to \infty$ if and only if*

$$\left| \frac{\lambda(t_{j+1}, \gamma_j)}{\lambda(t_j, \gamma_j)} \right| \leq \ell < 1 \tag{3.1}$$

for all $j > i(\tau)$.

Proof. Since $t \in [t_{i(t)}, t_{i(t)+1})$ and $\frac{\lambda(t,\gamma_{i(t)})}{\lambda(t_{i(t)},\gamma_{i(t)})}$ is continuous, the function $\frac{\lambda(t,\gamma_{i(t)})}{\lambda(t_{i(t)},\gamma_{i(t)})}$ is bounded for all t. The proof then follows easily from (2.2). ■

The next theorem gives necessary and sufficient conditions for the global asymptotic stability of zero solution of Eq.(2.18). To prove the last theorem we need the following assertion.
For $t_{j+1} - t_j \neq 2(\gamma_j - t_j)$, let

$$\varphi(a) := e^{a\cdot(t_{j+1}-t_j)} - 2e^{a\cdot(\gamma_j - t_j)} + 1, \tag{3.2}$$

if $\bar{a}$ is the nonzero solution of Eq.(3.2), we can check that $\frac{\varphi(a)}{a} > 0$ for $a > \bar{a}$ and $\frac{\varphi(a)}{a} < 0$ for $a < \bar{a}$.

Theorem 3.2 *Let $\bar{a}$ be the nonzero solution of Eq.(3.2) if $t_{j+1}-t_j \neq 2(\gamma_j - t_j)$, and $\bar{a} = 0$ if $t_{j+1} - t_j = 2(\gamma_j - t_j)$. The zero solution of Eq.(2.18) is global asymptotic stability as $t \to \infty$ if and only if any one of the following hypothesis is satisfied: for all $j > i(\tau)$,*

i) $a < \bar{a}, \quad a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}}+e^{at_j}} - 1\right)^{-1} < b \quad$ *or* $\quad b < -a;$

ii) $a > \bar{a}, \quad a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}}+e^{at_j}} - 1\right)^{-1} < b < -a;$

iii) $a = \bar{a}, \quad b < -a.$

Proof. If $a+b > 0$, then $\hat{\lambda}(t-\gamma_j)$ is increasing in I_j and assuming $\hat{\lambda}(t_j - \gamma_j) > 0$ leads to $\hat{\lambda}(t_{j+1}-\gamma_j) > \hat{\lambda}(t_j - \gamma_j)$, that is, $\frac{\hat{\lambda}(t_{j+1}-\gamma_j)}{\hat{\lambda}(t_j-\gamma_j)} > 1$. The conditions $a+b > 0$ and $\hat{\lambda}(t_j - \gamma_j) > 0$ can be written as

$$-a < b < \frac{a}{e^{a(\gamma_j - t_j)} - 1}.$$

In this case, the solution $y = 0$ is unstable. The case

$$a + b < 0, \quad \hat{\lambda}(t_j - \gamma_j) < 0$$

is impossible. Indeed, the inequalities

$$b < -a, \quad b > \frac{a}{e^{a(\gamma_j - t_j)} - 1}$$

are inconsistent because $-a < \frac{a}{e^{a(\gamma_j - t_j)} - 1}$. From

$$a + b > 0 \quad \text{and} \quad \hat{\lambda}(t_j - \gamma_j) < 0$$

it follows that

$$b > \frac{a}{e^{a(\gamma_j - t_j)} - 1} > 0. \tag{3.3}$$

The inequality $\frac{\hat{\lambda}(t_{j+1} - \gamma_j)}{\hat{\lambda}(t_j - \gamma_j)} < 1$ implies

$$e^{a(t_{j+1} - \gamma_j)} + \frac{b}{a}\left(e^{a(t_{j+1} - \gamma_j)} - 1\right) > e^{a(t_j - \gamma_j)} + \frac{b}{a}\left(e^{a(t_j - \gamma_j)} - 1\right)$$

which is equivalent to $a + b > 0$. On the other hand, $\frac{\hat{\lambda}(t_{j+1} - \gamma_j)}{\hat{\lambda}(t_j - \gamma_j)} > -1$ gives

$$e^{a(t_{j+1} - \gamma_j)} + \frac{b}{a}\left(e^{a(t_{j+1} - \gamma_j)} - 1\right) < -e^{a(t_j - \gamma_j)} - \frac{b}{a}\left(e^{a(t_j - \gamma_j)} - 1\right)$$

whence

$$1 < \frac{b}{a}\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}} + e^{at_j}} - 1\right) = -\frac{\varphi(a)}{a}\left(\frac{b}{e^{a(t_{j+1} - t_j)} + 1}\right).$$

If $a > \bar{a}$, we have $\frac{\varphi(a)}{a} > 0$, then

$$0 > -\frac{a}{\varphi(a)}\left(e^{a(t_{j+1} - t_j)} + 1\right) > b.$$

This contradicts (3.3). For $a < \bar{a}$, we have $\frac{\varphi(a)}{a} < 0$, then

$$-\frac{a}{\varphi(a)}\left(e^{a(t_{j+1} - t_j)} + 1\right) = a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}} + e^{at_j}} - 1\right)^{-1} < b$$

and since

$$\frac{a}{e^{a(\gamma_j - t_j)} - 1} < a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}} + e^{at_j}} - 1\right)^{-1},$$

hypothesis (i) ensures asymptotic stability of $y = 0$. Finally, the conditions $a + b < 0$ and $\hat{\lambda}(t_j - \gamma_j) > 0$ simply reduce to $b < -a$. The same result follows from the inequality $\hat{\lambda}(t_{j+1} - \gamma_j) < \hat{\lambda}(t_j - \gamma_j)$. Furthermore, from $\hat{\lambda}(t_{j+1} - \gamma_j) > -\hat{\lambda}(t_j - \gamma_j)$ we obtain

$$1 > -\frac{\varphi(a)}{a}\left(\frac{b}{e^{a(t_{j+1} - t_j)} + 1}\right).$$

For $a > \bar{a}$, this confirms hypothesis (ii). The case $a < \bar{a}$ again leads to $b < -a$. In the same way, if $a = \bar{a}$ we obtain condition (iii). ■

In view of the Theorem 3.2 and Corollary 2.3 we conclude that:

Corollary 3.1 *Let $\bar{a}$ be the nonzero solution of Eq.(3.2) if $t_{j+1}-t_j \neq 2(\gamma_j - t_j)$, and $\bar{a} = 0$ if $t_{j+1} - t_j = 2(\gamma_j - t_j)$. Then every oscillatory solution of Eq.(2.18) tends to zero if and only if any one of the following hypothesis is satisfied:*

i) $a < \bar{a}, \quad a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}}+e^{at_j}} - 1\right)^{-1} < b;$

ii) $a > \bar{a}, \quad a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}}+e^{at_j}} - 1\right)^{-1} < b < -\lim_{i\to\infty}\inf \frac{ae^{a(t_{i+1}-\gamma_i)}}{e^{a(t_{i+1}-\gamma_i)}-1}.$

4 Illustrative examples

We will introduce appropriate examples in this section. These examples will show the usefulness of our theory.

Consider the following scalar equations with a general piecewise constant argument:

Example 5.1. Let us consider the DEPCAG

$$y'(t) = (ln3)y(t) - 2y(\gamma(t)), \quad y(0) = y_0 \tag{4.1}$$

where $t_i = 3j$, $\gamma_j = 3j + 2$ for all $j \in \mathbb{N} \cup \{0\}$. Eq.(4.1) is a special case of Eq.(2.18) with $a = ln3$, $b = -2$. It is easy to see that $\hat{\lambda}\,(t_i - \gamma_i) = e^{-2} - \frac{2}{ln3}\,(e^{-2} - 1) \neq 0$ and $\bar{a} \approx 0.48121$ is the nonzero solution of the equation (3.2) with $a = ln3$, $t_{j+1} - \gamma_j = 1$ and $\gamma_j - t_j = 2$.

We calculate

$$-\frac{ae^{a(t_{j+1}-\gamma_j)}}{e^{a(t_{j+1}-\gamma_j)} - 1} = -\frac{3}{2}ln3 \approx -1.6479,$$

and

$$a\left(\frac{2e^{a\gamma_j}}{e^{at_{j+1}} + e^{at_j}} - 1\right)^{-1} = 3\left(\frac{2e^{\ln 3\cdot(3j+2)}}{e^{\ln 3\cdot 3(j+1)} + e^{\ln 3\cdot 3j}} - 1\right)^{-1} \approx -2.0102$$

for $j \geq i(0)$. In this case, the second hypotheses (2.19) of Corollary 2.3 holds. So, every solution of (4.1) is oscillatory. On the other hand, the hypotheses ii) of Theorem 3.2 is satisfied, we conclude that any solution of Eq.(4.1) goes to zero as $t \to \infty$ by oscillating.

Example 5.2. Let us consider the DEPCAG

$$y'(t) = -\frac{4}{e^2+1}y(t) + \frac{\sqrt{3}}{8}y\left(4\left[\frac{t+2}{4}\right]\right), \quad y(-2) = y_0 \tag{4.2}$$

where $t_i = 4j-2$, $\gamma_j = 4j$ for all $j \in \mathbb{N}\cup\{0\}$. It is easy to see that $\hat{\lambda}\,(t_i - \gamma_i) = \left(1 - \frac{\sqrt{3}(e^2+1)}{32}\right) e^{\frac{8}{e^2+1}} + \frac{\sqrt{3}(e^2+1)}{32} \neq 0$ and as $t_{j+1} - t_j = 2(\gamma_j - t_j) = 4$, we have $\bar{a} = 0$.
In this case,

$$-\frac{ae^{a(t_{j+1}-\gamma_j)}}{e^{a(t_{j+1}-\gamma_j)} - 1} = \frac{4}{e^2+1}\frac{e^{-\frac{8}{e^2+1}}}{e^{-\frac{8}{e^2+1}} - 1} \approx -0.29892,$$

and

$$\frac{a}{e^{a(\gamma_j - t_j)} - 1} = -\frac{4}{e^2+1}\frac{1}{e^{-\frac{8}{e^2+1}} - 1} \approx 0.77574$$

for $j \geq i(-2)$.

We can see that the hypotheses (2.20) of Corollary 2.4 holds. So, every solution of (4.2) is nonoscillatory. On the other hand, the hypotheses i) of Theorem 3.2 is satisfied, because, $b = \frac{\sqrt{3}}{8} < \frac{4}{e^2-1} = -a$. Then we conclude that zero solution of Eq.(4.2) is globally asymptotically stable.

Example 5.3. The solution of the DEPCAG

$$y'(t) = (2\pi + \cos t)y\left(2\pi\left[\frac{t+\pi}{2\pi}\right]\right), \; y(-\pi) = c_0, \tag{4.3}$$

is oscillatory, but zero solution is not global asymptotic stability.
Proof. According to (4.3), we have $\gamma(t) = 2\pi\left[\frac{t+\pi}{2\pi}\right]$, then $t_j = 2\pi j - \pi$, $\gamma_i = 2\pi j$, for all $j \in \mathbb{N}\cup\{0\}$. Replacing $b(t) = 2 + \cos t$ in (2.10), we have

$$\frac{1+\beta_j^-}{1+\beta\,(t_j)} = \frac{1+\int_{\gamma_j}^{t_{j+1}} b(s)ds}{1+\int_{\gamma_j}^{t_j} b(s)ds} = \frac{1+\int_{2\pi j}^{2\pi j+\pi}(2\pi+\cos s)\,ds}{1+\int_{2\pi j}^{2\pi j-\pi}(2\pi+\cos s)\,ds} = \frac{1+2\pi^2}{1-2\pi^2} < -1.$$

Then, $\{\frac{1+\beta_j^-}{1+\beta(t_j)}\}_{j\geq i(-\pi)}$ is not eventually positive. All assumptions of Theorem 2.3 are satisfied, then every solution of Eq.(4.3) is oscillatory. But the condition (3.1) is not fulfilled, so, due to Theorem 3.1, the zero solution of Eq.(4.3) is not global asymptotic stability. ∎

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